GGI School on Theoretical Aspects of Astroparticle Physics, Cosmology, and Gravitation 2023

Particle dark matter

Exercise sheet 2

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Problem 1: Including spin-statistical factors for 2-to-2 processes

Consider the collision operator for the number density from a process $12 \to 34$

$$\begin{aligned}\frac{C_{n,12\to 34}}{\kappa_{12\to 34}} = \int & \frac{\mathrm{d}^3p_1}{(2\pi)^3 2E_1}\frac{\mathrm{d}^3p_2}{(2\pi)^3 2E_2}\frac{\mathrm{d}^3p_3}{(2\pi)^3 2E_3}\frac{\mathrm{d}^3p_4}{(2\pi)^3 2E_4}(2\pi)^4\delta(\underline{p}_1+\underline{p}_2-\underline{p}_3-\underline{p}_4)|\mathcal{M}_{12\to 34}|^2 \\ & \times f_1 f_2 (1\pm f_3)(1\pm f_4)\,. \end{aligned} \tag{1}$$

The integrations here are over the whole $\mathbb{R}^3$ for all momenta; possible overcounting of physically identical regions is compensated with the symmetry factor $\kappa_{12\to 34}$. We aim to derive the most simplified expression for a general matrix element $|\mathcal{M}_{12\to 34}|^2$ and keeping the spin-statistical factors $1 \pm f_{3,4}$. Rotational invariance can be used to go to a coordinate system where

$$\mathbf{p}_1 = p_1(0,0,1)^T\ ,\ \ \mathbf{p}_2 = p_2(\sin\beta, 0, \cos\beta)^T\ ,\ \ \mathbf{p}_3 = p_3(\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)^T\ . \tag{2}$$

1. Rewrite the above integral in this coordinate system.
2. Remind yourself of the Mandelstam variables: In its most general form, on how many Lorentz-invariant variables does $|\mathcal{M}_{12\to 34}|^2$ depend? What are these variables in the above coordinate system?
3. Integrate over $\mathbf{p}_4$ and rewrite the remaining δ-distribution for integration over ϕ in the form $[(E_1+E_2-E_3)/(2p_2p_3\sin\beta\sin\theta)]\,\delta(\cos\phi - c_\phi)$.
4. Integrate over ϕ. As $-1 \le \cos\phi \le 1$, this integration restricts the remaining integrations, which can be dealt with by restricting the integration range over $\cos\theta$ and introducing a Heaviside-function for restricting the range in $\cos\beta$, i.e. s after variable transformation from $\cos\beta$ to s. Derive this restricted integration region.
   [Hint: Use $c_\phi^2 \le 1$ to find
   $$0 \ge \cos^2\theta + \frac{b}{a}\cos\theta + \frac{c}{a}\ , \tag{3}$$
   $$a = -4p_3^2[(E_1+E_2)^2 - s]\ , \tag{4}$$
   $$b = \frac{2p_3}{p_1}[s - 2E_1(E_1+E_2) + m_1^2 - m_2^2][s - 2E_3(E_1+E_2) + m_3^2 - m_4^2]\ , \tag{5}$$
   $$c = -[2E_3(E_1+E_2) - m_3^2 + m_4^2 - s]^2 - \frac{p_3^2}{p_1^2}(s - s_{12,-})(s - s_{12,+})\ , \tag{6}$$
   $$s_{12/34,\pm} = m_{1/3}^2 + m_{2/4}^2 + 2E_{1/3}E_{2/4} \pm 2p_{1/3}p_{2/4}\ . \tag{7}$$
   A useful relation is $b^2 - 4ac = 4(p_3^2/p_1^2)(s - s_{12,-})(s - s_{12,+})(s - s_{34,-})(s - s_{34,+})$. While it is a good exercise to proof the relations in this hint, you can also just use them for the tutorial.]
5. Perform a variable transformation to s, E_1, E_2, and E_3 and change integration regions such that integrations are over s, E_1, E_2, E_3, and $\cos\theta$ (from left to right, last to first integral).

Problem 2: Relic density calculations with DarkSUSY

You should all have a running version of DarkSUSY *at this point. In case of questions about how to use the code, remember to take a look at the tutorial at* ***https://darksusy.hepforge.org/tutorials/TOOLS_2021/DarkSUSY_getting_started.pdf*** *if you haven't done so yet.*

For concreteness, we will consider a simple toy model of DM composed of Dirac fermions ψ that couples to the standard model only via the 5D operator

$$\mathcal{L} \supset \Lambda^{-1} \bar{\psi}\psi \, |H|^2 \,, \tag{8}$$

where H is the SM Higgs doublet and Λ is a parameter with mass dimension one.

1. Compare this to the Scalar singlet model (see, e.g., Appendix D in the DarkSUSY paper) and convince yourself that the same type of diagrams are relevant in this case in order to describe DM annihilation. Concretely, how does Eq. (D.3) change? For the purpose of this exercise, you can assume that (D.5) receives the same overall modification (this is just an approximation, and with more time available you could of course compute the correction exactly).

2. Modify the ScalarSinglet model that ships with the code such that you can use it to compute annihilations for the model given in Eq. (8) instead; try to do this *without modifying the downloaded code*, i.e. by using the concept of replaceable functions (see the tutorial for details). Compare the value of Λ, as a function of m_ψ, that is necessary to obtain the correct relic density with the corresponding value of the coupling λ in the ScalarSinglet model.
[Hints: (i) Take a look at src_models/silveira_zee/an/dssigmavpartial.f. *(ii) Is the DM particle still self-conjugate in this model? How about the internal degrees of freedom? Try to identify where these changes need to be implemented.]*

3. Compute the value of $\langle \sigma v \rangle$ at freeze-out in the model above and compare it to *(i)* the corresponding value in the ScalarSinglet model and *(ii)* the corresponding value of the generic_wimp model. What do you notice?
[Hint: Take a look at examples/aux/ScalarSinglet_thermal_averages.f.*]*

4. You are now in a very good position to numerically explore further avenues of DM production that were discussed in the lecture. For example:

   - Compute the value of Λ, again as a function of m_ψ, that is necessary to obtain the correct relic density for *freeze-in* production!
   - Back to freeze-out calculations, it was mentioned in the lecture that the standard Boltzmann equation potentially breaks down close to resonances, due to *early kinetic decoupling*. In the model considered here, explore by how much the value of $\Omega_\psi h^2$ calculated in the standard approach can be off for $m_\psi \lesssim m_h/2$!
   - Now replace $|H|^2 \to \phi^2$ in Eq. (8), where ϕ is a scalar particle in an otherwise completely decoupled *dark sector*. Assume that $m_\phi \ll m_\psi$ and that the dark sector and the SM were in equilibrium for $T > T_c$ (e.g. through a $|H|^2 \phi^2$ term in $\mathcal{L}$), with $T_c \gg m_\psi$, but that $T_\chi \neq T$ for $T < T_c$. Compute the value of Λ necessary to provide the correct relic density in this case!

For all these problems, you can find simple demonstration programs in examples/aux *that will guide you in the correct direction. Note that most likely the time of the exercises will not suffice to attempt even one of the above suggestions – so they are really only meant as an indication of what you could try for further exploring the code. If at any point, even after the school, you have questions about this – please don't hesitate to get in touch!*