Particle interactions in Kinetic Field Theory

Application to cosmic structure formation

Christophe Pixius

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Institute for Theoretical Physics
Heidelberg University

Goal

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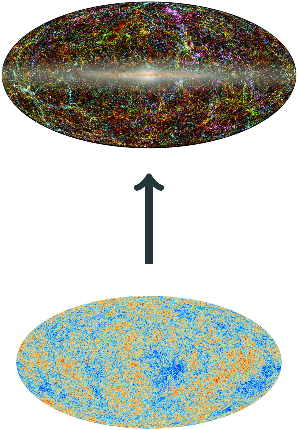
- Analytical description of non-linear structure formation



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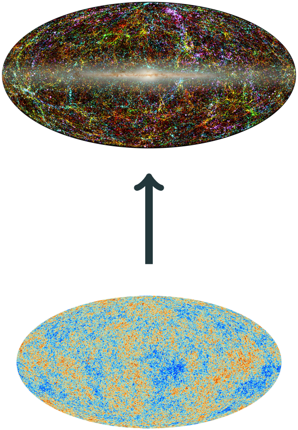

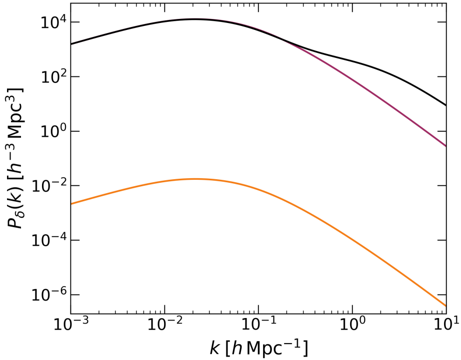

Non-linear power spectrum from Smith et al., Stable clustering, the halo model and nonlinear cosmological power spectra, 2002

Why analytical methods?

- Can yield a proper understanding of physical mechanisms, leading to e. g.
  - universality of cosmic structures
  - halo density profiles

Why analytical methods?

- Can yield a proper understanding of physical mechanisms, leading to e. g.
  - universality of cosmic structures
  - halo density profiles
- Can cover large theory spaces at low computational cost
  - Investigate different models of dark energy and dark matter
  - Probe alternative gravity models

Standard Perturbation Theory (SPT) - Overview

Idea

- Model the content of the universe as a fluid
- Initial conditions are set by inflation and are assumed to be Gaussian
- The evolution of the cosmic fluid is governed by the equations of (ideal) hydrodynamics

$$\partial_\tau \delta + \nabla \cdot [(1+\delta)u] = 0 \qquad \text{(continuity)}$$

$$\partial_\tau u + \tfrac{1}{2} u + (u \cdot \nabla) u + \nabla \Phi = 0 \qquad \text{(Euler)}$$

$$\nabla^2 \Phi = \tfrac{3}{2} \delta \qquad \text{(Poisson)}$$

Standard Perturbation Theory

$$\partial_\tau \delta_k + \theta_k = -\int_{k_1} \alpha(k_1, k - k_1)\theta_{k_1}\delta_{k-k_1}$$

$$\partial_\tau \theta_k + \frac{1}{2}\theta_k + \frac{3}{2}\delta_k = -\int_{k_1} \beta(k_1, k - k_1)\theta_{k_1}\theta_{k-k_1}$$

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- Expand around linear solution for δ_k and θ_k (LHS)
- Include RHS perturbatively for higher order solutions (loop corrections)

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$$P_\delta(k, \tau) = \int_{k'} \langle \delta_k(\tau)\delta_{k'}(\tau) \rangle$$

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- But: no access to fully non-linear scales (single stream approximation)

Kinetic Field Theory (KFT)

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- Treat content of universe as **classical point particles**
- Particles follow **classical phase-space trajectories**
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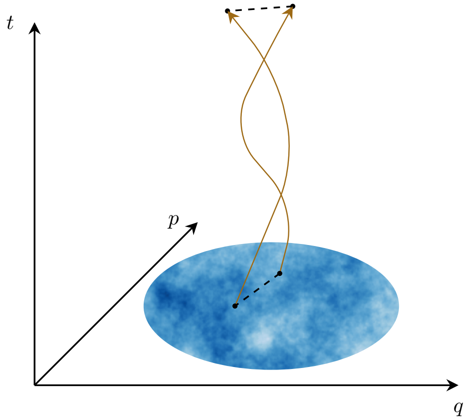

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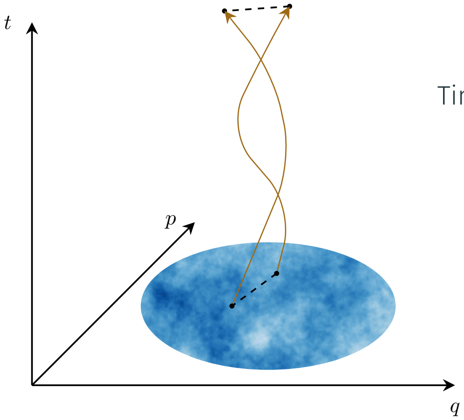

Time evolution:

$$\dot{q} = \frac{\partial \mathcal{H}}{\partial p}$$

$$\dot{p} = -\frac{\partial \mathcal{H}}{\partial q}$$

- Probability distribution in phase-space

$$\mathcal{P}(\mathbf{x}) = \int \mathrm{d}\mathbf{x}^{(\mathrm{i})} \mathcal{P}(\mathbf{x}^{(\mathrm{i})})\ \mathcal{P}(\mathbf{x}|\mathbf{x}^{(\mathrm{i})}), \qquad \mathbf{x} = (\mathbf{q}, \mathbf{p})^{\mathrm{T}}$$

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$$\mathcal{P}(\mathbf{x}) = \int \mathrm{d}\mathbf{x}^{(\mathrm{i})} \mathcal{P}(\mathbf{x}^{(\mathrm{i})})\ \delta_D\Big(\mathbf{x} - \mathbf{x}^{(\mathrm{cl})}(\tau; \mathbf{x}^{(\mathrm{i})})\Big), \qquad \mathbf{x} = (\mathbf{q}, \mathbf{p})^{\mathrm{T}}$$

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- Resulting generating functional:

$$Z[\mathbf{J}] = \mathrm{e}^{\mathrm{i}\hat{S}_I} Z_0[\mathbf{J}]$$

Field theory formulation - The generating functional

- Probability distribution in phase-space

$$\mathcal{P}(\mathbf{x}) = \int \mathrm{d}\mathbf{x}^{(\mathrm{i})}\, \mathcal{P}(\mathbf{x}^{(\mathrm{i})})\ \delta_D\Big(\mathbf{x} - \mathbf{x}^{(\mathrm{cl})}(\tau; \mathbf{x}^{(\mathrm{i})})\Big), \qquad \mathbf{x} = (\mathbf{q}, \mathbf{p})^{\mathrm{T}}$$

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$$Z[\mathbf{J}] = \mathrm{e}^{\mathrm{i}\hat{S}_I}\, Z_0[\mathbf{J}]$$

- Calculate density correlations, e. g. a two-point function:

$$\langle \rho(k, \tau)\rho(k', \tau) \rangle = \hat{\rho}(k, \tau)\hat{\rho}(k', \tau) Z[\mathbf{J}]\big|_{\mathbf{J}=0}$$

KFT Perturbation Theory

- Interaction operator can't be evaluated in the full exponential form

- Series expansion:

$$Z[\mathbf{J}] = \mathrm{e}^{\mathrm{i}\hat{S}_I} Z_0[\mathbf{J}]$$

Expansion of the interaction operator

- Interaction operator can't be evaluated in the full exponential form

- Series expansion:

$$
\begin{aligned}
Z[\mathbf{J}] &= e^{i\hat{S}_I} Z_0[\mathbf{J}] \\
&= \left(1 + i\hat{S}_I + \frac{1}{2}(i\hat{S}_I)^2 + \dots\right) Z_0[\mathbf{J}]
\end{aligned}
$$

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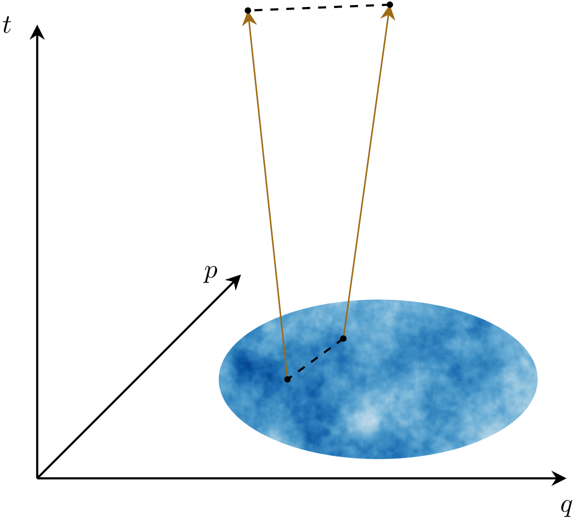
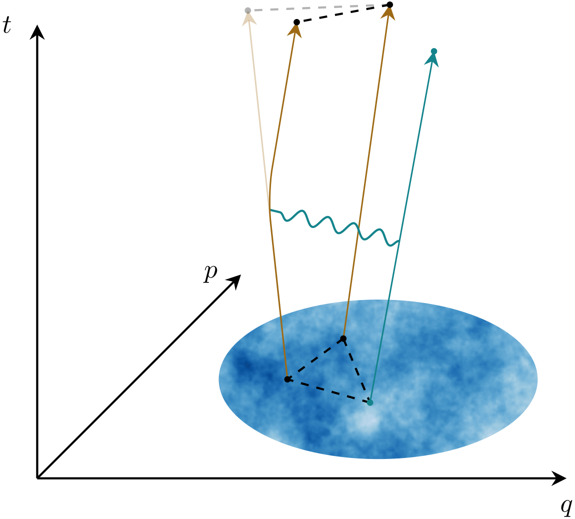
- Two-point function in perturbation theory:

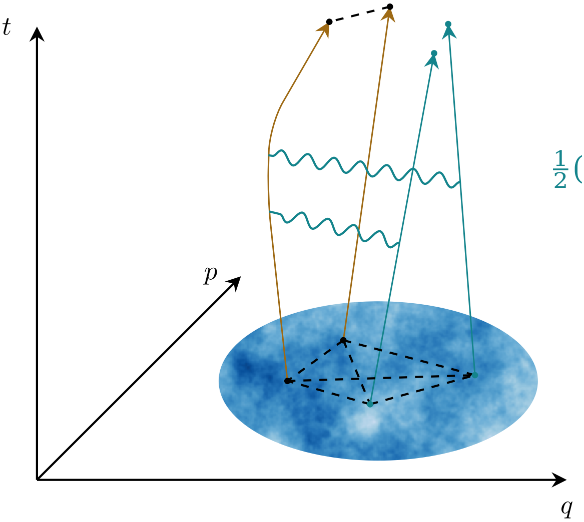
$$\langle \rho(k)\rho(k')\rangle = \left(1 + i\hat{S}_I + \frac{1}{2}(i\hat{S}_I)^2 + \dots\right)\hat{\rho}\hat{\rho} Z_0[\mathbf{J}]$$



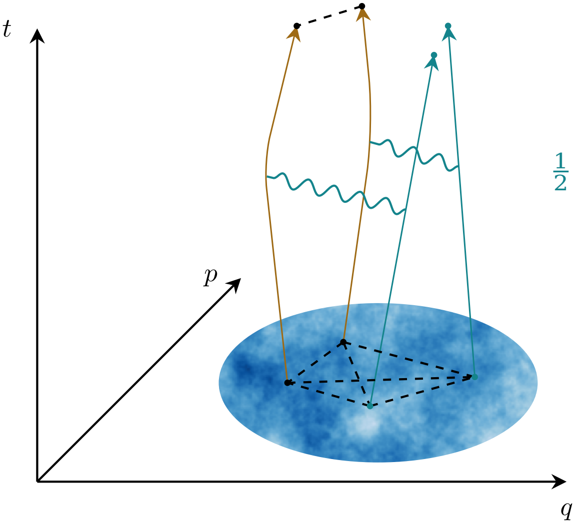
$\hat{\rho}\,\hat{\rho}\,Z_0[\mathbf{J}]$



$$\mathrm{i}\hat{S}_I \hat{\rho}\hat{\rho} Z_0[\mathbf{J}]$$



$$\frac{1}{2}(\mathrm{i}\hat{S}_I)^2\hat{\rho}\hat{\rho}Z_0[\mathbf{J}]$$



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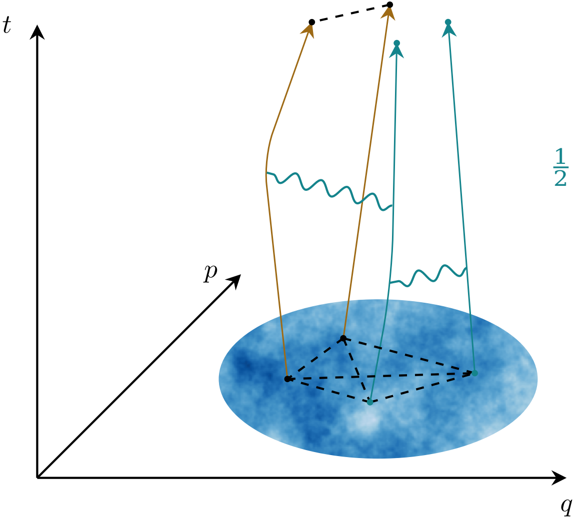

$$\frac{1}{2}\left(\mathrm{i}\hat{S}_I\right)^2\hat{\rho}\,\hat{\rho}\,Z_0[\mathbf{J}]$$

Systematics of perturbation theory

- Effect of the interaction operator can be represented in terms of **diagrams**
- **Number of diagrams increases** with each order
- **Number of terms per diagram increases** with each order

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Systematics of perturbation theory

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⇒ Calculations by hand become unfeasible for orders $n > 2$

- **But:** logic can be translated into a symbolic code
- At present this allows to go up to **7th order** in the interaction operator

$$e^{i\hat{S}_I} \; Z_0[\mathbf{J}]$$

$$1 + i\hat{S}_I + \frac{1}{2}\left(i\hat{S}_I\right)^2 + \ldots$$

up to 7th order

$\mathrm{e}^{\mathrm{i}\hat{S}_I}\ Z_0[\mathbf{J}]$

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expand in powers of $P_\delta^{(\mathrm{i})}$

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up to 7th order

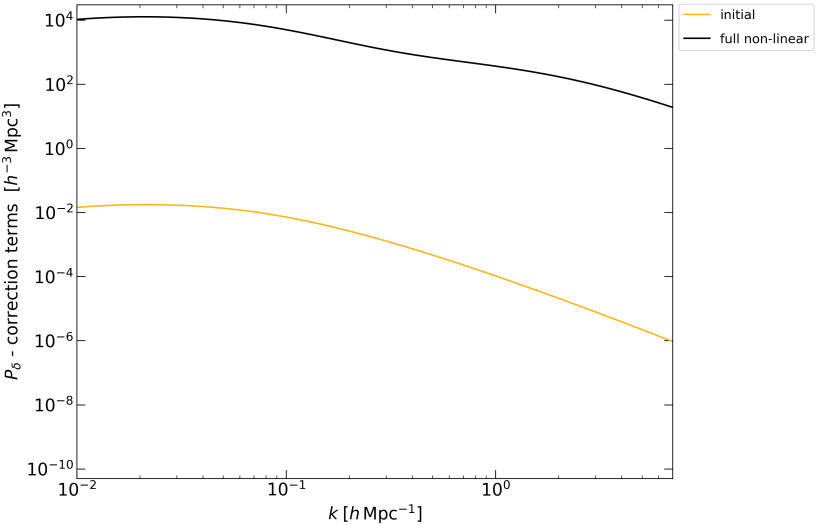
expand in powers of $P_\delta^{(i)}$

up to second order

(linear and quadratic (mode coupling) in $P_\delta^{(i)}$)

Results

Linear and mode coupling results



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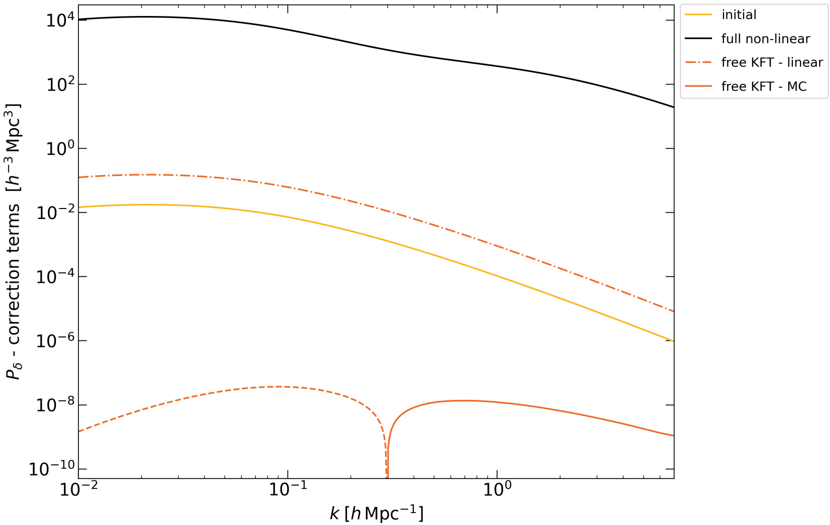

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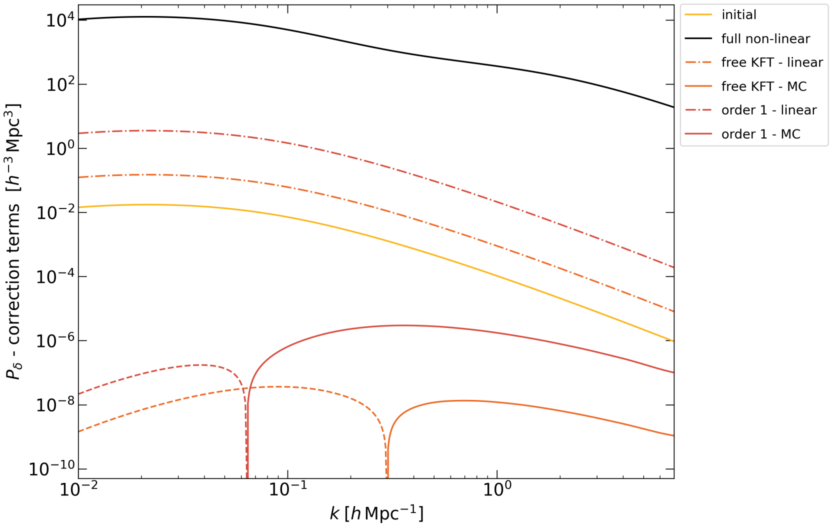

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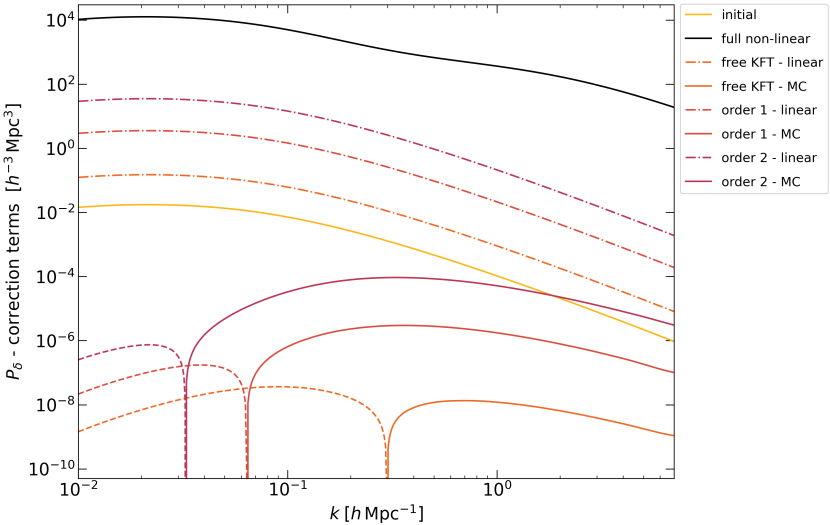

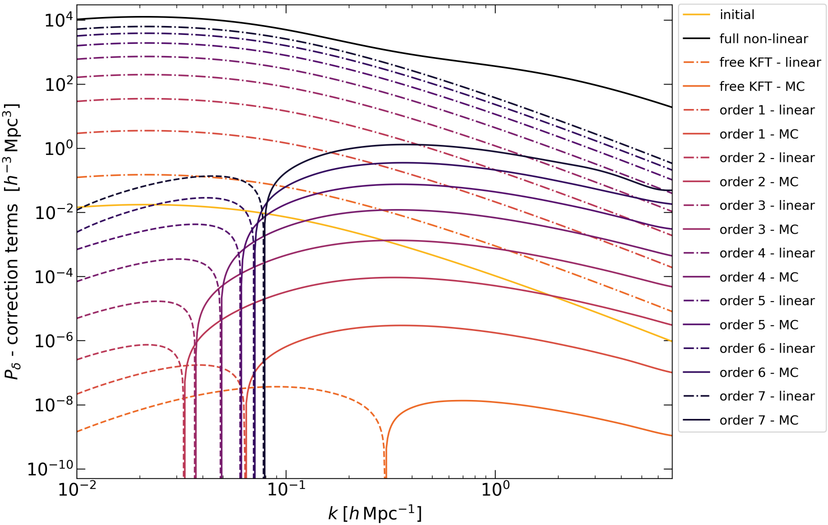
initial
full non-linear
free KFT - linear
free KFT - MC
order 1 - linear
order 1 - MC
order 2 - linear
order 2 - MC
order 3 - linear
order 3 - MC
order 4 - linear
order 4 - MC
order 5 - linear
order 5 - MC
order 6 - linear
order 6 - MC
order 7 - linear
order 7 - MC
P_δ - correction terms $[h^{-3}\,\mathrm{Mpc}^3]$
$k\,[h\,\mathrm{Mpc}^{-1}]$
10^{4}
10^{2}
10^{0}
10^{-2}
10^{-4}
10^{-6}
10^{-8}
10^{-10}
10^{-2}
10^{-1}
10^{0}

Conclusion & Outlook

- KFT allows for **analytical calculations of correlation functions** of cosmic structures **from first principles**
- Perturbation theory in KFT is **conceptually very different from SPT**
- **Higher order perturbations are needed** to calculate the non-linear dark matter power spectrum
- Alternatively, choose a **different splitting between free and interacting theory**

  $\Rightarrow$ improve convergence of perturbation theory

Thank you!

Density operators

Density in real and Fourier space, and as an operator

$$\rho(q_1, t_1) = \sum_{s=1}^{N} \delta_D(q_1 - q_s(t_1))$$

$$\rho(k_1, t_1) = \sum_{s=1}^{N} \exp\left(-\mathrm{i}k_1 \cdot q_s(t_1)\right)$$

$$\hat{\rho}(k_1, t_1) = \sum_{s=1}^{N} \exp\left(-\mathrm{i}k_1 \cdot \frac{\delta}{\mathrm{i}\delta J_{q_s}(t_1)}\right)$$

Effect of density operators on the generating functional

$$\hat{\rho}(k_1, t_1) Z_0[\mathbf{J}] = \sum_{s=1}^{N} Z_0[\mathbf{J} + \mathbf{L}] \qquad \text{with} \quad L_s(t) = -\begin{pmatrix} k_1 \\ 0 \end{pmatrix} \delta_D(t - t_1)$$

Free generating functional - initial momentum correlations

$$Z_0[\mathbf{L}] = V^{-n}(2\pi)^3\delta_D\Big(\sum_{j=1}^{n} L_{q_j}\Big)\mathrm{e}^{-Q_D}\prod_{2\leq b<a}^{n}\int_{k_{ab}}\prod_{1\leq i<j}^{n}\big(\mathcal{P}_{ij}(k_{ij}) +$$

The factors of the generating functional $\mathcal{P}_{ij}$ and the damping factor Q_D are defined by

$$\mathcal{P}_{ij}(k_{ij}) = \int_q \left(e^{-L_{p_i}^{\mathrm{T}} C_{pp}(q) L_{p_j}} - 1\right) e^{\mathrm{i}k_{ij}\cdot q}$$

$$Q_D = \frac{\sigma_p^2}{2}\sum_{j=1}^{n} L_{p_j}^2 .$$