# The impact of General Relativity on Galactic Dynamics


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DarkCosmoGrav - Pisa

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# What we know

The speeds of the stars are non relativistic

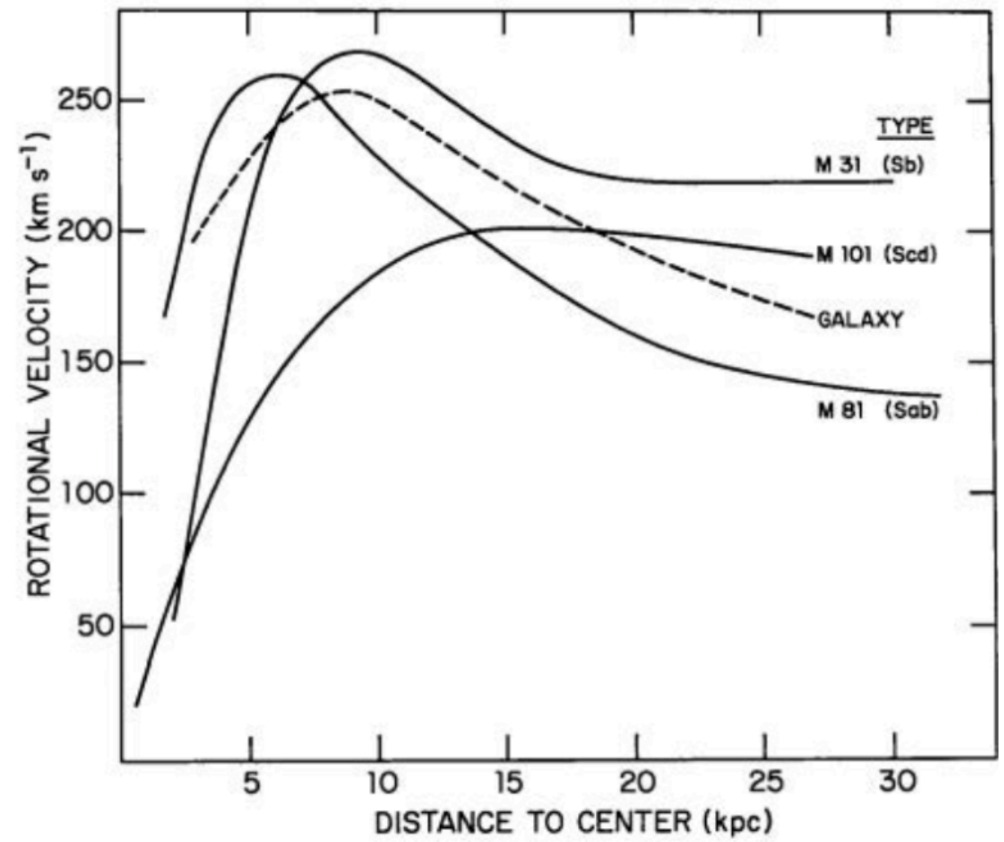
**Galaxies are *Newtonian* objects**

The gravitational field is weak for the great part of their extension

**Nonetheless…**

The flatness of the rotation curves at large galactocentric distances supports the existence of dark matter

In other words, Newtonian models alone cannot explain what we observe and  extra matter is need for purely dynamical (and also stability reasons)




But what is dark matter?

# An open issue and some recent claims

**what if we used GR?** → There are GR effects without Newtonian counterpart (i.e. gravitational-waves, gravitomagnetic effects…)

$$ds^2 = -(dt - N\,d\phi)^2 + r^2\,d\phi^2 + e^{\nu}(dr^2 + dz^2)$$

- Cooperstock-Tieu 2005
- Balasin-Grumiller (BG) 2006

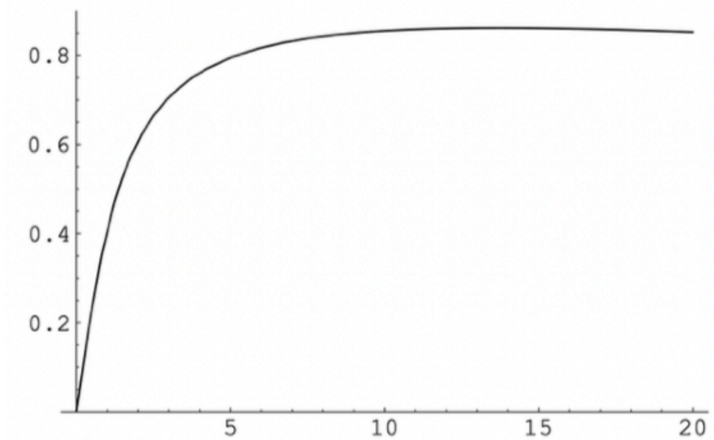
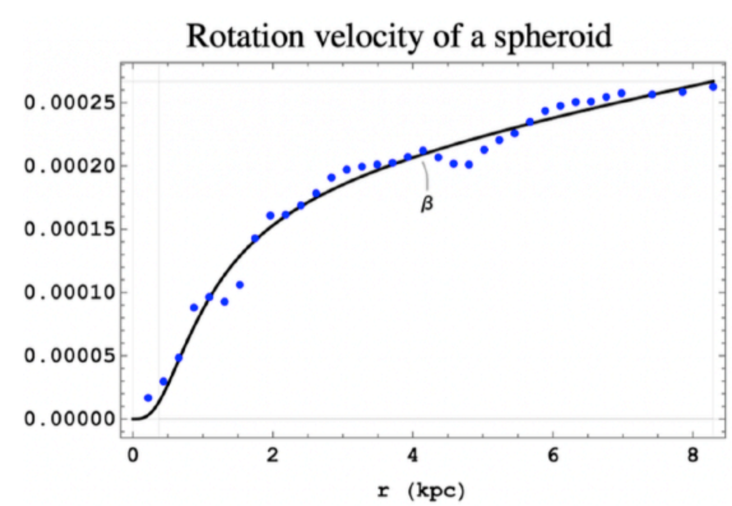



Fig. 3. $V(r,0)/V_0$ plotted up to 20 $kpc$

- Ludwig 2021

Ludwig uses the slow-motion and weak-field approximation of GR that leads to the analogy with Maxwell Equations (GEM)



# An open issue and some recent claims

- Crosta et al. 2020

They use an exact solution (BG) of GR field equation to show that it can explain the rotation curve of the Milky way without dark matter

**These papers share an idea**

GR adds a new degree of freedom, that has no Newtonian counterpart, and is determined by mass currents (magnetic-like or gravitomagnetic terms)

But are these effects big enough?

# An intuitive picture

**A quite general spacetime metric**

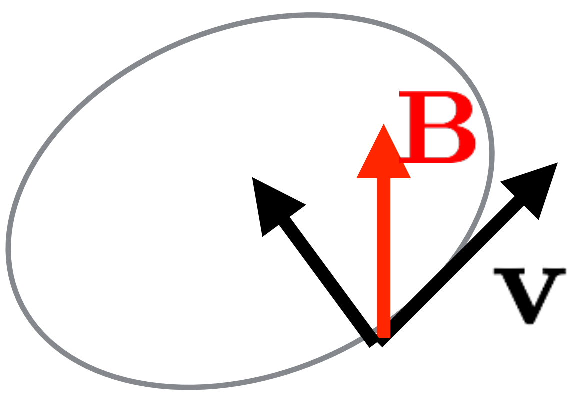
$$ds^2 = g_{00}(x)dt^2 + 2g_{0i}(x)dx^i dt + g_{ij}(x)dx^i dx^j$$

Newtonian field

$$g_{0i} \to \mathbf{A}, \quad \mathbf{B} = \nabla \wedge \mathbf{A}$$

$$m\frac{d\mathbf{v}}{dt} = m\mathbf{g} - 2m\frac{\mathbf{v}}{c} \wedge \mathbf{B}$$

**extra centripetal force**




The presence of the off-diagonal terms may produce an extra "force" term which mimics a "stronger" gravitational attraction, and increase the rotation speed

# A (very) simple model for a galaxy

## The symmetries and the exact solution

The model we consider is constituted by neutral, stationary and axisymmetric self-gravitating dust coupled to Einstein's equations.

Using a set of cylindrical coordinates adapted to inertial asymptotical observers:

$$T^{\mu\nu}(r,z) = \rho(r,z) u^{\mu}(r,z) u^{\nu}(r,z)$$

$$u^{\mu}(r,z) =\propto (1,0,0,\Omega(r,z))$$

$$\Omega(r,z) = \frac{d\phi}{dt} = \frac{u^{\phi}}{u^{t}}$$

$$g_{tt} = \frac{(H-\eta\Omega)^2 - r^2\Omega^2}{H},$$

$$g_{t\phi} = \frac{\eta^2 - r^2}{(-H)}\Omega + \eta,$$

$$g_{\phi\phi} = \frac{r^2-\eta^2}{(-H)}.$$

**metric coefficients**

field equations

$$\mathcal{F} = 2\eta + r^2 \int \frac{H'}{H}\frac{d\eta}{\eta} - \int \frac{H'}{H}\eta d\eta.$$

$$\mathcal{F}_{,rr} - \frac{1}{r}\mathcal{F}_{,r} + \mathcal{F}_{,zz} = 0,$$

$$\Omega = \frac{1}{2}\int H' \frac{d\eta}{\eta}.$$

# Relevant low energy limits I: GEM

**The gravitoelectromagnetic (GEM) limit**

The standard weak-field and slow-motion approximation of Einstein's field equation leads to the gravitoelectromagnetic limit, where we have a magnetic-like field in addition to the Newtonian one

geodesic equations

$$\left\{\begin{aligned} &\frac{V}{r}\partial_z\psi = \partial_z\Phi, \\ &V^2 = r(-\partial_r\Phi) + V\partial_r\psi, \end{aligned}\right.$$

A self-gravitating Newtonian dust can be in equilibrium only if it possesses cylindrical symmetry (Bonnor, 1977)

$$\left.\begin{aligned} &\partial_{zz}\Phi + \partial_{rr}\Phi + \frac{\partial_r\Phi}{r} = -4\pi G\rho, \\ &\partial_{zz}\psi + \partial_{rr}\psi - \frac{\partial_r\psi}{r} = \frac{4\pi G}{c}\rho V. \end{aligned}\right\}$$

sources equations

# Relevant low energy limits I: GEM

**The gravitoelectromagnetic (GEM) limit**

Standard approach:

$$ds^2 = -c^2\left(1 - 2\frac{\Phi}{c^2}\right)dt^2 - \frac{4}{c}(\mathbf{A}\cdot d\mathbf{x})dt + \left(1 + 2\frac{\Phi}{c^2}\right)\delta_{ij}dx^i dx^j$$

geodesic equations

$$\mathbf{a} = \mathbf{g} - 2\frac{\mathbf{v}}{c}\wedge\mathbf{B},$$

$$\nabla^2\Phi = -4\pi G\,\rho,$$

$$\nabla^2\mathbf{A} = -\frac{8\pi G}{c}\,\mathbf{j},$$

sources equations

The coupling between the gravitomagnetic field **B** and the fluid vorticity $\boldsymbol{\omega}$ influences the relation between matter density and velocity

$$4\pi G\rho + \frac{2}{c}\mathbf{B}\cdot\omega = -\nabla\cdot\left[\left(\mathbf{v}\cdot\nabla\right)\mathbf{v}\right].$$

**extra contribution to density**

# Relevant low energy limits II: Strong GEM

**The strong GEM limit**

$$ds^2 = -c^2\left(1 - \frac{2\Phi}{c^2} - \frac{\psi^2}{r^2c^2}\right)dt^2 - 2\frac{\psi}{c}cdtd\phi + r^2\left(1 + \frac{2\Phi}{c^2}\right)d\phi^2 + e^{\Psi}\left(dr^2 + dz^2\right)$$

The dragging (gravitomagnetic, off-diagonal) terms are of the same order as the gravitational potential

geodesic equations

$$\left\{\begin{aligned} \frac{V}{r}\partial_z\psi &= \partial_z\Phi + \frac{\psi}{r^2}\partial_z\psi, \\ \frac{V}{r}\partial_r\psi &= \partial_r\Phi + \frac{V^2}{r} + \frac{\psi}{r}\partial_r\frac{\psi}{r} \end{aligned}\right.$$

non linear terms

$$\partial_{rr}\psi + \partial_{zz}\psi - \frac{\partial_r\psi}{r} = 0,$$

$$\rho = -\frac{1}{4\pi G}\left[\nabla^2\Phi + \frac{(\partial_z\psi)^2 + (\partial_r\psi - 2\frac{\psi}{r})^2}{2r^2}\right]$$

**homogenous equation**

# Recap

**Exact solution for a rotating dust**
**axially symmetric, stationary**

**rigid rotation**

$$\Omega(r,z) = \Omega_0$$

**Balasin-Grumiller**

**low energy limits**

**strong GEM** $\psi \simeq c^0$

**GEM** $\psi \simeq c^{-1}$

All relevant models considered in the literature are summarised by this approach

# Are these effects big enough?

$$\nabla^2 \Phi = -4\pi G\,\rho,$$
$$\nabla^2 \mathbf{A} = -\frac{8\pi G}{c}\mathbf{j},$$

$$\frac{A}{\Phi} \simeq \frac{v}{c}$$

Magnetic-like effects are always smaller than Newtonian ones…

$$\mathbf{a} = \mathbf{g} - 2\frac{\mathbf{v}}{c} \wedge \mathbf{B},$$

$$\frac{a_{GM}}{a_N} \simeq \frac{v^2}{c^2}$$

The gravitomagnetic correction to the Newtonian acceleration is very small, i.e. in the order of $10^{-6}$!

See the recent paper by Luca Ciotti (ApJ 2022) for a detailed analysis

So, what we (they) went wrong?

# The Homogenous Solution (HS)

**In the GEM limit**

Gravitomagnetic effects are determined by mass currents, through the Poisson equation:

$$\nabla^2 \mathbf{A} = -\frac{8\pi G}{c}\mathbf{j}$$

The solution can be generally written in the form

$$\mathbf{A} = \mathbf{A}^{NHS} + \mathbf{A}^{HS}$$

$$\nabla^2 \mathbf{A} = 0$$

**The gravitomagnetic field is**

$$\mathbf{B} = \mathbf{B}^{NHS} + \mathbf{B}^{HS}$$

The gravitomagnetic field arising from the HS is not related to the mass currents, hence its effects are not negligibly small, in general, with respect to the Newtonian ones.

$$\psi = \underbrace{\psi^{NHS}}_{\sim c^{-1}} + \underbrace{\psi^{HS}}_{\sim c^{0}}$$

# The Homogenous Solution (HS)

**In the GEM limit**

This system describes a non trivial modification of Newtonian dynamics

$$\partial_{zz}\Phi + \partial_{rr}\Phi + \frac{\partial_r \Phi}{r} = -4\pi G\rho,$$

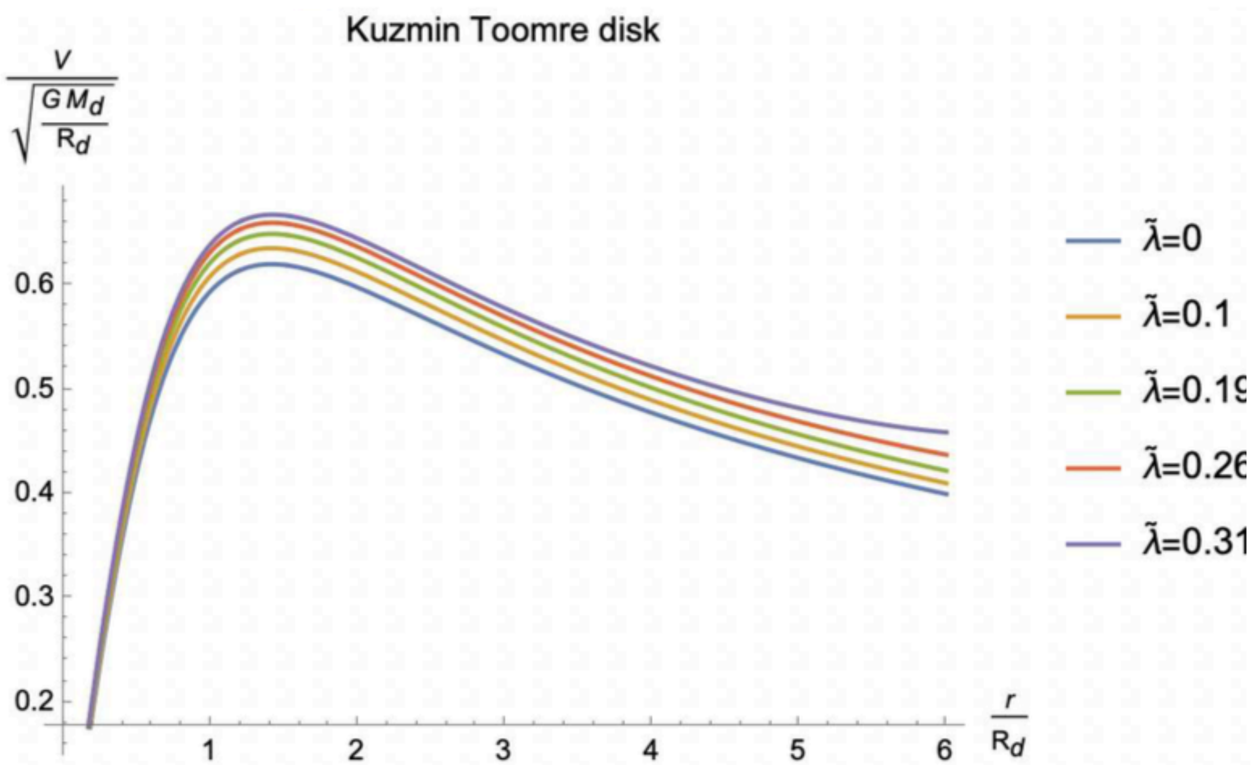
$$\partial_{zz}\psi + \partial_{rr}\psi - \frac{\partial_r \psi}{r} = 0,$$

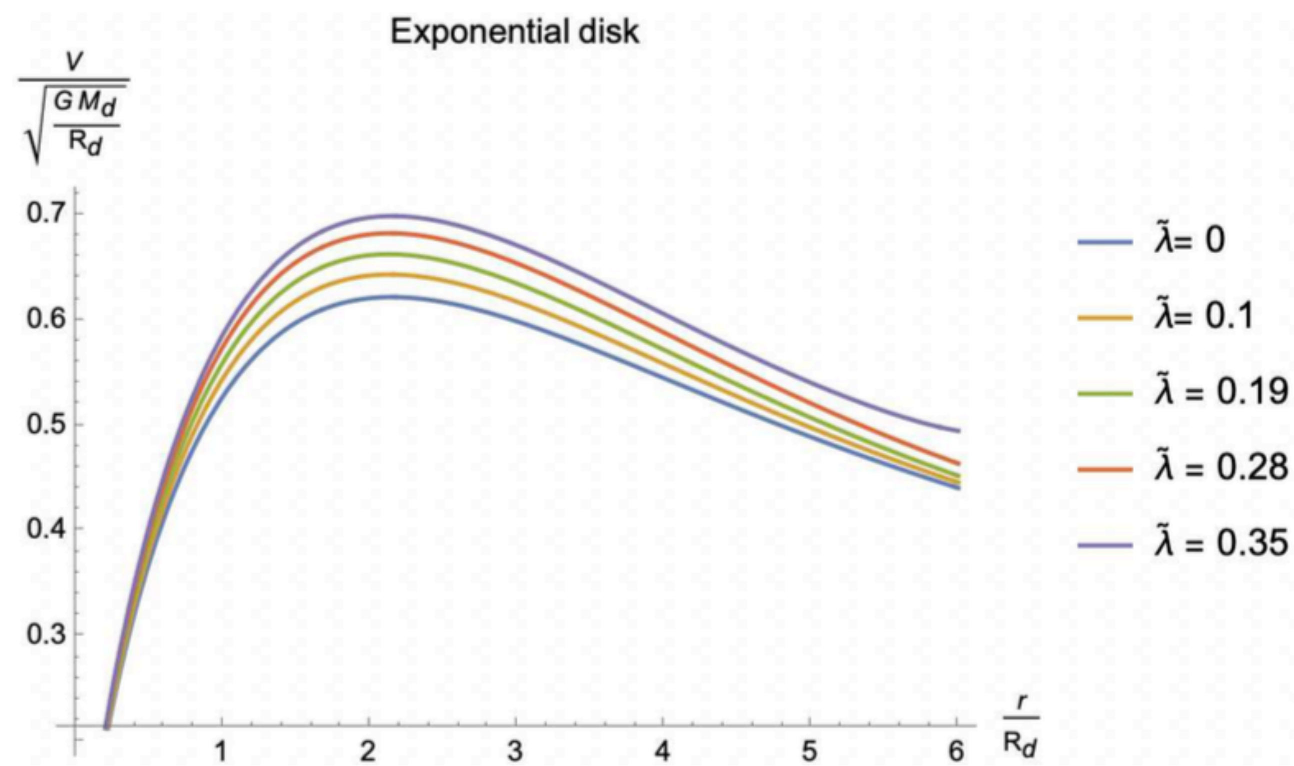
$$\frac{V}{r}\partial_z\psi = \partial_z\Phi,$$

$$V^2 = r(-\partial_r\Phi) + V\partial_r\psi$$

1) New solutions → toy model
2) Comparison with known results → e.g. Ludwig, BG

# Kuzmin-Toomre and Exponential Disk




gravitational potential

$$\Phi(r,0) = \frac{GM_d}{\sqrt{(r^2+R_d^2)}},$$

$$\Phi(r,0) = \frac{GM_d\, r\left[I_0\left(\frac{r}{2R_d}\right)K_1\left(\frac{r}{2R_d}\right) - I_1\left(\frac{r}{2R_d}\right)K_0\left(\frac{r}{2R_d}\right)\right]}{2R_d^2}$$

surface density

$$\sigma(r) = \frac{M_d R_d}{2\pi(r^2+R_d^2)^{3/2}}$$

$$\sigma(r) = \frac{M_d}{2\pi R_d^2} e^{-r/R_d}$$

# Comparison with Ludwig and BG

**Ludwig's results**

They are consistently reproduced by considering the contribution of the HS solution only: namely the non homogenous solutions are negligible in its calculations

**Balasin-Grumiller and related models**

$$r\nu_z + N_r N_z = 0\,,$$
$$2r\nu_r + N_r^2 - N_z^2 = 0\,,$$
$$\nu_{rr} + \nu_{zz} + \frac{1}{2r^2}\left(N_r^2 + N_z^2\right) = 0\,,$$
$$N_{rr} - \frac{1}{r}N_r + N_{zz} = 0\,,$$
$$\frac{1}{r^2}\left(N_r^2 + N_z^2\right) = \kappa\rho e^{\nu}\,.$$

homogenous solution!

# Conclusions and Perspectives

- ☑ Simplified model of a galaxy as a self-gravitating and stationary rotating dust
- ☑ Exact solution of Einstein field equations and suitable low energy limits (GEM, and strong GEM)
- ☑ In the strong GEM limit gravitomagnetic effects are of the same order as the Newtonian ones
- ☑ The homogenous solutions have a crucial role in explaining recent models fitting galactic rotation curves

- ☐ *Use the strong GEM equations to build models of galactic dynamics*
- ☐ *Better understanding of the meaning of these homogenous solutions*

arbitrary function for the solution of Einstein equations

$$\mathcal{F}_{,rr} - \frac{1}{r}\mathcal{F}_{,r} + \mathcal{F}_{,zz} = 0,$$

# Related publications

- D. Astesiano, MLR, **Galactic dark matter effects from purely geometrical aspects of General Relativity**, Phys.Rev.D 106 (2022) 4, 04406

- D. Astesiano, MLR, **Can General relativity play a role in galactic dynamics?**, Phys.Rev.D 106 (2022) 12, L121501

- MLR, A. Ortolan, C.C. Speake, **Galactic dynamics in general relativity: the role of gravitomagnetism**, Class. Quantum Grav. 39 225015

**Thanks for your attention**