GGI School on Theoretical Aspects of Astroparticle Physics, Cosmology, and Gravitation 2023

Particle dark matter

Exercise sheet 1

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Discussed on: Monday, March 20, 2023

Problem 1: The free Boltzmann equation

1. Show that the free Boltzmann equation $\partial f/\partial t - Hp\,\partial f/\partial p = 0$ is solved by $f(t,p) = g(p\,a(t))$, where a is the scale factor and g is an arbitrary function.

2. In which cases is the Fermi-Dirac or Bose-Einstein distribution function a solution of the free Boltzmann equation?
[Hint: For these functions to be a solution, there must exist a function h such that $(E-\mu)/T = h(p\,a(t))$, where μ and T are functions of t, but not of p. Consider the ultra-relativistic and the non-relativistic limits with the first order including p. What happens when you include higher-order terms?]

Problem 2: Boltzmann equation in equilibrium

Consider the sum of collision operators from a reaction r and inverse reaction r_{inv} with the set of initial-state particles $\mathcal{I}_r$ and final-state particle $\mathcal{F}_r$ ($\mathcal{I}_{r_{\text{inv}}} = \mathcal{F}_r$, $\mathcal{F}_{r_{\text{inv}}} = \mathcal{I}_r$)[1] for the phase-space distribution f_x of a particle x with $x \in \mathcal{F}_r$

$$\hat{C}_r[f_x] + \hat{C}_{r_{\text{inv}}}[f_x] = \frac{\kappa_r}{2g_x} \int \prod_{k\in(\mathcal{I}_r\cup\mathcal{F}_r)\setminus\{x\}} \frac{\mathrm{d}^3p_k}{(2\pi)^3 2E_k} (2\pi)^4 \delta\left(\sum_{i\in\mathcal{I}_r} \underline{p}_i - \sum_{j\in\mathcal{F}_r} \underline{p}_j\right) \times \left(|\mathcal{M}_r|^2 \prod_{i\in\mathcal{I}_r} f_i \prod_{j\in\mathcal{F}_r}(1\pm f_j) - |\mathcal{M}_{r_{\text{inv}}}|^2 \prod_{j\in\mathcal{I}_{r_{\text{inv}}}} f_j \prod_{i\in\mathcal{F}_{r_{\text{inv}}}}(1\pm f_i)\right) . \quad (1)$$

We assume that the particles are in equilibrium such that detailed balance holds, i.e. the integrand vanishes pointwise via

$$|\mathcal{M}_r|^2 \prod_{i\in\mathcal{I}_r} f_i \prod_{j\in\mathcal{F}_r}(1\pm f_j) - |\mathcal{M}_{r_{\text{inv}}}|^2 \prod_{j\in\mathcal{I}_{r_{\text{inv}}}} f_j \prod_{i\in\mathcal{F}_{r_{\text{inv}}}}(1\pm f_i) = 0 . \quad (2)$$

1. Assuming CP-invariance $|\mathcal{M}_r|^2 = |\mathcal{M}_{r_{\text{inv}}}|^2$ show that there is an additive conservation law

$$\sum_{i\in\mathcal{I}_r} \ln\left(\frac{f_i}{1\pm f_i}\right) = \sum_{j\in\mathcal{F}_r} \ln\left(\frac{f_j}{1\pm f_j}\right) . \quad (3)$$

2. Each summand here can only be a linear combination of additively conserved quantities. From Eq. (1) these are energy, three-momentum, and (up to prefactors) particle number. Why can there be no dependence on the three-momentum?

3. Write down the linear combination for all $k \in (\mathcal{I}_r \cup \mathcal{F}_r)$. What does this imply for f_k?

[1] κ_r is a symmetry factor taking into account that the momentum integration should only go over *physically distinct* regions, i.e. regions differing by the exchange of identical particles are only counted once. With this factor the integration can go over all momenta and we only need to calculate it for a given reaction, e.g. $\kappa_r = 1/2$ if there are two identical particles in the initial or final state.

4. Use Eq. (3) with the linear combination from 3. to derive relations for the temperature and the chemical potentials.
5. Parametrizing CP-violation as $|\mathcal{M}_{r_{\rm inv}}|^2 = (1+\epsilon)|\mathcal{M}_r|^2$ with a constant real ϵ how do these relations for the temperature and chemical potentials change?
6. You showed in problem 1 that the Fermi-Dirac and Bose-Einstein distribution functions are only solutions of the free Boltzmann equation in the ultra- and non-relativistic case. Still, detailed balance seems to imply that for these distributions the collision operator vanishes. What needs to happen such that the particles obey these distributions when going from an ultra- to a non-relativistic phase in equilibrium?

Problem 3: Collision operator for the number density

The Boltzmann equation for the number density of a particle χ can be obtained from the one for the phase-space density by integration over all momenta and reads[2]

$$\dot{n}_\chi + 3Hn_\chi = g_\chi \int \frac{\mathrm{d}^3 p_\chi}{(2\pi)^3} \frac{\hat{C}[f_\chi]}{E_\chi} = C_n[f_\chi] \,, \tag{4}$$

1. Show that elastic scatterings of the form $\chi + \psi \to \psi + \chi$ for some other particle ψ do not contribute to $C_n[f_\chi]$.
2. Assume that χ is self-conjugate and there are annihilation reactions $\chi\chi \leftrightarrow \psi_1\psi_2$ with $|\mathcal{M}_{\chi\chi\to\psi_1\psi_2}|^2 = |\mathcal{M}_{\psi_1\psi_2\to\chi\chi}|^2 = |\mathcal{M}|^2$ into particles ψ_1 and ψ_2, which are part of a heat bath with temperature T and vanishing chemical potential. Further assume that for all relevant times, elastic scatterings are efficient enforce detailed balance,[3] and that for $T \gtrsim m_\chi$ with m_χ the mass of χ, the annihilations are in equilibrium. Derive the Boltzmann equation for the number density for $m_\chi \gg T$ (an $m_\chi > \mathcal{O}(\text{few})T$ is typically enough) starting from

$$\begin{aligned} C_n[f_\chi] =&\kappa_{\psi_1\psi_2} \int \frac{\mathrm{d}^3 p_{\chi,1}}{(2\pi)^3 2E_{\chi,1}} \frac{\mathrm{d}^3 p_{\chi,2}}{(2\pi)^3 2E_{\chi,2}} \frac{\mathrm{d}^3 p_{\psi_1}}{(2\pi)^3 2E_{\psi_1}} \frac{\mathrm{d}^3 p_{\psi_2}}{(2\pi)^3 2E_{\psi_2}} \delta(\underline{p}_{\chi,1} + \underline{p}_{\chi,2} - \underline{p}_{\psi_1} - \underline{p}_{\psi_2}) \\ &\times (2\pi)^4 |\mathcal{M}|^2 (f_{\psi_1} f_{\psi_2} (1 \pm f_{\chi,1})(1 \pm f_{\chi,2}) - f_{\chi,1} f_{\chi,2} (1 \pm f_{\psi_1})(1 \pm f_{\psi_2})) \,, \end{aligned} \tag{5}$$

where the symmetry factor $\kappa_{\psi_1\psi_2}$ only takes into account if ψ_1 and ψ_2 are identical particles and all integrations are over the entire $\mathbb{R}^3$.

   (a) Convince yourself that $(m_\chi - \mu_\chi)/T \gg 1$ and therefore χ follows a Maxwell-Boltzmann distribution, which can be written as $f_\chi \simeq n_\chi / n_{\chi,\rm eq} \exp(-E_\chi/T)$ and satisfies $f_\chi \ll 1$, where $n_{\chi,\rm eq}$ is the number density of χ for vanishing chemical potential μ_χ.

   (b) Why can you also approximate $f_{\psi,1/2} \simeq \exp(-E_{\psi,1/2}/T)$? Show that in the above integral

$$\begin{aligned} &f_{\psi_1} f_{\psi_2} (1 \pm f_{\chi,1})(1 \pm f_{\chi,2}) - f_{\chi,1} f_{\chi,2} (1 \pm f_{\psi_1})(1 \pm f_{\psi_2}) \\ &\qquad \simeq \frac{n_{\chi,\rm eq}^2 - n_\chi^2}{n_{\chi,\rm eq}^2} \exp(-[E_{\chi,1} + E_{\chi,2}]/T) \,. \end{aligned} \tag{6}$$

   (c) Use the definition of the cross-section

$$4 g_\chi^2 \sigma v E_{\chi,1} E_{\chi,2} = \int \frac{\mathrm{d}^3 p_{\psi_1}}{(2\pi)^3 2E_{\psi_1}} \frac{\mathrm{d}^3 p_{\psi_2}}{(2\pi)^3 2E_{\psi_2}} (2\pi)^4 \delta(\underline{p}_{\chi,1} + \underline{p}_{\chi,2} - \underline{p}_{\psi_1} - \underline{p}_{\psi_2}) |\mathcal{M}|^2 \tag{7}$$

[2] The collision operator $\hat{C}[f_\chi]$ can be obtained from Eq. (1) by summing over all possible reactions, weighting the terms for the reaction (inverse reaction) with the times χ appears in the final (initial) state.

[3] This is called kinetic equilibrium.

with the Møller velocity $v = \sqrt{(\underline{p}_{\chi,1} \cdot \underline{p}_{\chi,2})^2 - m_\chi^4}/(E_{\chi,1}E_{\chi,2})$ to arrive at

$$\dot{n}_\chi + 3Hn_\chi \simeq \langle \sigma v \rangle (n_{\chi,\mathrm{eq}}^2 - n_\chi^2) \, , \tag{8}$$

where

$$\langle \sigma v \rangle = \frac{g_\chi^2}{n_{\chi,\mathrm{eq}}^2} \int \frac{\mathrm{d}^3 p_{\chi,1}}{(2\pi)^3} \frac{\mathrm{d}^3 p_{\chi,2}}{(2\pi)^3} \exp(-[E_{\chi,1} + E_{\chi,2}]/T) \sigma v \, . \tag{9}$$

(d) Show that

$$\langle \sigma v \rangle = \frac{4x}{K_2^2(x)} \int_1^\infty \mathrm{d}\tilde{s} \, (\tilde{s} - 1) \sqrt{\tilde{s}} K_1(2\sqrt{\tilde{s}} x) \sigma \, , \tag{10}$$

where K_i the modified Bessel function of second kind and order i, $\tilde{s} = s/(4m_\chi^2)$ and $x = m_\chi/T$. *[Hint: First choose a coordinate system such that the integrations in Eq. (9) are over $E_{\chi,1}$, $E_{\chi,2}$, and $\cos\theta$, where θ is the angle between $\mathbf{p}_{\chi,1}$ and $\mathbf{p}_{\chi,2}$. Then make a coordinate transformation to $E_+ = E_{\chi,1} + E_{\chi,2}$, $E_- = E_{\chi,1} - E_{\chi,2}$, and $s = 2m_\chi^2 + 2E_{\chi,1}E_{\chi,2} - 2p_{\chi,1}p_{\chi,2}\cos\theta$. The integration region becomes $s \geq 4m_\chi^2$, $E_+ \geq \sqrt{s}$, and $|E_-| \leq \sqrt{1 - 4m_\chi^2/s}\sqrt{E_+^2 - s}$. While it is instructive to show this, you can just use this result for the purpose of this exercise.]*

3. What changes in the last exercise if χ is not self-conjugate, but instead annihilates with a particle $\bar{\chi}$ in a reaction $\bar{\chi}\chi \leftrightarrow \psi_1\psi_2$?
   [Hint: Why does the symmetry factor in Eq. (5) miss a factor 1/2 from χ being self-conjugate?]

Preparation for the second exercise session

DarkSUSY is a widely used numerical package to calculate all kinds of dark matter observables, and during the next exercise session we want to explore some of its functionalities for relic density calculations. For this to work in practice, we need you to come prepared – so please do the following **before** Wednesday:

1. Go to `https://www.darksusy.org`, download the most recent version (6.4.0) and follow the instructions on the webpage to install it.

2. Go to /examples/aux, and compile and run the example program oh2_generic_wimp. Plot the output in the $\langle \sigma v \rangle$ vs m_{DM} plane, and interpret it. Try to identify the DarkSUSY functions that *i)* initialize a model with a given set of model parameters and *ii)* calculate the relic density for this parameter point.

3. Do the same with the example program oh2_ScalarSinglet.

4. If time allows, also take a glance at the tutorial at `https://darksusy.hepforge.org/tutorials/TOOLS_2021/DarkSUSY_getting_started.pdf` in order to familiarize yourself with some of the key principles of how the code is organized.

If you encounter problems at any of the steps above, please let us know ASAP for support – either by email or directly on site. Note that for the purpose of the exercises on Wednesday, it will be sufficient to install the 'light' version of the code.