Percolation(s) on Complex Networks

"a physicist's guide to prepare a *critically good* coffee"

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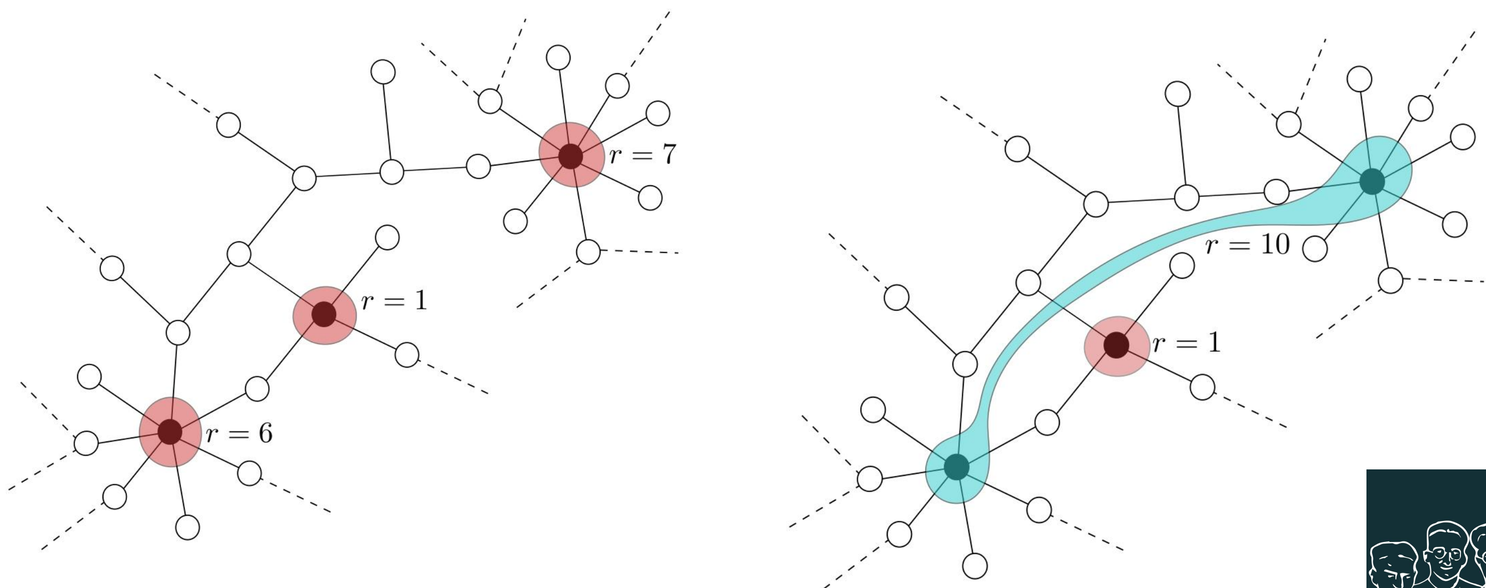
- **Goal:** explore the basic notions of percolation theory and complex networks theory, and how they can come together



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Percolation

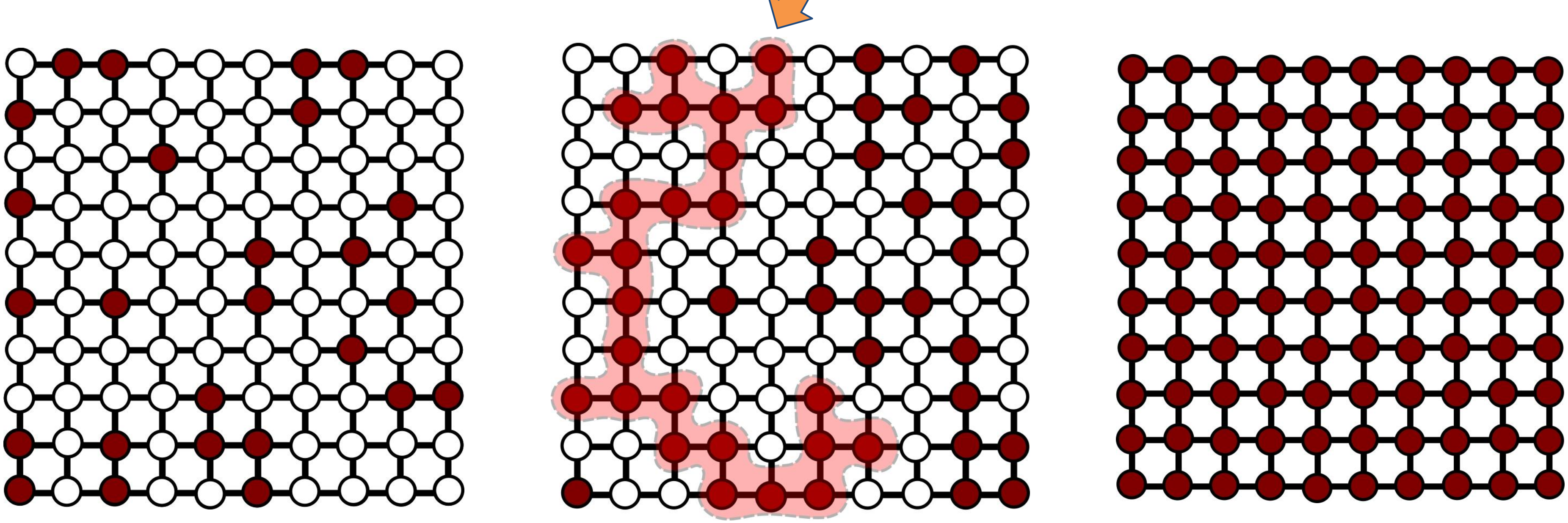
suggested read

Introduction to percolation theory, D. Stauffer

= **active** node

= **inactive** node

Something is happening here: “spanning” cluster



Fraction of active nodes

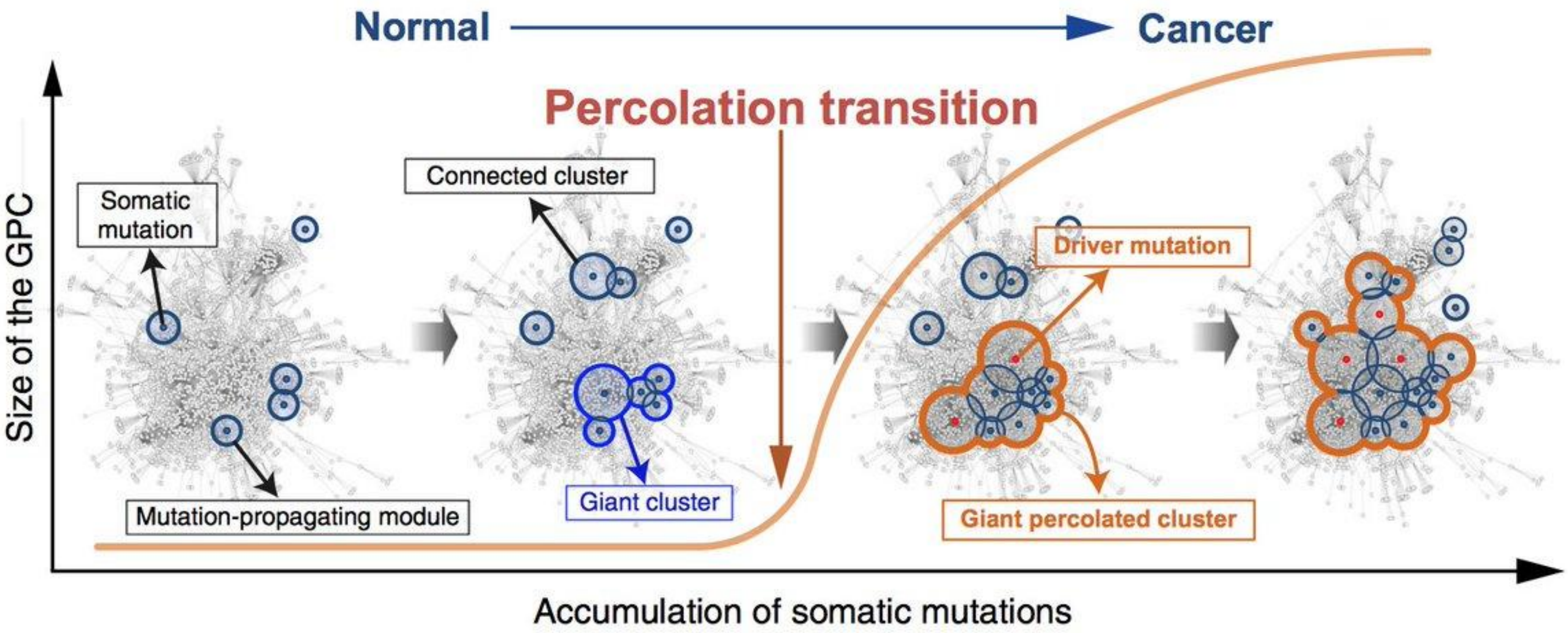
Percolation theory

the existence of large clusters

→ studies how the structural properties of networks change if we remove (or add, switch on/off, burn, activate/deactivate, occupy...) some nodes

→ **Applications** in epidemic spreading, forest fires, flow in porous media, tumorigenesis...

→ simplest model that exhibits a phase transition



Percolation and cancer (source: Shin, D., et al., Nat Commun 8, 1270 (2017))

Basic definitions and standard tools

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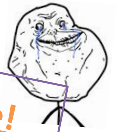

Order parameter: size of the largest component

$$S = \frac{\#\text{nodes in the LC}}{N}$$

Control parameter: fraction of active nodes ϕ

Critical point: where we observe a “phase transition” ϕ_c

Critical exponent: $\beta \longrightarrow S \sim (\phi - \phi_c)^\beta$

Task: understand how the order parameter varies if we vary the control parameter

$$S(\phi) \text{ vs } \phi \qquad S(\phi)$$

A “typical” plot in percolation theory

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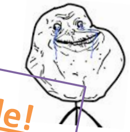

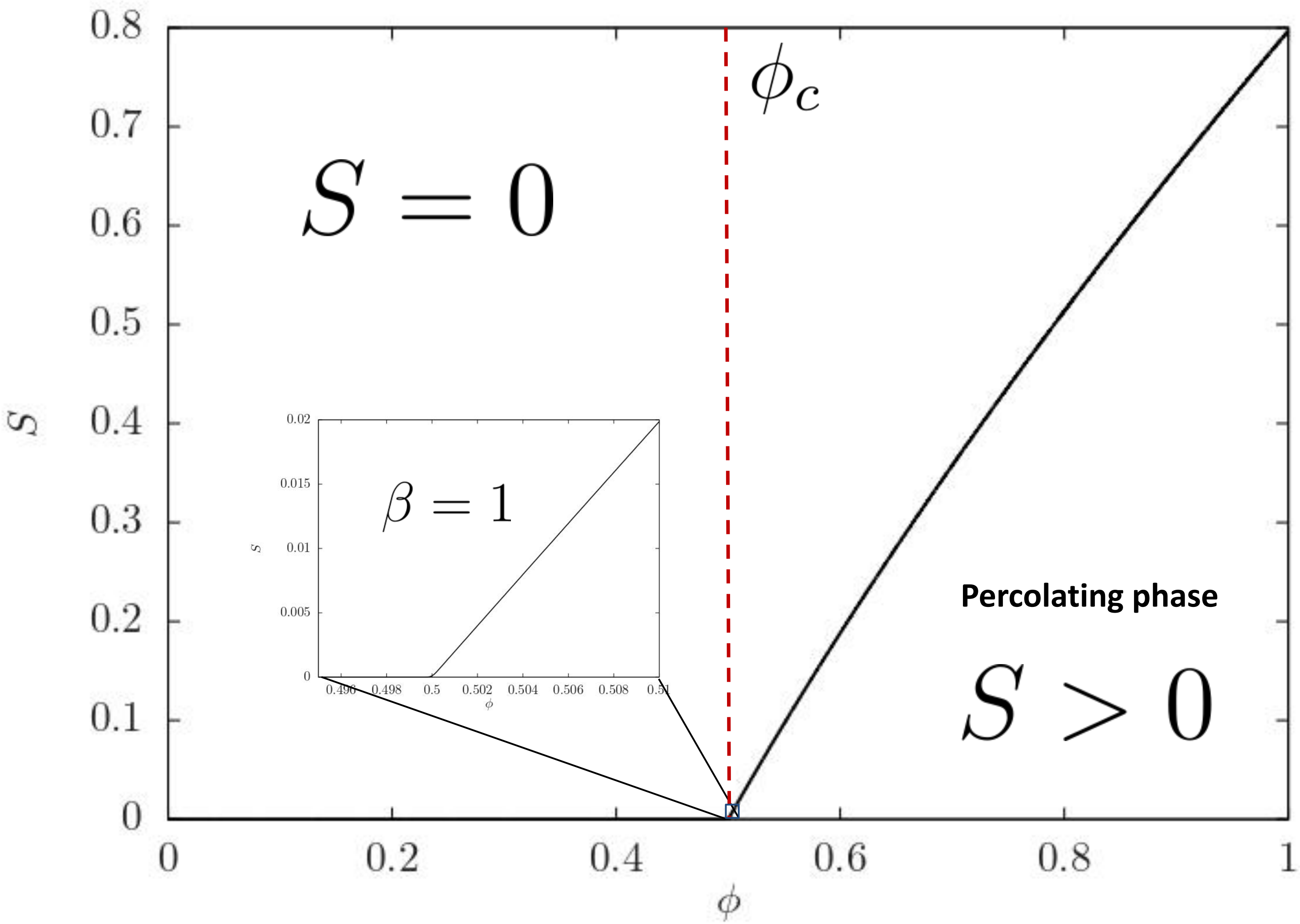

Largest vs giant component

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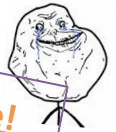
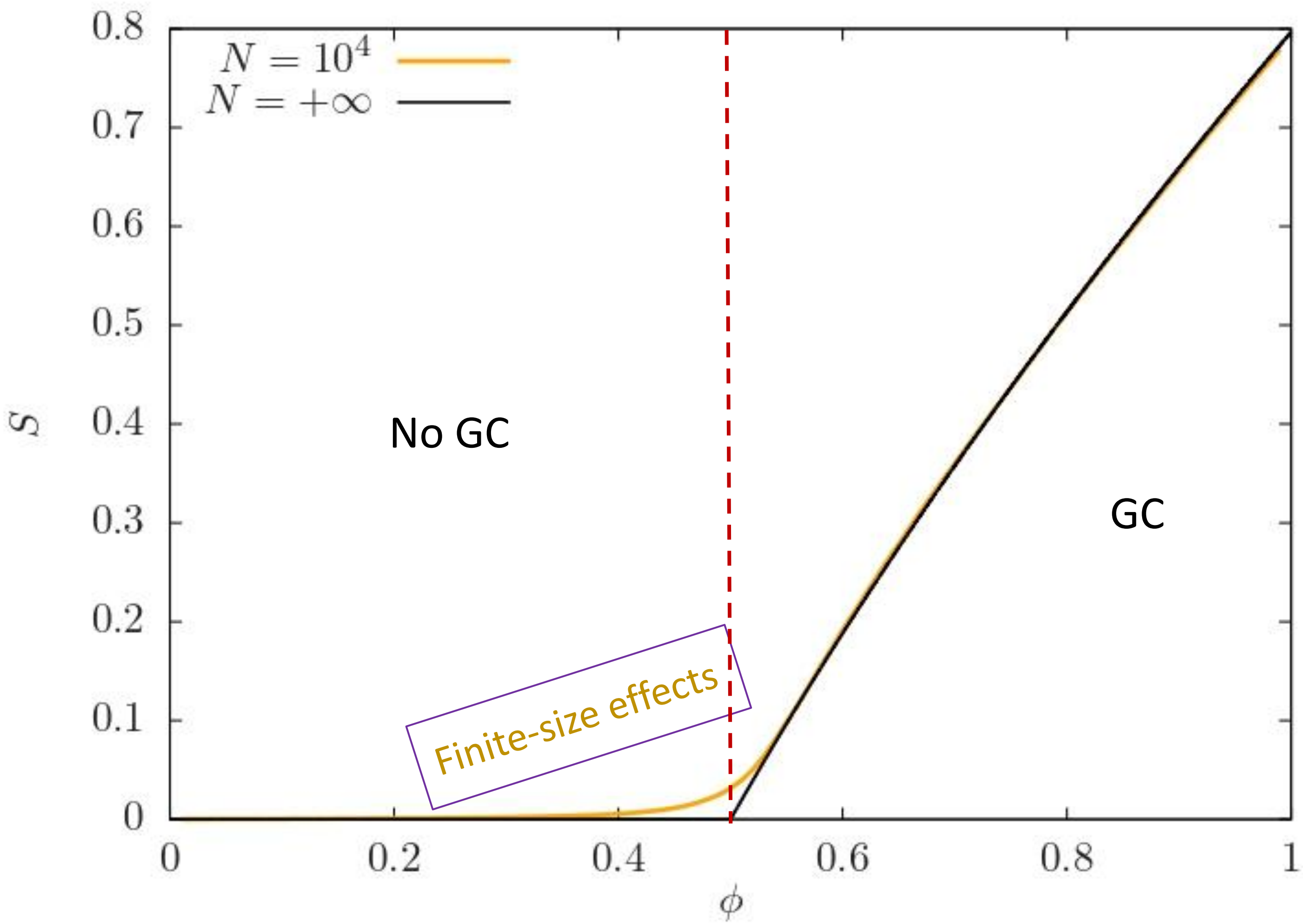


Phase transition only in the "thermodynamic limit" $N \to +\infty$

$\to$ we call the largest component a "giant component" above the critical point

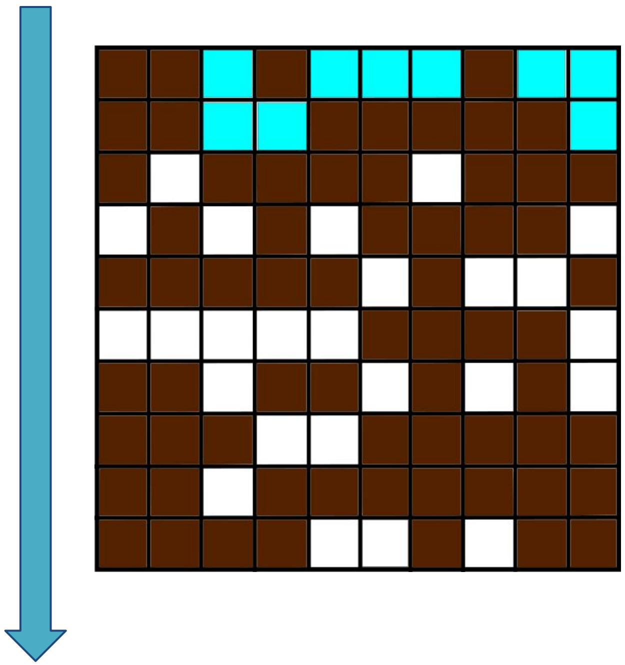


Now we can make the perfect coffee!

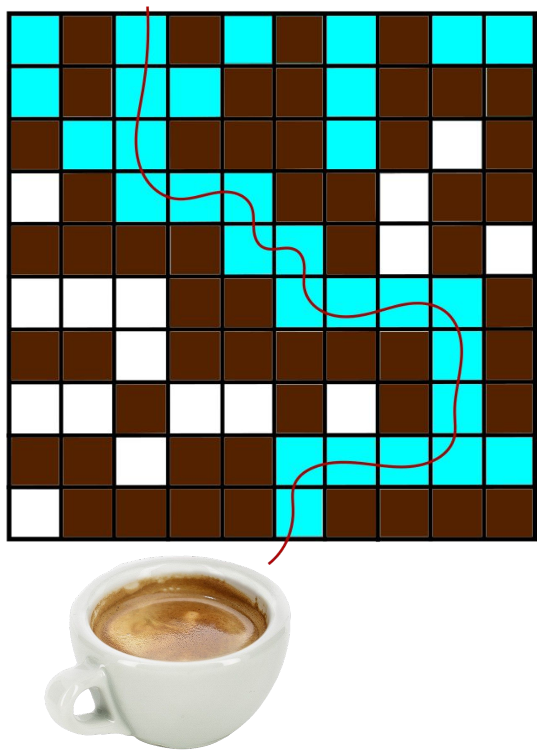
active node = water flows

("Flatland's" coffee just for the sake of simplicity)

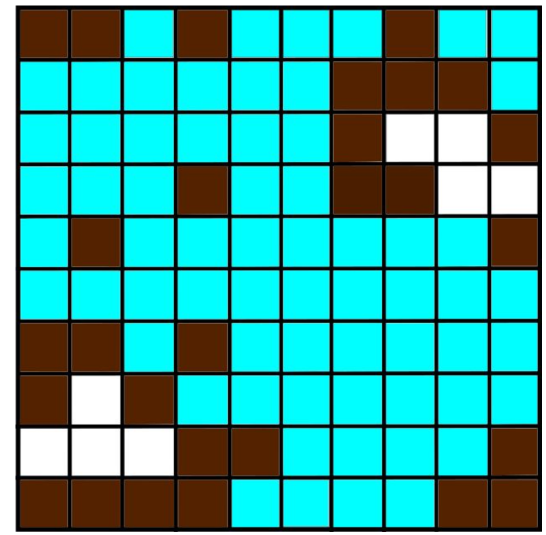
Sub-critical (=**no coffee**!)



Critical (=***perfect coffee***!)



Super-critical (=**watery coffee**!)



At the critical point we maximize the ratio between surface of coffee grains and volume of water flow

Fractals!

Complex Networks

suggested read → *Networks: an introduction*, M. E. J. Newman

→ non-trivial topological features

- that do **not** occur in **simple** networks such as lattices or random graphs but often occur in networks representing real systems (typically with a power-law degree distribution) like:
  - Internet
  - Human brain
  - Social networks
  - Neural Networks(→machine learning)
  - Epidemic Networks
  - ….

$$p(k) \sim k^{-\gamma}$$

degree = number of neighbors

Heterogeneity

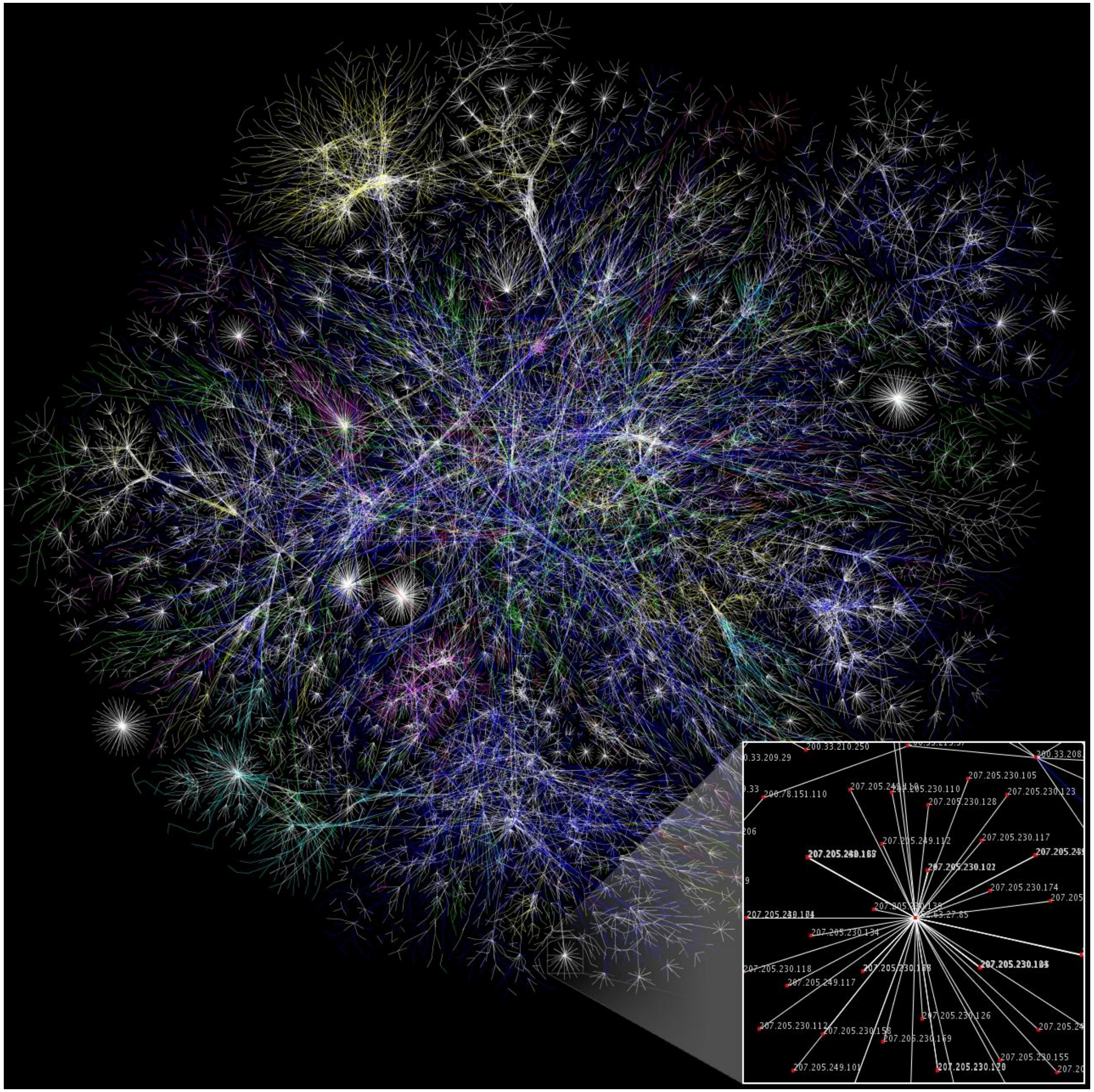
"small-world" effect

"scale-free"

High-clustering

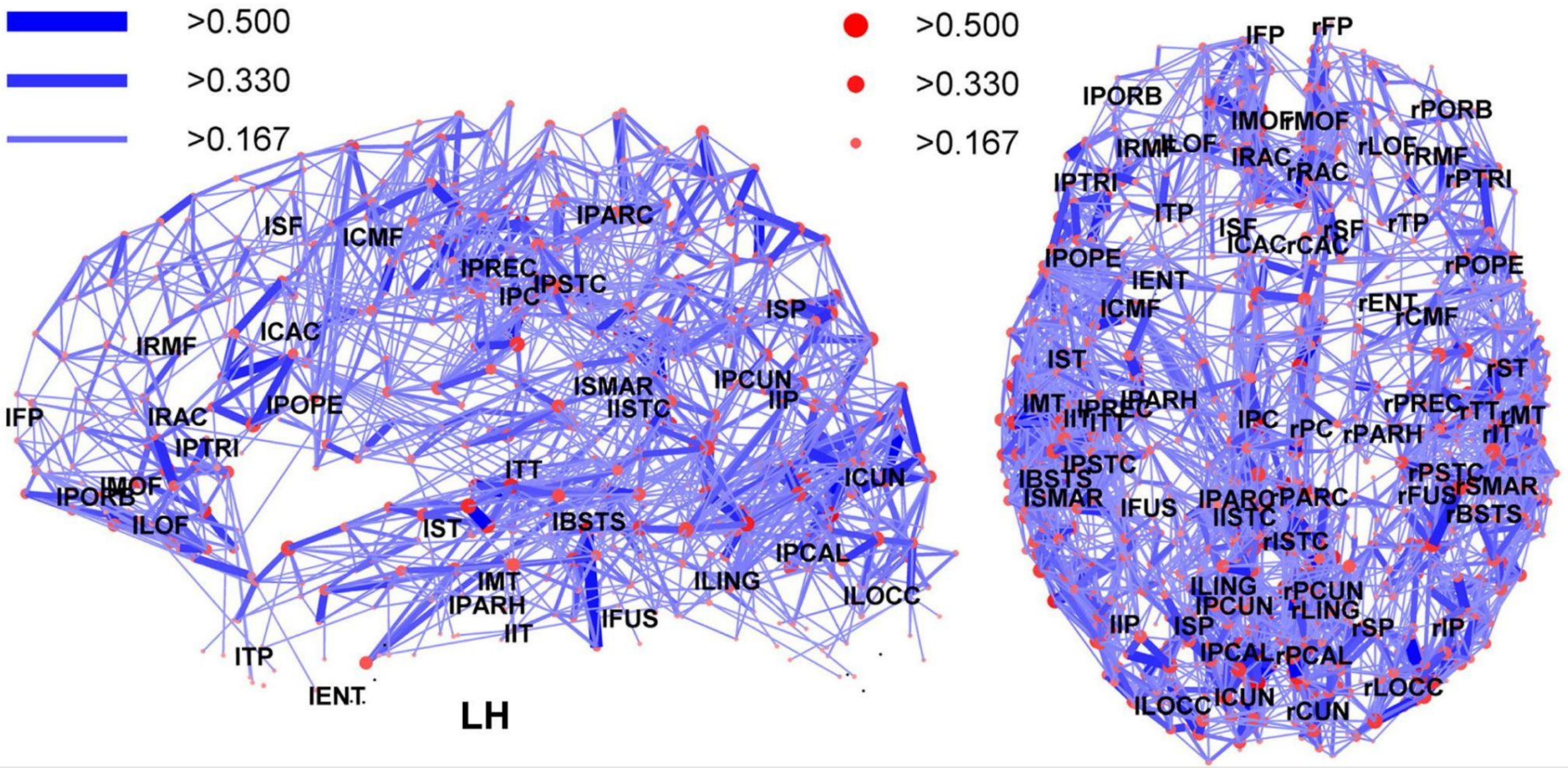
assortativity

Examples of complex networks: internet



Computer Network (source: Wikipedia)

Examples of complex networks: the brain



Human Brain Network (source: Hagmann, P., et al., (2008), PLoS biology, 6(7), e159)

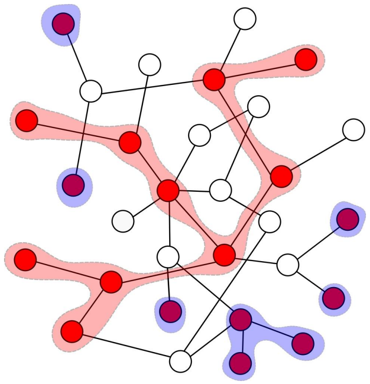
Examples of complex networks: social networks

A social network (source: Wikipedia)

Percolation + complex networks

Question: is the network robust with respect to the failure of some nodes?

Answer: percolation theory!



- Epidemic threshold (failure = vaccination)
- Computer virus spreading (failure = infection)
- Power grid
- Road networks (failure = road closing)

→ percolation transition

Many real-world applications

Random Networks

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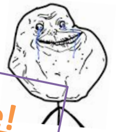


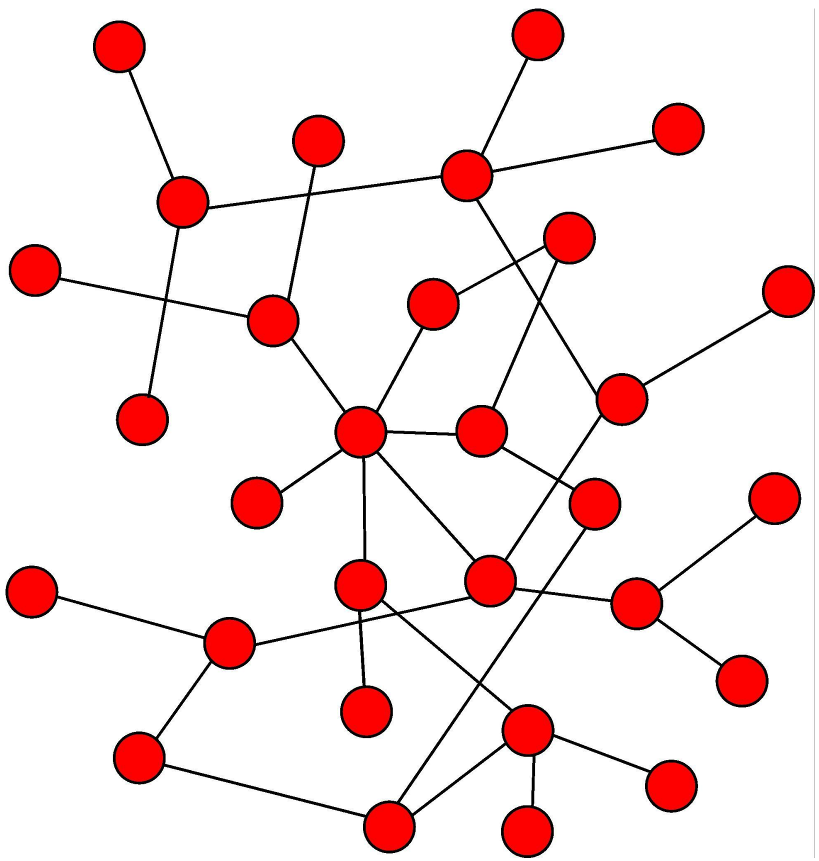
We need **models**

simple enough to be tractable, but
complex enough to describe reality

How can we "model" real-world complex networks?

Degree distribution

$$p(k) \sim k^{-\gamma}$$



We generate random networks which share some of the features we observe

Necessary to obtain analytical results and to perform numerical simulations

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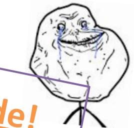
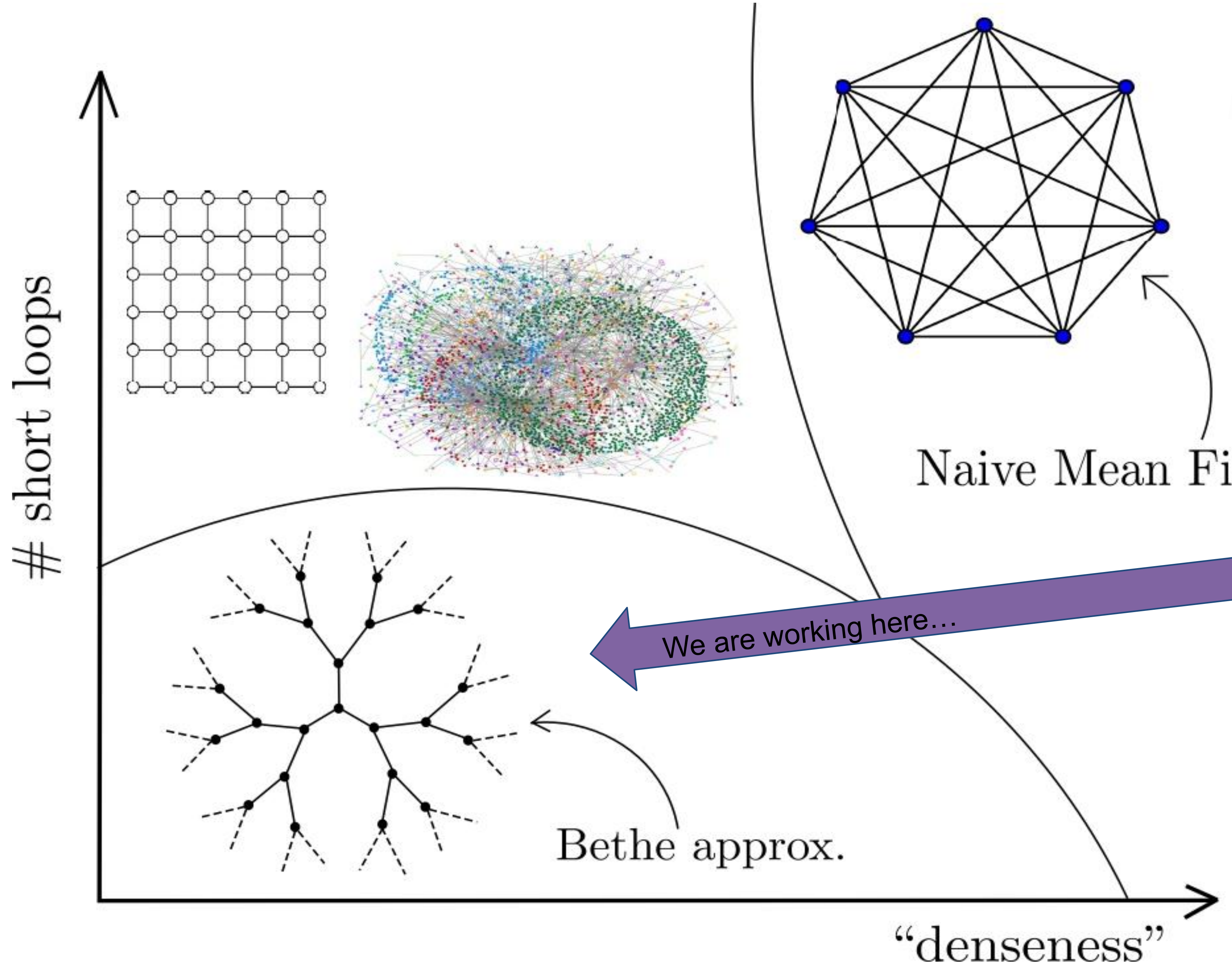


Which networks should we use?



Generating functions formalism

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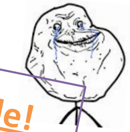


Callaway, D. S., Newman, M. E., Strogatz, S. H., & Watts, D. J. (2000), *PRL 85*(25), 5468.

Analytical solution for uncorrelated random graph with a given degree distribution

$$S(\phi) = \phi(1 - g_0(u))$$
$$u = \phi g_1(u) + 1 - \phi$$

$$g_0(u) = \sum_{k=0}^{\infty} p_k u^k$$

$$g_1(u) = g_1'(u)/g_1'(1)$$

probability that following a random link we **do not** reach the GC

We can compute

ϕ_c and β

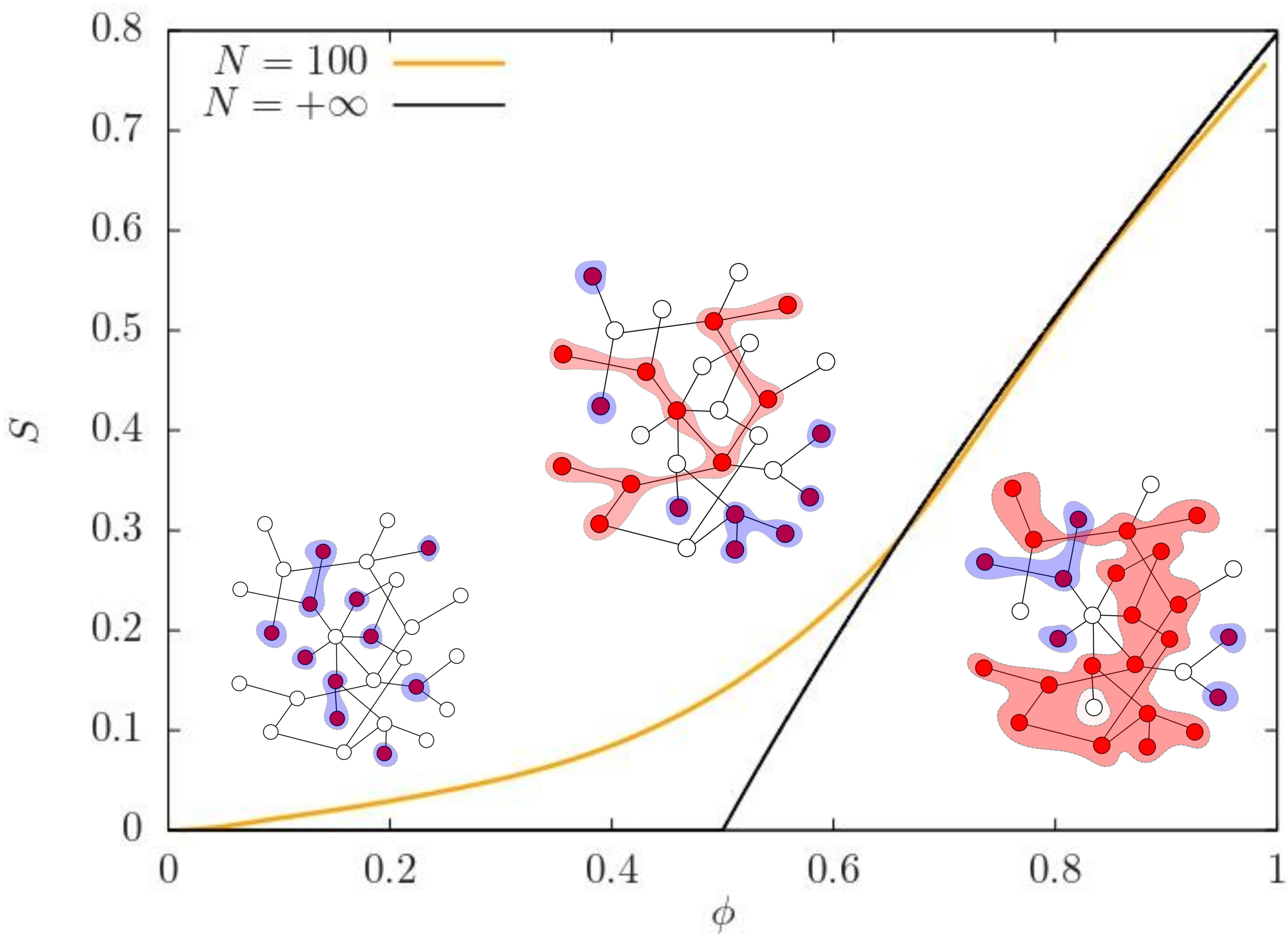
Generating functions: a powerful tool in probability and combinatorics

depends only on the shape of the degree distribution → universality

Percolation on random networks

= **active** node

= **inactive** node



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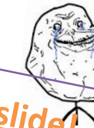


Cumulative Merging Percolation (CMP)

PRE 105, 054310 (2022)

- A "**long-range**" model
  - growing interaction range

    →arbitrary distant nodes can interact

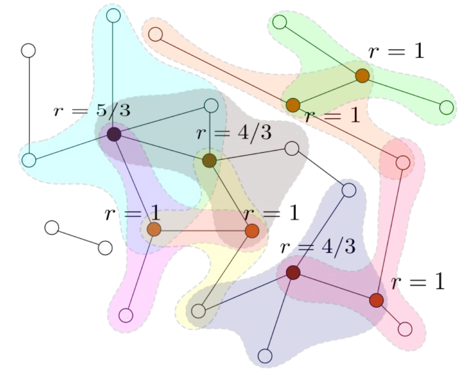
    The generating functions formalism fails!
  - Mass of the clusters
  - Percolation occurs in a degree ordered way

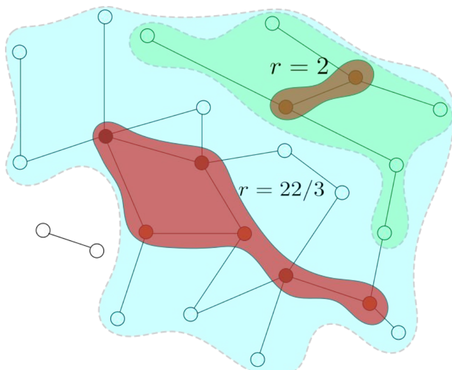
    → all nodes with degree larger than the threshold are active
  - Algebraic growth:

$$r(m) = (m/k_a)^{\alpha}$$

  - Logarithmic growth

$$r(m) = 1 + \delta \ln(m/k_a)$$





- **Scaling arguments** + numerical simulations
  - Two competing mechanisms→ their behaviour depends on the parameters

→ New critical exponents

α and δ

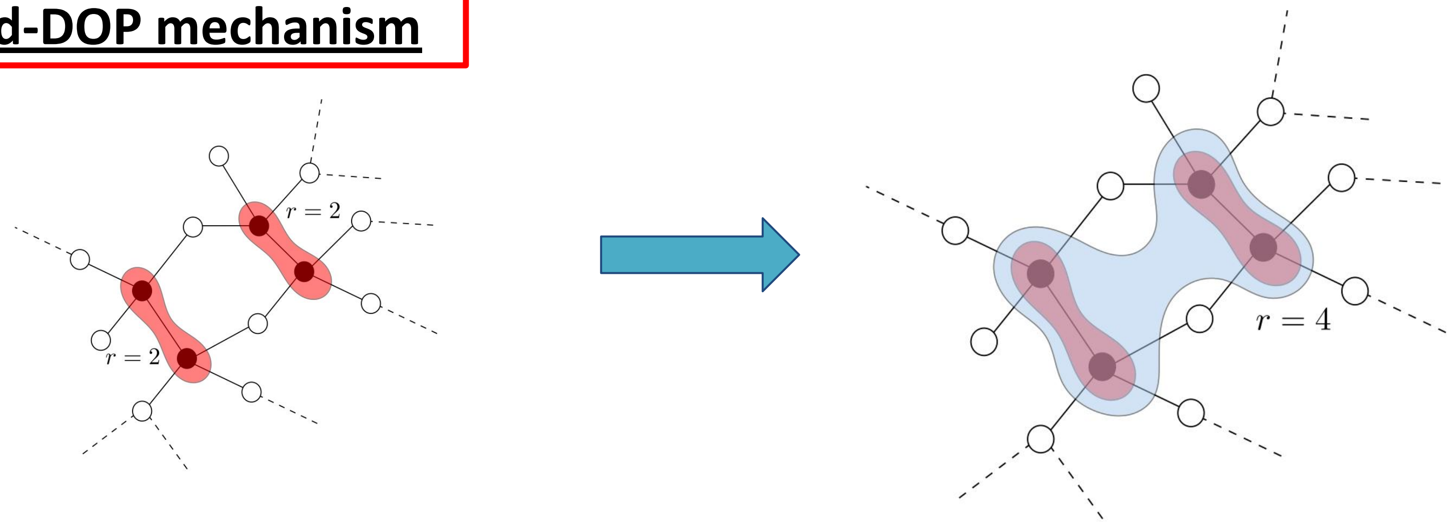
Application to epidemic spreading

(epidemic transition of the SIS model on complex networks)

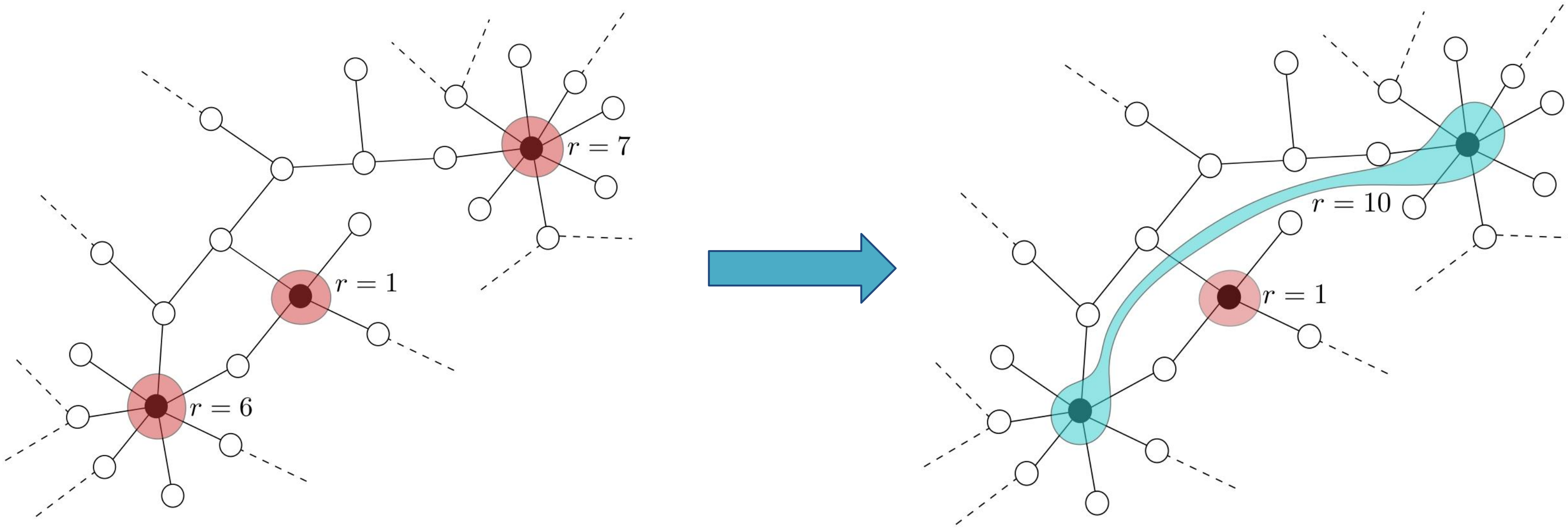
C.Castellano, R.Pastor-Satorras PRX 10, 011070 (2020)

CMP: the two mechanisms at work

I. Extended-DOP mechanism



II. Distant isolated nodes merging



CMP: main results

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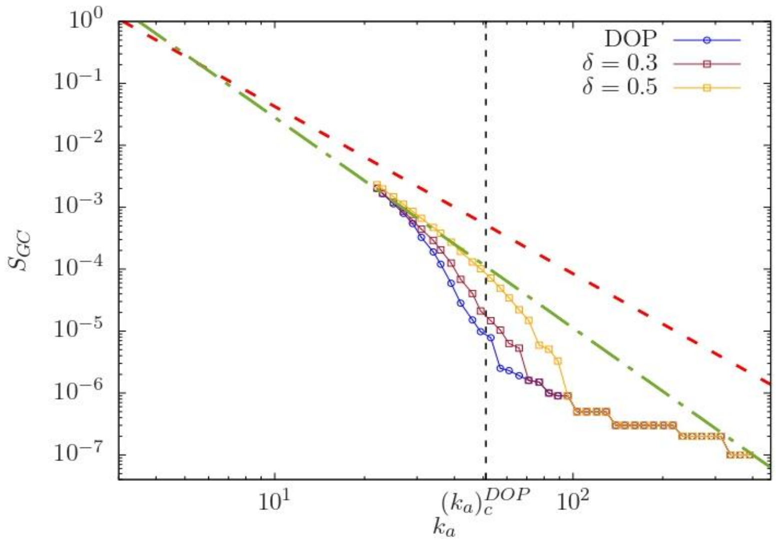
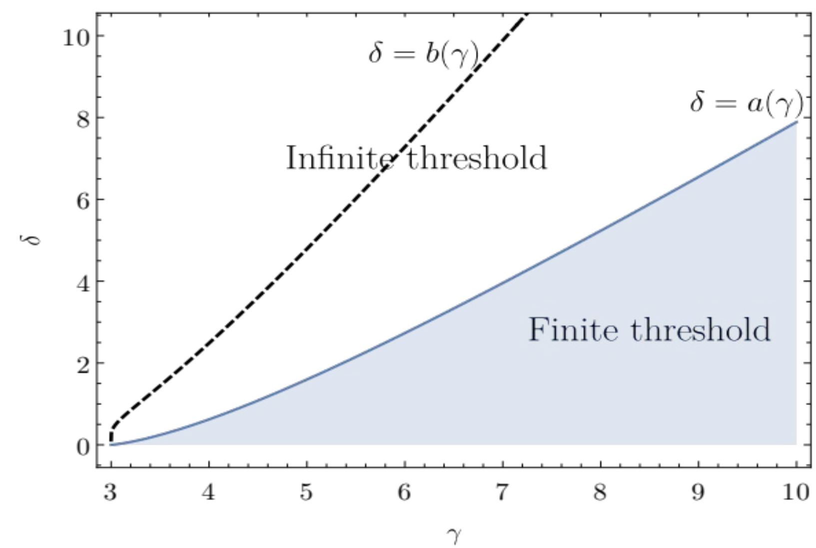
- New critical exponents for logarithmic growth
  - **Phase-transition** from an "finite-threshold" phase to a "infinite-threshold" phase





- Future research:
  - Study the cluster merging process for uniform percolation
  - Study the process on simpler networks topology

Thank you for your time!