toy model example [2208.09229]

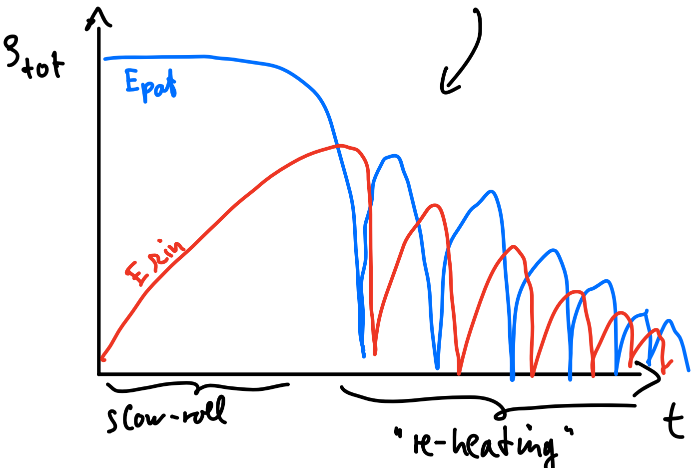
$$S = \int d^4x \sqrt{|g|} \left\{ \frac{M_{Pl}^2}{2} R + \mathcal{L}_\phi + \mathcal{L}_\chi \right\}$$

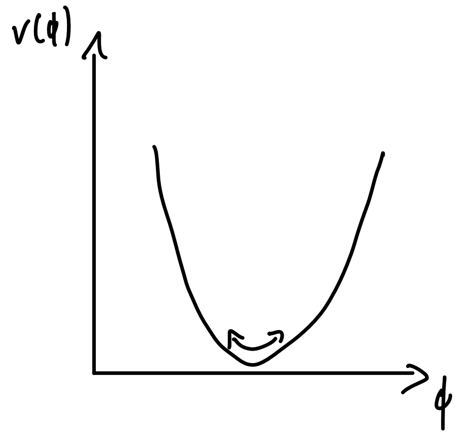
inflaton; $\rho \sim \rho_\phi$

$\frac{1}{2} g^{\mu\nu}(\partial_\mu \phi)(\partial_\nu \phi) - V(\phi)$

$\mathcal{L}_\chi = \frac{1}{2} g^{\mu\nu}(\partial_\mu \chi)(\partial_\nu \chi) - \frac{1}{2} m_\chi^2 \chi^2$

$\rho_\chi \ll \rho_\phi$





$\rightsquigarrow$ rapid oscillations, essential for production of χ

$d\eta \equiv a^{-1}(t)\, dt$ "conformal time"

$\Rightarrow g_{\mu\nu} = a(\eta)\, \eta_{\mu\nu}$

$\tilde{\chi} \equiv a^{-1} \chi$

$$\Rightarrow \sqrt{|g|}\, \mathcal{L}_\chi = \frac{1}{2}(\tilde{\chi}')^2 - \frac{1}{2} \tilde{\chi}\, \omega^2 \tilde{\chi} \quad ; \quad \omega^2 = \underbrace{-\nabla^2}_{\to +k^2} + a^2 m_\chi^2 - \underbrace{\frac{a''}{a}}_{>0} > 0$$

$$\pi = \frac{\partial \mathcal{L}_\chi}{\partial \tilde{\chi}'} = \tilde{\chi}' \Rightarrow H = \int d^3x \left\{ \tilde{\chi}' \tilde{\pi} - \mathcal{L}_\chi \right\}$$

$$= \int d^3x \left\{ \frac{1}{2} \tilde{\pi}^2 + \frac{1}{2} \tilde{\chi}\, \omega^2 \tilde{\chi} \right\}$$

$\omega = \omega(\eta) \Rightarrow H = H(\eta) \Rightarrow$ particle production!

$\hat{\chi}(x) = \int \frac{d^3k}{(2\pi)^3} e^{i\vec{k}\vec{x}} \tilde{\chi}_k(t)$; $\tilde{\chi}_k = f_k a_{\vec{k}} + f_k^* a^\dagger_{-\vec{k}}$ $\quad$ $f \hat{=} u$ above

$\hat{\pi}(x) = \int \frac{d^3k}{(2\pi)^3} e^{i\vec{k}\vec{x}} \tilde{\pi}_k(t)$; $\tilde{\pi}_k = g_k a_{\vec{k}} + g_k^* a^\dagger_{-\vec{k}}$ $\quad$ $g \,\hat{\neq}\, v$ above!

ansatz : $f_k = \frac{\tilde{\alpha}_k(t)}{\sqrt{2\omega_k}} + \frac{\tilde{\beta}_k(t)}{\sqrt{2\omega_k}}$ $\quad$ [why ?

$\rightsquigarrow$ correct flat space normalization

$g_k = -i\sqrt{\frac{\omega_k}{2}}\,\tilde{\alpha}_k + i\sqrt{\frac{\omega_k}{2}}\,\tilde{\beta}_k$ $\quad$ for $\tilde{\alpha}_{\vec{k}} = e^{-i\omega_k \eta}$; $\tilde{\beta}_{\vec{k}} = 0$]

two conditions:

i) $\tilde{\pi}_k = \tilde{\chi}_k'$

ii) $\tilde{\pi}_k' = -\frac{\partial \pi}{\partial \tilde{\chi}_k} = \omega_k^2 \tilde{\chi}_k$

[e.o.m.]

$$\Rightarrow \boxed{\begin{aligned} \tilde{\alpha}_k' &= -i\omega_k \tilde{\alpha}_k + \frac{\omega_k'}{2\omega_k}\tilde{\beta}_k \\ \tilde{\beta}_k' &= i\omega_k \tilde{\beta}_k + \frac{\omega_k'}{2\omega_k}\tilde{\alpha}_k \end{aligned}}$$

(fully general, no assumptions)

initial condition: $\tilde{\alpha}_k(\eta=0) = 1$

$\tilde{\beta}_k(\eta=0) = 0$

- $\omega_k' \simeq 0 \quad \Rightarrow \quad \tilde{\alpha}_k = e^{-i\omega_k \eta}$, $\tilde{\beta}_k = 0$ ✓

(ordinary plane waves)

$$\rightsquigarrow (\tilde{\alpha}, \tilde{\beta}) \equiv \underbrace{e^{-i\omega_k \eta}}_{u_k \cdot \sqrt{2\omega_k}} (\alpha, \beta)$$

in notation of intro

- $|\tilde{\beta}_k|^2 \overset{!}{\ll} 1$ $\quad \Rightarrow \quad \boxed{\tilde{\beta}_k \simeq \int_0^\eta d\eta' \, \frac{\omega_k'}{2\omega_k} \, e^{-2i\int_0^{\eta'} d\eta'' \omega_k(\eta'')}}$

(occupation number remains small)

$\omega_k \in a, H$; use $\phi(t) = \Phi \sin(m_\phi t)$

WKB ansatz, stationary phase ...

$\frac{k}{a} \gtrsim H_e \sim m_\phi$: $|\beta_k|^2 \propto k^{-9/2}$ ⌈fully analytical result, including prefactors⌋

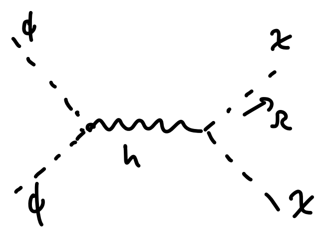
high energy / sub-horizon @ end of inflation

$\frac{k}{a} \lesssim H_e \sim m_\phi$: $|\beta_k|^2 \propto \begin{cases} k^{-6} & \text{for } m_\chi = 0 \\ k^{-3} & \text{for } m_\chi \sim 0.1\, m_\phi \end{cases}$

So... what does "really" produce χ ?

$$g_{\mu\nu} \simeq \eta_{\mu\nu} + \frac{2 h_{\mu\nu}}{M_{pl}} \rightsquigarrow S \simeq \int d^4x \left\{ \mathcal{L}_{h,kin} + \overbrace{\mathcal{L}_\phi + \mathcal{L}_\chi}^{\text{Minkowski!}} - \frac{1}{M_{pl}} h^{\mu\nu} \left(T^\phi_{\mu\nu} + T^\chi_{\mu\nu} \right) \right\}$$

=> production through



$\xrightarrow{L[f]=C[f]}$ $f_\chi(k) \propto k^{-9/2}$

$\equiv |\beta_k|^2$ for $E_k > m_\phi$!

⇝ take-home :

- very complementary viewpoints
- perturbative regime – can use both approaches ($k > m_\phi$) ⌈but Boltzmann typically much easier in practice!⌋
- non- " " – must use Bogoliubov

II.3 Primordial black holes

density fluctuations

FLRW: $\rho(t) \longrightarrow \bar{\rho}(t)\,(1+\delta(t,\vec{x}))$

isotropic: $\delta(\vec{k}) = \delta_k$

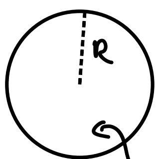
uncorrelated: $\langle \delta^*(\vec{k})\,\delta(\vec{k}')\rangle \propto \delta^{(3)}(\vec{k}-\vec{k}')$

$$\equiv \frac{2\pi^2}{k^3} P(k)\, \delta^{(3)}(\vec{k}-\vec{k}')$$

"spectrum"

average density contrast at scale R:

$\langle\rho\rangle = \bar{\rho}$



$\langle\rho\rangle = \bar{\rho}\,(1+\delta_R)$

$$\delta_R(\vec{x}) \equiv V^{-1} \int W\left(\frac{|\vec{x}-\vec{x}'|}{R}\right) \delta(\vec{x}')\, d^3x'$$

window function: Gaussian, top hat, ...

$\Rightarrow$ variance

$$\sigma_R^2 \equiv \langle \delta_R^2 \rangle = \int_0^\infty \frac{dk}{k}\, W^2(kR)\, P(k)$$

e.g. $P(k) \propto k^{n+3}$ $\Rightarrow$ at horizon entry $\left(R = \frac{2\pi}{k} = aH^{-1}\right)$:

$$\sigma_H^2 \equiv \sigma_R^2\big|_{t_R} \propto k^{n-1}$$

CMB scales + below: (= all observable scales consistent w/ the following)

$\sigma_H \sim 10^{-4}$, $n \lesssim 1$, Gaussian fluctuations: $p(\delta_R) = \frac{1}{\sqrt{2\pi}\,\sigma_R}\, e^{-\frac{1}{2}\frac{\delta_R^2}{\sigma_R^2}}$

$\delta_H \sim 10^{-5} \equiv \frac{(aH)^2}{k^2} P^{1/2}$

gravitational collapse

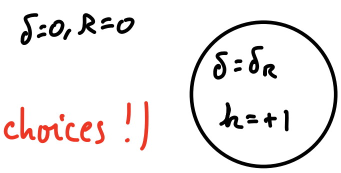
energy to overcome pressure $\sim p R^2 \cdot R = w \rho R^3$

potential energy $\sim M_{Pl}^{-2} (\rho R^3)^2 / R$

$\Rightarrow$ collapse if $\underset{(\#)}{R \gtrsim} \sqrt{w}\, \underbrace{M_{pl}\, \rho^{-1/2}}_{H^{-1} \text{ @ turnaround}} =$ Jeans length ($\rho \sim \rho_{crit}$)

more proper GR treatment :

2 independent FLRW patches



($\rightsquigarrow$ careful with matching / coordinate choices !)

identify $\quad \dot{a}(t_c) = 0$ for $k = +1$ patch

$\Rightarrow t_c = t_c(\delta_{R,i})$ [similar during MD to derive $\delta_c \simeq 1.68\ldots$]

(#)
$\Rightarrow \boxed{\delta_R \gtrsim w}$ at $t = t_R$ ($R = aH^{-1}$)

$$\Rightarrow m_{PBH} \simeq \frac{4\pi}{3} \rho R^3 \Big|_{t_c} \sim M_H \equiv \frac{4\pi}{3} \bar{\rho} H^{-3} \sim 10^{15} g \left(\frac{t}{10^{-23} s} \right)$$

[very shortly after horizon entry

$\Rightarrow$ PBHs can be "arbitrarily" small !

($\leftrightarrow$ BHs from stellar collapse : $M_{BH} \gtrsim 2\text{-}3\, M_\odot$)

= non-baryonic DM ? (if formation $t < t_{BBN}$: yes !)

numerically : collapse ~ critical phenomenon

$m_{PBH} \simeq M_H (\delta - \delta_c)^\eta \qquad \delta_c \sim 0.4, \quad \eta \sim 0.35$ (for RD)

abundance today

Probability to form a BH of mass $m_{PBH} \simeq m_H$

$\simeq$ " " collapse a region of size $R = aH^{-1}$

$$\equiv \beta(M_H) = \frac{1}{\sqrt{2\pi}\,\sigma_H(t_R)} \int_{\delta_{min}}^{\infty} e^{-\frac{\delta^2}{2\sigma_H^2(t_R)}} d\delta \approx \frac{\sigma_H(t_R)}{\sqrt{2\pi}\,\delta_{min}} e^{-\frac{\delta_{min}^2}{2\sigma_H^2(t_R)}}$$

Gaussian fluctuations

expectation based on current observations of $P(R)$:

$\beta(M_H) \sim 10^{-4} e^{-10^7} \sim$ very small ... "worse than cc problem"

"But they start to redshift like $\rho_{PBH} \propto a^{-3}$ very early"

yes: $\Omega_{PBH} = \beta(M_H)\underbrace{(1+z)}_{\substack{\sim \text{ gain of } 1+z \text{ during RD} \\ \sim \text{ const. during MD}}}\Omega_\gamma \approx \frac{\beta}{10^{-8}}\left(\frac{M_H}{M_\odot}\right)^{-1/2}$

=> default expectation (still ...) : $\Omega_{PBH} \lll \Omega_{DM}$

BSM (!) ways out :
- blue-tilted spectra ?
  -> quickly over produce PBHs @ even smaller scales
- small-scale peak in $P(R)$
  motivation ? Hardly possible without invoking complex spectrum of inflaton fields
- non-Gaussianities ?
  -> strong generic constraints. Again exotic inflation models required to make it work.

observations ?

$$f = \frac{\Omega_{PBH}}{\Omega_{DM}} = 1 \quad \text{largely excluded!}$$

(with some caveats for extended mass distribution)

- $m_{PBH} \lesssim 10^{16}\,g \,/\, 10^{-17} M_\odot$:
  radiation due to $T = \frac{M_{pl}^2}{8\pi\, m_{PBH}} \Rightarrow \frac{dm}{dt} \propto R_s^2 T^4$
  $\left[M \lesssim 10^{15} g \Rightarrow t_{PBH} \sim \frac{m_{PBH}^3}{M_{pl}^4} < t_{universe} \right]$

- $m_{PBH} \gtrsim 10^{-12} M_\odot$: micro-lensing ~ MACHO searches
  [both MACHO collab. + EROS etc.]

- $m_{PBH} \gtrsim 10^{-1} M_\odot$: GWs, accretion ⇝ CMB, Ly-α
  (future: 21 cm)

⇝ open window for "asteroid-sized" PBHs!

$$m_{PBH} \sim (10^{-17} - 10^{-12})\, M_\odot$$