Hadronic contributions to $(g-2)_\mu$

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FOR FUNDAMENTAL PHYSICS

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Outline

Introduction: $(g-2)_\mu$ in the Standard Model

Hadronic Vacuum Polarization contribution to $(g-2)_\mu$
- Data-driven approach
- Lattice approach: BMW result and its consequences
- Window quantity: Lattice vs. data-driven

Radiative corrections
- Forward-backward asymmetry in $e^+e^- \to \pi^+\pi^-$
- Dispersive approach to FSR in $e^+e^- \to \pi^+\pi^-$

Conclusions and Outlook

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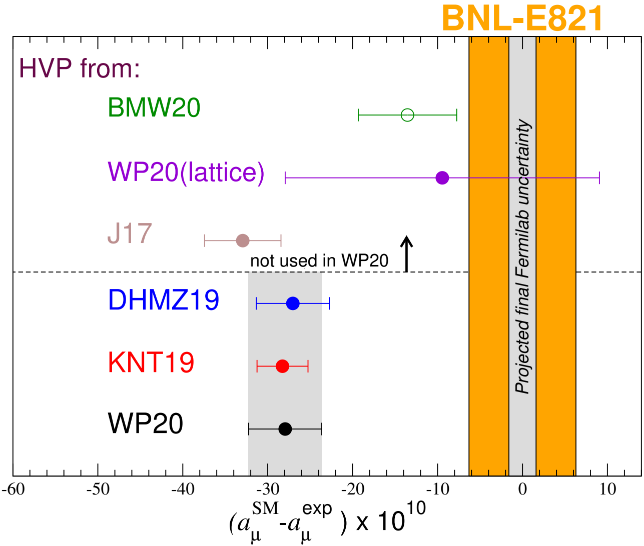
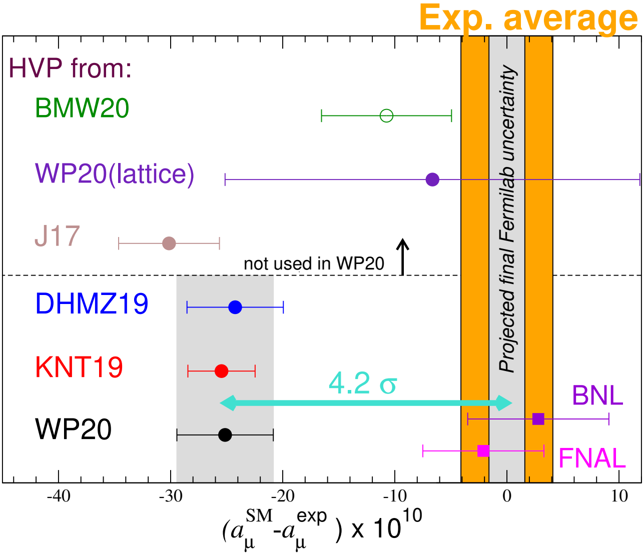
Present status of $(g-2)_\mu$: experiment vs SM

Before



Present status of $(g-2)_\mu$: experiment vs SM

After the Fermilab result



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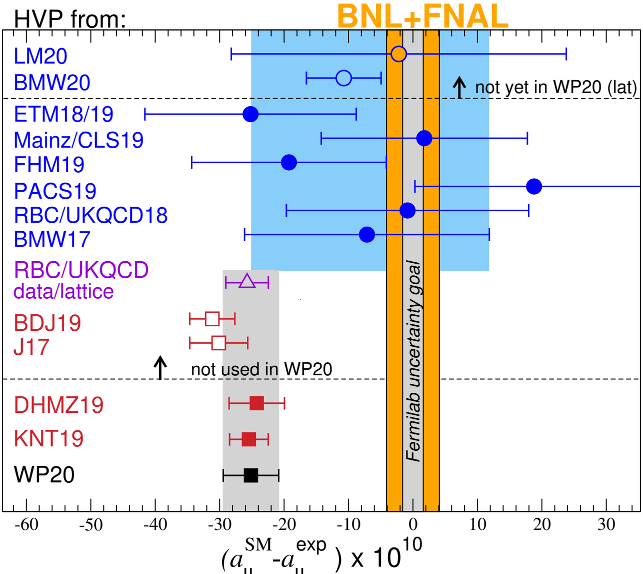

White Paper (2020): $(g-2)_\mu$, experiment vs SM

Contribution	Value $\times 10^{11}$
HVP LO (e^+e^-)	6931(40)
HVP NLO (e^+e^-)	$-98.3(7)$
HVP NNLO (e^+e^-)	12.4(1)
HVP LO (lattice , *udsc*)	7116(184)
HLbL (phenomenology)	92(19)
HLbL NLO (phenomenology)	2(1)
HLbL (lattice, *uds*)	79(35)
HLbL (phenomenology + lattice)	90(17)
QED	116 584 718.931(104)
Electroweak	153.6(1.0)
HVP (e^+e^-, LO + NLO + NNLO)	6845(40)
HLbL (phenomenology + lattice + NLO)	92(18)
Total SM Value	116 591 810(43)
Experiment	116 592 061(41)
Difference: $\Delta a_\mu := a_\mu^{\text{exp}} - a_\mu^{\text{SM}}$	251(59)

White Paper (2020): $(g-2)_\mu$, experiment vs SM

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HVP LO (e^+e^-)	6931(40)
HVP NLO (e^+e^-)	−98.3(7)
HVP NNLO (e^+e^-)	12.4(1)
HVP LO (lattice BMW(20), *udsc*)	7075(55)
HLbL (phenomenology)	92(19)
HLbL NLO (phenomenology)	2(1)
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White Paper (2020): $(g-2)_\mu$, experiment vs SM

White Paper:
T. Aoyama et al. Phys. Rep. 887 (2020) = WP(20)

Muon $g-2$ Theory Initiative
Steering Committee:
GC
Michel Davier (vice-chair)
Aida El-Khadra (chair)
Martin Hoferichter
Laurent Lellouch
Christoph Lehner (vice-chair)
Tsutomu Mibe (J-PARC E34 experiment)
Lee Roberts (Fermilab E989 experiment)
Thomas Teubner
Hartmut Wittig

White Paper (2020): $(g-2)_\mu$, experiment vs SM

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Muon $g-2$ Theory Initiative
Workshops:

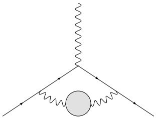
- 1st plenary meeting, Q-Center (Fermilab), 3-6 June 2017
- HVP WG workshop, KEK (Japan), 12-14 February 2018
- HLbL WG workshop, U. of Connecticut, 12-14 March 2018
- 2nd plenary meeting, Mainz, 18-22 June 2018
- 3rd plenary meeting, Seattle, 9-13 September 2019
- Lattice HVP workshop, virtual, 16-20 November 2020
- 4th plenary meeting, KEK (virtual), 28 June-02 July 2021
- 5th plenary meeting, Higgs Center Edinburgh, 5-9 Sept. 2022
- 6th plenary meeting, Bern, 4-8 Sept. 2023

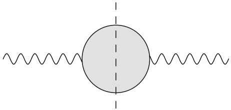
White Paper executive summary (my own)

- QED and EW known and stable, negligible uncertainties
- HVP dispersive: consensus number, conservative uncertainty (KNT19, DHMZ19, CHS19, HHK19)
- HVP lattice: consensus number, $\Delta a_\mu^{\rm HVP,latt} \sim 5\,\Delta a_\mu^{\rm HVP,disp}$ (Fermilab-HPQCD-MILC18,20, BMW18, RBC/UKQCD18, ETM19,SK19, Mainz19, ABTGJP20)
- HVP BMW20: central value $\rightarrow$ discrepancy $< 2\sigma$; $\Delta a_\mu^{\rm HVP,BMW} \sim \Delta a_\mu^{\rm HVP,disp}$ published 04/21 $\rightarrow$ not in WP
- HLbL dispersive: consensus number, w/ recent improvements $\Rightarrow$ $\Delta a_\mu^{\rm HLbL} \sim 0.5\,\Delta a_\mu^{\rm HVP}$
- HLbL lattice: single calculation, agrees with dispersive ($\Delta a_\mu^{\rm HLbL,latt} \sim 2\,\Delta a_\mu^{\rm HLbL,disp}$) $\rightarrow$ final average (RBC/UKQCD20)

Theory uncertainty comes from hadronic physics

- Hadronic contributions responsible for most of the theory uncertainty
- Hadronic vacuum polarization (HVP) is $\mathcal{O}(\alpha^2)$, dominates the total uncertainty, despite being known to $< 1\%$





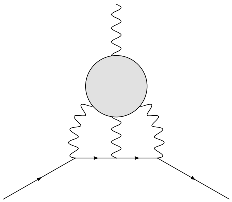
  - unitarity and analyticity $\Rightarrow$ dispersive approach
  - $\Rightarrow$ direct relation to experiment: $\sigma_{\text{tot}}(e^+e^- \to \text{hadrons})$
  - e^+e^- Exps: BaBar, Belle, BESIII, CMD2/3, KLOE2, SND
  - alternative approach: lattice, becoming competitive

(BMW, ETMC, Fermilab, HPQCD, Mainz, MILC, RBC/UKQCD)

$\longrightarrow$ talks by Lehner, Sanfilippo, Szabo, Tavella

Theory uncertainty comes from hadronic physics

- Hadronic contributions responsible for most of the theory uncertainty
- Hadronic vacuum polarization (HVP) is $\mathcal{O}(\alpha^2)$, dominates the total uncertainty, despite being known to $< 1\%$
- Hadronic light-by-light (HLbL) is $\mathcal{O}(\alpha^3)$, known to $\sim 20\%$, second largest uncertainty (now subdominant)



- earlier: model-based—uncertainties difficult to quantify
- recently: dispersive approach $\Rightarrow$ data-driven, systematic treatment
- lattice QCD is becoming competitive

(Mainz, RBC/UKQCD)

$\longrightarrow$ talks by Lehner, Verplanke

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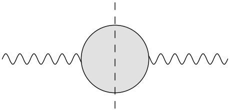
HVP contribution: Master Formula

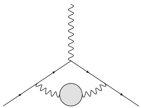
Unitarity relation: simple, same for all intermediate states



$$\mathrm{Im}\bar{\Pi}(q^2) \propto \sigma(e^+e^- \to \mathrm{hadrons}) = \sigma(e^+e^- \to \mu^+\mu^-)R(q^2)$$

Analyticity $\left[\bar{\Pi}(q^2) = \frac{q^2}{\pi}\int ds \frac{\mathrm{Im}\bar{\Pi}(s)}{s(s-q^2)}\right] \Rightarrow$ Master formula for HVP

Bouchiat, Michel (61)



$$\Leftrightarrow \qquad a_\mu^{\rm hvp} = \frac{\alpha^2}{3\pi^2}\int_{s_{th}}^{\infty}\frac{ds}{s}K(s)R(s)$$

$K(s)$ known, depends on m_μ and $K(s) \sim \frac{1}{s}$ for large s

Comparison between DHMZ19 and KNT19

	DHMZ19	KNT19	Difference
$\pi^+\pi^-$	507.85(0.83)(3.23)(0.55)	504.23(1.90)	3.62
$\pi^+\pi^-\pi^0$	46.21(0.40)(1.10)(0.86)	46.63(94)	-0.42
$\pi^+\pi^-\pi^+\pi^-$	13.68(0.03)(0.27)(0.14)	13.99(19)	-0.31
$\pi^+\pi^-\pi^0\pi^0$	18.03(0.06)(0.48)(0.26)	18.15(74)	-0.12
K^+K^-	23.08(0.20)(0.33)(0.21)	23.00(22)	0.08
K_SK_L	12.82(0.06)(0.18)(0.15)	13.04(19)	-0.22
$\pi^0\gamma$	4.41(0.06)(0.04)(0.07)	4.58(10)	-0.17
Sum of the above	626.08(0.95)(3.48)(1.47)	623.62(2.27)	2.46
$[1.8, 3.7]$ GeV (without $c\bar{c}$)	33.45(71)	34.45(56)	-1.00
J/ψ, $\psi(2S)$	7.76(12)	7.84(19)	-0.08
$[3.7, \infty)$ GeV	17.15(31)	16.95(19)	0.20
Total $a_\mu^{\mathrm{HVP,\,LO}}$	$694.0(1.0)(3.5)(1.6)(0.1)_\psi(0.7)_{\mathrm{DV+QCD}}$	692.8(2.4)	1.2

2π: comparison with the dispersive approach

The 2π channel can itself be described dispersively $\Rightarrow$ more constrained theoretically

Ananthanarayan, Caprini, Das (19), GC, Hoferichter, Stoffer (18)

Energy range	ACD18	CHS18	DHMZ19	KNT19
≤ 0.6 GeV		110.1(9)	110.4(4)(5)	108.7(9)
≤ 0.7 GeV		214.8(1.7)	214.7(0.8)(1.1)	213.1(1.2)
≤ 0.8 GeV		413.2(2.3)	414.4(1.5)(2.3)	412.0(1.7)
≤ 0.9 GeV		479.8(2.6)	481.9(1.8)(2.9)	478.5(1.8)
≤ 1.0 GeV		495.0(2.6)	497.4(1.8)(3.1)	493.8(1.9)
$[0.6, 0.7]$ GeV		104.7(7)	104.2(5)(5)	104.4(5)
$[0.7, 0.8]$ GeV		198.3(9)	199.8(0.9)(1.2)	198.9(7)
$[0.8, 0.9]$ GeV		66.6(4)	67.5(4)(6)	66.6(3)
$[0.9, 1.0]$ GeV		15.3(1)	15.5(1)(2)	15.3(1)
≤ 0.63 GeV	132.9(8)	132.8(1.1)	132.9(5)(6)	131.2(1.0)
$[0.6, 0.9]$ GeV		369.6(1.7)	371.5(1.5)(2.3)	369.8(1.3)
$[\sqrt{0.1}, \sqrt{0.95}]$ GeV		490.7(2.6)	493.1(1.8)(3.1)	489.5(1.9)

WP(20)

Combination method and final result

Complete analyses DHMZ19 and KNT19, as well as CHS19 (2π) and HHK19 (3π), have been so combined:

- central values are obtained by simple averages (for each channel and mass range)
- the largest experimental and systematic uncertainty of DHMZ and KNT is taken
- $1/2$ difference DHMZ$-$KNT (or BABAR$-$KLOE in the 2π channel, if larger) is added to the uncertainty

Final result:

$$a_\mu^{\text{HVP, LO}} = 693.1(2.8)_{\text{exp}}(2.8)_{\text{sys}}(0.7)_{\text{DV+QCD}} \times 10^{-10}$$
$$= 693.1(4.0) \times 10^{-10}$$

WP(20)

The BMW result

Borsanyi et al. Nature 2021, ⟶ talk by Szabo

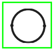
State-of-the-art lattice calculation of $a_\mu^{\text{HVP, LO}}$ based on

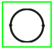
- current-current correlator, summed over all distances, integrated in time with appropriate kernel function (TMR)
- using staggered fermions on an $L \sim 6$ fm lattice ($L \sim 11$fm used for finite volume corrections)
- at (and around) physical quark masses
- including isospin-breaking effects

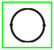
The BMW result

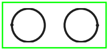
Borsanyi et al. Nature 2021, ⟶ talk by Szabo

Isospin-symmetric

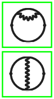

Connected light
$633.7(2.1)_{\mathrm{stat}}(4.2)_{\mathrm{syst}}$

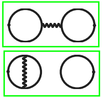

Connected strange
$53.393(89)_{\mathrm{stat}}(68)_{\mathrm{syst}}$


Connected charm
$14.6(0)_{\mathrm{stat}}(1)_{\mathrm{syst}}$

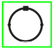

Disconnected
$-13.36(1.18)_{\mathrm{stat}}(1.36)_{\mathrm{syst}}$

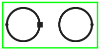
QED isospin breaking: valence


Connected $-1.23(40)_{\mathrm{stat}}(31)_{\mathrm{syst}}$

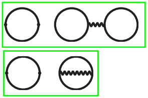

Disconnected $-0.55(15)_{\mathrm{stat}}(10)_{\mathrm{syst}}$

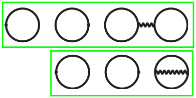
Strong–isospin breaking


Connected
$6.60(63)_{\mathrm{stat}}(53)_{\mathrm{syst}}$


Disconnected
$-4.67(54)_{\mathrm{stat}}(69)_{\mathrm{syst}}$

QED isospin breaking: sea


Connected $0.37(21)_{\mathrm{stat}}(24)_{\mathrm{syst}}$

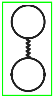

Disconnected $-0.040(33)_{\mathrm{stat}}(21)_{\mathrm{syst}}$

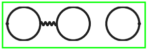
Other

Bottom; higher-order; perturbative

$0.11(4)_{\mathrm{tot}}$

QED isospin breaking: mixed


Connected $-0.0093(86)_{\mathrm{stat}}(95)_{\mathrm{syst}}$


Disconnected $0.011(24)_{\mathrm{stat}}(14)_{\mathrm{syst}}$

Finite-size effects

Isospin-symmetric

$18.7(2.5)_{\mathrm{tot}}$

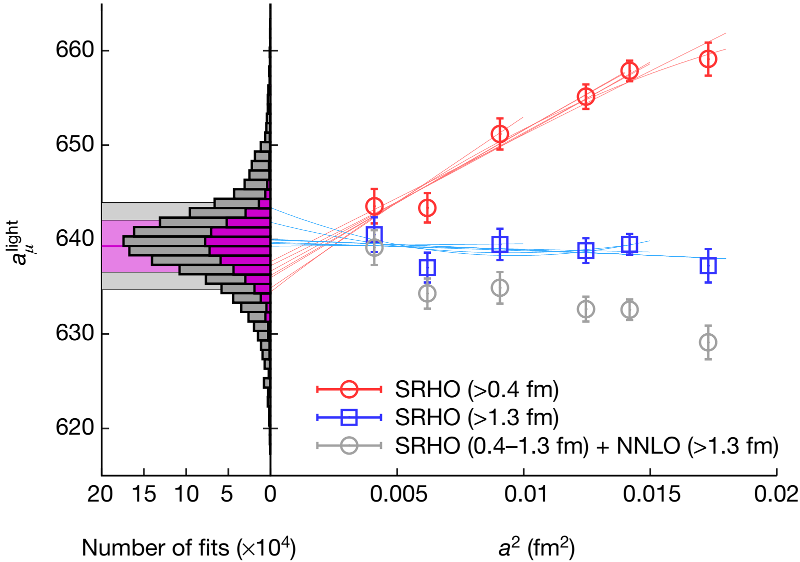
Isospin-breaking

$0.0(0.1)_{\mathrm{tot}}$

$$a_\mu^{\mathrm{LO\text{-}HVP}}\,(\times 10^{10}) = 707.5(2.3)_{\mathrm{stat}}(5.0)_{\mathrm{syst}}(5.5)_{\mathrm{tot}}$$

The BMW result

Borsanyi et al. Nature 2021, ⟶ talk by Szabo



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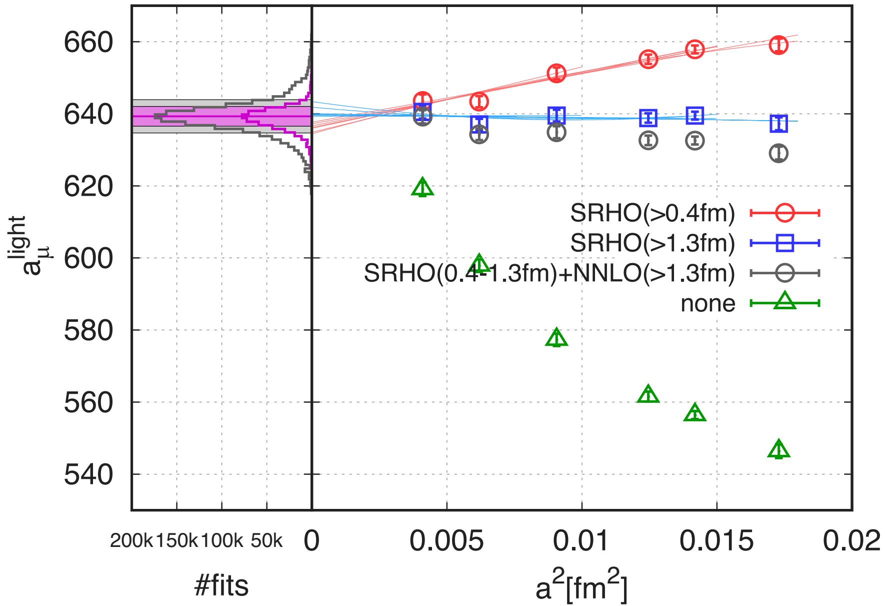
Borsanyi et al. Nature 2021, ⟶ talk by Szabo



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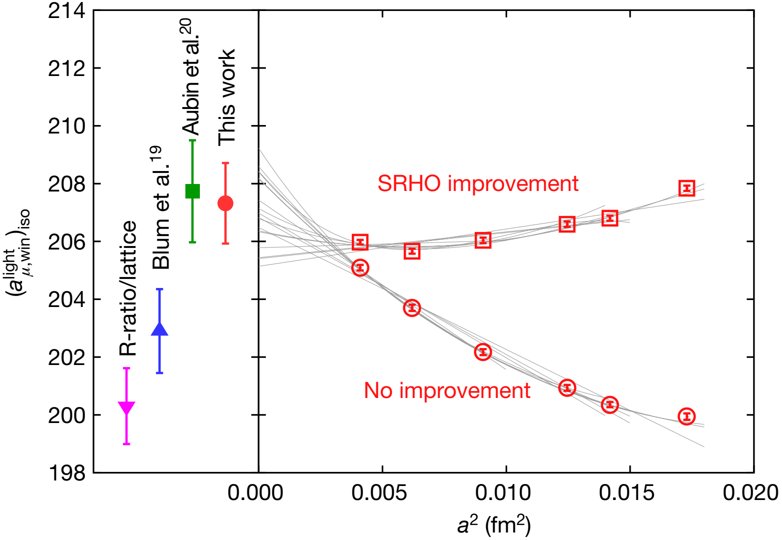
Borsanyi et al. Nature 2021, ⟶ talk by Szabo

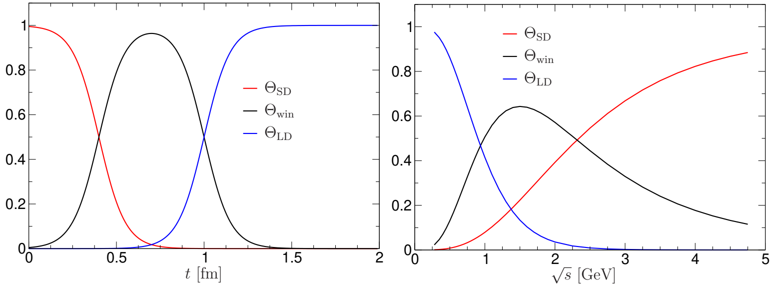
Article



The BMW result

Borsanyi et al. Nature 2021, ⟶ talk by Szabo

Weight functions for window quantities

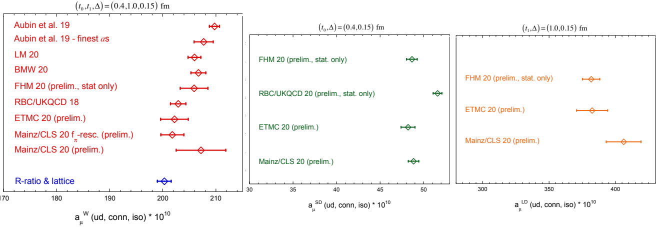


Present status of the window quantities

Lattice calculations of $a_\mu^{\rm win}$, circa 2021

Summary: *ud* contribution

f	$a_\mu^{SD}(f)\cdot 10^{10}$	$a_\mu^{W}(f)\cdot 10^{10}$	$a_\mu^{LD}(f)\cdot 10^{10}$
ud	48.2 (0.8)	202.2 (2.6)	382.5 (11.7)



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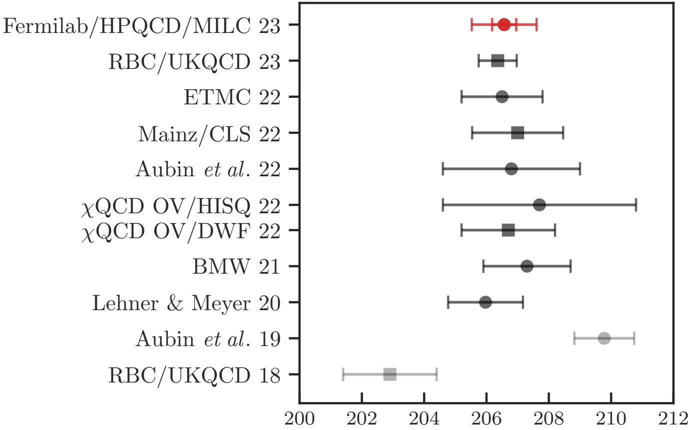
D. Giusti, talk at Lattice 2021

Present status of the window quantities

Now several lattice calculations confirm BMW's result



arXiv:2301.08274, Fermilab Lattice-HPQCD-MILC (23)

⟶ talk by Sanfilippo

Present status of the window quantities

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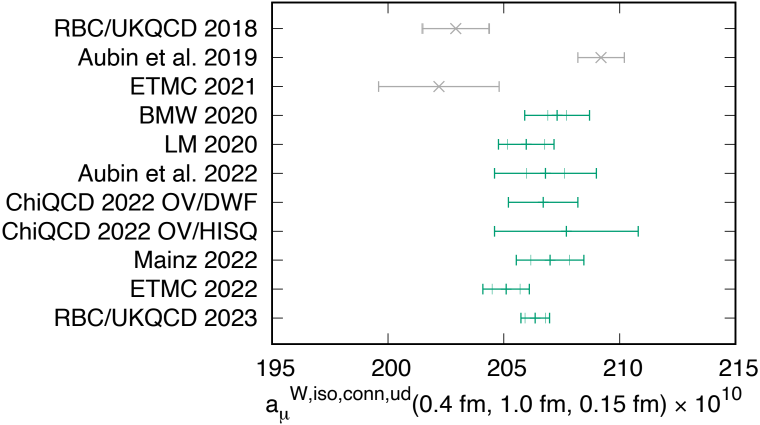

arXiv:2301.08696 RBC/UKQCD (23)

⟶ talk by Sanfilippo

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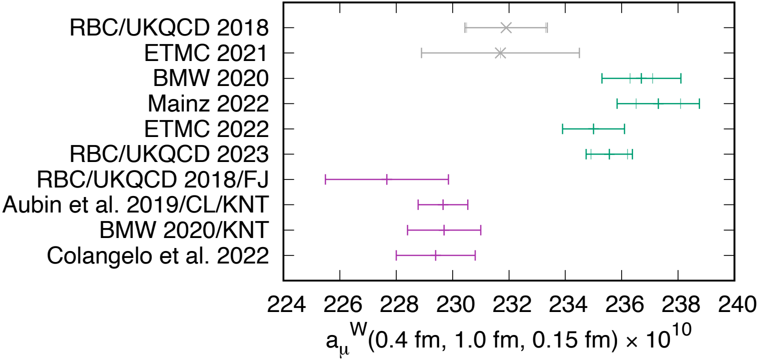


arXiv:2301.08696 RBC/UKQCD (23)

⟶ talk by Sanfilippo

Individual-channel contributions to $a_\mu^{\rm win}$

Channel	total	window
$\pi^+\pi^-$	504.23(1.90)	144.08(49)
$\pi^+\pi^-\pi^0$	46.63(94)	18.63(35)
$\pi^+\pi^-\pi^+\pi^-$	13.99(19)	8.88(12)
$\pi^+\pi^-\pi^0\pi^0$	18.15(74)	11.20(46)
K^+K^-	23.00(22)	12.29(12)
K_SK_L	13.04(19)	6.81(10)
$\pi^0\gamma$	4.58(10)	1.58(4)
Sum of the above	623.62(2.27)	203.47(78)
$[1.8, 3.7]$ GeV (without $c\bar{c}$)	34.45(56)	15.93(26)
J/ψ, $\psi(2S)$	7.84(19)	2.27(6)
$[3.7, \infty)$ GeV	16.95(19)	1.56(2)
WP(20) / GC, El-Khadra *et al.* (22)	693.1(4.0)	229.4(1.4)
BMWc	707.5(5.5)	236.7(1.4)
Mainz/CLS		237.3(1.5)
ETMc		235.0(1.1)
RBC/UKQCD		235.6(0.8)

Numbers for the channels refer to KNT19 — thanks to Alex Keshavarzi for providing them

$$\Delta a_\mu^{\text{HVP, LO}} = 14.4(6.8)\,(2.1\sigma), \qquad \Delta a_\mu^{\rm win} \sim 6.5(1.5)\,(\sim 4.3\sigma)$$

Consequences of the BMW result

A shift in the value of $a_\mu^{\text{HVP, LO}}$ would have consequences:

- $\Delta a_\mu^{\text{HVP, LO}} \quad \Leftrightarrow \quad \Delta\sigma(e^+e^- \to \text{hadrons})$
- $\Delta\alpha_{\text{had}}(M_Z^2)$ is determined by an integral of the same $\sigma(e^+e^- \to \text{hadrons})$ (more weight at high energy)
- changing $a_\mu^{\text{HVP, LO}}$ necessarily implies a shift in $\Delta\alpha_{\text{had}}(M_Z^2)$: size depends on the energy range of $\Delta\sigma(e^+e^- \to \text{hadrons})$
- a shift in $\Delta\alpha_{\text{had}}(M_Z^2)$ has an impact on the EW-fit
- to save the EW-fit $\Delta\sigma(e^+e^- \to \text{hadrons})$ must occur below ~ 1 (max 2) GeV

  Crivellin, Hoferichter, Manzari, Montull (20)/Keshavarzi, Marciano, Passera, Sirlin (20)/Malaescu, Schott (20)
- or the need for BSM physics would be moved elsewhere

$\sigma(e^+e^- \to \pi^+\pi^-)$ and $F_\pi^V(s)$

- Below $1-2$ GeV only one significant channel: $\pi^+\pi^-$
- Strongly constrained by analyticity and unitarity ($F_\pi^V(s)$)
- $F_\pi^V(s)$ parametrization which satisfies these
  $\Rightarrow$ small number of parameters GC, Hoferichter, Stoffer (18)
- $\Delta a_\mu^{\text{HVP, LO}} \quad \Leftrightarrow \quad$ shifts in these parameters
  analysis of the corresponding scenarios GC, Hoferichter, Stoffer (21)

Vector form factor of the pion

$$\langle \pi^i(p')|V_\mu^k(0)|\pi^l(p)\rangle = i\epsilon^{ikl}(p'+p)_\mu F_\pi^V(s) \qquad s=(p'-p)^2$$

Analyticity:

$$e^{-i\delta(s)}F_\pi^V(s) \in \mathbb{R} \text{ for } s+i\varepsilon,\, 4M_\pi^2 \leq s < \infty$$

Exact solution: Omnès (58)

$$F_\pi^V(s) = P(s)\Omega(s) = P(s)\exp\left\{\frac{s}{\pi}\int_{4M_\pi^2}^\infty \frac{ds'}{s'}\frac{\delta(s')}{s'-s}\right\} \quad ,$$

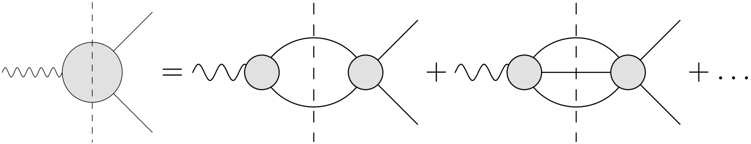
$P(s)$ a polynomial $\Leftrightarrow$ behaviour of $F_\pi^V(s)$ for $s\to\infty$ (or zeros)

- normalization fixed by gauge invariance:

$$F_\pi^V(0)=1 \qquad \overset{\text{no zeros}}{\Longrightarrow} \qquad P(s)=1$$

- $e^+e^- \to \pi^+\pi^-$ data $\Rightarrow$ free parameters in $\Omega(t)$

Omnès representation including isospin breaking



Omnès representation including isospin breaking

- Omnès representation

$$F_\pi^V(s) = \exp\left[\frac{s}{\pi}\int_{4M_\pi^2}^\infty ds' \frac{\delta(s')}{s'(s'-s)}\right] \equiv \Omega(s)$$

- Split elastic ($\leftrightarrow \pi\pi$ phase shift, δ_1^1) from inelastic phase

$$\delta = \delta_1^1 + \delta_{\text{in}} \quad \Rightarrow \quad F_\pi^V(s) = \Omega_1^1(s)\Omega_{\text{in}}(s)$$

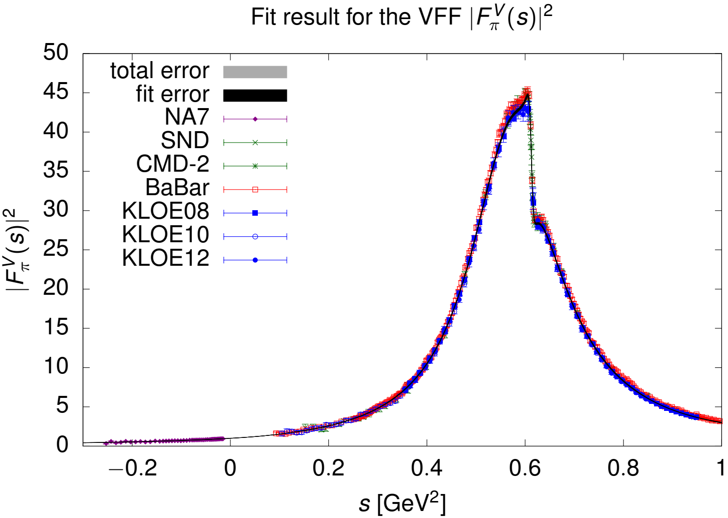
Eidelman-Lukaszuk: unitarity bound on δ_{in}

$$\sin^2\delta_{\text{in}} \leq \frac{1}{2}\Big(1-\sqrt{1-r^2}\Big)\ ,\ \ r = \frac{\sigma^{I=1}_{e^+e^-\to\neq 2\pi}}{\sigma_{e^+e^-\to 2\pi}} \Rightarrow s_{\text{in}} = (M_\pi + M_\omega)^2$$

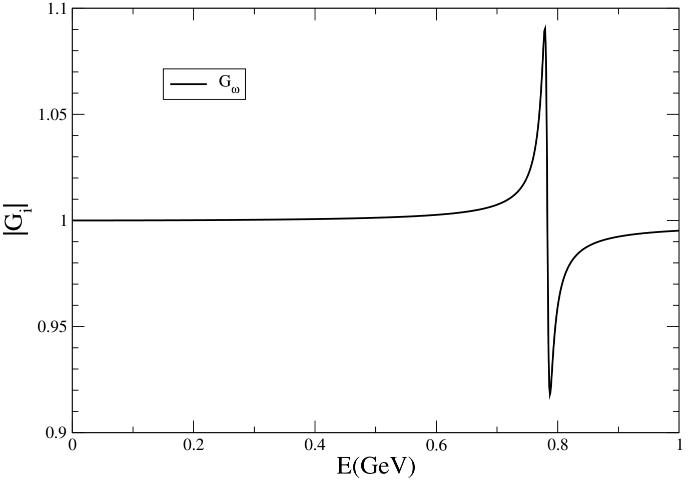
- $\rho - \omega$–mixing $\qquad F_V(s) = \Omega_{\pi\pi}(s)\cdot\Omega_{\text{in}}(s)\cdot G_\omega(s)$

$$G_\omega(s) = 1 + \epsilon\frac{s}{s_\omega - s} \qquad \text{where} \qquad s_\omega = (M_\omega - i\Gamma_\omega/2)^2$$

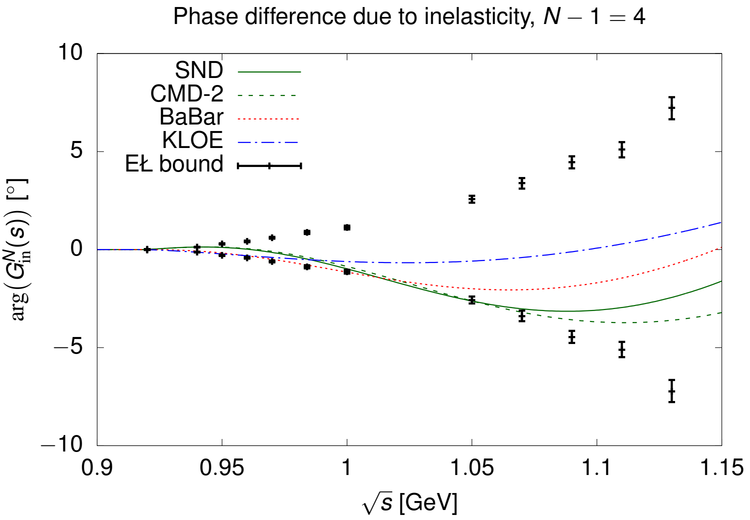
Fit results


Fit result for the VFF $|F_\pi^V(s)|^2$
total error
fit error
NA7
SND
CMD-2
BaBar
KLOE08
KLOE10
KLOE12
$|F_\pi^V(s)|^2$
s [GeV^2]
0
5
10
15
20
25
30
35
40
45
50
−0.2
0
0.2
0.4
0.6
0.8
1

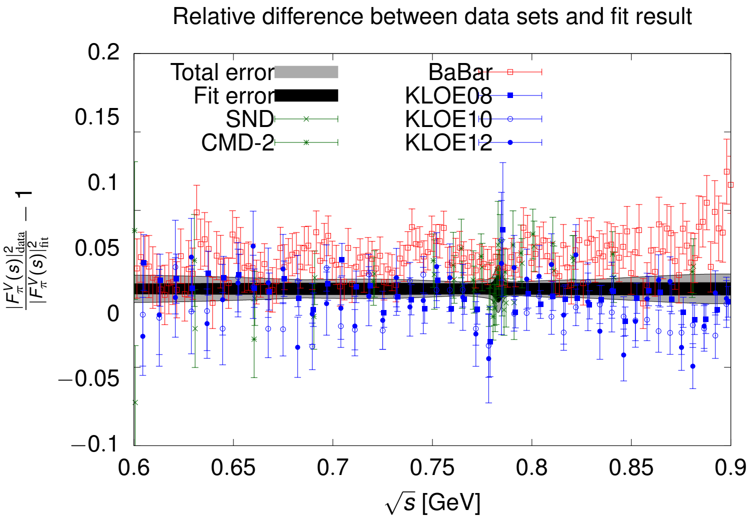
Fit results



Fit results

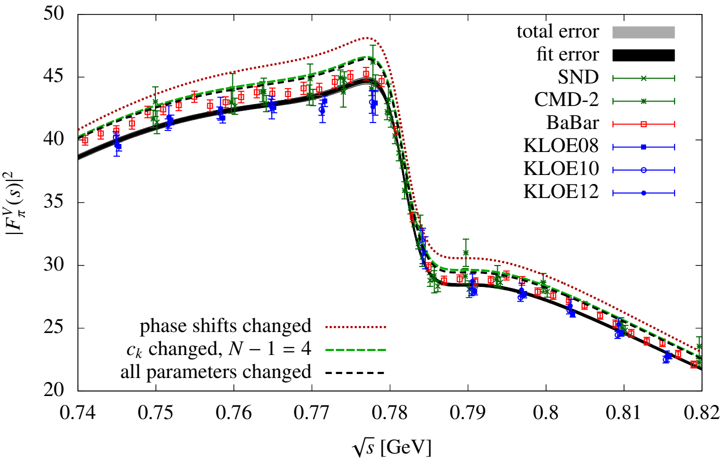

Phase difference due to inelasticity, $N-1=4$
10
5
0
−5
−10
0.9
0.95
1
1.05
1.1
1.15
$\sqrt{s}$ [GeV]
$\arg\left(G_{\text{in}}^{N}(s)\right)$ [°]
SND
CMD-2
BaBar
KLOE
EŁ bound

Fit results



Relative difference between data sets and fit result
Total error
Fit error
SND
CMD-2
BaBar
KLOE08
KLOE10
KLOE12
$\frac{|F_\pi^V(s)|^2_{\text{data}}}{|F_\pi^V(s)|^2_{\text{fit}}} - 1$
$\sqrt{s}$ [GeV]
0.2
0.15
0.1
0.05
0
−0.05
−0.1
0.6
0.65
0.7
0.75
0.8
0.85
0.9

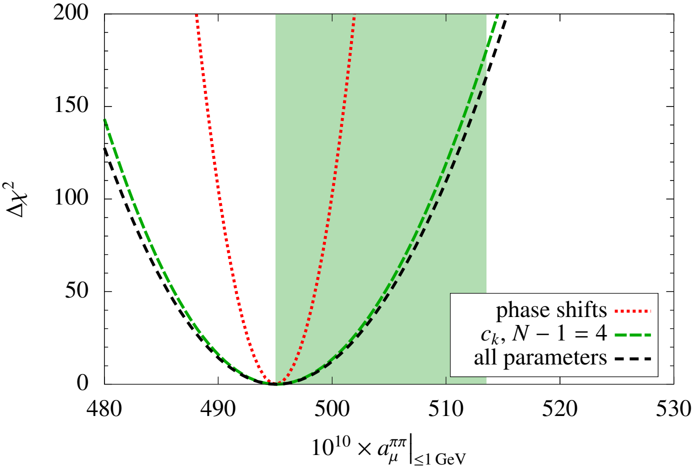
Change $\sigma(e^+e^- \to \pi^+\pi^-)|_{\sqrt{s}<1\text{GeV}}$ to agree w/ BMW



GC, Hoferichter, Stoffer (21)

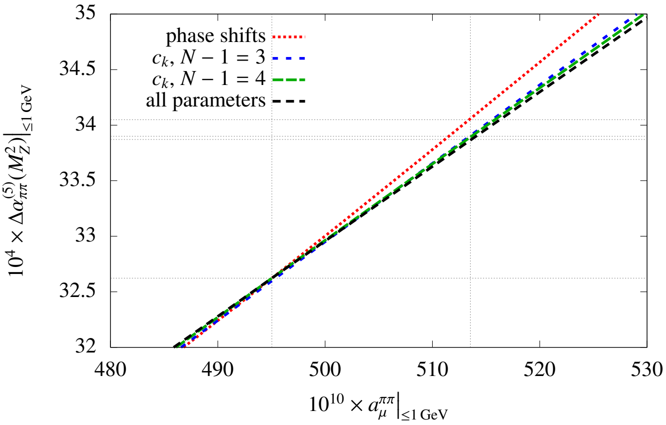
Tension [BMW20 vs e^+e^- data] stronger for KLOE than for BABAR

Change $\sigma(e^+e^- \to \pi^+\pi^-)|_{\sqrt{s}<1\text{GeV}}$ to agree w/ BMW



GC, Hoferichter, Stoffer (21)

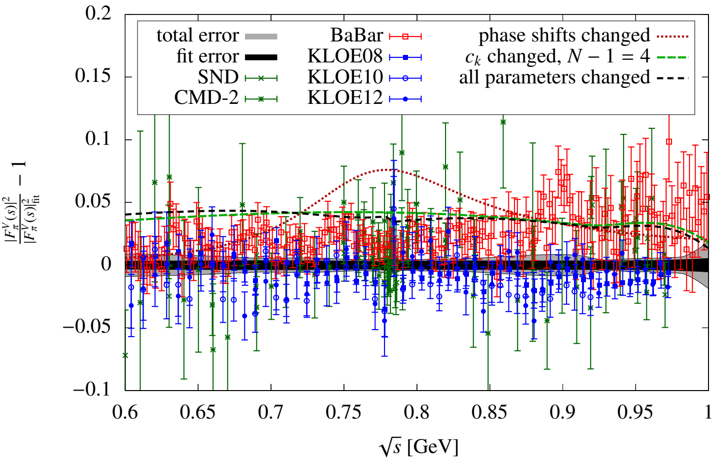
Change $\sigma(e^+e^- \to \pi^+\pi^-)|_{\sqrt{s}<1\text{GeV}}$ to agree w/ BMW



GC, Hoferichter, Stoffer (21)

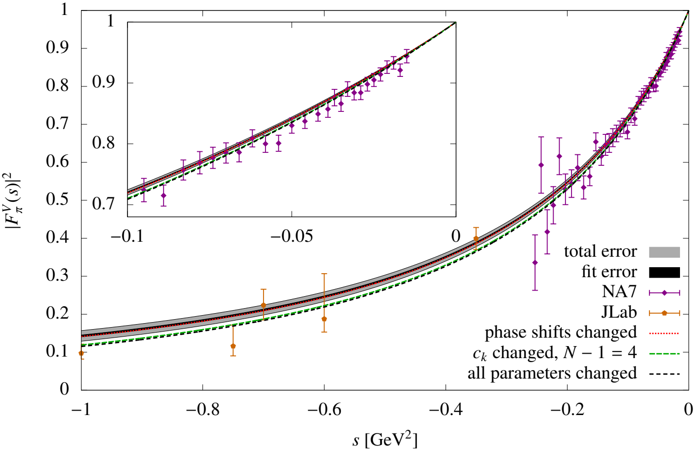
$$10^4 \Delta\alpha^{(5)}_{\text{had}}(M_Z^2) = \begin{cases} 272.2(4.1) & \text{EW fit} \\ 276.1(1.1) & \sigma_{\text{had}}(s) \end{cases}$$

Change $\sigma(e^+e^- \to \pi^+\pi^-)|_{\sqrt{s}<1\text{GeV}}$ to agree w/ BMW



GC, Hoferichter, Stoffer (21)

Change $\sigma(e^+e^- \to \pi^+\pi^-)|_{\sqrt{s}<1\mathrm{GeV}}$ to agree w/ BMW



GC, Hoferichter, Stoffer (21)

How does the change in $(2\pi, < 1\text{GeV})$ affect $a_\mu^{\rm win}$?

Channel	total	window
$\pi^+\pi^-$	504.23(1.90)	144.08(49)
$\pi^+\pi^-\pi^0$	46.63(94)	18.63(35)
$\pi^+\pi^-\pi^+\pi^-$	13.99(19)	8.88(12)
$\pi^+\pi^-\pi^0\pi^0$	18.15(74)	11.20(46)
K^+K^-	23.00(22)	12.29(12)
$K_S K_L$	13.04(19)	6.81(10)
$\pi^0\gamma$	4.58(10)	1.58(4)
Sum of the above	623.62(2.27)	203.47(78)
$[1.8, 3.7]$ GeV (without $c\bar{c}$)	34.45(56)	15.93(26)
J/ψ, $\psi(2S)$	7.84(19)	2.27(6)
$[3.7, \infty)$ GeV	16.95(19)	1.56(2)
WP(20) / GC, El-Khadra *et al.* (22)	693.1(4.0)	229.4(1.4)
BMWc	707.5(5.5)	236.7(1.4)
Mainz/CLS		237.3(1.5)
ETMc		235.0(1.1)
RBC/UKQCD		235.6(0.8)

Numbers for the channels refer to KNT19 — thanks to Alex Keshavarzi for providing them

$$\Delta a_\mu^{\text{HVP, LO}} = 14.4(6.8)\,(2.1\sigma), \qquad \Delta a_\mu^{\rm win} \sim 6.5(1.5)\,(\sim 4.3\sigma)$$

How does the change in $(2\pi, < 1\text{GeV})$ affect $a_\mu^{\rm win}$?

Channel	total	window
$\pi^+\pi^-$	518.6	148
$\pi^+\pi^-\pi^0$	46.63(94)	18.63(35)
$\pi^+\pi^-\pi^+\pi^-$	13.99(19)	8.88(12)
$\pi^+\pi^-\pi^0\pi^0$	18.15(74)	11.20(46)
K^+K^-	23.00(22)	12.29(12)
K_SK_L	13.04(19)	6.81(10)
$\pi^0\gamma$	4.58(10)	1.58(4)
Sum of the above	638.0	207.5
$[1.8, 3.7]$ GeV (without $c\bar{c}$)	34.45(56)	15.93(26)
J/ψ, $\psi(2S)$	7.84(19)	2.27(6)
$[3.7, \infty)$ GeV	16.95(19)	1.56(2)
WP(20) / GC, El-Khadra *et al.* (22)	707.5	233.4
BMWc	707.5(5.5)	236.7(1.4)
Mainz/CLS		237.3(1.5)
ETMc		235.0(1.1)
RBC/UKQCD		235.6(0.8)

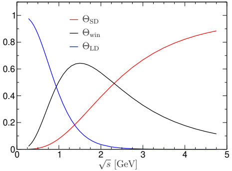
Numbers for the channels refer to KNT19 — thanks to Alex Keshavarzi for providing them

$$\Delta a_\mu^{\text{HVP, LO}} = 0, \qquad \Delta a_\mu^{\rm win} \sim 2.5$$

What does this tell us about $\sqrt{s}$-range of $\Delta\sigma(e^+e^-)$?

- $\Delta\sigma(e^+e^-)$ all < 1 GeV does not allow one to satisfy simultaneously $\Delta a_\mu^{\text{HVP, LO}} = 0$ and $\Delta a_\mu^{\rm win} = 0$
- $\Delta\sigma(e^+e^-)$ must happen < 2 GeV (EWFit)

Weight function $\in (0.5, 0.65)$



- assume reasonable shape of $\Delta\sigma(e^+e^-)$ (no negative shifts)

$\Rightarrow$ at least 40% of $\Delta a_\mu^{\text{HVP, LO}} = 14.4$ from above 1 GeV

$\longrightarrow$ see also talk by De Santis

What does this tell us about $\sqrt{s}$-range of $\Delta\sigma(e^+e^-)$?

Channel	total	window
$\pi^+\pi^-$	504.23(1.90)	144.08(49)
$\pi^+\pi^-\pi^0$	46.63(94)	18.63(35)
$\pi^+\pi^-\pi^+\pi^-$	13.99(19)	8.88(12)
$\pi^+\pi^-\pi^0\pi^0$	18.15(74)	11.20(46)
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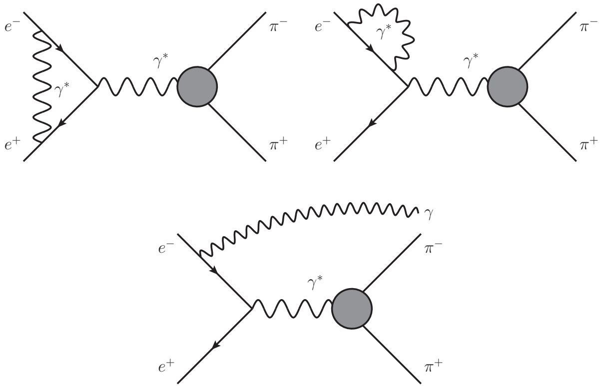
Numbers for the channels refer to KNT19 — thanks to Alex Keshavarzi for providing them

Fermilab/HPQCD/MILC result

Outline

Radiative corrections to $e^+e^- \to \pi^+\pi^-$

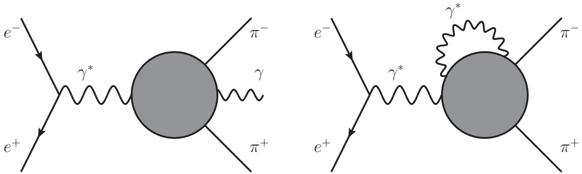
Initial State Radiation:



can be calculated in QED in terms of $F_\pi^V(s)$

Radiative corrections to $e^+e^- \to \pi^+\pi^-$

Final State Radiation:

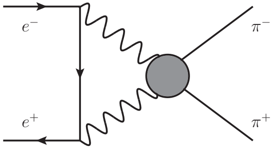


requires hadronic matrix elements beyond $F_\pi^V(s)$
known in ChPT to one loop

Kubis, Meißner (01)

Radiative corrections to $e^+e^- \to \pi^+\pi^-$

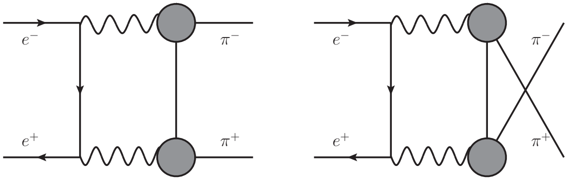
Interference terms:



also require hadronic matrix elements beyond $F_\pi^V(s)$

Radiative corrections to $e^+e^- \to \pi^+\pi^-$

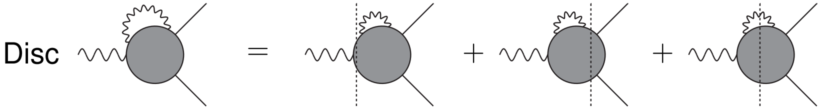
Interference terms:



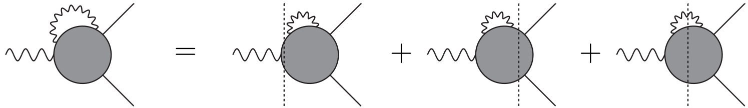
also require hadronic matrix elements beyond $F_\pi^V(s)$
other than in the 1π-exchange approximation;

do not contribute to the total cross section because C-odd
but to the forward-backward asymmetry

Dispersive approach to FSR



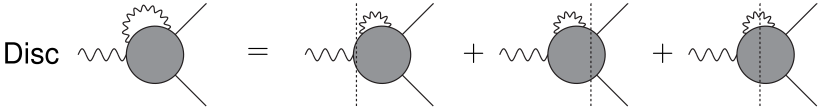
Dispersive approach to FSR



Neglecting intermediate states beyond 2π, unitarity reads

$$\frac{\mathrm{Disc}F_\pi^{V,\alpha}(s)}{2i} = \frac{(2\pi)^4}{2}\int d\Phi_2 F_\pi^V(s)\times T_{\pi\pi}^{\alpha *}(s,t) + \frac{(2\pi)^4}{2}\int d\Phi_2 F_\pi^{V,\alpha}(s)\times T_{\pi\pi}^{*}(s,t) + \frac{(2\pi)^4}{2}\int d\Phi_3 F_\pi^{V,\gamma}(s,t) T_{\pi\pi}^{\gamma *}(s,\{t_i\})$$

Dispersive approach to FSR

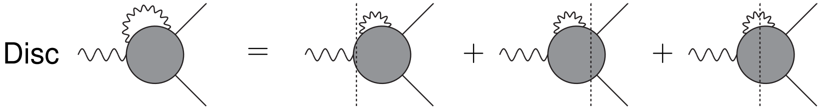


Neglecting intermediate states beyond 2π, unitarity reads

$$\frac{\mathrm{Disc} F_\pi^{V,\alpha}(s)}{2i} = \frac{(2\pi)^4}{2}\int d\Phi_2 F_\pi^V(s) \times T_{\pi\pi}^{\alpha *}(s,t)$$
$$+ \frac{(2\pi)^4}{2}\int d\Phi_2 F_\pi^{V,\alpha}(s) \times T_{\pi\pi}^{*}(s,t)$$
$$+ \frac{(2\pi)^4}{2}\int d\Phi_3 F_\pi^{V,\gamma}(s,t) T_{\pi\pi}^{\gamma *}(s,\{t_i\})$$

$\Rightarrow$ need $T_{\pi\pi}^{\alpha}$ as well as $T_{\pi\pi}^{\gamma}$ and $F_\pi^{V,\gamma}$ as input

Dispersive approach to FSR



Neglecting intermediate states beyond 2π, unitarity reads

$$\frac{\mathrm{Disc} F_\pi^{V,\alpha}(s)}{2i} = \frac{(2\pi)^4}{2}\int d\Phi_2 F_\pi^V(s) \times T_{\pi\pi}^{\alpha *}(s,t) + \frac{(2\pi)^4}{2}\int d\Phi_2 F_\pi^{V,\alpha}(s) \times T_{\pi\pi}^{*}(s,t) + \frac{(2\pi)^4}{2}\int d\Phi_3 F_\pi^{V,\gamma}(s,t) T_{\pi\pi}^{\gamma *}(s,\{t_i\})$$

$\Rightarrow$ need $T_{\pi\pi}^\alpha$ as well as $T_{\pi\pi}^\gamma$ and $F_\pi^{V,\gamma}$ as input

The DR for $F_\pi^{V,\alpha}(s)$ takes the form of an integral equation

Forward-backward asymmetry

$$\frac{d\sigma_0}{dz} = \frac{\pi\alpha^2\beta^3}{4s}(1-z^2)\left|F_\pi^V(s)\right|^2, \qquad \beta = \sqrt{1-\frac{4M_\pi^2}{s}}, \qquad z = \cos\theta$$

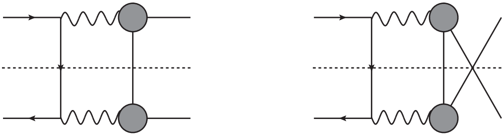
$$A_{\text{FB}}(z) = \frac{\frac{d\sigma}{dz}(z) - \frac{d\sigma}{dz}(-z)}{\frac{d\sigma}{dz}(z) + \frac{d\sigma}{dz}(-z)}$$

$$\left.\frac{d\sigma}{dz}\right|_{C\text{-odd}} = \frac{d\sigma_0}{dz}\left[\delta_{\text{soft}}(m_\gamma^2,\Delta) + \delta_{\text{virt}}(m_\gamma^2)\right] + \left.\frac{d\sigma}{dz}\right|_{\text{hard}}(\Delta)$$

$$\delta_{\text{soft}} = \frac{2\alpha}{\pi}\left\{\log\frac{m_\gamma^2}{4\Delta^2}\log\frac{1+\beta z}{1-\beta z} + \log(1-\beta^2)\log\frac{1+\beta z}{1-\beta z} + \ldots\right\}$$

Calculation of δ_{virt} in the 1π-exchange approximation

- cut the diagrams in the t (or u) channel



- represent the subamplitude $e^+e^- \to \pi^+\pi^-$ dispersively

$$\frac{F_\pi^V(s)}{s} = \frac{1}{s-m_\gamma^2} - \frac{1}{\pi}\int_{4M_\pi^2}^\infty ds' \frac{\mathrm{Im}F_\pi^V(s')}{s'}\frac{1}{s-s'}$$

- which leads to

GC, Hoferichter, Monnard, Ruiz de Elvira (22)

$$\delta_{\text{virt}} = \bar{\delta}_{\text{virt}}(m_\gamma^2, m_\gamma^2) - \frac{1}{\pi}\int_{4M_\pi^2}^\infty ds' \frac{\mathrm{Im}F_\pi^V(s')}{s'}\left[\bar{\delta}_{\text{virt}}(s', m_\gamma^2) + \bar{\delta}_{\text{virt}}(m_\gamma^2, s')\right]$$
$$+ \frac{1}{\pi}\int_{4M_\pi^2}^\infty ds' \frac{\mathrm{Im}F_\pi^V(s')}{s'}\frac{1}{\pi}\int_{4M_\pi^2}^\infty ds'' \frac{\mathrm{Im}F_\pi^V(s'')}{s''}\bar{\delta}_{\text{virt}}(s', s''),$$

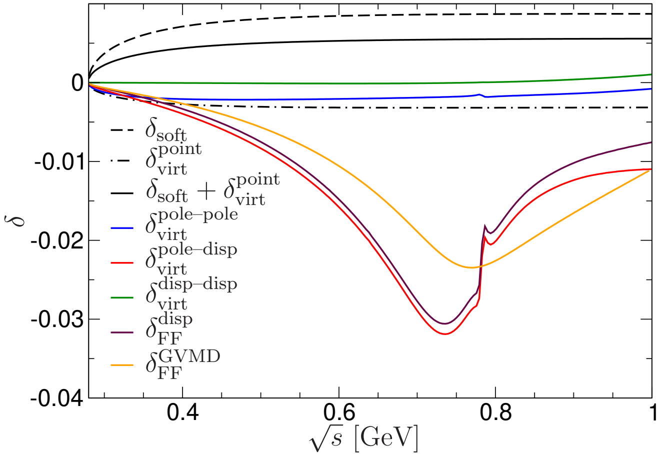
Calculation of δ_{virt} in the 1π-exchange approximation

GC, Hoferichter, Monnard, Ruiz de Elvira (22)

$$\bar{\delta}_{\text{virt}} = -\frac{\mathrm{Re}F_\pi^V(s)}{2\beta^2 s(1-z^2)|F_\pi^V(s)|^2}\frac{\alpha}{\pi}$$
$$\times \mathrm{Re}\Big[4t(M_\pi^2-t)\Big(C_0(m_e^2,t,M_\pi^2,s',m_e^2,M_\pi^2)+C_0(m_e^2,t,M_\pi^2,s'',m_e^2,M_\pi^2)\Big)$$
$$-4t\Big(sC_0(m_e^2,s,m_e^2,m_e^2,s',s'')-tC_0(M_\pi^2,s,M_\pi^2,M_\pi^2,s',s'')\Big)$$
$$+4(M_\pi^2-t)\Big((M_\pi^2-t)^2+M_\pi^4+t(s'+s''-u)\Big)$$
$$\times D_0(m_e^2,m_e^2,M_\pi^2,M_\pi^2,s,t,s',m_e^2,s'',M_\pi^2)-(t\leftrightarrow u)\Big]$$
$$+(\mathrm{Re}\to\mathrm{Im})$$

Numerical analysis

GC, Hofeichter, Monnard, Ruiz de Elvira (22)



GVMD describes well preliminary CMD3 data Ignatov, Lee (22)

Numerical analysis

GC, Hoferichter, Monnard, Ruiz de Elvira (22)

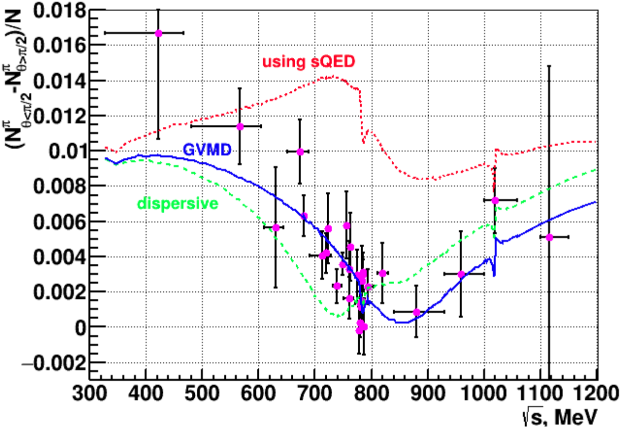


Figure courtesy of F. Ignatov

Dispersive treatment of FSR in $e^+e^- \to \pi^+\pi^-$

$$\frac{\mathrm{Disc}F_\pi^{V,\alpha}(s)}{2i} = \frac{(2\pi)^4}{2}\int d\Phi_2 F_\pi^V(s) \times T_{\pi\pi}^{\alpha *}(s,t) + \frac{(2\pi)^4}{2}\int d\Phi_2 F_\pi^{V,\alpha}(s) \times T_{\pi\pi}^{*}(s,t) + \frac{(2\pi)^4}{2}\int d\Phi_3 F_\pi^{V,\gamma}(s,t) T_{\pi\pi}^{\gamma *}(s,\{t_i\})$$

Long digression $\Rightarrow$ $T_{\pi\pi}^{\alpha}$

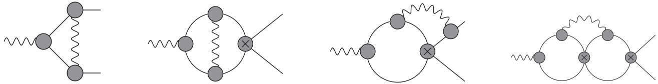
Approximation: only 2π intermediate states for $F_\pi^{V,\gamma}$ and $T_{\pi\pi}^{\gamma}$:

All subamplitudes known $\Rightarrow$ $F_\pi^{V,\gamma}$ and $T_{\pi\pi}^{\gamma}$ ✓

Evaluation of $F_\pi^{V,\alpha}$

Having evaluated all the following diagrams

J. Monnard, PhD thesis 2021



Evaluation of $F_\pi^{V,\alpha}$

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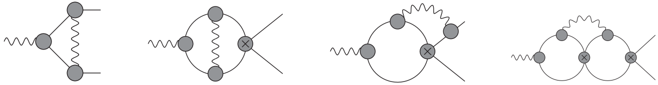
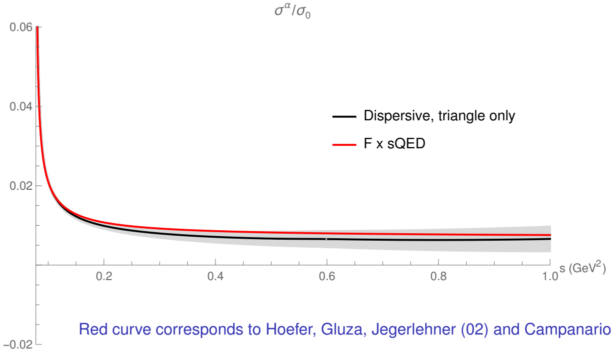


the results for $\sigma(e^+e^- \to \pi^+\pi^-(\gamma))$ look as follows:

Preliminary!



Red curve corresponds to Hoefer, Gluza, Jegerlehner (02) and Campanario et al. (19) (?)

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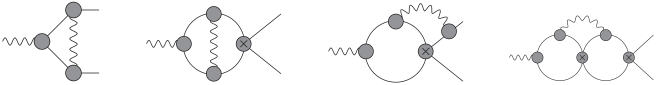

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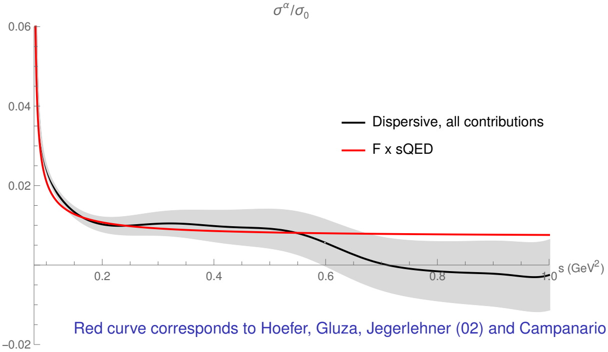

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Impact on a_μ^{HVP}

Ideally: use calculated RC in the data analysis (future?).

Quick estimate of the impact: thanks to M. Hoferichter and P. Stoffer

1. remove RC from the measured $\sigma(e^+e^- \to \pi^+\pi^-(\gamma))$
2. fit with the dispersive representation for $F_\pi^V(s)$
3. insert back the RC

The impact on a_μ^{HVP} is evaluated by comparing to the result obtained by removing RC with $\eta(s)$ calculated in sQED

$$10^{11}\Delta a_\mu^{\mathrm{HVP}} = \begin{cases} 10.2 \pm 0.5 \pm 5 & \text{FsQED} \\ 10.5 \pm 0.5 \pm (?) & \text{triangle} \\ 13.2 \pm 0.5 & \text{full} \end{cases}$$

Outline

Conclusions

- Current status of the $(g-2)_\mu$: 4.2σ discrepancy between SM (WP20) and experiment (BNL+FNAL)

- Data-driven evaluation of the HVP contribution (WP20): 0.6% error $\Rightarrow$ dominates the theory uncertainty

- Recent lattice calculation [BMW(20)] has reached a similar precision but differs from the dispersive one (=from e^+e^- data). If confirmed $\Rightarrow$ discrepancy with experiment $\searrow$ below 2σ

- For the intermediate window BMW has been confirmed by other lattice collaborations (Aubin et al., Mainz, ETMc, RBC/UKQCD, Fermilab-HPQCD-MILC) Disagreement with data-driven worse than for the total: puzzle!

- Evaluation of the HLbL contribution based on the dispersive approach: 20% accuracy. Two recent lattice calculations [RBC/UKQCD(20), Mainz(21)] agree with it

Outlook

- The Fermilab experiment aims to reduce the BNL uncertainty by a factor four $\Rightarrow$ potential 7σ discrepancy

- Improvements on the SM theory/data side:

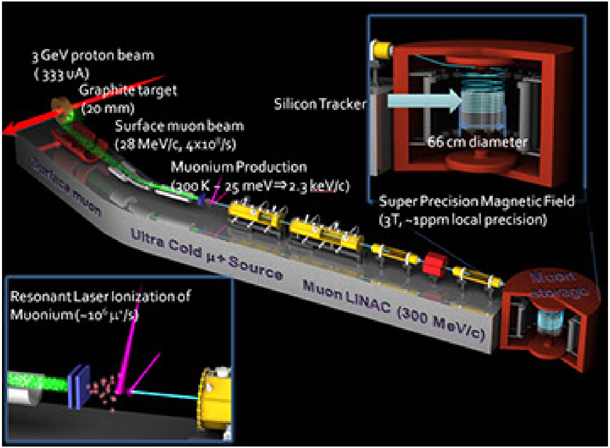
  - HVP data-driven:
    Other e^+e^- experiments are available or forthcoming:
    SND, BaBar, Belle II, BESIII, CMD3 $\Rightarrow$ Error reduction
    Model-independent evaluation of RadCorr underway
    MuonE will provide an alternative way to measure HVP

  - HVP lattice:
    calculations with precision $\sim$ BMW for $a_\mu^{\text{HVP, LO}}$ are awaited
    Discrepancy with data-driven for a_μ^{win} must be understood

  - HLbL data-driven: goal of $\sim$ 10% uncertainty within reach

  - HLbL lattice: RBC/UKQCD $\Rightarrow$ similar precision as Mainz.
    Good agreement with data-driven evaluation.

Future: Muon $g-2$/EDM experiment @ J-PARC



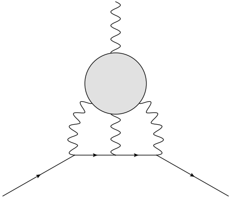
Credit: J-PARC

Backup Slides

Calculating the HLbL contribution

The HLbL contribution is a very complex quantity

- 4-point function of em currents in QCD



- early on, it has been calculated with models

  Hayakawa-Kinoshita-Sanda/Bijnens-Pallante-Prades (96), Knecht, Nyffeler (02), Melnikov, Vainshtein (04)

- a data-driven approach, like for HVP, has only recently been developed and used

  GC, Hoferichter, Procura, Stoffer=CHPS (14,15,17), Hoferichter, Hoid, Kubis, Leupold, Schneider (18)

- lattice QCD is becoming competitive RBC/UKQCD (20), Mainz (21)

HLbL contribution: Master Formula

$$a_\mu^{\mathrm{HLbL}} = \frac{2\alpha^3}{48\pi^2} \int_0^\infty dQ_1 \int_0^\infty dQ_2 \int_{-1}^1 d\tau \sqrt{1-\tau^2} \sum_{i=1}^{12} T_i(Q_1, Q_2, \tau) \bar{\Pi}_i(Q_1, Q_2, \tau)$$

Q_i^μ are the Wick-rotated four-momenta and τ the four-dimensional angle between Euclidean momenta:

$$Q_1 \cdot Q_2 = |Q_1||Q_2|\tau$$

The integration variables $Q_1 := |Q_1|$, $Q_2 := |Q_2|$.

GC, Hoferichter, Procura, Stoffer (15)

- T_i: known kernel functions
- $\bar{\Pi}_i$ are amenable to a dispersive treatment:
  imaginary parts are related to measurable subprocesses

Improvements obtained with the dispersive approach

Contribution	PdRV(09) *Glasgow consensus*	N/JN(09)	J(17)	WP(20)
π^0, η, η'-poles	114(13)	99(16)	95.45(12.40)	93.8(4.0)
π, K-loops/boxes	−19(19)	−19(13)	−20(5)	−16.4(2)
S-wave $\pi\pi$ rescattering	−7(7)	−7(2)	−5.98(1.20)	−8(1)
subtotal	88(24)	73(21)	69.5(13.4)	69.4(4.1)
scalars	–	–	–	} − 1(3) (scalars + tensors)
tensors	–	–	1.1(1)	
axial vectors	15(10)	22(5)	7.55(2.71)	6(6)
u, d, s-loops / short-distance	–	21(3)	20(4)	15(10)
c-loop	2.3	–	2.3(2)	3(1)
total	105(26)	116(39)	100.4(28.2)	92(19)

- significant reduction of uncertainties in the first three rows
  CHPS (17), Masjuan, Sánchez-Puertas (17) Hoferichter, Hoid et al. (18), Gerardin, Meyer, Nyffeler (19)
- $1-2$ GeV resonances affected by basis ambiguity and large uncertainties Danilkin, Hoferichter, Stoffer (21)]
- asymptotic region recently addressed, Melnikov, Vainshtein (04), Nyffeler (09), WP still work in progress Bijnens et al. (20,21), Cappiello et al. (20), Leutgeb, Rebhan (19,21)

Situation for HLbL