Lattice calculation of the R-ratio smeared with Gaussian kernels

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Extended Twisted Mass Collaboration

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Probing the R-ratio on the lattice

arXiv:2212.08467

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(Extended Twisted Mass Collaboration (ETMC))

- Setup and strategy
- Results and current status of the analysis
- Outlook and conclusions

$$a_\mu^{\text{HVP,LO}} = \frac{\alpha^2}{3\pi^2} \int_{2m_\pi}^{\infty} \mathrm{d}s \frac{K(s)}{s} \boldsymbol{R(s)} \qquad s = E^2$$

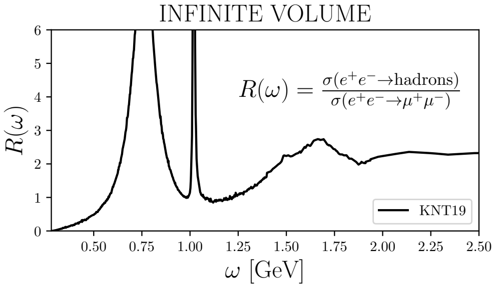
*"There is a **tension** between our result and those obtained by the **R-ratio** method."*
BMW 2020 (arXiv:2002.12347)

*"Our accurate lattice results in the short and intermediate windows point to a possible **deviation** of the $\mathbf{e^+e^-}$ **cross section data** with respect to Standard Model predictions in the low and intermediate energy regions."*
ETMC 2022 (arXiv:2206.15084v3)

*"The tension for the intermediate window between lattice QCD and the dispersive result needs to be addressed in future work. As it stands, this tension may be interpreted as a yet to be understood **new physics contribution to hadronic $\mathbf{e^+e^-}$ decays.**"*
RBC-UKQCD 2023 (arXiv:2301.08696)

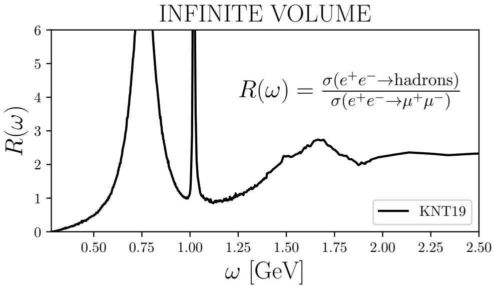
We want to investigate $R(E)$ on the lattice and not a_μ.

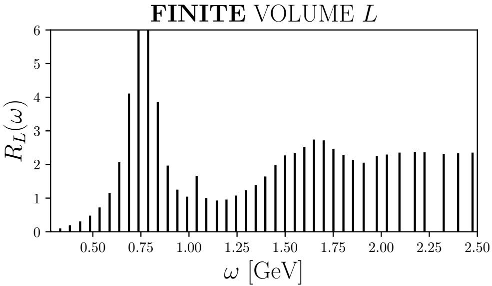
$$\underbrace{\left\langle \frac{1}{3}\int \mathrm{d}^3x \sum_i \hat{J}_i(t,\boldsymbol{x})\hat{J}_i(0)\right\rangle}_{V_L(t)} = \int_{2m_\pi}^{\infty} \mathrm{d}\omega\, e^{-\omega t}\, \underbrace{\omega^2 R_L(\omega)}_{\rho_L(\omega)}$$



Difficulties of the problem

$$\underbrace{\left\langle \frac{1}{3}\int \mathrm{d}^3x \sum_i \hat{J}_i(t,\boldsymbol{x})\hat{J}_i(0) \right\rangle}_{\color{red}{V_L(t)}} = \int_{2m_\pi}^{\infty} \mathrm{d}\omega\, e^{-\omega t}\, \underbrace{\omega^2 \color{blue}{R_L(\omega)}}_{\rho_L(\omega)}$$

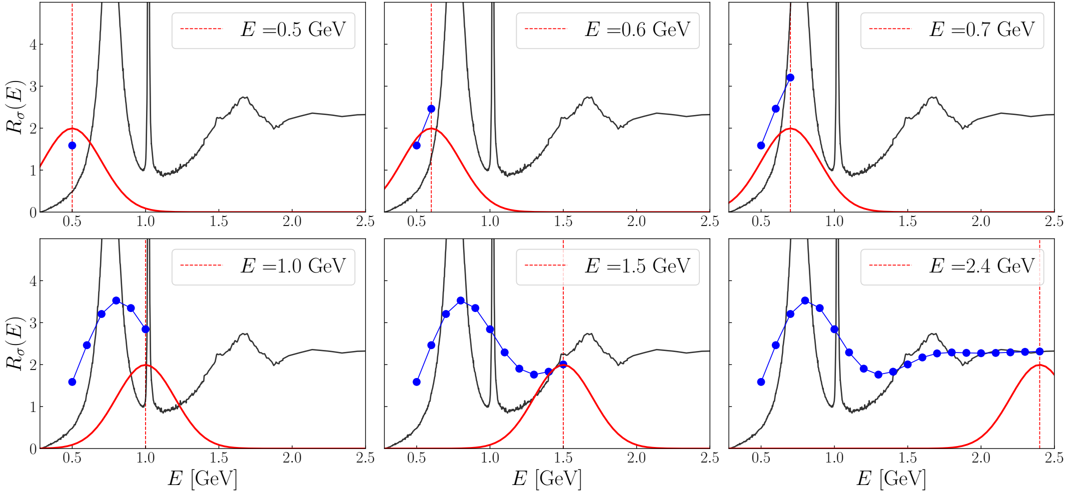


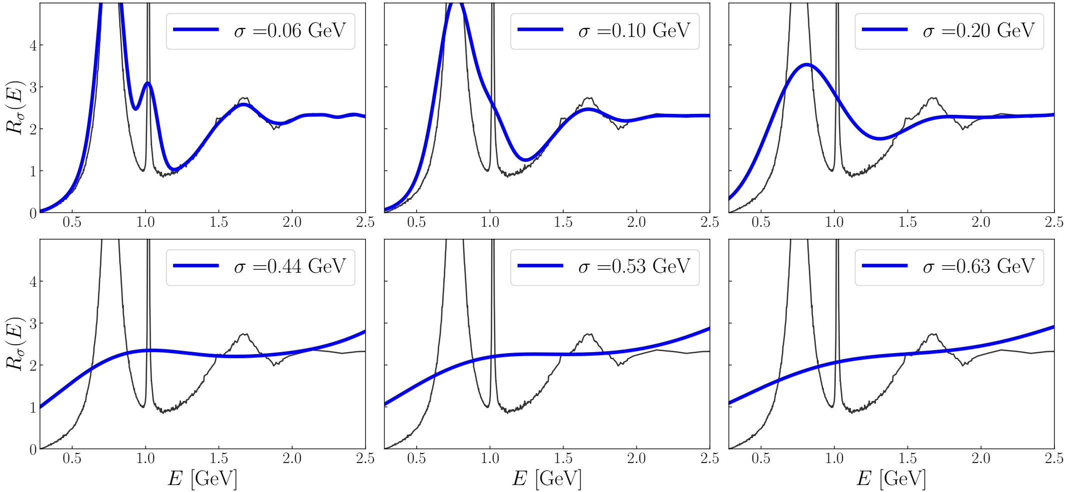


- ▷ $V_L(t) + \delta V(t)\,, \qquad t/a = 1, 2, \cdots \mapsto$ **Ill-posed** inverse problem
- ▷ $\rho_L(\omega) \sim \sum_n c_n \delta(\omega - \omega_n)$ is a **distribution** for $L < \infty$

Solution: consider a **smeared** spectral density

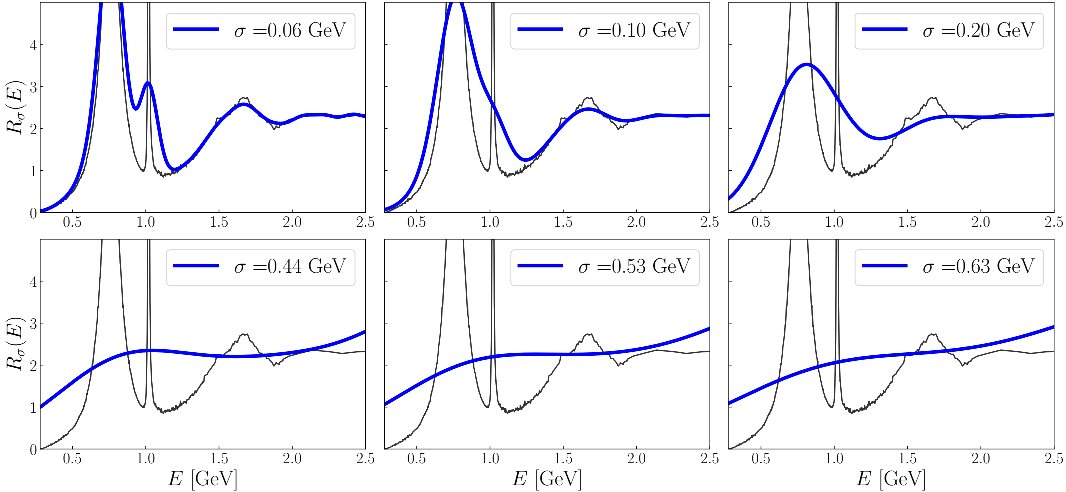
$$R_\sigma(E) = \int_{2m_\pi}^{\infty} d\omega\, G_\sigma(\omega, E) R(\omega) \qquad G_\sigma(\omega, E) = \frac{1}{\sqrt{2\pi}\sigma} \exp\left(-\frac{(E-\omega)^2}{2\sigma^2}\right)$$





The smearing is **well defined** also **for finite volume spectral densities**:

$$R(E) = \lim_{\sigma \to 0} \lim_{L \to \infty} R_{L,\sigma}(E)$$



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$$\cancel{R(E) = \lim_{\sigma \to 0} \lim_{L \to \infty} R_{L,\sigma}(E)}$$

$$\boxed{\text{Compare} \quad R_\sigma(E) \quad \text{with} \quad R_\sigma^{\text{exp}}(E) \quad \text{at} \quad \sigma > 0}$$

1) $$V(t) = \int_{2m_\pi}^{\infty} \mathrm{d}\omega\, e^{-\omega t} \omega^2 R(\omega)$$

2) $$R_\sigma(E) = \int_{2m_\pi}^{\infty} \mathrm{d}\omega\, \frac{G_\sigma(E,\omega)}{\omega^2} \omega^2 R(\omega)$$

3) $$\frac{G_\sigma(E,\omega)}{\omega^2} \sim \sum_{n=1}^{T} g_n\, e^{-\omega t_n}$$

4) $$\Longrightarrow \quad R_\sigma(E) \sim \sum_{n=1}^{T} g_n \int_{2m_\pi}^{\infty} \mathrm{d}\omega\, e^{-\omega t_n} \omega^2 R(\omega) = \sum_{n=1}^{T} g_n\, V(t_n)$$

HLT approach: M. Hansen, A. Lupo, N. Tantalo HLT (arXiv:1903.06476), variation of Backus-Gilbert method
Extensively investigated in arXiv:2111.12774

- Choose the $\boldsymbol{g}$ coefficients minimizing

$$W[\lambda, \boldsymbol{g}] = (1-\lambda)A[\boldsymbol{g}] + \lambda B[\boldsymbol{g}]$$

- Accuracy of the reconstructed kernel

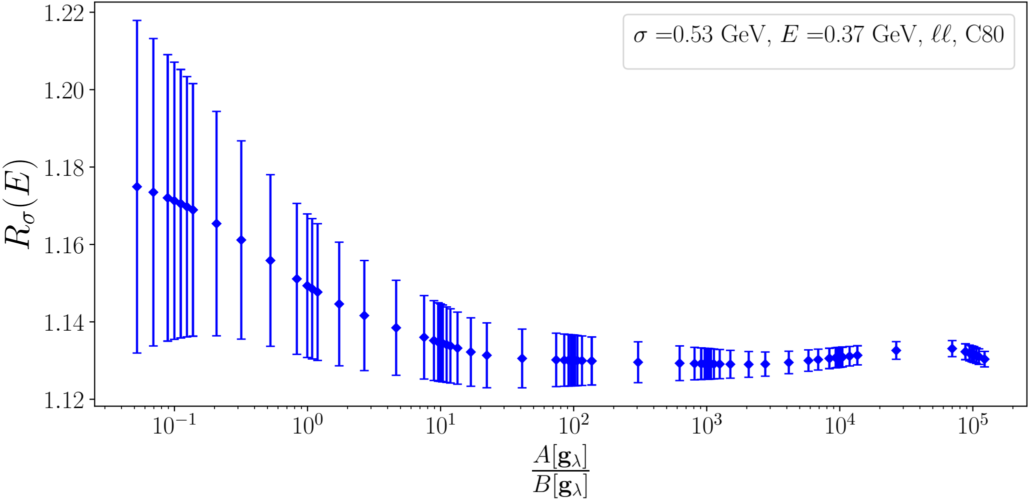
$$A[\boldsymbol{g}] = \frac{\int_{2m_\pi}^{\infty} \mathrm{d}\omega \left\{ \frac{G_\sigma(\omega, E)}{\omega^2} - \sum_{n=1}^{T} g_n e^{-\omega t_n} \right\}^2 \cdot e^{\alpha\omega}}{\int_{2m_\pi}^{\infty} \mathrm{d}\omega \left\{ \frac{G_\sigma(\omega, E)}{\omega^2} \right\}^2 \cdot e^{\alpha\omega}} \qquad \alpha = 2^{-}$$

- Suppression of the statistical error

$$B[\boldsymbol{g}] \propto \boldsymbol{g}^T \cdot \widehat{\mathrm{COV}}\left[V(t)\right] \cdot \boldsymbol{g} \equiv \sigma^2_{R_\sigma}$$

We look for stability w.r.t. changing of relative functional magnitudes

$$W[\lambda, \boldsymbol{g}] = (1-\lambda)A[\boldsymbol{g}] + \lambda B[\boldsymbol{g}] \qquad \rightarrow \qquad \boldsymbol{g}_\lambda$$



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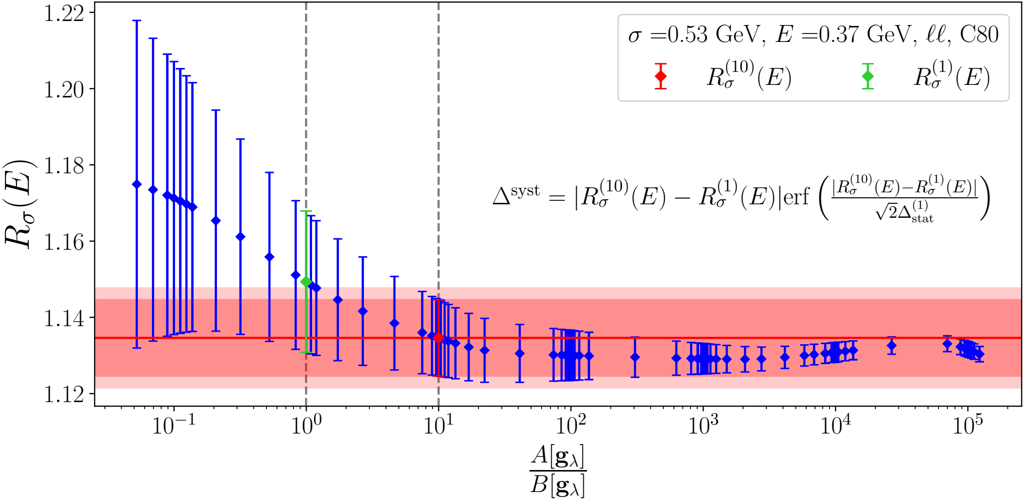

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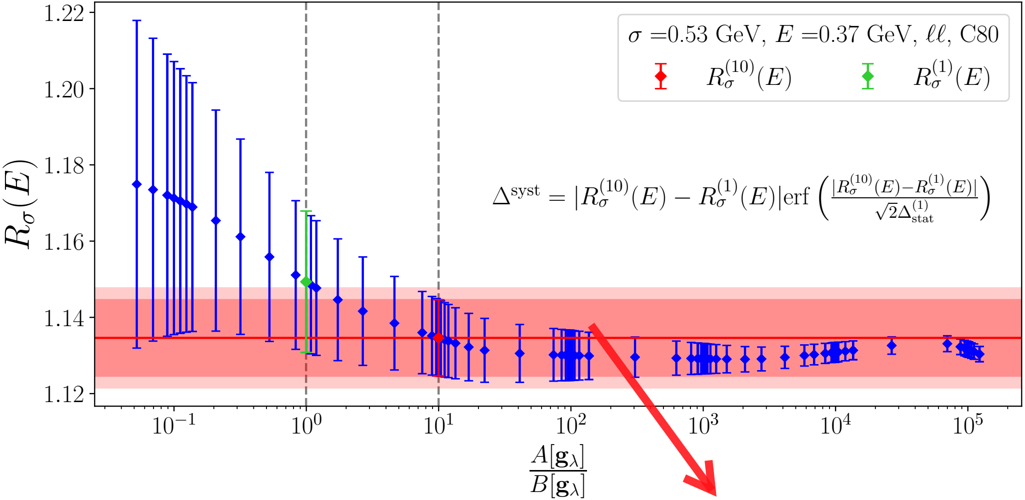
$$W[\lambda, \boldsymbol{g}] = (1-\lambda)A[\boldsymbol{g}] + \lambda B[\boldsymbol{g}] \qquad \to \qquad \boldsymbol{g}_\lambda$$



CONSERVATIVE ERROR ESTIMATION

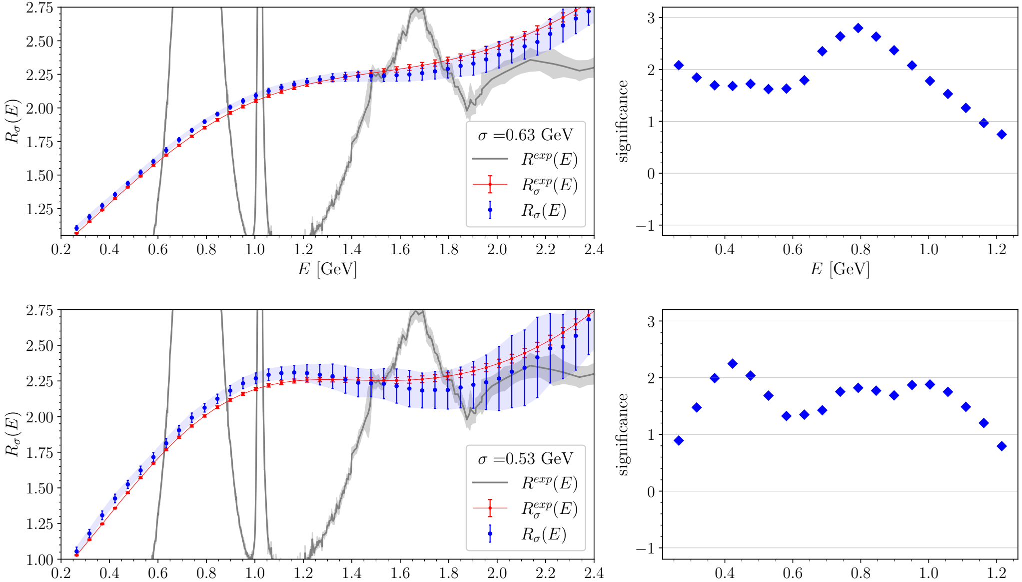
(arXiv:2206.15084) $N_f = 2+1+1$ employed ensembles:

ensemble	$L^3 \cdot T$	a (fm)	L (fm)	M_π (MeV)	β
B64	$64^3 \cdot 128$	0.07961(13)	5.09	135.2(2)	1.778
B96	$96^3 \cdot 192$	0.07961(13)	7.64	135.2(2)	1.778
C80	$80^3 \cdot 160$	0.06821(12)	5.46	134.9(3)	1.836
D96	$96^3 \cdot 192$	0.05692(10)	5.46	135.1(3)	1.900

The analysis includes:

- Procedure applied to $\sigma = \{0.63, 0.53, 0.44\}$ GeV and $E \in [0.25, 2.4]$ GeV $\mapsto \boldsymbol{\Delta_\sigma^{\text{rec}}(E)}$
- **Both connected and disconnected** contributions to $V(t)$
- Two regularizations considered (Twisted-Mass and Osterwalder-Seiler)
- Constrained and unconstrained linear **continuum extrapolation** $\mapsto \boldsymbol{\Delta_\sigma^{a}(E)}$
- **Data-driven finite volume effects** estimate $\mapsto \boldsymbol{\Delta_\sigma^{L}(E)}$

Final results



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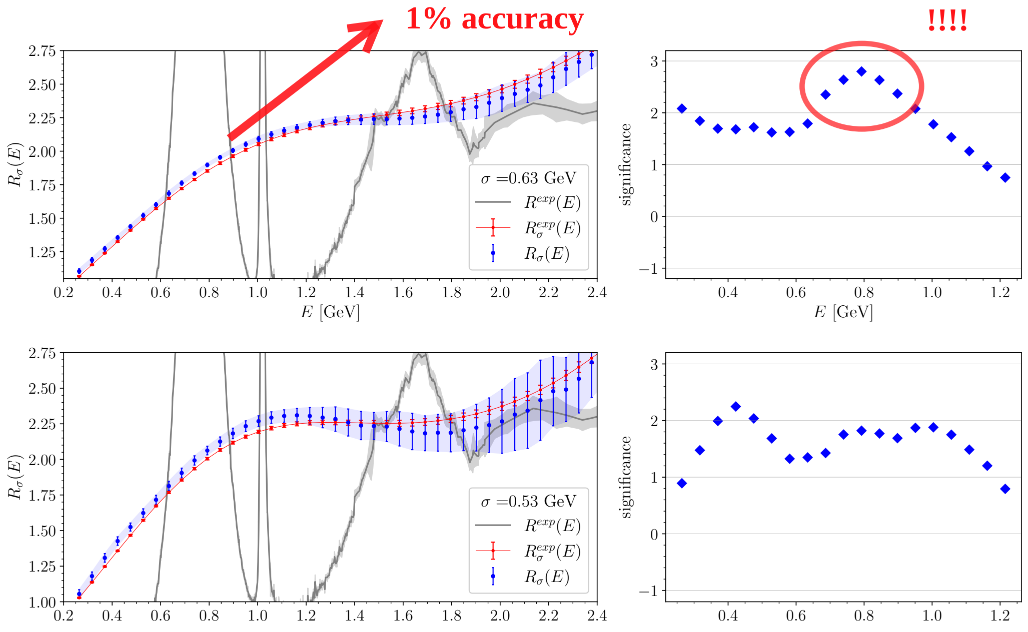
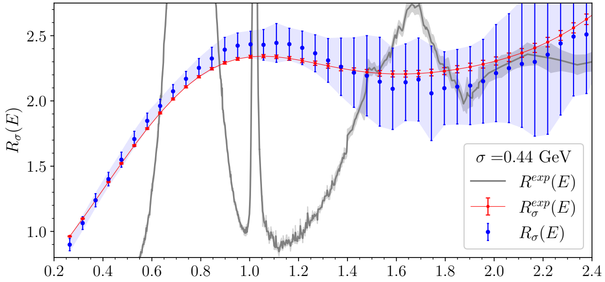
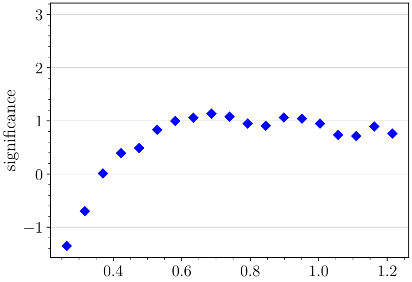
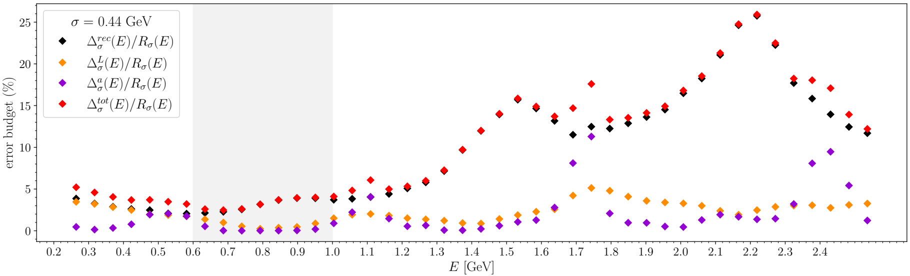


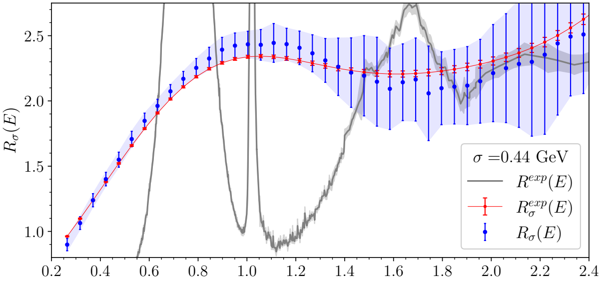
Future prospectives - reduction of the error

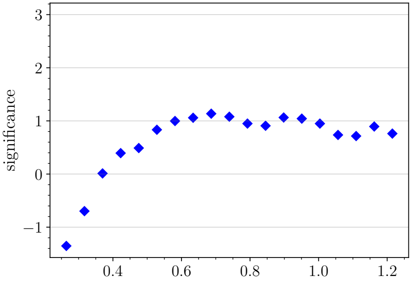




The overall error increases when σ gets smaller. Error budget:







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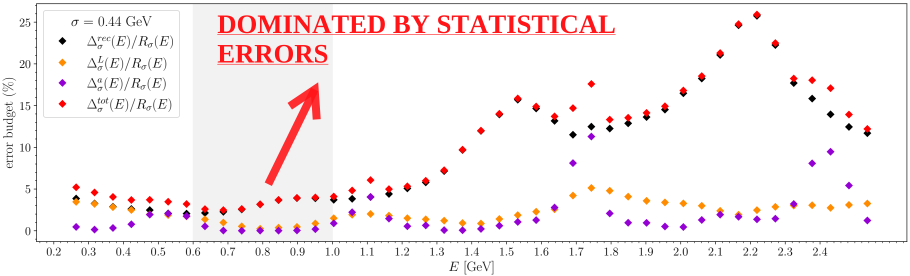
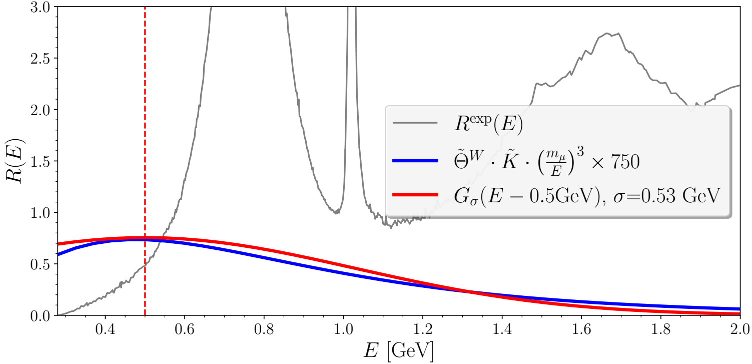


- Gaussian kernels are not much different from the kernel providing $a_\mu^{\mathrm{W}} \to a_\mu^{\mathrm{W}}(\mathrm{IB}) \sim 0.2\%$
- However $\dfrac{R_\sigma(E)}{R_\sigma^{\exp}(E)} - 1 \sim \mathcal{O}(5\%)$ at $E = 0.5$ GeV



Isospin-breaking effects may become relevant when increasing the resolution in the energy (decreasing σ)

Isospin-breaking effects will be be computed from first principles

Outlook and conclusions

- Finite volume spectral densities require a **regulator** $\mapsto$ **smearing**
- The **comparison** between theoretical and experimental results can be done **at fixed regulator**
- Smeared spectral densities can be calculated on the lattice through **spectral reconstruction techniques** (HLT)
- We calculated $R_\sigma(E)$ for $\sigma = 0.44, 0.53, 0.63$ GeV and achieved 1% **accuracy** around $E = 0.8$ GeV
- Our result is in tension with the experimental one at the level of **2-3 standard deviations** around 0.8 GeV
- This tension was expected from a_μ^{W} calculation but it is **not yet understood**
- Reduce the error on the theoretical side and solve the tension among e^+e^- experiments will hopefully clarify the situation
- The study of observables related to $R(E)$ and **localized in the energy** could provide a valuable probe to unravel new physics BSM.

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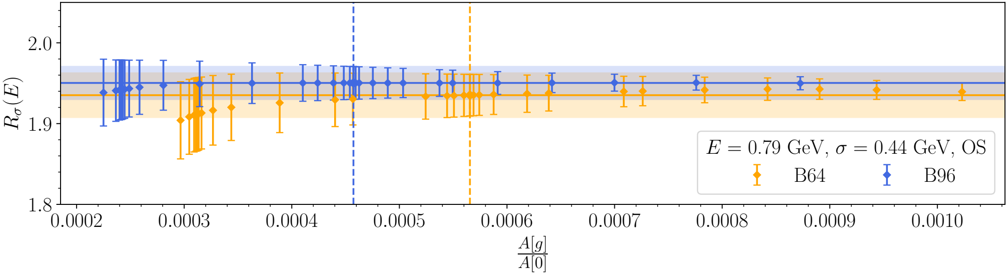
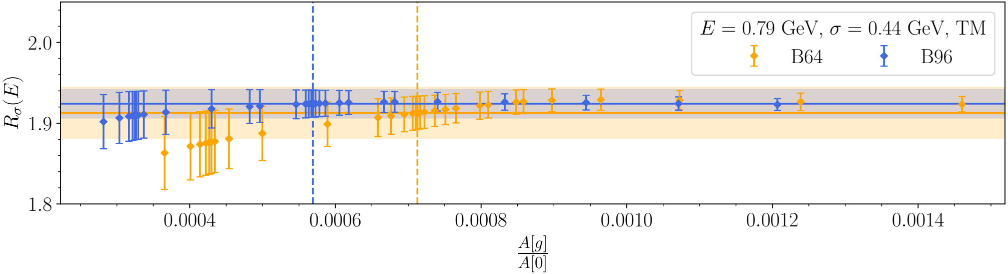
Thank you for the attention!

Backup slides



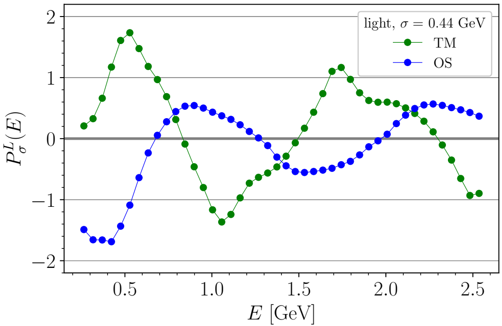
$E = 0.79$ GeV, $\sigma = 0.44$ GeV, OS
B64
B96
2.0
1.9
1.8
$R_\sigma(E)$
0.0002
0.0003
0.0004
0.0005
0.0006
0.0007
0.0008
0.0009
0.0010
$\frac{A[g]}{A[0]}$

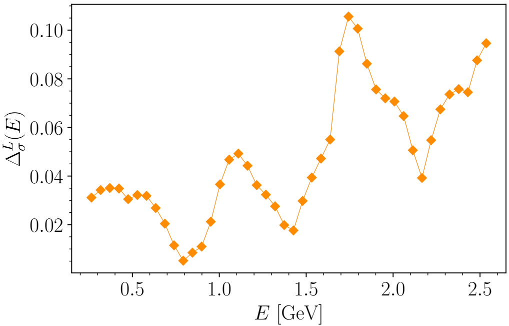


$E = 0.79$ GeV, $\sigma = 0.44$ GeV, TM
B64
B96
2.0
1.9
1.8
$R_\sigma(E)$
0.0004
0.0006
0.0008
0.0010
0.0012
0.0014
$\frac{A[g]}{A[0]}$

Data-drive estimation of **finite volume effects** (reg=TM/OS)

$$P^L_{\sigma,\mathrm{reg}}(E) = \frac{R_{\mathrm{reg}}(B96) - R_{\mathrm{reg}}(B64)}{\sqrt{\Delta^{\mathrm{rec}}_{\mathrm{reg}}(B96)^2 + \Delta^{\mathrm{rec}}_{\mathrm{reg}}(B64)^2}} \qquad \Delta^L_\sigma(E) = \max_{\mathrm{reg}} |R_{\mathrm{reg}}(B96) - R_{\mathrm{reg}}(B64)| \,\mathrm{erf}\left(\frac{P^L_{\sigma,\mathrm{reg}}(E)}{\sqrt{2}}\right)$$



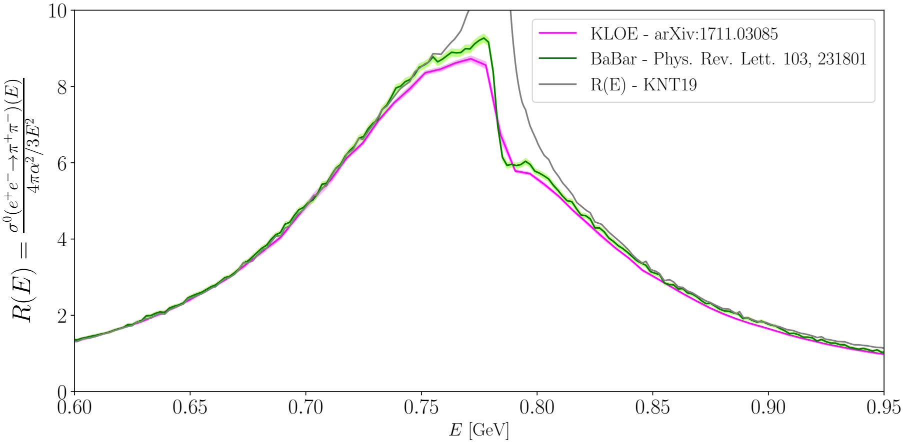


On the experimental side ...

More than 30 exclusive channels in low energy region (table from KNT19)

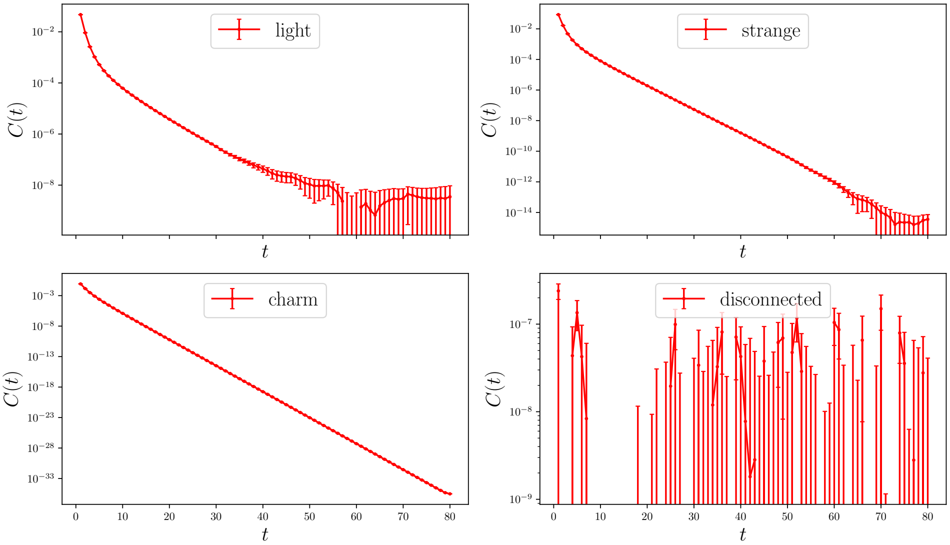
Channel	$a_e^{\text{had, LO VP}} \times 10^{14}$	$a_\mu^{\text{had, LO VP}} \times 10^{10}$	$a_\tau^{\text{had, LO VP}} \times 10^{8}$	$\Delta\alpha_{\text{had}}^{(5)}(M_Z^2) \times 10^4$	$\Delta\nu_{\text{Mu}}^{\text{had, VP}}$ (Hz)
Chiral perturbation theory (ChPT) threshold contributions					
$\pi^0\gamma$	0.04 ± 0.00	0.12 ± 0.01	0.03 ± 0.00	0.00 ± 0.00	0.04 ± 0.00
$\pi^+\pi^-$	0.31 ± 0.01	0.87 ± 0.02	0.11 ± 0.00	0.01 ± 0.00	0.25 ± 0.01
$\pi^+\pi^-\pi^0$	0.00 ± 0.00	0.01 ± 0.00	0.00 ± 0.00	0.00 ± 0.00	0.00 ± 0.00
$\eta\gamma$	0.00 ± 0.00	0.00 ± 0.00	0.00 ± 0.00	0.00 ± 0.00	0.00 ± 0.00
Exclusive channels ($\sqrt{s} \leq 1.937$ GeV)					
$\pi^0\gamma$	1.19 ± 0.03	4.46 ± 0.10	1.75 ± 0.04	0.36 ± 0.01	1.45 ± 0.03
$\pi^+\pi^-$	138.59 ± 0.54	503.46 ± 1.91	172.84 ± 0.61	34.29 ± 0.12	159.64 ± 0.60
$\pi^+\pi^-\pi^0$	12.29 ± 0.25	46.73 ± 0.94	20.47 ± 0.39	4.69 ± 0.09	15.48 ± 0.31
$\pi^+\pi^-\pi^+\pi^-$	3.67 ± 0.05	14.87 ± 0.20	11.50 ± 0.16	4.02 ± 0.05	5.58 ± 0.08
$\pi^+\pi^-\pi^0\pi^0$	4.80 ± 0.19	19.39 ± 0.78	14.56 ± 0.58	5.00 ± 0.20	7.22 ± 0.29
$(2\pi^+2\pi^-\pi^0)_{\text{no }\eta\omega}$	0.24 ± 0.02	0.98 ± 0.09	0.84 ± 0.08	0.32 ± 0.03	0.38 ± 0.03
$(\pi^+\pi^-3\pi^0)_{\text{no }\eta}$	0.15 ± 0.03	0.62 ± 0.11	0.54 ± 0.10	0.21 ± 0.04	0.24 ± 0.04
$(3\pi^+3\pi^-)_{\text{no }\omega}$	0.06 ± 0.00	0.23 ± 0.01	0.21 ± 0.01	0.09 ± 0.01	0.09 ± 0.01
$(2\pi^+2\pi^-2\pi^0)_{\text{no }\eta}$	0.33 ± 0.04	1.35 ± 0.17	1.24 ± 0.15	0.51 ± 0.06	0.53 ± 0.07
$(\pi^+\pi^-4\pi^0)_{\text{no }\eta}$	0.05 ± 0.05	0.21 ± 0.21	0.19 ± 0.19	0.08 ± 0.08	0.08 ± 0.08
$(3\pi^+3\pi^-\pi^0)$	0.00 ± 0.00	0.00 ± 0.01	0.00 ± 0.00	0.00 ± 0.00	0.00 ± 0.00
⋮	⋮	⋮	⋮	⋮	⋮
$\Upsilon(4S)$	0.00 ± 0.00	0.01 ± 0.00	0.02 ± 0.00	0.10 ± 0.01	0.00 ± 0.00
pQCD ($\sqrt{s} > 11.199$ GeV)	0.48 ± 0.00	2.07 ± 0.00	5.33 ± 0.00	124.79 ± 0.09	1.34 ± 0.00
Total ($< \infty$ GeV)	186.08 ± 0.66	692.78 ± 2.42	332.81 ± 1.39	276.09 ± 1.12	232.04 ± 0.82

Tension between experiments: whose fault is it?

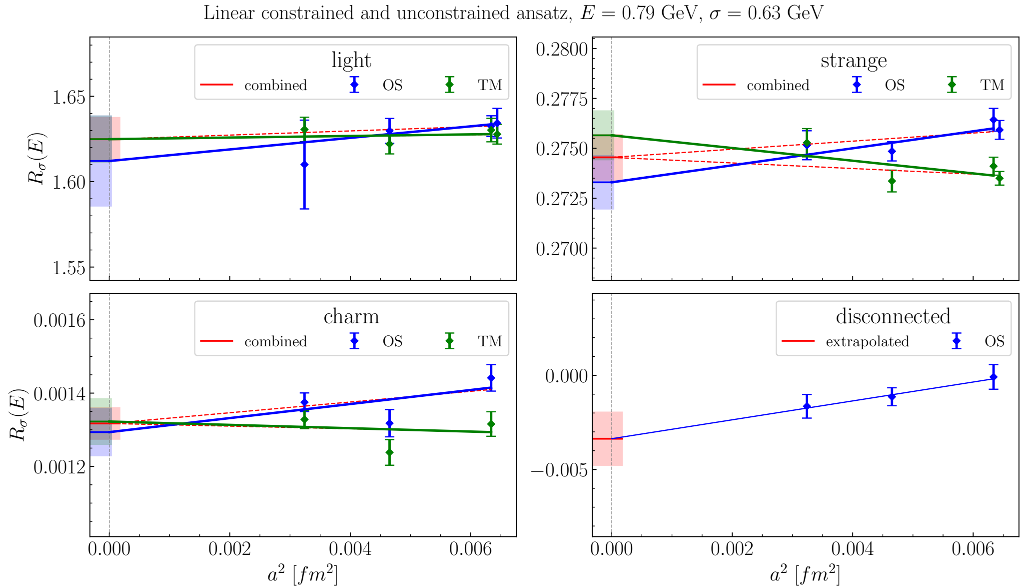


- Our results seem to prefer BaBar data...
- ... but this experimental discrepancy seems not to be enough to explain our observed tension

Correlators in the C80 ensemble



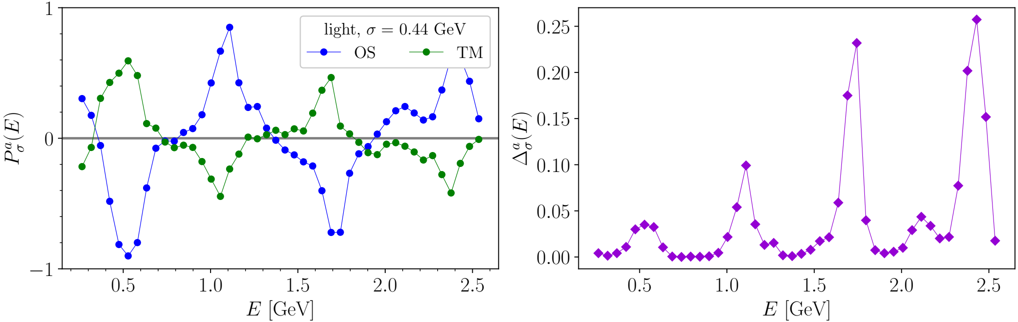
$$C(t) = C^{\text{light}}(t) + C^{\text{strange}}(t) + C^{\text{charm}}(t) + C^{\text{disconnected}}(t)$$



Linear constrained and unconstrained ansatz, $E = 0.79$ GeV, $\sigma = 0.63$ GeV
light
combined
OS
TM
strange
combined
OS
TM
charm
combined
OS
TM
disconnected
extrapolated
OS
$R_\sigma(E)$
$R_\sigma(E)$
1.65
1.60
1.55
0.2800
0.2775
0.2750
0.2725
0.2700
0.0016
0.0014
0.0012
0.000
−0.005
0.000
0.002
0.004
0.006
0.000
0.002
0.004
0.006
a^2 $[fm^2]$
a^2 $[fm^2]$

Data-drive estimation of error associated with continuum extrapolation

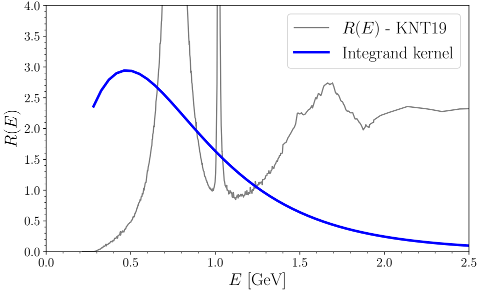
$$P_{\sigma,\mathrm{reg}}(E) = \frac{R^{\mathrm{comb}} - R^{\mathrm{reg}}}{\sqrt{\Delta^2_{\mathrm{comb}} + \Delta^2_{\mathrm{reg}}}} \qquad \Delta^a_\sigma(E) = \mathrm{max}_{\mathrm{reg}} |R^{\mathrm{comb}} - R^{\mathrm{reg}}| \mathrm{erf}\left(\frac{P^a_{\sigma,\mathrm{reg}}(E)}{\sqrt{2}}\right)$$



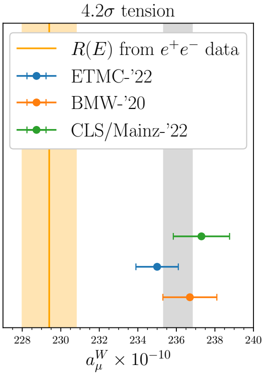
Did we expect such result?

Some hints already come from the intermediate-window observable (RBC/UKQCD)

$$a_\mu^{W,\text{disp.}} \propto \int_{E_{\text{thr}}}^{\infty} \mathrm{d}E \left(\frac{m_\mu}{E}\right)^3 \tilde{K}\left(\frac{E}{m_\mu}\right) \tilde{\Theta}^W(E) R(E) = 229.51(87)$$

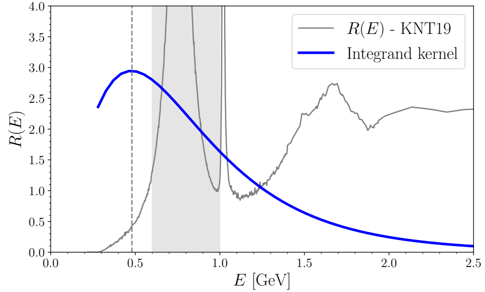


From LQCD:



The tension on a_μ^W arises from the low energy region of $R(E)$

Isospin-Breaking Bffects are responsible for the $\rho - \omega$ mixing around 0.8 GeV but ..



BMW 2020, Lattice calculation

$$a_\mu^W = 236.7(1.4)$$

$$a_\mu^W(IB) = 0.43(4) \sim 0.2\%$$

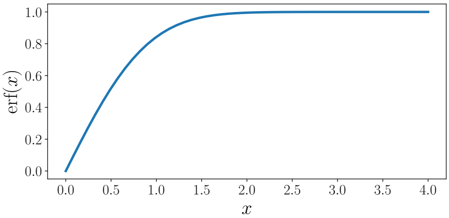
$$\frac{R_\sigma(E) - R_\sigma^{\text{exp}}(E)}{R_\sigma^{\text{exp}}(E)} \sim 2.5\% \quad \text{for } E \text{ in } [0.6, 1]\text{GeV}$$

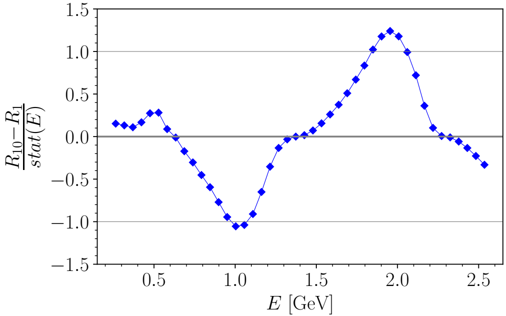
$$\frac{a_\mu^W - a_\mu^{W,\text{disp.}}}{a_\mu^{W,\text{disp.}}} \sim 3\%$$

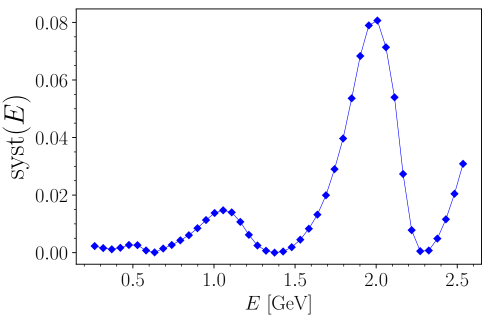
- The Window kernel is slowly decreasing
- But also our Gaussian kernels are still quite broad... potentially large local IB effects are smeared out
- Anyway electromagnetic effects must be computed from first principles

Error associated with the reconstruction

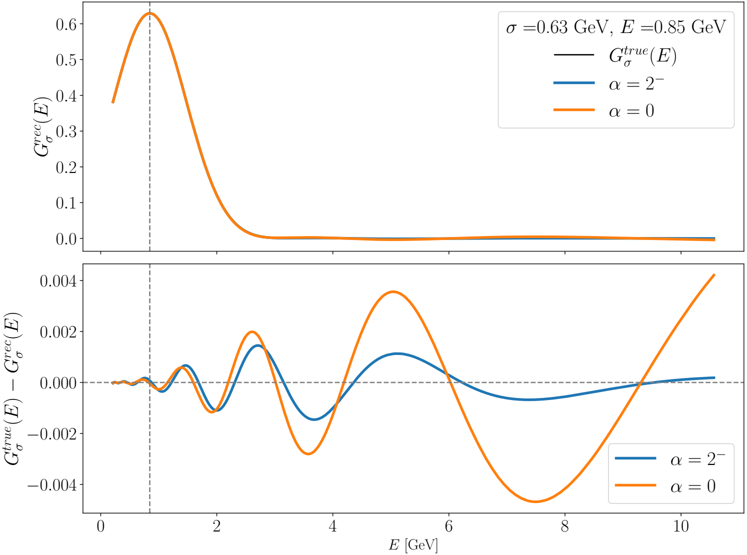
$$\mathrm{syst}(E) = |R_{10} - R_1|\,\mathrm{erf}\left(\frac{|R_{10} - R_1|}{\mathrm{stat}(E)\sqrt{2}}\right)$$



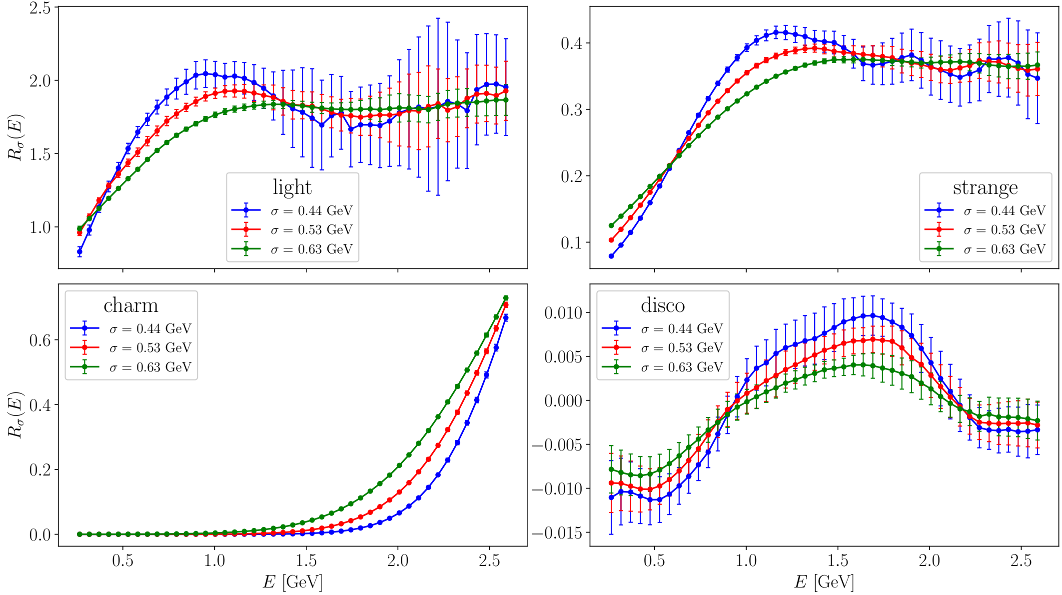




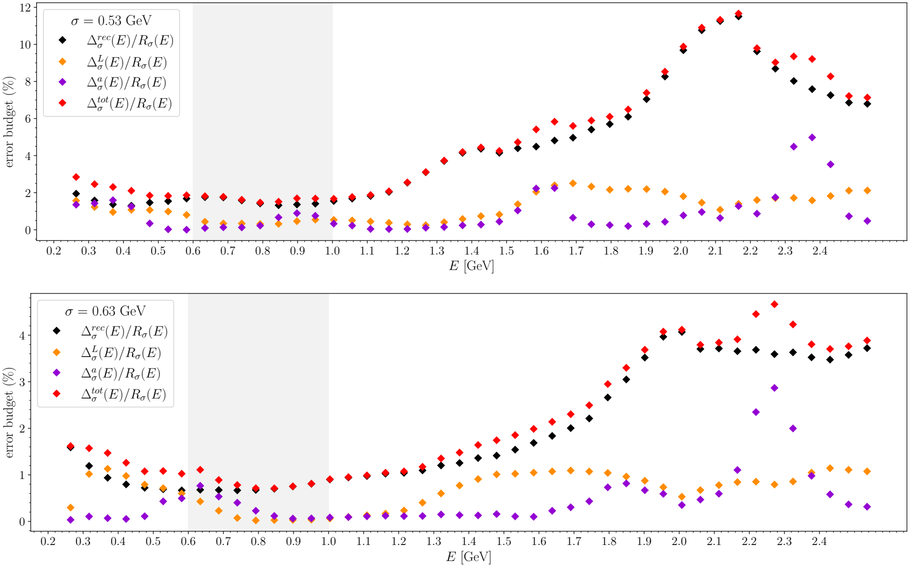
Reconstructed kernels



Final results (separated by flavour)



Error budget



Backup: the functional $A_\alpha[\boldsymbol{g}]$

$$W_\lambda[\boldsymbol{g}] = (1-\lambda)\frac{A_\alpha[\boldsymbol{g}]}{A_\alpha[0]} + \lambda B[\boldsymbol{g}] \qquad K^{\text{rec}}(\omega, E) = \sum_{\tau=1}^{\tau_{\max}} g_\tau(K, E, \cdots)e^{-\tau\omega}$$

where

$$A_\alpha[\boldsymbol{g}] = \int_{E_0}^{\infty} d\omega \left\{K^{\text{true}}(\omega, E) - K^{\text{rec}}(\omega, E)\right\}^2 e^{\alpha\omega} = \boldsymbol{g}^T \cdot \hat{A}\boldsymbol{g} - 2\boldsymbol{g}^T \cdot \boldsymbol{f} + A_\alpha[0]$$

$$\rho[K]^{\text{true}}(E) - \rho[K]^{\text{rec}}(E) = \int_{E_0}^{\infty} d\omega\, \rho(\omega)\left[K^{\text{true}}(\omega, E) - K^{\text{rec}}(\omega, E)\right]$$

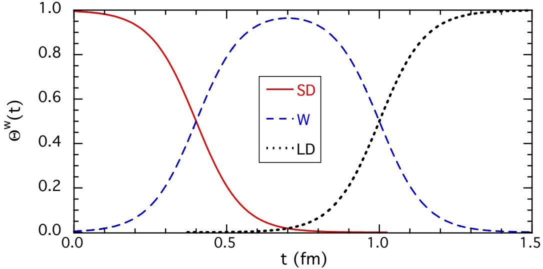
- $\rho(\omega)$ in general increases as a power of the energy (Axiomatic QFT)
- $\left[K(\omega, E) - K^{\text{rec}}(\omega, E)\right]$ is forced to decrease exponentially thanks to $e^{\alpha\omega}$ with $\alpha > 0$

$$\boldsymbol{g}^T \cdot \hat{A}\boldsymbol{g} = \sum_{\tau_1,\tau_2=1}^{\tau_{\max}} g_{\tau_1} g_{\tau_2} \frac{e^{\omega(\alpha-\tau_1-\tau_2)}}{\alpha - \tau_1 - \tau_2}\Bigg|_{E_0}^{\infty} \quad \text{convergent if} \quad \alpha < \tau_1 + \tau_2 < 2$$

$$-2\boldsymbol{g}^T \cdot \boldsymbol{f} = \sum_{\tau=1}^{\tau_{\max}} g_\tau \int_{E_0}^{\infty} d\omega\, e^{\omega(\alpha-\tau)} K(\omega, E)$$

As a measure of the reconstruction accuracy we consider

$$\boxed{\frac{A_0[g_{\alpha,\lambda}]}{A_0[0]} = \frac{\int_{E_0}^{\infty} d\omega \left\{K(\omega, E) - \boldsymbol{g}_{\alpha,\lambda}\cdot \boldsymbol{b}\right\}^2}{\int_{E_0}^{\infty} d\omega \left\{K(\omega, E)\right\}^2}} \quad B[\boldsymbol{g}] = \sum_{\tau,\tau'=1}^{\tau_{\max}} g_\tau \frac{\text{Cov}(C(\tau), C(\tau'))}{C(1)^2} g_{\tau'} E^{-2[g]}$$



$$a_\mu^w = 2\alpha_{\text{em}}^2 \int_0^\infty dt\, t^2 K(m_\mu t) \Theta^w(t) C(t) \qquad a_\mu^{\text{HVP}} = a_\mu^{\text{SD}} + a_\mu^{\text{W}} + a_\mu^{\text{LD}}$$

$$\Theta^{SD}(t) = 1 - \frac{1}{1 + e^{-2(t-t_0)/\Delta}}$$

$$\Theta^{W}(t) = \frac{1}{1 + e^{-2(t-t_0)/\Delta}} - \frac{1}{1 + e^{-2(t-t_1)\Delta}}$$

$$\Theta^{LD}(t) = 1 - \frac{1}{1 + e^{-2(t-t_1)/\Delta}}$$

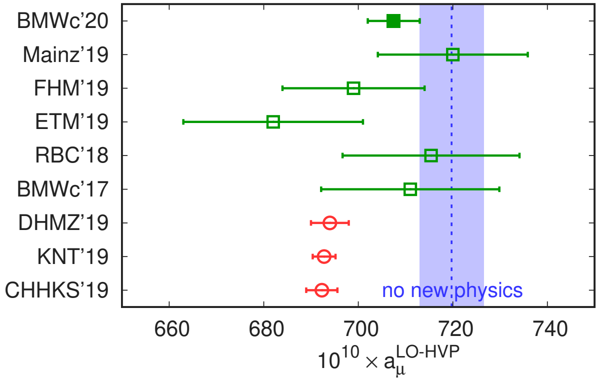
$t_0 = 0.4$ fm, $t_1 = 1$ fm, $\Delta = 0.15$ fm (arXiv:1801.07224)

From experimental R-ratio

$$a_\mu^{\mathrm{HVP}} \propto \alpha_{\mathrm{em}}^2 \int_0^\infty \frac{\mathrm{d}\omega}{\omega^3} \tilde{K}(\omega) R(\omega)$$

From lattice QCD

$$a_\mu^{\mathrm{HVP}} \propto \alpha_{\mathrm{em}}^2 \int_0^\infty \mathrm{d}t\, t^2 K(t) C(t)$$



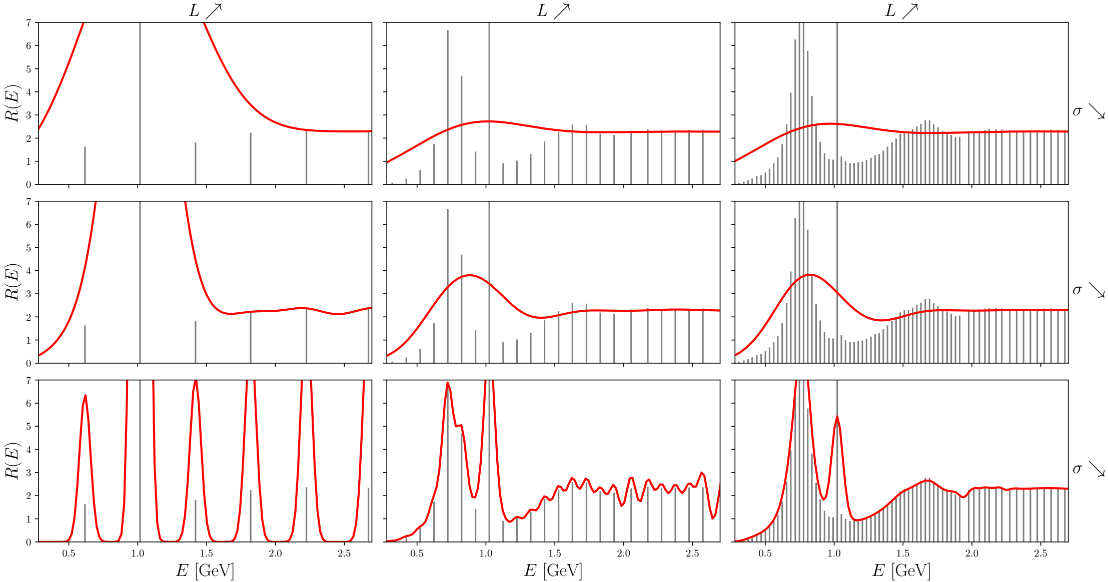
Why $\quad a_\mu^{\mathrm{HVP}}(\text{R-ratio}) \neq a_\mu^{\mathrm{HVP}}(\text{lattice QCD})$?

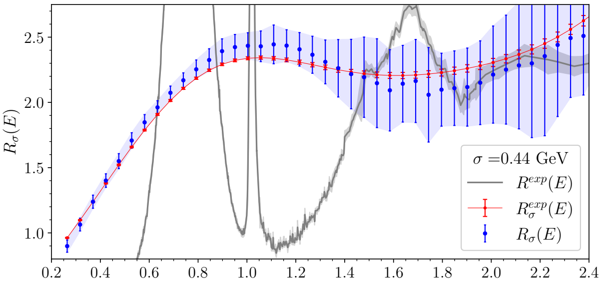
The anomalous magnetic moment of the muon in the Standard Model (arXiv:2006.04822)

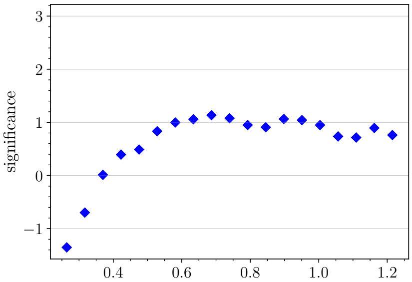
Contribution	value $\times 10^{10}$
Experiment (E821)	11 659 208.9(6.3) (0.54 ppm)
QED	11 658 471.893(10)
Electroweak	15.4(0.1)
Hadronic VP (from R-ratio)	684.5(4.0)
HLbL	9.2(1.8)
Total SM	11 659 181.0(4.3)
$a_\mu^{\rm exp} - a_\mu^{\rm SM}$	27.9(7.6) ($\sim$3 σ tension)

Hadronic contribution is the most challenging and imprecise.

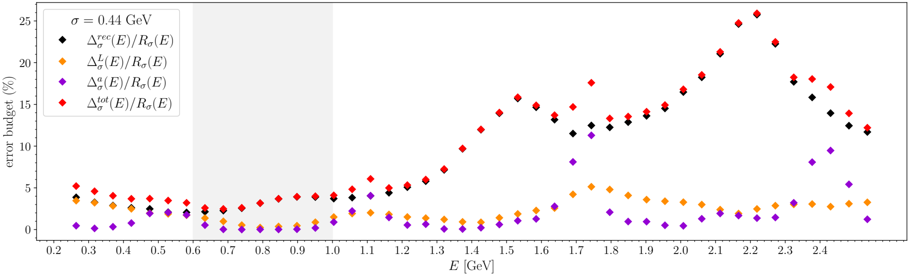
Order of limits, $R(E) = \lim_{\sigma \to 0} \lim_{L \to \infty} R_\sigma(E)$





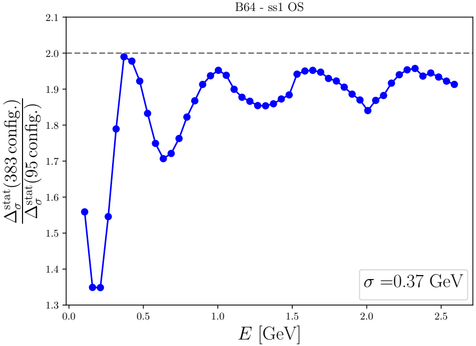


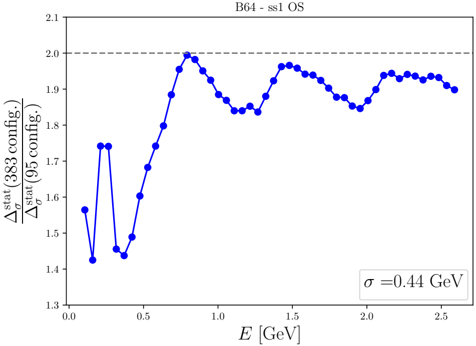
The overall error increases when σ gets smaller. Error budget:



We can reduce the current number of configurations to figure out the impact

$$\frac{\Delta_\sigma^{\text{stat.}}(N/4)}{\Delta_\sigma^{\text{stat.}}(N)} \approx \sqrt{\frac{4N}{N}} = 2$$





Aim $\mapsto$ reduce the error by a factor ≈ 2