Topological susceptibility of $2d$ CP^1 or $O(3)$ non-linear σ-model: is it divergent or not?

Claudio Bonanno, Massimo D'Elia, **Francesca Margari**⋆

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⋆Università di Pisa and INFN Sezione di Pisa, Largo B. Pontecorvo 3, I-56127 Pisa, Italy - f.margari@studenti.unipi.it

Introduction

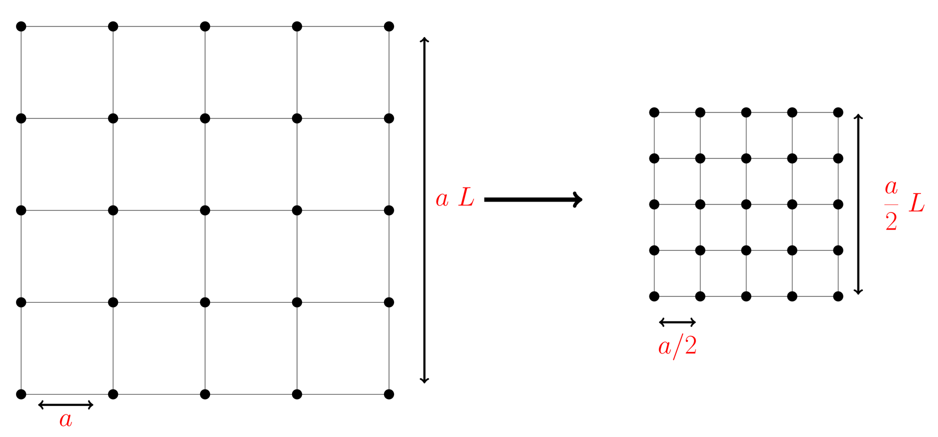
The study of the θ-dependent vacuum energy density is of particular relevance in QCD and in $SU(N)$ gauge theories:

- The QCD θ-term enters in hadronic phenomenology: θ-dependence of pure Yang–Mills theories enters in the η' physics.
- The θ-term introduces a violation of the CP-symmetry $\Rightarrow$ the strong CP problem and the Peccei-Quinn axion.

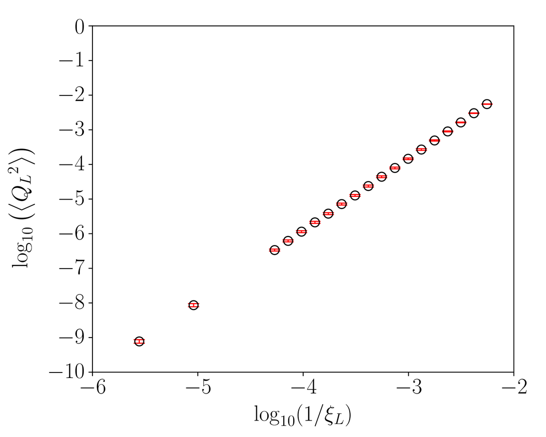
The lattice approach is a natural first-principle tool to study the non-perturbative properties of gauge theories. In the literature there are many numerical studies devoted to the study of simpler toy models as a test-bed for new algorithms and numerical methods for the study of gauge theories. An example is provided by the $2d$ CP^{N-1} models. Many important features of QCD are shared with the CP^{N-1} models, such as asymptotic freedom, confinement and non-trivial topological structure. Unlike QCD, which involves $SU(3)$ matrices, CP^{N-1} are vector models and more accurate computations can be carried on with **less numerical effort**. A peculiarity of these models is the pathological behavior in the limit $N \to 2$. In the case of CP^1, the topological susceptibility χ is expected, based on perturbative computations, to develop a divergence. The divergence is due to the dominance of instantons of arbitrarily small size and its detection by numerical lattice simulations is notoriously difficult, because it is logarithmic in the lattice spacing.

Numerical methods

New strategy: we study the behavior of the model when **the volume is fixed in dimensionless lattice units**. Approaching the continuum limit at fixed volume in lattice units = keep the number of sites L^2 fixed.

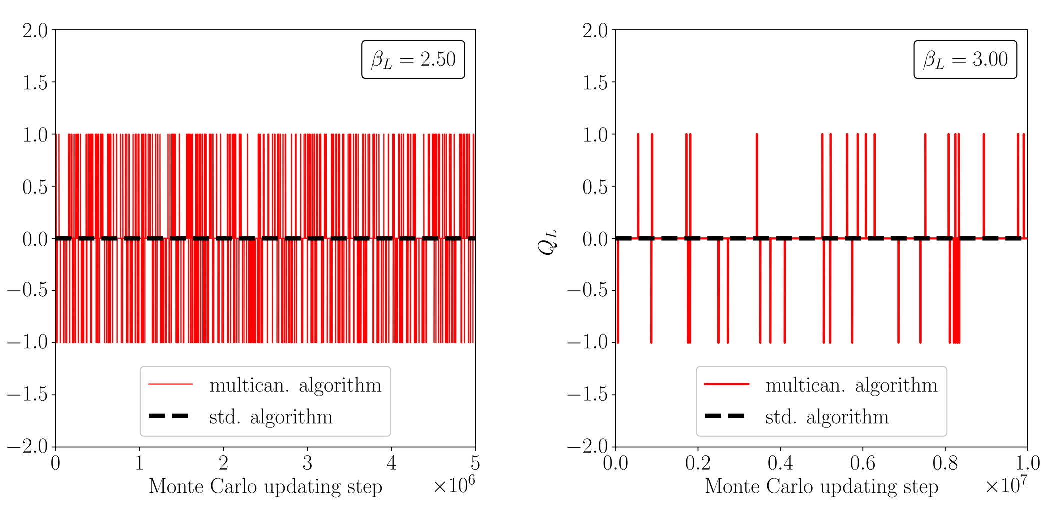


Multicanonical algorithm



On the adopted lattices, $\langle Q^2 \rangle = V\chi \ll 1$, i.e., we have to face the same problem of high-temperature QCD.

We implemented the **multicanonical algorithm** [1–4]. With Monte Carlo (MC) methods, simulations are carried out by extracting lattice gauge configurations with a given statistical weight. The general idea is to add a bias potential to the action, so that the probability of visiting suppressed topological sectors is enhanced. The MC averages with respect to the original distribution are then obtained by means of a standard reweighting procedure. We show MC histories of the cooled geometric topological charge for the smallest lattice spacings with the standard and the multicanonic algorithm.



Continuum limit at fixed volume in lattice units

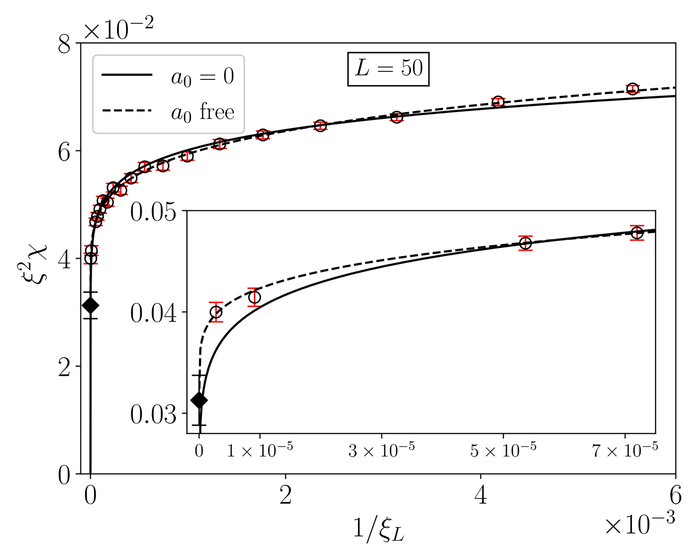
In some regimes, topological properties can be described in terms of a weakly-interacting gas of Instantons and Anti-instantons. Field configuration with $n = n_I + n_A$ instantons has $Q = n_I - n_A$. In this approximation, given the number of instantons per volume and size unit, $d_I(\rho) \propto \rho^{N-3}$, for $N = 2$ the topological susceptibility $\chi \sim \int_{\rho_{\min}}^{\rho_{\max}} d_I(\rho) d\rho$ is divergent when $a \to 0$ because of the dominant contribution of instantons with $\rho \sim a \to 0$. We study the behavior of χ approaching the continuum limit keeping the volume fixed in lattice units. Using this approach, we have:

$$\xi^2\chi = \xi^2 \frac{\langle Q^2 \rangle}{l^2} \propto \begin{cases} a^{N-2}\left[\left(\frac{L}{R}\right)^{N-2} - 1\right] & (N > 2), \\ \log\left(\frac{L}{R}\right) & (N = 2), \end{cases}$$

where R is an effective parameter accounting for the ratio between $\rho_{\max}$ and $\rho_{\min}$ which can live on the same lattice, which is expected in this case to be proportional to $l = aL$ (since $L \ll \xi_L$), i.e., $\rho_{\max}/\rho_{\min} \equiv L/R$, with R independent of L.

If the continuum of χ computed on lattices with $L/\xi_L \gg 1$ is **finite**, the continuum limit of χ at fixed L is expected to vanish: when $a \to 0$ at fixed L, the physical lattice size vanishes proportionally to a, and any topological fluctuation on physical scales disappears.
If the continuum limit of χ taken at fixed $L/\xi_L \gg 1$ is **logarithmically divergent**, we expect to approach a constant and finite value for χ when, instead, the continuum limit is taken at fixed L.

Numerical results

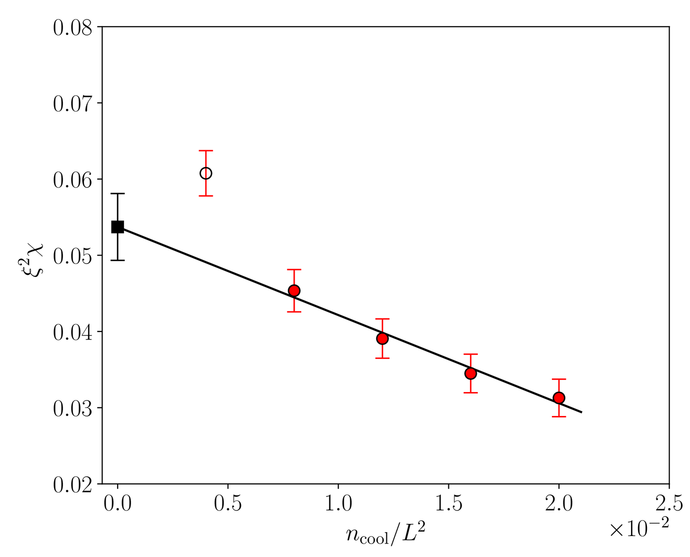


Continuum limit at fixed L

We extrapolated the quantity $\xi^2\chi$ towards the continuum limit fitting the ξ_L-dependence of $\xi^2\chi$ according to the fit function

$$f(x) = a_0 + a_1 x^c, \qquad x = 1/\xi_L,$$

where c is a free exponent. The best fit yields $a_0 = 0.031(2)$, $c = 0.20(2)$ and $\tilde{\chi}^2/\mathrm{dof} = 8.0/16$. The best fit performed fixing $a_0 = 0$ yields a $\tilde{\chi}^2/\mathrm{dof} = 42.7/17$, thus clearly providing a bad description of our data.

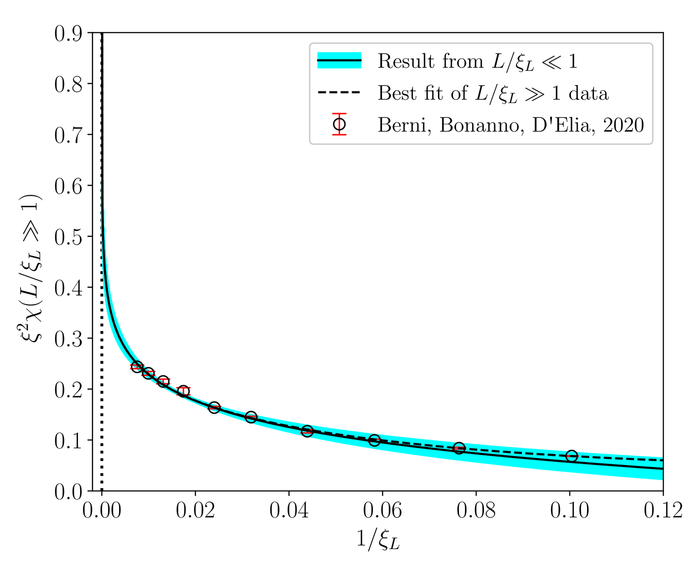


Zero cooling extrapolation at fixed L

We observe a systematic drift of our continuum extrapolations for $\xi^2\chi$ as n_{cool} is increased. Since we took $a \to 0$ fixing $L = l/a$, the quantity $n_{\mathrm{cool}}/L^2 = (r_s/l)^2$ is kept constant in our continuum extrapolation and does not disappear from the game. We extrapolate continuum results towards the $n_{\mathrm{cool}} \to 0$ limit:

$$\xi^2\chi\left(\frac{n_{\mathrm{cool}}}{L^2}\right) = \xi^2\chi\left(\frac{n_{\mathrm{cool}}}{L^2} = 0\right) + A\frac{n_{\mathrm{cool}}}{L^2}.$$

The best fit yields $\xi^2\chi(n_{\mathrm{cool}} = 0) = 0.054(4)$ and $\tilde{\chi}^2/\mathrm{dof} = 0.32/2$.



Checking the thermodynamic limit

$\xi^2\chi(\xi_L) \underset{\xi_L\to\infty}{\sim} C\log\left(\frac{\xi_L}{R}\right)$ if $L \gg \xi_L$ and $\xi^2\chi(L) \underset{\xi_L\to\infty}{\sim} C\log\left(\frac{L}{\overline{R}}\right)$ if $L \ll \xi_L$.
where R and $\overline{R}$ are two different parameters, while the pre-factor C is the same, as we checked. On top of the $L/\xi_L \gg 1$ determinations for $\xi^2\chi$ of Ref. [5], we plot the curve $C\log(\xi_L/R) + \log(R/\overline{R})$, using $C = 0.074(11)$ coming from the logarithmic best fit of the fixed-L results obtained in this work. The two curves collapse on top of each other.

Conclusions

Our results show that the continuum limit of χ of the CP^1 model obtained at fixed L, and after extrapolation to zero cooling steps, is non-vanishing, as predicted by semiclassical computations. Thanks to the multicanonical algorithm, we were able to measure χ for a $P(Q)$ with $\langle Q^2 \rangle$ as low as $\sim 10^{-9}$. The adoption of the multicanonical algorithm allows to reduce the computational effort by up to two orders of magnitude for the smallest explored lattice spacings. At the end, we provide evidence that our investigation at fixed L is perfectly consistent with what would be obtained in the thermodynamic infinite volume limit.

References

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