On sum rules for double and triple parton distribution functions and Pythia's model of multiple parton interactions

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Based upon 2208.08197

# How little we know about protons?

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(Honestly, very little) (We do not really understand proton mass, proton spin and proton structure)

### How little we know about protons?





The most popular MC cartoon ever, 1411.4085

# Why MPI exist: charged multiplicity





Charged multiplicity distribution, from 1101.2599

# Why MPI exist: Theory vs. CMS data





Comparison of theoretic predictions (MC and resummation) against recent CMS data for the Jet Thrust angularity,  $p_{T,jet} \in [120, 150]$  GeV

Data: 2109.03340; Theory: 2104.06920



#### What is double parton scattering?

By *double parton scattering* (DPS) we mean a particular case of the MPI process when two hard interactions occur per one hadron-hadron collision.









Assuming that hard processes factorize one can write

$$\sigma_{AB} = \sum_{i,j,k,l} \int \prod_{a=1}^{4} dx_a \, d^2 \mathbf{b} \, \hat{\sigma}_{ij \to A} \, \hat{\sigma}_{kl \to B} \, \mathsf{\Gamma}_{ik} \left( x_1, x_2, \mathbf{b}, \, Q_A, \, Q_B \right) \, \mathsf{\Gamma}_{jl} \left( x_3, x_4, \mathbf{b}, \, Q_A, \, Q_B \right)$$

where functions  $\Gamma_{ik}(x_1, x_2, \mathbf{b}, Q_A, Q_B)$  are called *generalized parton* distribution functions (gPDFs) and give a probability to find two partons, separated by transverse distance **b**, in a hadron (in case of *bare* gPDFs).





#### Assumption 1

Assuming  $\Gamma_{ik}(x_1, x_2, \mathbf{b}, Q_A, Q_B) \simeq D_p^{ik}(x_1, x_2, Q_A, Q_B) F(\mathbf{b})$  one can write

$$\sigma_{AB} = \frac{1}{(1+\delta_{AB})\sigma_{eff}} \sum_{i,j,k,l} \int \prod_{a=1}^{4} dx_a D_p^{ik} \left(x_1, x_2, Q_A, Q_B\right) D_p^{jl} \left(x_3, x_4, Q_A, Q_B\right) \hat{\sigma}_{ij \rightarrow A} \hat{\sigma}_{kl \rightarrow B}$$

#### Assumption 2

Assuming  $D_p^{ij}(x_1, x_2, Q_A, Q_B) \approx f_i(x_1, Q_A)f_j(x_2, Q_B)$  one can write

$$\sigma_{AB} = \frac{1}{1 + \delta_{AB}} \frac{\sigma_A \sigma_B}{\sigma_{eff}}$$

where we defined

$$\frac{1}{\sigma_{eff}} \equiv \int d^2 \mathbf{b} \left[ F(\mathbf{b}) \right]^2$$

Note that now we can estimate the value of  $\sigma_{eff}$  (yielding ~ 30 mb).





Different measurements of  $\sigma_{eff}$ , 1811.11094. See also recent *pp* 1909.06265, CMS-PAS-SMP-21-013 and *pA* 2007.06945 measurements. Also TPS was recently observed 2111.05370!

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#### The main issues are

- Double counting between SPS and DPS (1v1 and 1v2 terms).
- Our (almost complete!) ignorance of the nPDFs.
- Correlations in spin, colour and flavour. Longitudinal and transverse correlations.
- Incorporation of the existing first principles QCD results into general-purpose MC event generators.

#### Several different contributions are possible





#### DPS

- Double counting 1702.06486.
- Lattice QCD 2106.03451
- dShower 1906.04669, 2008.01442
- Sum rules 1811.00289, 2001.10428

#### TPS

- Formal solution of the evolution equations exist 1603.08187
- Sum rules 2208.08197
- In general, still "terra incognita"!

#### Several different contributions are possible





1v2





### Holy Trinity of Monte Carlo









SHERPA



A good concrete illustration of the Blessed Trinity is an equilateral triangle. Such a triangle has three stides aqual in every way, and yet distinct from each other. There are three stides, but only one Person is different from the other two, but all three are God. Each one is God, distinct from the two others, and yet one with them. The three Persons are equal in every way, with one nature and one

# Sum rules for double and triple PDFs

# Master formulas for DPS and TPS



A master formula for DPS can be schematically written as

$$\begin{split} \sigma_{hh'}^{\mathrm{DPS}} &= \sum_{\mathrm{partons}} \int \prod_{i=1}^{2} dx_{i} \, dx_{i}' \, d^{2}b \times \\ &\times \quad \Gamma_{h}(\{x_{i}\}, \mathbf{b}, \{Q_{i}\}) \, \Gamma_{h'}(\{x_{i}'\}, \mathbf{b}, \{Q_{i}\}) \, [\cdots] \,, \end{split}$$

which for the TPS becomes

$$\begin{split} \sigma_{hh'}^{\mathrm{TPS}} &= \sum_{\mathrm{partons}} \int \prod_{i=1}^{3} dx_{i} \, dx_{i}' \, d^{2}b_{i} \, d^{2}b \times \\ &\times \quad \Gamma_{h}\left(\{x_{i}\}, \mathbf{b}_{i}, \{Q_{i}\}\right) \Gamma_{h'}\left(\left\{x_{i}'\right\}, \left\{\mathbf{b}_{i} - \mathbf{b}\right\}, \{Q_{i}\}\right) \left[\cdots\right]. \end{split}$$

A standard assumption on factorization into longitudinal and transverse parts reads  $\Gamma_h(\{x_i\}, \mathbf{b}_i, \{Q_i\}) \approx T_h(\{x_i\}, \{Q_i\}) \sum_i f(\mathbf{b}_i).$ 



#### The sum rules for dPDFs and tPDFs are

$$\sum_{j_2} \int_{0}^{1-x_1} dx_2 x_2 D_{j_1j_2}(x_1, x_2, Q) = (1 - x_1) f_{j_1}(x_1, Q),$$

$$\sum_{j_3} \int_{0}^{1-x_1-x_2} dx_3 x_3 T_{j_1j_2j_3}(x_1, x_2, x_3, Q) = (1 - x_1 - x_2) D_{j_1j_2}(x_1, x_2, Q),$$

$$\int_{0}^{1-x_1} dx_2 D_{j_1j_{2\nu}}(x_1, x_2, Q) = (N_{j_{2\nu}} - \delta_{j_1j_2} + \delta_{j_1j_2}) f_{j_1}(x_1, Q),$$

$$\int_{0}^{1-x_1-x_2} dx_3 T_{j_1j_2j_{3\nu}}(x_1, x_2, x_3, Q) = (N_{j_{3\nu}} - \delta_{j_3j_1} - \delta_{j_3j_2} + \delta_{\overline{j}_3j_1} + \delta_{\overline{j}_3j_2}) \times$$

$$\times D_{j_1j_2}(x_1, x_2, Q).$$

# GS sum rules for tPDFs



#### The Light-cone formalism for "bare" PDFs implies

$$D_{j_{1}j_{2}}(x_{1}, x_{2}) = \sum_{N, \{\beta_{i}\}} \int [dz]_{N} [d^{2} \mathbf{k}]_{N} |\Phi_{N}(\{\beta_{i}, z_{i}, \mathbf{k}_{i}\})|^{2} \times \\ \times \sum_{i}^{N} \delta(x_{1} - z_{i}) \delta_{j_{1}p_{i}} \sum_{k \neq i}^{N} \delta(x_{2} - z_{k}) \delta_{j_{2}p_{k}},$$

$$T_{j_{1}j_{2}j_{3}}(x_{1}, x_{2}, x_{3}) = \sum_{N, \{\beta_{i}\}} \int [dz]_{N} [d^{2} \mathbf{k}]_{N} |\Phi_{N}(\{\beta_{i}, z_{i}, \mathbf{k}_{i}\})|^{2} \times \\ \times \sum_{i}^{N} \delta(x_{1} - z_{i}) \delta_{j_{1}p_{i}} \sum_{k \neq i}^{N} \delta(x_{2} - z_{k}) \delta_{j_{2}p_{k}} \sum_{l \neq i, k}^{N} \delta(x_{3} - z_{l}) \delta_{j_{3}p_{l}},$$

where

$$\begin{bmatrix} dz \end{bmatrix}_{N} \equiv \prod_{i=1}^{N} dz_{i} \, \delta \left( 1 - \sum_{i}^{N} z_{1} \right),$$
$$\begin{bmatrix} d^{2} \mathbf{k} \end{bmatrix}_{N} \equiv \prod_{i=1}^{N} d^{2} \mathbf{k}_{i} \, \delta^{2} \left( \sum_{i}^{N} \mathbf{k}_{i} \right).$$

# GS sum rules for tPDFs



#### Which can be written

$$\sum_{j_{3}} \int_{0}^{1-x_{1}-x_{2}} dx_{3} x_{3} T_{j_{1}j_{2}j_{3}}(x_{1}, x_{2}, x_{3}) = \sum_{N, \{\beta_{i}\}} \int [dz]_{N} [d^{2} \mathbf{k}]_{N} |\Phi_{N}(\{\beta_{i}, z_{i}, \mathbf{k}_{i}\})|^{2} \times \sum_{j_{3}} \int_{0}^{1-x_{1}-x_{2}} dx_{3} \sum_{i}^{N} \delta(x_{1}-z_{i}) \delta_{j_{1}p_{i}} \sum_{k\neq i}^{N} \delta(x_{2}-z_{k}) \delta_{j_{2}p_{k}} \sum_{l\neq i,k}^{N} \delta(x_{3}-z_{l}) \delta_{j_{3}p_{l}} = \sum_{N, \{\beta_{i}\}} \int [dz]_{N} [d^{2} \mathbf{k}]_{N} |\Phi_{N}(\{\beta_{i}, z_{i}, \mathbf{k}_{i}\})|^{2} \sum_{i}^{N} \delta(x_{1}-z_{i}) \delta_{j_{1}p_{i}} \times \sum_{k\neq i}^{N} \delta(x_{2}-z_{k}) \delta_{j_{2}p_{k}} \sum_{j_{3}} \sum_{l\neq i,k}^{N} z_{l} \delta_{j_{3}p_{l}}.$$

The last sum in equation above can be written as

$$\sum_{j_3}\sum_{l \neq i,k}^N z_l \ \delta_{j_3p_l} = \sum_l^N z_l - z_i - z_k = 1 - x_1 - x_2,$$

which allows to recover expression for the momentum sum rule!



#### RGE can be used to prove the sum rules for renormalized tPDFs

$$\begin{split} T_{j_1 j_2 j_3}(x_1, x_2, x_3, Q) &= \sum_{j'_1 j'_2 j'_3} Z_{j_1 j'_1}(Q) \otimes_1 Z_{j_2 j'_2}(Q) \otimes_2 Z_{j_3 j'_3}(Q) \otimes_3 T^B_{j'_1 j'_2 j'_3}(z_1, z_2, z_3) \\ &+ \sum_{j'_1 j'_2} Z_{j_1 j'_1}(Q) \otimes_1 Z_{j_2 j_3, j'_2}(Q) \otimes_{23} D^B_{j'_1 j'_2}(z_1, z_2) + (\text{permutations}) \\ &+ \sum_{j'_1} Z_{j_1 j_2 j_3, j'_1}(Q) \otimes_{123} f^B_{j'_1}(z_1) \end{split}$$

where

$$A \otimes_{1} B = \int \frac{dz}{z} A\left(\frac{x_{1}}{z}\right) B(z),$$
  

$$A \otimes_{12} B = \int \frac{dz}{z^{2}} A\left(\frac{x_{1}}{z}, \frac{x_{2}}{z}\right) B(z),$$
  

$$A \otimes_{123} B = \int \frac{dz}{z^{3}} A\left(\frac{x_{1}}{z}, \frac{x_{2}}{z}, \frac{x_{3}}{z}\right) B(z).$$

# Using PYTHIA8 to create nPDFs satisfying the sum rules



#### Master formula to compute cross $2 \rightarrow 2$ section

$$\frac{d\sigma}{dp_{\perp}^{2}} = \sum_{j_{1}, j_{2}, j_{3}, j_{4}} \int dx_{1} dx_{2} d\hat{t} f_{j_{1}}(x_{1}, Q) f_{j_{2}}(x_{2}, Q) \frac{d\sigma_{j_{1}, j_{2} \to j_{3}, j_{4}}}{d\hat{t}} \delta\left(p_{\perp}^{2} - \frac{\hat{t}\hat{u}}{\hat{s}}\right)$$



Two different ways to generate  $(dg \rightarrow dg) \otimes (\bar{d}g \rightarrow \bar{d}g)$  DPS process in the PYTHIA8 event generator: a) the initial state *d*-quark comes from a perturbative  $g \rightarrow d\bar{d}$  splitting process, b) the initial state *d*-quark comes from a raw *d*-quark sPDF.

Companion quark contribution and modification of a valence quark

$$q_c(x, x_s) = C P_{g \to q\bar{q}}\left(\frac{x_s}{x_s + x}\right) \frac{g(x_s + x)}{x_s + x}; q_{fvn}(x) = \frac{N_{fvn}}{N_{fv0}} \frac{1}{X_n} q_{fv0}\left(\frac{x}{X_n}\right)$$



Two different ways to generate  $(dg \rightarrow dg) \otimes (\bar{d}g \rightarrow \bar{d}g)$  DPS process in the PYTHIA8 event generator: a) the initial state *d*-quark comes from a perturbative  $g \rightarrow d\bar{d}$  splitting process, b) the initial state *d*-quark comes from a raw *d*-quark sPDF.



#### According to the PYTHIA8 model

$$\begin{aligned} T_{j_1 j_2 j_3}(x_1, x_2, x_3, Q) &= f_{j_1}^r(x_1, Q) f_{j_2}^{m \leftarrow j_1, x_1}(x_2, Q) f_{j_3}^{m \leftarrow j_1, x_1, j_2, x_2}(x_3, Q), \\ D_{j_1 j_2}(x_1, x_2, Q) &= f_{j_1}^r(x_1, Q) f_{j_2}^{m \leftarrow j_1, x_1}(x_2, Q). \end{aligned}$$

- To construct tPDFs one needs to access sPDFs used in PYTHIA8 at different generation stages.
- The tPDFs are constructed in a Monte Carlo way by taking an average over a large sample of "events".
- The approach can be applied to construct nPDFs as well!

#### As a baseline we use "naive" approach to tPDFs and dPDFs



#### Let's check momentum rule first

<i>x</i> <sub>1</sub>	<i>x</i> <sub>2</sub>	<i>j</i> 1	<b>j</b> 2	Pythia tPDFs	"Naive" tPDFs
10 <sup>-6</sup>	$10^{-4}$	u	и	0.996	0.996
10 <sup>-3</sup>	$10^{-4}$	u	и	0.997	0.997
10 <sup>-1</sup>	$10^{-4}$	u	и	1.007	1.096
0.2	$10^{-4}$	u	u	1.008	1.195
0.4	$10^{-4}$	u	u	1.007	1.390
0.8	$10^{-4}$	u	u	1.002	1.626

Test of the momentum sum rule for the tPDFs.

Note that the factor  $\theta(1 - x_1 - x_2 - x_3)$  in the definition of "naive" tPDFs does not imply that tPDFs obey the momentum sum rule!



#### Now let's check number rule

We define

$$R_{j_1 j_2 j_3}(x_1, x_2, x_3, Q) \equiv x_3 \, \frac{T_{j_1 j_2 j_3}(x_1, x_2, x_3, Q) - T_{j_1 j_2 \overline{j_3}}(x_1, x_2, x_3, Q)}{D_{j_1 j_2}(x_1, x_2, Q)},$$

which can be seen as a response of the valence sPDF  $f_{j_{3\nu}}(x_3, Q)$  to the first two interactions.

## Check of the number rule





The responses of the valence *u*-quark sPDF  $R_{u\bar{u}u}(x_1, x_2, x_3, Q)$  as function of  $x_3$  for  $x_1 \in [10^{-6}, 0.8]$  and  $x_2 = 10^{-4}$ . The response functions are averaged over  $10^7$  function calls.



#### The numerical integration over the response function yields

x <sub>1</sub>	x <sub>2</sub>	$N_{u_v}$ Pythia	$N_{u_v}$ "Naive"
10 <sup>-6</sup>	10 <sup>-4</sup>	2.019	2.006
10 <sup>-3</sup>	10 <sup>-4</sup>	2.005	2.006
10 <sup>-1</sup>	10 <sup>-4</sup>	2.001	2.005
0.2	10 <sup>-4</sup>	2.000	2.005
0.4	10 <sup>-4</sup>	1.999	1.997
0.8	10 <sup>-4</sup>	1.995	1.708

Integration over  $R_{u\bar{u}u}$  response function with respect to  $x_3$  at fixed  $x_1$ ,  $x_2$ .

- Similar checks can be made for other flavour combinations.
- PYTHIA8 tPDFs preserve the sum rules at about 1% accuracy level.
- PYTHIA8 tPDFs do not obey DGLAP evolution equation and are asymmetric.



#### Summary and possible next steps

- We generalized the GS sum rules for the case of tPDFs.
- The sketch of proof of sum rules for "bare" and renormalized tPDFs is given.
- We demonstrated how one can construct asymmetric dPDFs, tPDFs (and nPDFs!) using PYTHIA8 code.
- Our attempt to construct symmetric tPDFs was not successful. However, the largest violations of the sum rules by symmetric tPDFs appear in the "deep valence region" (x > 0.4).
- Try to modify PYTHIA8 model to suppress large violations of the sum rules by symmetrized PYTHIA8 tPDFs.
- Make a phenomenological study with a tPDFs for different TPS production states. (e.g. to extend 4-jet DPS predictions of 2008.08347 to the TPS case using PYTHIA8 tPDFs).

Those who pursue the scientific way In a different language display Their ignorance and the way they pray. They too one day shall be dust and clay. Omar Khayyam, Rubaiyat

# Thank you!

### Backup: symmetrization





The responses of the valence *u*-quark sPDF  $R_{u\bar{u}u}(x_1, x_2, x_3, Q)$  as functions of  $x_3$  for  $x_1 \in [10^{-6}, 0.8]$  and  $x_2 = 10^{-4}$ . The green lines are symmetrized PYTHIA8 tPDFs.



#### The numerical integration over the response function yields

<i>x</i> <sub>1</sub>	<i>x</i> <sub>2</sub>	$N_{u_v}$ Pythia	$N_{u_v}$ Pythia sym.	$N_{u_v}$ "Naive"
10 <sup>-6</sup>	$10^{-4}$	2.019	2.542	2.006
10 <sup>-3</sup>	$10^{-4}$	2.005	2.154	2.006
10 <sup>-1</sup>	$10^{-4}$	2.001	2.188	2.005
0.2	$10^{-4}$	2.000	2.189	2.005
0.4	$10^{-4}$	1.999	2.161	1.997
0.8	$10^{-4}$	1.995	2.079	1.708

Integration over  $R_{u\bar{u}u}$  response function with respect to  $x_3$  at fixed  $x_1$ ,  $x_2$ .

### Backup: However, sometimes





The responses of the "valence" *s*-quark sPDF  $R_{s\bar{s}s}(x_1, x_2, x_3, Q)$  as functions of  $x_3$  for  $x_1 \in [10^{-6}, 0.8]$  and  $x_2 = 10^{-4}$ . The green lines are symmetrized PYTHIA8 tPDFs.