GGI School on Theoretical Aspects of Astroparticle Physics, Cosmology, and Gravitation 2023

Particle dark matter
Solution for exercise sheet 1

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Problem 1: The free Boltzmann equation

1. Plugging in gives

$$\frac{\partial f(t,p)}{\partial p} = p\,a(t)\frac{\mathrm{d}g(p\,a(t))}{\mathrm{d}(p\,a(t))}\ , \tag{1}$$

$$\frac{\partial f(t,p)}{\partial t} = p\,\dot{a}(t)\frac{\mathrm{d}g(p\,a(t))}{\mathrm{d}(p\,a(t))} = p\,H(t)\frac{\partial f(t,p)}{\partial p}\ . \tag{2}$$

2. The Fermi-Dirac or Bose-Einstein distribution function are given by

$$f(t,p) = \frac{1}{\exp([E-\mu]/T) \pm 1}\ , \tag{3}$$

where neither μ nor T are functions of p and may only be functions of t. As we have shown above, for these distributions to be a solution of the free Boltzmann equation, there must be a function g such that $f(t,p) = g(p\,a(t))$. This is equivalent to the existence of a function h such that

$$\frac{E-\mu}{T} = h(p\,a(t))\ . \tag{4}$$

(a) Ultra-relativistic limit:

$$E \simeq p \Rightarrow \frac{p}{T} - \frac{\mu}{T} = h(p\,a(t)) \tag{5}$$

This can only be fulfilled if

$$T \propto 1/a(t) \quad \text{and} \quad \mu \propto T \propto 1/a(t)\ . \tag{6}$$

Taking into account the next order in the expansion of E we find

$$E \simeq p + \frac{m^2}{2p} \Rightarrow \frac{p}{T} + \frac{m^2}{2pT} - \frac{\mu}{T} = h(p\,a(t))\ . \tag{7}$$

Here, it is not possible to find a t-dependence of T and μ such that the first two terms *only* depend on $p\,a(t)$ and otherwise have no p- or t-dependence.

(b) Non-relativistic limit:

$$E \simeq m + \frac{p^2}{2m} \Rightarrow \frac{p^2}{2mT} + \frac{m-\mu}{T} = h(p\,a(t)) \tag{8}$$

This can only be fulfilled if

$$T \propto 1/a^2(t) \quad \text{and} \quad m - \mu \propto T \propto 1/a^2(t)\ . \tag{9}$$

Taking into account the next order in the expansion of E we find

$$E \simeq m + \frac{p^2}{2m} - \frac{p^4}{8m^3} \Rightarrow \frac{p^2}{2mT} - \frac{p^4}{8m^3T} + \frac{m-\mu}{T} = h(p\,a(t)) \tag{10}$$

Again, it is not possible to find a t-dependence of T and μ such that the first two terms *only* depend on $p\,a(t)$ and otherwise have no p- or t-dependence.

Clearly, without finding a possible t-dependence of T and μ when going from ultra- to non-relativistic, the Fermi-Dirac and Bose-Einstein distribution functions are only solutions of the free Boltzmann equation in the fully ultra- or non-relativistic limit.

Problem 2: Boltzmann equation in equilibrium

This is a short solution of the exercise. If you are interested in more details you can take a look in Sec. 2.2.4 of my PhD thesis `Link`.

1. After division of

$$|\mathcal{M}_r|^2 \prod_{i\in\mathcal{I}_r} f_i \prod_{j\in\mathcal{F}_r} (1\pm f_j) - |\mathcal{M}_{r_{\text{inv}}}|^2 \prod_{j\in\mathcal{I}_{r_{\text{inv}}}} f_j \prod_{i\in\mathcal{F}_{r_{\text{inv}}}} (1\pm f_i) = 0 \,. \tag{11}$$

by $|\mathcal{M}_r|^2 = |\mathcal{M}_{r_{\text{inv}}}|^2$ and due to $\mathcal{I}_{r_{\text{inv}}} = \mathcal{F}_r$, $\mathcal{F}_{r_{\text{inv}}} = \mathcal{I}_r$ we have

$$\prod_{i\in\mathcal{I}_r} f_i \prod_{j\in\mathcal{F}_r} (1\pm f_j) = \prod_{j\in\mathcal{F}_r} f_j \prod_{i\in\mathcal{I}_r} (1\pm f_i) \tag{12}$$

and therefore

$$\sum_{i\in\mathcal{I}_r} \ln\left(\frac{f_i}{1\pm f_i}\right) = \sum_{j\in\mathcal{F}_r} \ln\left(\frac{f_j}{1\pm f_j}\right) \,. \tag{13}$$

2. There can be no dependence on the three-momentum as the phase-space distributions in equilibrium cannot depend on the direction of the three-momentum. Note that the absolute value of the three-momentum is *not* additively conserved, i.e. in general $\sum_{i\in\mathcal{I}_r} p_i \neq \sum_{k\in\mathcal{F}_r} p_j$.

3. The linear combination is given by

$$\ln\left(\frac{f_k}{1\pm f_k}\right) = \alpha_k - \beta_k E_k \tag{14}$$

for all $k \in (\mathcal{I}_r \cup \mathcal{F}_r)$ where the first term corresponds to the term from particle number and the second term is from energy.

4. Inversion of the linear combination gives

$$f_k = \frac{1}{\exp(-\alpha_k + \beta_k E_k) \mp 1} \tag{15}$$

such that we can identify the chemical potential μ_k and temperature T_k

$$\alpha_k = \mu_k \quad \text{and} \quad \beta_k = 1/T_k \,. \tag{16}$$

As Eq. (13) applies to all E_k, we find

$$T_k = T \;\forall\; k \in (\mathcal{I}_r \cup \mathcal{F}_r) \,, \tag{17}$$

$$\sum_{i\in\mathcal{I}_r} \mu_i = \sum_{j\in\mathcal{F}_r} \mu_j \,. \tag{18}$$

5. Here, we find

$$T_k = T \;\forall\; k \in (\mathcal{I}_r \cup \mathcal{F}_r) \,, \tag{19}$$

$$\sum_{i\in\mathcal{I}_r} \mu_i = \ln(1+\epsilon_r) + \sum_{j\in\mathcal{F}_r} \mu_j \,. \tag{20}$$

6. When going from an ultra- to a non-relativistic phase in equilibrium particles need to constantly re-distribute in phase-space to maintain detailed balance. This is typically ensured by having sufficiently large interaction rates. Note that in numerical calculations this can lead to problems as the Boltzmann equation becomes a stiff integro-differential equation.

Problem 3: Collision operator for the number density

1. The matrix element in the case of elastic scatterings is equal for the reaction and inverse reaction by crossing symmetry. The contribution of elastic scatterings to the collision operator can therefore be written as

$$\begin{aligned} C_n[f_\chi] =& \kappa \int \frac{\mathrm{d}^3 p_{\chi,1}}{(2\pi)^3 2E_{\chi,1}} \frac{\mathrm{d}^3 p_{\chi,2}}{(2\pi)^3 2E_{\chi,2}} \frac{\mathrm{d}^3 p_{\psi_1}}{(2\pi)^3 2E_{\psi_1}} \frac{\mathrm{d}^3 p_{\psi_2}}{(2\pi)^3 2E_{\psi_2}} \delta(\underline{p}_{\chi,1} + \underline{p}_{\psi_1} - \underline{p}_{\chi,2} - \underline{p}_{\psi_2}) \\ & \times (2\pi)^4 |\mathcal{M}|^2 (f_{\chi 1} f_{\psi_1} (1 \pm f_{\chi,2})(1 \pm f_{\psi,2}) - f_{\chi,2} f_{\psi,2} (1 \pm f_{\chi,1})(1 \pm f_{\psi_1})) \,. \end{aligned} \tag{21}$$

This evaluates to zero as the two terms in the brackets with phase-space distribution functions are equal in magnitude but opposite in sign after integration.

2. Assume that χ is self-conjugate and there are annihilation reactions $\chi\chi \leftrightarrow \psi_1\psi_2$ with $|\mathcal{M}_{\chi\chi\to\psi_1\psi_2}|^2 = |\mathcal{M}_{\psi_1\psi_2\to\chi\chi}|^2 = |\mathcal{M}|^2$ into particles ψ_1 and ψ_2, which are part of a heat bath with temperature T and vanishing chemical potential. Further assume that for all relevant times, elastic scatterings are efficient enforce detailed balance,[1] and that for $T \gtrsim m_\chi$ with m_χ the mass of χ, the annihilations are in equilibrium. Derive the Boltzmann equation for the number density for $m_\chi \gg T$ (an $m_\chi > \mathcal{O}(\text{few})T$ is typically enough) starting from

$$\begin{aligned} C_n[f_\chi] =& \kappa_{\psi_1\psi_2} \int \frac{\mathrm{d}^3 p_{\chi,1}}{(2\pi)^3 2E_{\chi,1}} \frac{\mathrm{d}^3 p_{\chi,2}}{(2\pi)^3 2E_{\chi,2}} \frac{\mathrm{d}^3 p_{\psi_1}}{(2\pi)^3 2E_{\psi_1}} \frac{\mathrm{d}^3 p_{\psi_2}}{(2\pi)^3 2E_{\psi_2}} \delta(\underline{p}_{\chi,1} + \underline{p}_{\chi,2} - \underline{p}_{\psi_1} - \underline{p}_{\psi_2}) \\ & \times (2\pi)^4 |\mathcal{M}|^2 (f_{\psi_1} f_{\psi_2} (1 \pm f_{\chi,1})(1 \pm f_{\chi,2}) - f_{\chi,1} f_{\chi,2} (1 \pm f_{\psi_1})(1 \pm f_{\psi_2})) \,, \end{aligned} \tag{22}$$

where the symmetry factor $\kappa_{\psi_1\psi_2}$ only takes into account if ψ_1 and ψ_2 are identical particles and all integrations are over the entire $\mathbb{R}^3$.

   (a) As long as detailed balance of the annihilations holds, one has $\mu_\chi = 0$ since ψ_1 and ψ_2 have vanishing chemical potential. Since we assume that this is true for $T \gtrsim m_\chi$, i.e. annihilations only fall out of equilibrium for $T \ll m_\chi$, it is clear that $(m_\chi - \mu_\chi)/T \gg 1$. Insertion into Bose-Einstein/Fermi-Dirac distribution functions gives

$$f_\chi = \frac{1}{\exp([E_\chi - \mu_\chi]/T) \pm 1} \simeq \exp(-[E_\chi - \mu_\chi]/T) \,. \tag{23}$$

   Note that this directly also implies $f_\chi \ll 1$. The number density is given by

$$n_\chi = g_\chi \int \frac{\mathrm{d}^3 p_\chi}{(2\pi)^3} f_\chi \simeq \mathrm{e}^{\mu_\chi/T} \frac{g_\chi}{2\pi^2} m_\chi^2 T K_2(m_\chi/T) \tag{24}$$

   with K_2 the modified Bessel function of second type and second order. With $n_{\chi,\mathrm{eq}} = \exp(-\mu_\chi/T) n_\chi$ the number density assuming zero chemical potential this gives

$$f_\chi \simeq \frac{n_\chi}{n_{\chi,\mathrm{eq}}} \mathrm{e}^{-E_\chi/T} \,. \tag{25}$$

   (b) As $E_{\psi,1} + E_{\psi,2} = E_{\chi,1} + E_{\chi,2} \geq 2m_\chi \gg T$ and $\mu_{\psi,1} = \mu_{\psi,2} = 0$, one has

$$f_{\psi,1/2} = \frac{1}{\exp(E_{\psi,1/2}/T) \pm 1} \simeq \exp(-E_{\psi,1/2}/T) \ll 1 \tag{26}$$

[1] This is called kinetic equilibrium.

inside the integral for C_n, i.e. for all $E_{\psi,1/2}$ that are kinematically accessible.[2] Since this gives

$$f_{\psi,1} f_{\psi,2} \simeq \exp(-[E_{\psi,1} + E_{\psi,2}]/T) = \exp(-[E_{\chi,1} + E_{\chi,2}]/T) \,, \tag{27}$$

we arrive at

$$\begin{aligned} f_{\psi_1} f_{\psi_2} (1 \pm f_{\chi,1})(1 \pm f_{\chi,2}) - f_{\chi,1} f_{\chi,2} (1 \pm f_{\psi_1})(1 \pm f_{\psi_2}) \\ \simeq \frac{n_{\chi,\mathrm{eq}}^2 - n_\chi^2}{n_{\chi,\mathrm{eq}}^2} \exp(-[E_{\chi,1} + E_{\chi,2}]/T) \,. \end{aligned} \tag{28}$$

(c) This is easily verified by entering the above approximations and the definition of the cross-section.

(d) We start from

$$\langle \sigma v \rangle = \frac{g_\chi^2}{n_{\chi,\mathrm{eq}}^2} \int \frac{\mathrm{d}^3 p_{\chi,1}}{(2\pi)^3} \frac{\mathrm{d}^3 p_{\chi,2}}{(2\pi)^3} \exp(-[E_{\chi,1} + E_{\chi,2}]/T) \sigma v \,. \tag{29}$$

and consider only the integral for now. We can choose coordinates such that $\mathbf{p}_{\chi,1} = p_{\chi,1}(0,0,1)$, $\mathbf{p}_{\chi,2} = p_{\chi,2}(\sin\theta, 0, \cos\theta)$ and therefore

$$\begin{aligned} &\int \frac{\mathrm{d}^3 p_{\chi,1}}{(2\pi)^3} \frac{\mathrm{d}^3 p_{\chi,2}}{(2\pi)^3} \exp(-[E_{\chi,1} + E_{\chi,2}]/T) \sigma v \\ &= \frac{1}{8\pi^4} \int_0^\infty \mathrm{d}p_{\chi,1} \int_0^\infty \mathrm{d}p_{\chi,2} \int_{-1}^1 \mathrm{d}\cos\theta \, p_{\chi,1}^2 p_{\chi,2}^2 \exp(-[E_{\chi,1} + E_{\chi,2}]/T) \sigma v \\ &= \frac{1}{8\pi^4} \int_{m_\chi}^\infty \mathrm{d}E_{\chi,1} \int_{m_\chi}^\infty \mathrm{d}E_{\chi,2} \int_{-1}^1 \mathrm{d}\cos\theta \, E_{\chi,1} p_{\chi,1} E_{\chi,2} p_{\chi,2} \exp(-[E_{\chi,1} + E_{\chi,2}]/T) \sigma v \,. \end{aligned} \tag{30}$$

Next, we change coordinates to $E_+ = E_{\chi,1} + E_{\chi,2}$, $E_- = E_{\chi,1} - E_{\chi,2}$, and $s = 2m_\chi^2 + 2E_{\chi,1}E_{\chi,2} - 2p_{\chi,1}p_{\chi,2}\cos\theta$. The integration measure transforms to (the "$-$"-sign switches integration direction such that s is increasing)

$$\mathrm{d}E_{\chi,1} \, \mathrm{d}E_{\chi,2} \, \mathrm{d}\cos\theta = -\frac{1}{4 p_{\chi,1} p_{\chi,2}} \mathrm{d}E_+ \, \mathrm{d}E_- \, \mathrm{d}s \,. \tag{31}$$

The integration region goes to $s \geq 4m_\chi^2$, $E_+ \geq \sqrt{s}$, and $|E_-| \leq \sqrt{1 - 4m_\chi^2/s}\sqrt{E_+^2 - s} = E_{-,\mathrm{max}}$. Hence,

$$\begin{aligned} &\int \frac{\mathrm{d}^3 p_{\chi,1}}{(2\pi)^3} \frac{\mathrm{d}^3 p_{\chi,2}}{(2\pi)^3} \exp(-[E_{\chi,1} + E_{\chi,2}]/T) \sigma v \\ &= \frac{1}{32\pi^4} \int_{4m_\chi^2}^\infty \mathrm{d}s \int_{\sqrt{s}}^\infty \mathrm{d}E_+ \int_{E_{-,\mathrm{max}}}^{E_{-,\mathrm{max}}} \mathrm{d}E_- \, E_{\chi,1} E_{\chi,2} \exp(-E_+/T) \sigma v \,. \end{aligned} \tag{32}$$

Note that by the definition of the Møller velocity $v E_{\chi,1} E_{\chi,2} = \sqrt{(\underline{p}_{\chi,1} \cdot \underline{p}_{\chi,2})^2 - m_\chi^4} = (1/2)\sqrt{s^2 - 4sm_\chi^2}$. Therefore, the integration over E_- becomes trivial

$$\begin{aligned} &\int \frac{\mathrm{d}^3 p_{\chi,1}}{(2\pi)^3} \frac{\mathrm{d}^3 p_{\chi,2}}{(2\pi)^3} \exp(-[E_{\chi,1} + E_{\chi,2}]/T) \sigma v \\ &= \frac{1}{32\pi^4} \int_{4m_\chi^2}^\infty \mathrm{d}s (s - 4m_\chi^2) \sigma \int_{\sqrt{s}}^\infty \mathrm{d}E_+ \sqrt{E_+^2 - s} \exp(-E_+/T) \,. \end{aligned} \tag{33}$$

[2] Note that momentum conservation typically does not allow for $E_{\psi,1} \ll E_{\psi,2}$ (or vice versa) unless going to very large $E_{\chi,1} + E_{\chi,2}$, which then comes with a very large (exponential) suppression from the phase-space distribution functions.

The integral over E_+ can be expressed by the modified Bessel function K_1 of second kind and first order

$$\int \frac{d^3p_{\chi,1}}{(2\pi)^3}\frac{d^3p_{\chi,2}}{(2\pi)^3}\exp(-[E_{\chi,1}+E_{\chi,2}]/T)\sigma v$$
$$= \frac{T}{32\pi^4}\int_{4m_\chi^2}^{\infty} ds(s-4m_\chi^2)\sqrt{s}K_1(\sqrt{s}/T)\sigma \; . \quad (34)$$

Inserting the expression for $n_{\chi,\mathrm{eq}}$ from above we find

$$\langle\sigma v\rangle = \frac{1}{8m_\chi^4 T K_2(m_\chi/T)^2}\int_{4m_\chi^2}^{\infty} ds(s-4m_\chi^2)\sqrt{s}K_1(\sqrt{s}/T)\sigma \quad (35)$$
$$= \frac{4x}{K_2^2(x)}\int_1^{\infty} d\tilde{s}\,(\tilde{s}-1)\sqrt{\tilde{s}}K_1(2\sqrt{\tilde{s}}x)\sigma \; , \quad (36)$$

where K_i the modified Bessel function of second kind and order i, $\tilde{s} = s/(4m_\chi^2)$ and $x = m_\chi/T$.

3. In the above case, the symmetry factor $\kappa_{\psi_1\psi_2}$ does not include the DM particle as these would give a factor 1/2, which is cancelled by the fact that the reaction must be counted *twice*, since each annihilation removes two particles χ. The calculation can then be carried out as above,[3] and with the corresponding cross-section one finds

$$\dot{n}_\chi + 3Hn_\chi \simeq \langle\sigma v\rangle(n_{\chi,\mathrm{eq}}n_{\bar{\chi},\mathrm{eq}} - n_\chi n_{\bar{\chi}}) \; . \quad (37)$$

Assuming that there is no asymmetry such that $n_\chi = n_{\bar{\chi}}$, one has for the total number density $n_{\chi,\mathrm{tot}} = n_\chi + n_{\bar{\chi}} = 2n_\chi$

$$\dot{n}_{\chi,\mathrm{tot}} + 3Hn_{\chi,\mathrm{tot}} = \frac{\langle\sigma v\rangle}{2}(n_{\chi,\mathrm{eq,tot}}^2 - n_{\chi,\mathrm{tot}}^2) \; . \quad (38)$$

We can directly see that the value of $\langle\sigma v\rangle$ needed to obtain the observed dark matter relic density for a non-self-conjugate particle is around twice as large as the corresponding value for a self-conjugate particle, as expected from $\Omega_\chi \propto 1/\langle\sigma v\rangle$ needs to be half as large. Note that this factor is not exact as there is a logarithmic dependence of the freeze-out temperature on $\langle\sigma v\rangle$ and this also enters into Ω_χ.

Preparation for the second exercise session

2. The program oh2_generic_wimp assumes a constant value of σv and calculates the required value to obtain the observed DM relic density for different final states $\nu_e\bar{\nu}_e$, $\tau^-\tau^+$, $t\bar{t}$, and W^-W^+. Finding the value of σv is done in the function findsv in this program. The model initialization is done with the function dsgivemodel_generic_wimp for self-conjugate DM (assumed in this program, cf. line 46) and dsgivemodel_generic_wimp_aDM for non-self-conjugate DM, followed by a call to dsmodelsetup. Relic density calculations are performed with the function dsrdomega. The plot of σv (equalling $\langle\sigma v\rangle$ in this model) can be found in the left panel of Fig. 1.

3. The example program oh2_ScalarSinglet assumes that DM is a real scalar and calculates the value of the coupling λ between the DM particle S and the SM Higgs, for $\mathcal{L} \supset -(\lambda/2)S^2|H|^2$, required to obtain the observed DM relic density (as well as several other quantities). Here, finding the value of λ is done directly in the main program. Model initialization is performed with dsgivemodel_silveira_zee, as usual followed by a call to dsmodelsetup, and relic density calculations are, as for any model, done with dsrdomega. The plot of λ can be found in the right panel of Fig. 1.

[3] As long as there is no *extremely* large asymmetry such that $\mu_\chi \gg \mu_{\bar{\chi}}$ or vice versa such that the approximation of Maxwell-Boltzmann distributions is not valid anymore.

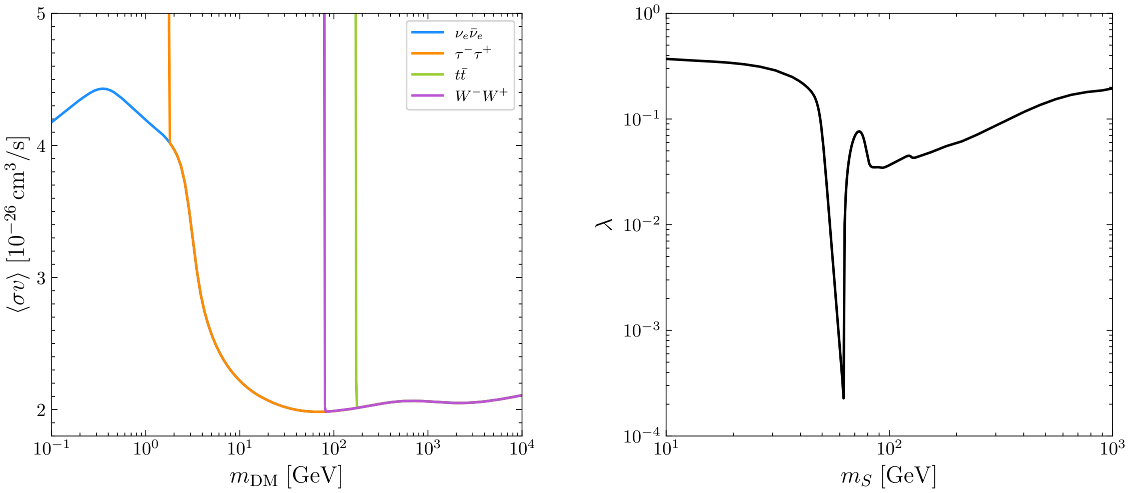

Figure 1: *Left:* $\langle\sigma v\rangle = \sigma v = const.$ required to obtain the observed DM relic density for different DM mass m_{DM} and final states. *Right:* λ required to obtain the observed DM relic density in the real scalar singlet model for different DM mass m_S.