Holographic Realization of the Prime Number Quantum Potential

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Work in collaboration with
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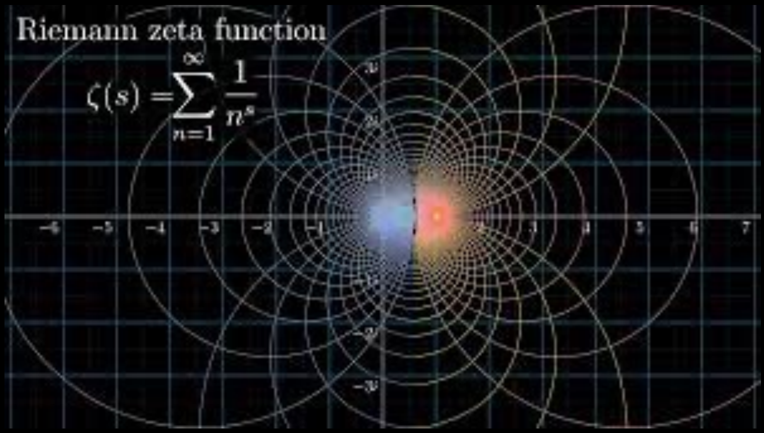
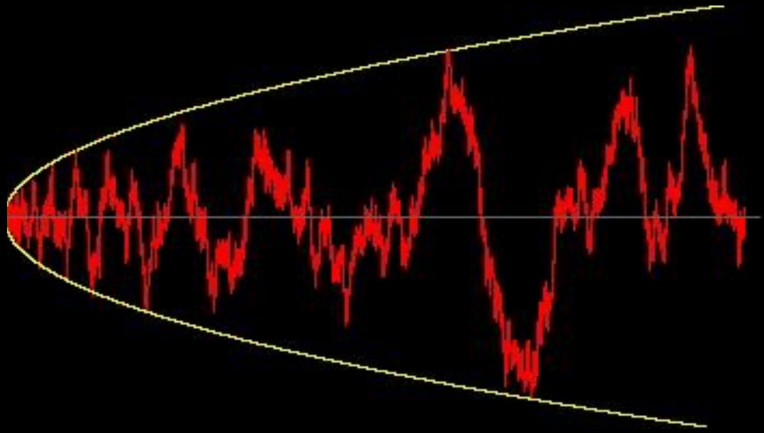
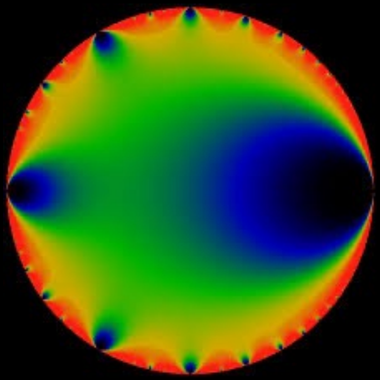
Topics of the seminar

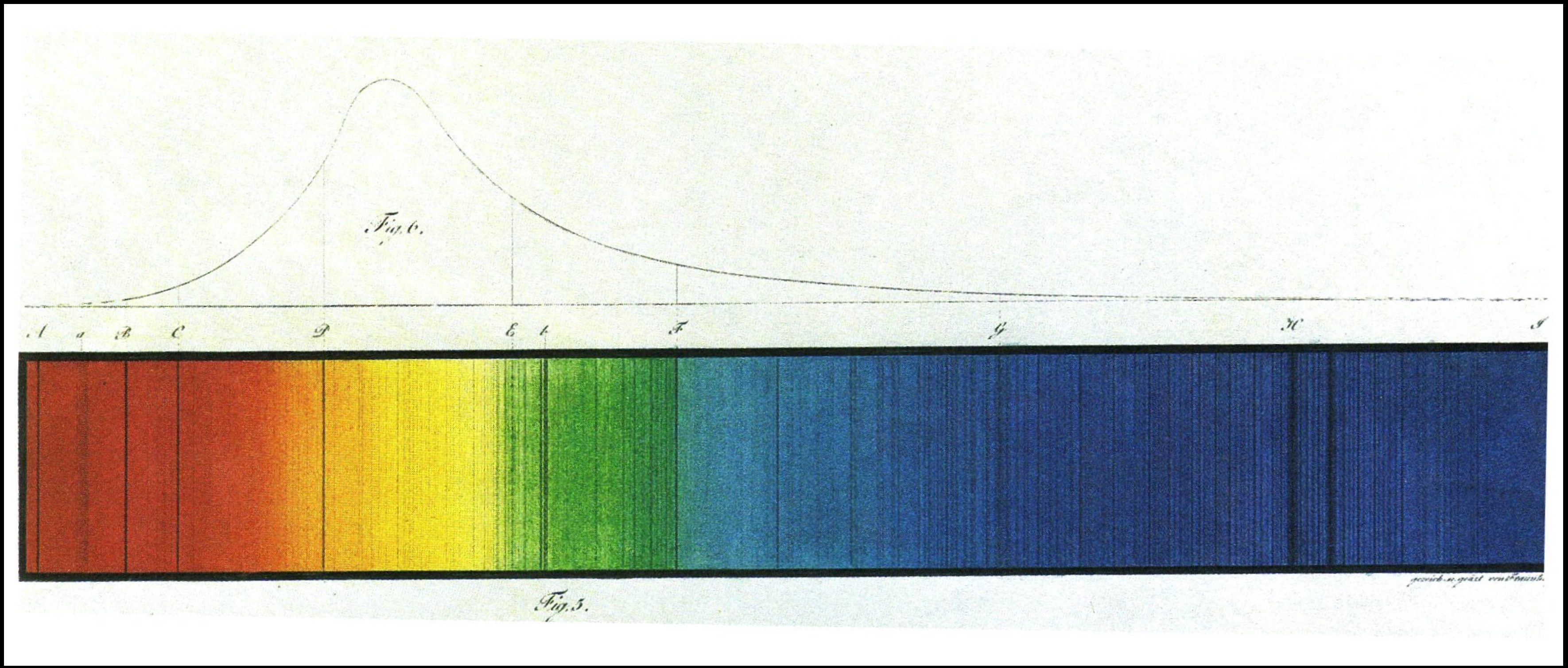
- Number Theory and Physics
- Quantum Abacus
- Prime numbers
- Experimental realization of the quantum potential for the primes
- New perspectives

There has been increasing interest for the profound and engaging links recently discovered between Number Theory and Physics









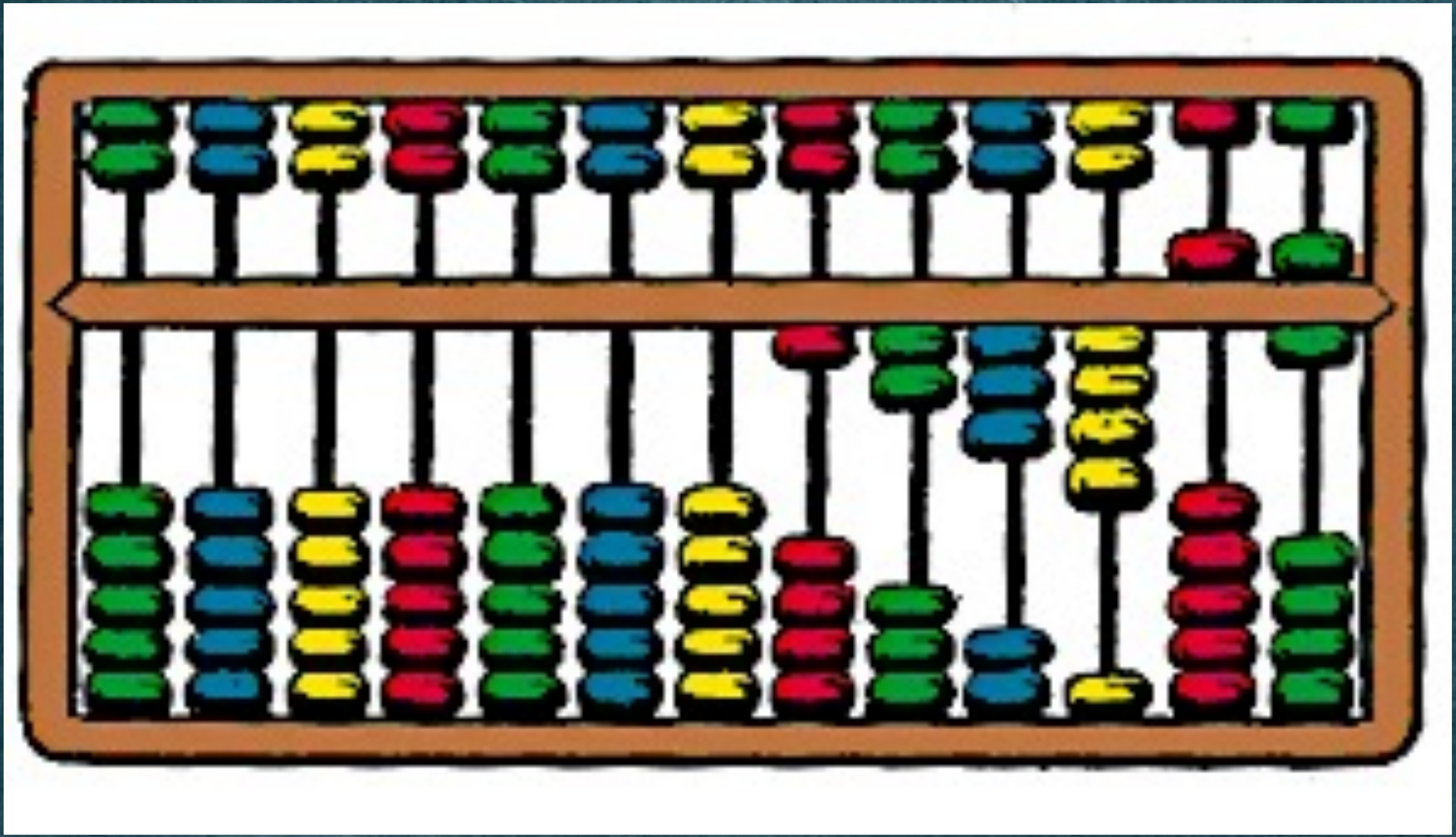
$$\nu_{n,m} = E_n - E_m$$

Natural questions

- Given an arithmetic sequence $\{S_n\}$, does a quantum system which has this sequence as a spectrum exist?

- Is the Hamiltonian of such a system unique?

Quantum Abacus



Quantum Abacus

We would like to **realise** quantum systems with energy levels related in a controllable way to arithmetic sequences

This would allow us to approach in the most efficient way problems of highest complexity, such as

- Primality test
- Factorization
- Several open conjectures, …

Today our attention is focused on quantum abacus based on a one-dimensional Hamiltonian of the form

$$H = \frac{p^2}{2m} + V(x)$$

Fibonacci numbers

$$F_{n+2} = F_{n+1} + F_n$$

$\{F_n\}$ =1, 1, 2, 3, 5, 8, 13, 21, 34, 55,…

$$\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}^n = \begin{pmatrix} F_{n+1} & F_n \\ F_n & F_{n-1} \end{pmatrix}$$

$$F_n = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right]$$

- Unfortunately, it does not exist a quantum system which has the Fibonacci numbers as spectrum…

- The reason is that their sequence grows too fast

- Similarly, it does not exist a quantum Schroedinger Hamiltonian with a spectrum given, for instance, by the Mersenne numbers or the perfect numbers

$$M_n = 2^n - 1$$

$$P_n = 2^{n-1} M_n$$

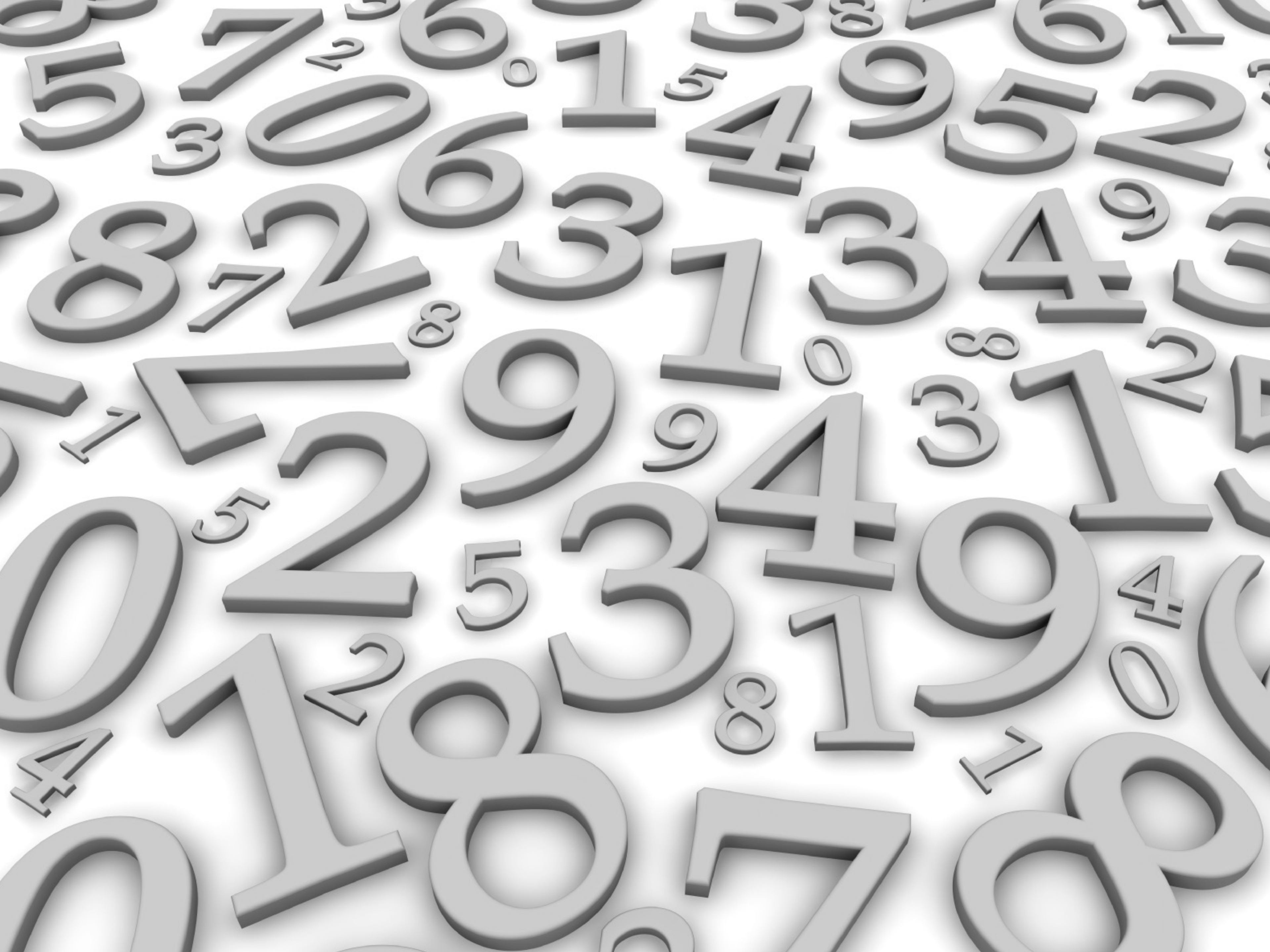
Bound on the growth of eigenvalues

- For a one-dimensional Hamiltonian of the form

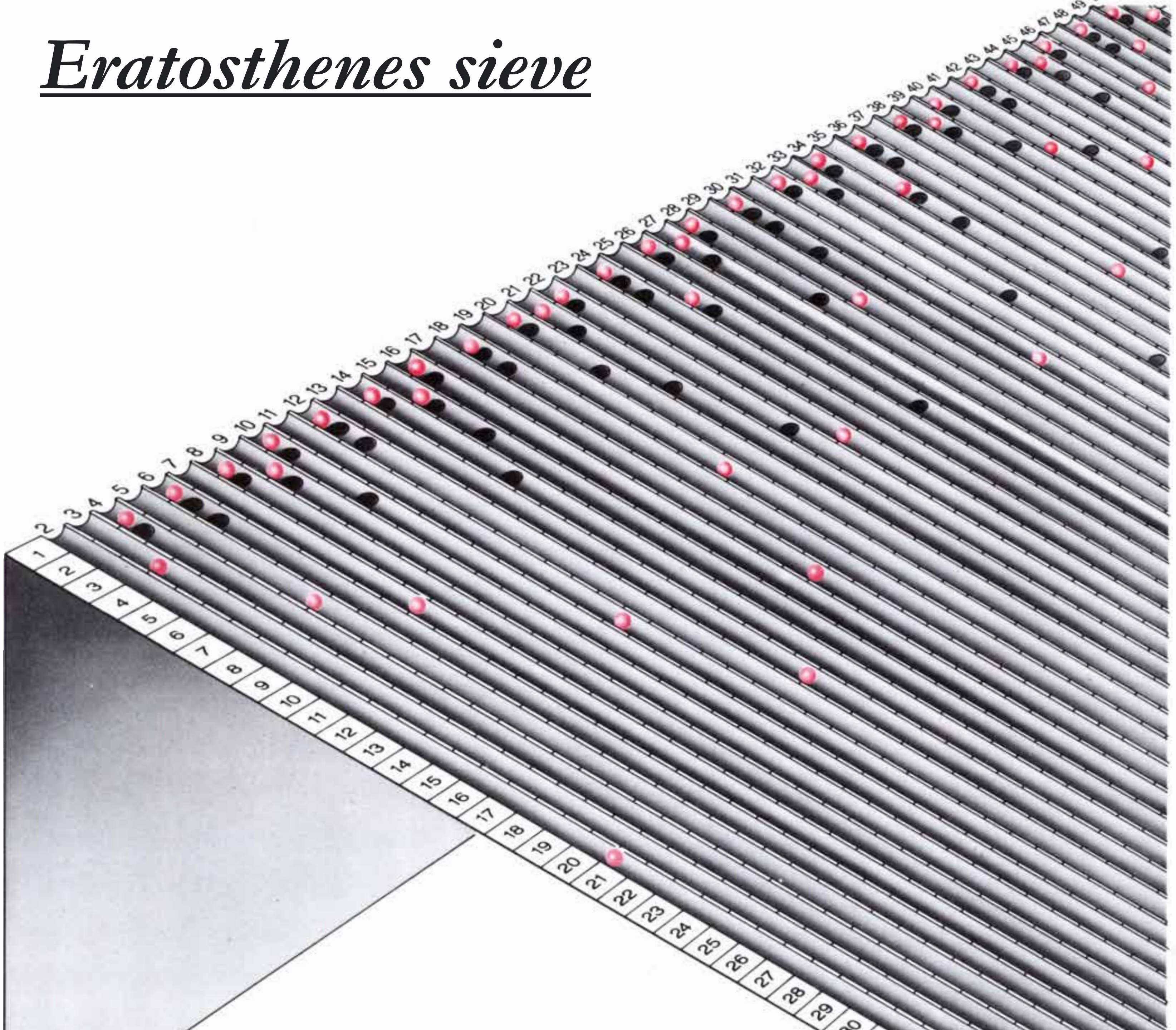
$$H = \frac{p^2}{2m} + V(x)$$

the sequence of energy eigenvalues must satisfy

$$E_n \leq n^2$$



Eratosthenes sieve



Dr. Jekyll and Mr. Hyde

- On a large scale, primes have extremely smooth distribution

$$n \leq p_n < 2n$$

- On a small scale, however, primes have highly unpredictable and irregular behavior

$$p_{n+1} < 2p_n$$

Example: Gap between the primes

1. Many (infinite?) twins of primes

(11,13) (17,19) (41,43) … (347,349) …

2. Arbitrarily large interval without a single prime!!

$$(10^{12}+1)! + n \quad , \quad n = 1, 2, 3, \ldots 10^{12}+1$$

It does not exist a close formula for the n-th prime number

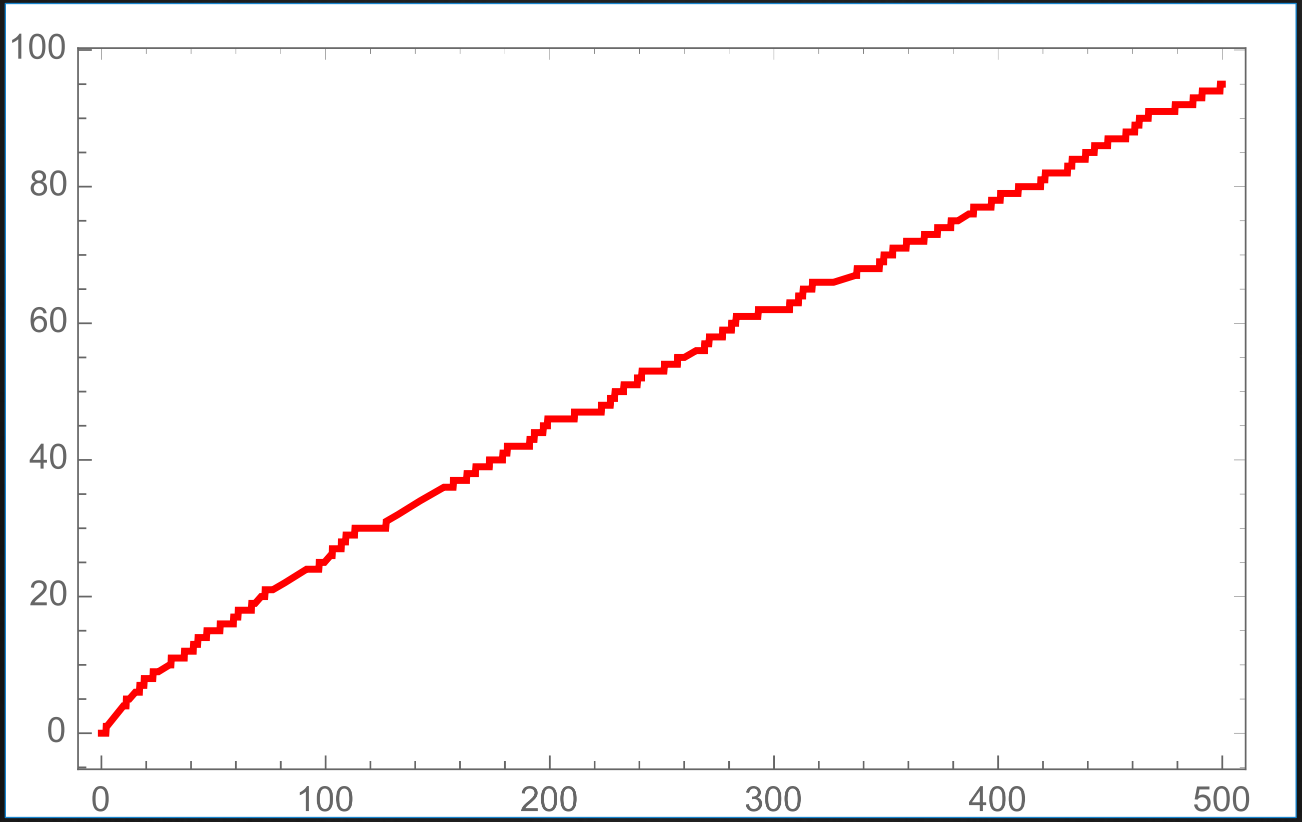
However their scaling law is captured by

$$p_n \simeq n \, \log n$$

Hence, there must exist a quantum Hamiltonian that has the primes as eigenvalues!

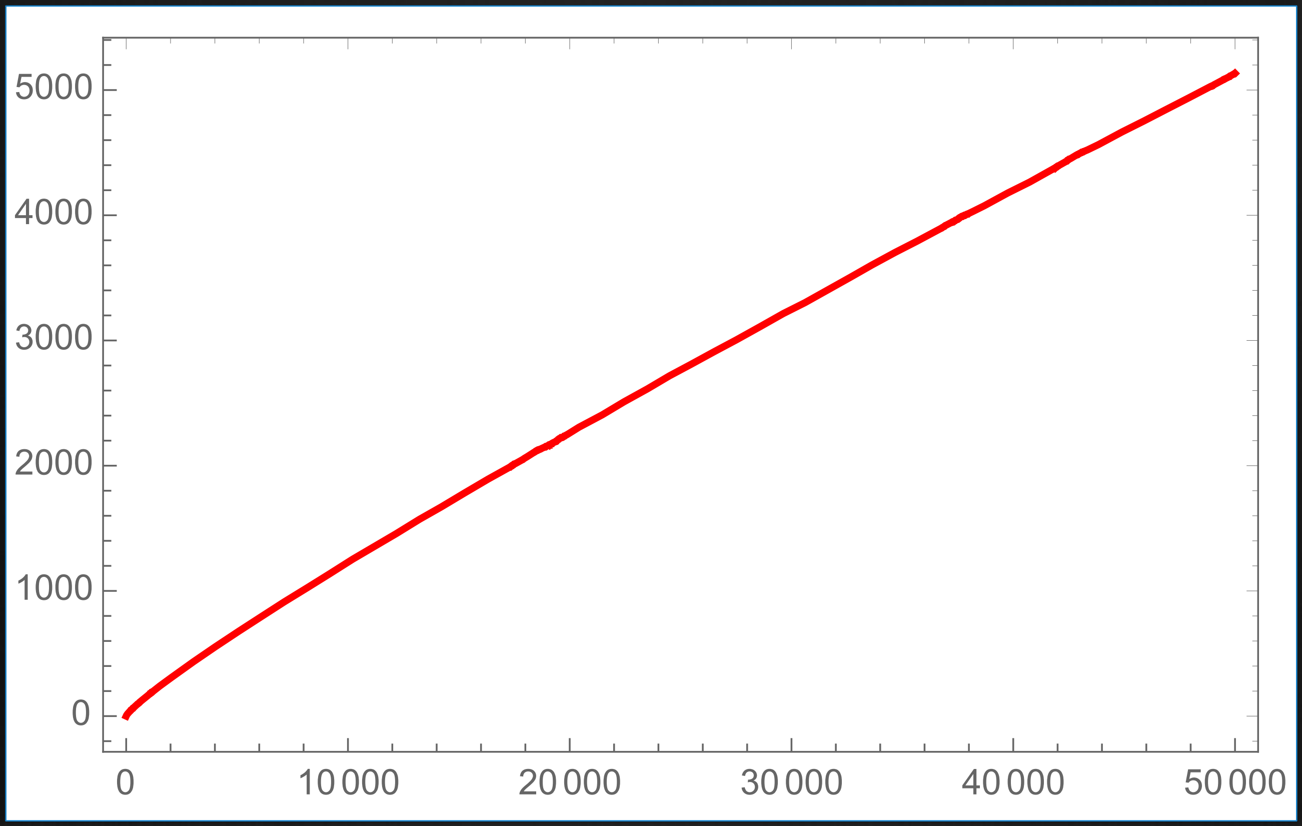
Counting the Primes

$\pi(x)$: gives the number of primes less or equal to x



Counting the Primes

$\pi(x)$: gives the number of primes less or equal to x



Prime Number Theorem: Gauss

$$\pi(x) \simeq Li(x) \equiv \int_1^x \frac{dt}{\log t} \simeq \frac{x}{\log x}$$

A simple proof

- assuming no correlation between numbers, 1/p is the probability that a number x is divisible by the prime p
- $$W(x) \simeq \left(1-\frac{1}{2}\right)\left(1-\frac{1}{3}\right)\left(1-\frac{1}{5}\right)\cdots = \prod_{p<x}\left(1-\frac{1}{p}\right)$$
- $$\log W(x) \simeq \sum_{p<x} \log\left(1-\frac{1}{p}\right) \simeq -\sum_{p} \frac{1}{p} \simeq -\int_1^x dn \frac{W(n)}{n}$$
- $$\frac{W'(x)}{W(x)} = \frac{W(x)}{x} \longrightarrow W(x) = \frac{1}{\log x}$$

x	$\pi(x) = \#\{\text{primes } \leq x\}$	Overcount: $\mathrm{Li}(x) - \pi(x)$
10^8	5761455	753
10^9	50847534	1700
10^{10}	455052511	3103
10^{11}	4118054813	11587
10^{12}	37607912018	38262
10^{13}	346065536839	108970
10^{14}	3204941750802	314889
10^{15}	29844570422669	1052618
10^{16}	279238341033925	3214631
10^{17}	2623557157654233	7956588
10^{18}	24739954287740860	21949554
10^{19}	234057667276344607	99877774
10^{20}	2220819602560918840	222744643
10^{21}	21127269486018731928	597394253
10^{22}	201467286689315906290	1932355207

The Riemann zeta Function

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_p \frac{1}{1 - \frac{1}{p^s}}$$

$$= \prod_p \left(1 + \frac{1}{p^s} + \frac{1}{p^{2s}} + \cdots \frac{1}{p^{ns}} + \right)$$

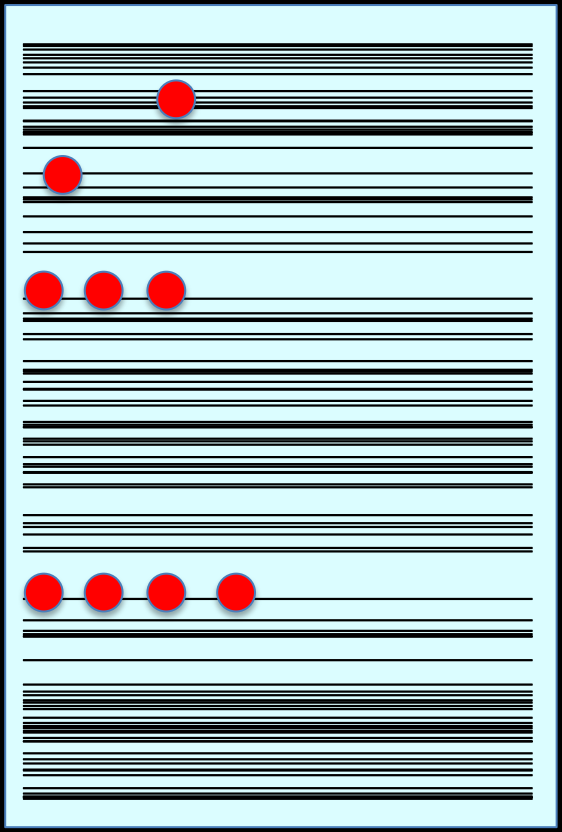
$$n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_k^{\alpha_k}$$

The Riemann zeta Function

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_p \frac{1}{1 - \frac{1}{p^s}}$$

Free bosonic system

$$E_p = \log p$$



The Riemann zeta Function

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_p \frac{1}{1 - \frac{1}{p^s}}$$

Grand canonical Ensemble

Micro canonical Ensemble

Connection between the primes and the Riemann ζ(s) function

$$\log \zeta(s) = s \int_1^\infty \frac{\pi(x)}{x(x^s - 1)} dx$$

Connection between the primes and the Riemann ζ(s) function

$$\pi(x) = J(x) - \frac{1}{2}J(x^{1/2}) - \frac{1}{3}J(x^{1/3}) - \frac{1}{5}J(x^{1/5}) + \frac{1}{6}J(x^{1/6}) + \cdots$$

$$= \sum_{n=1}^{\infty} \frac{\mu(n)}{n} J(x^{1/n})$$

$$\mu(n) = \begin{cases} 1 & \text{if n is squarefree with an even number of prime factors} \\ -1 & \text{if n is squarefeee with an odd number of prime factors} \\ 0 & \text{if n has a squared prime factor} \end{cases}$$

Connection between the primes and the Riemann ζ(s) function

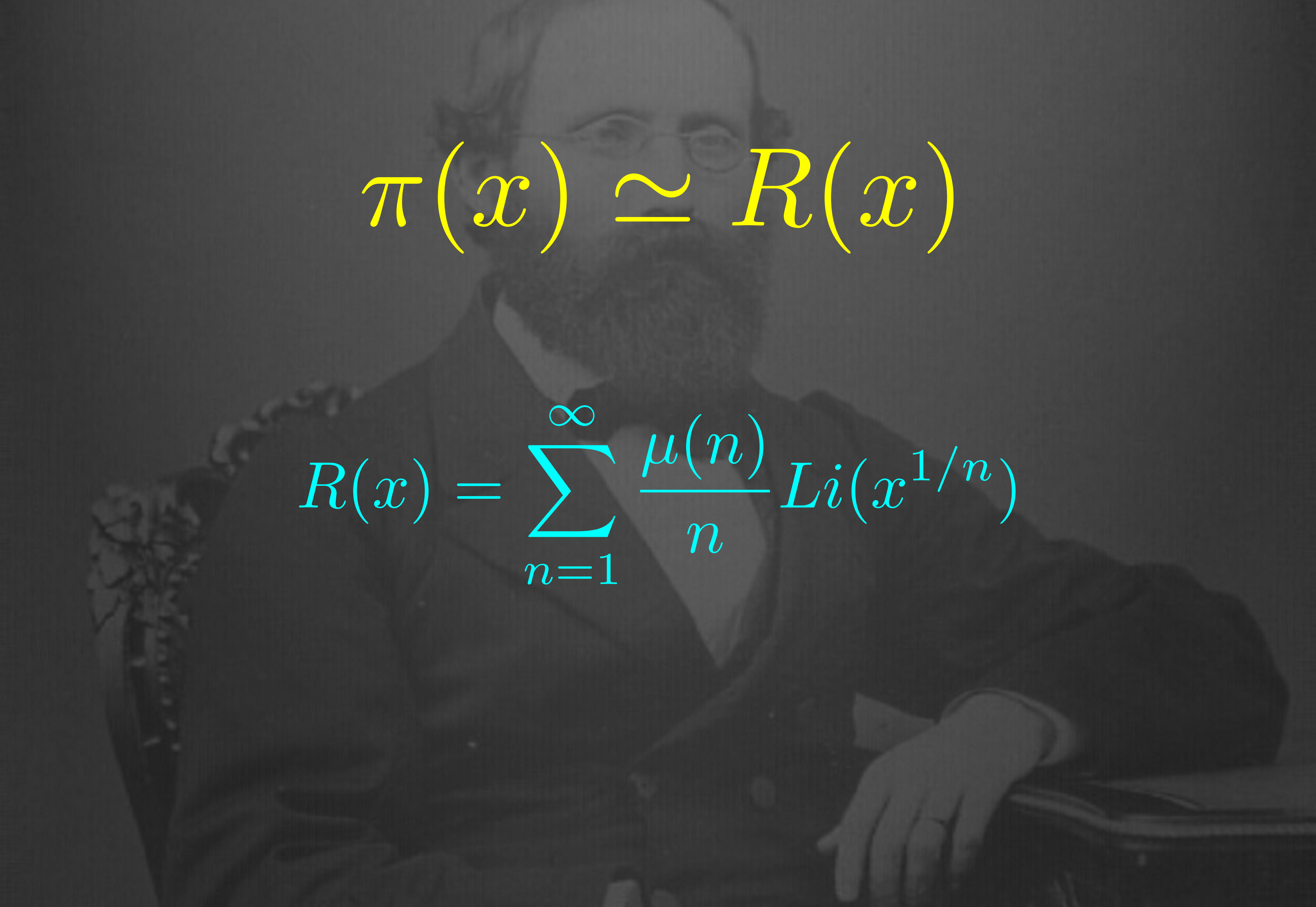
$$\pi(x) = J(x) - \frac{1}{2}J(x^{1/2}) - \frac{1}{3}J(x^{1/3}) - \frac{1}{5}J(x^{1/5}) + \frac{1}{6}J(x^{1/6}) + \cdots$$

$$= \sum_{n=1}^{\infty} \frac{\mu(n)}{n} J(x^{1/n})$$

$$J(x) = \frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty} \log\zeta(s)\, x^s \,\frac{ds}{s}$$

$$= Li(x) - \sum_{\rho} Li(x^{\rho}) - \log 2 + \int_x^{\infty} \frac{dt}{t(t^2-1)\log t}$$

Prime Number Theorem: Riemann

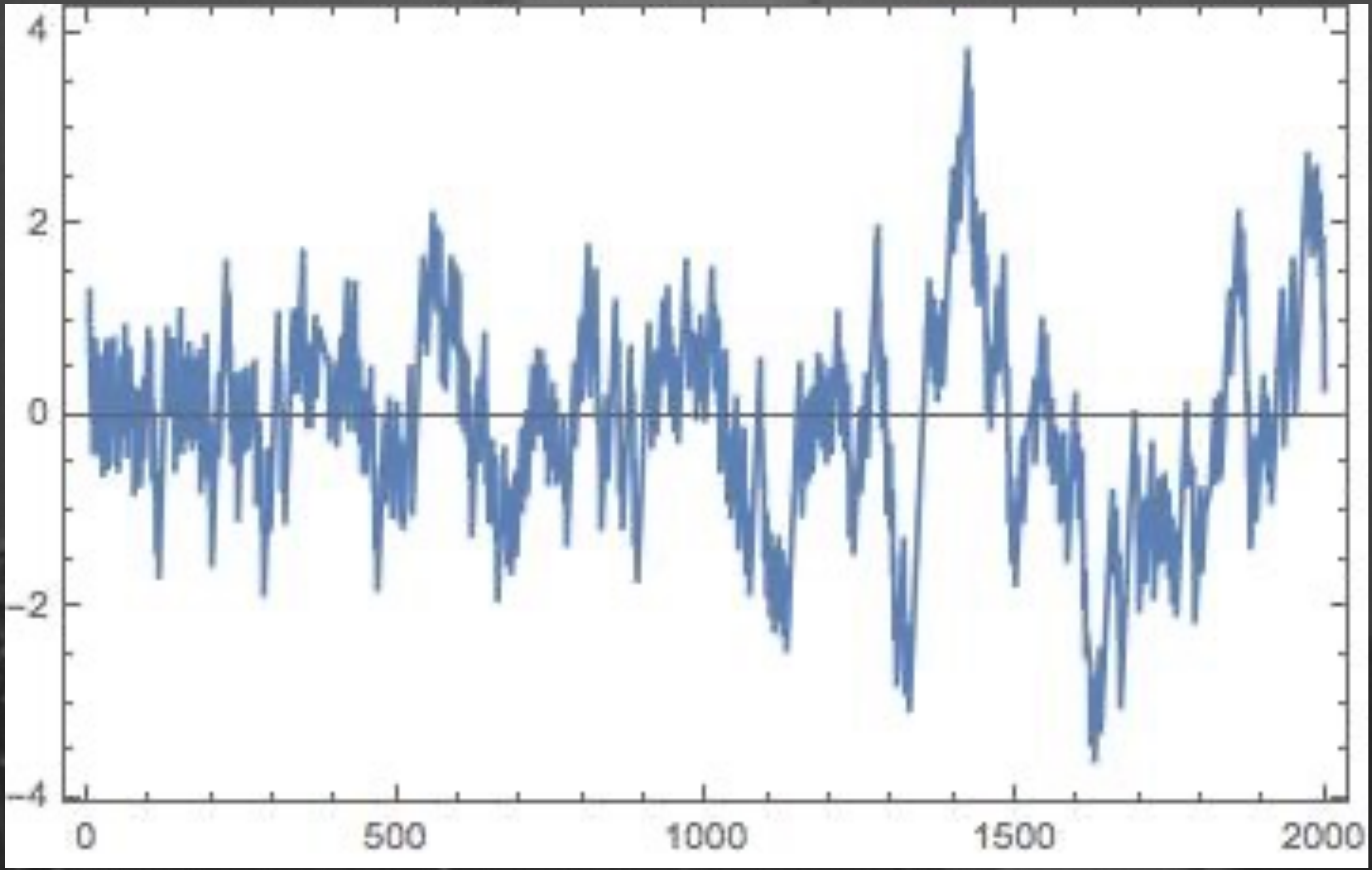


$$\pi(x) \simeq R(x)$$

$$R(x) = \sum_{n=1}^{\infty} \frac{\mu(n)}{n} Li(x^{1/n})$$

x	#{primes $\leq x$}	Gauss's overcount	Riemann's overcount
10^8	5761455	753	131
10^9	50847534	1700	−15
10^{10}	455052511	3103	−1711
10^{11}	4118054813	11587	−2097
10^{12}	37607912018	38262	−1050
10^{13}	346065536839	108970	−4944
10^{14}	3204941750802	314889	−17569
10^{15}	29844570422669	1052618	76456
10^{16}	279238341033925	3214631	333527
10^{17}	2623557157654233	7956588	−585236
10^{18}	24739954287740860	21949554	−3475062
10^{19}	234057667276344607	99877774	23937697
10^{20}	2220819602560918840	222744643	−4783163
10^{21}	21127269486018731928	597394253	−86210244
10^{22}	201467286689315906290	1932355207	−126677992

"The music of the primes"



Summary

- The Riemann function has to do with the counting of the primes

- If the Riemann hypothesis is true, we have a clear estimate of the error

$$|\pi(x) - Li(x)| < \frac{1}{8\pi}\sqrt{x}\,\log x$$

- The scaling law of the prime numbers is

$$p_n \simeq n \log n + n(\log \log n - 1) + \cdots$$

Inverse problems

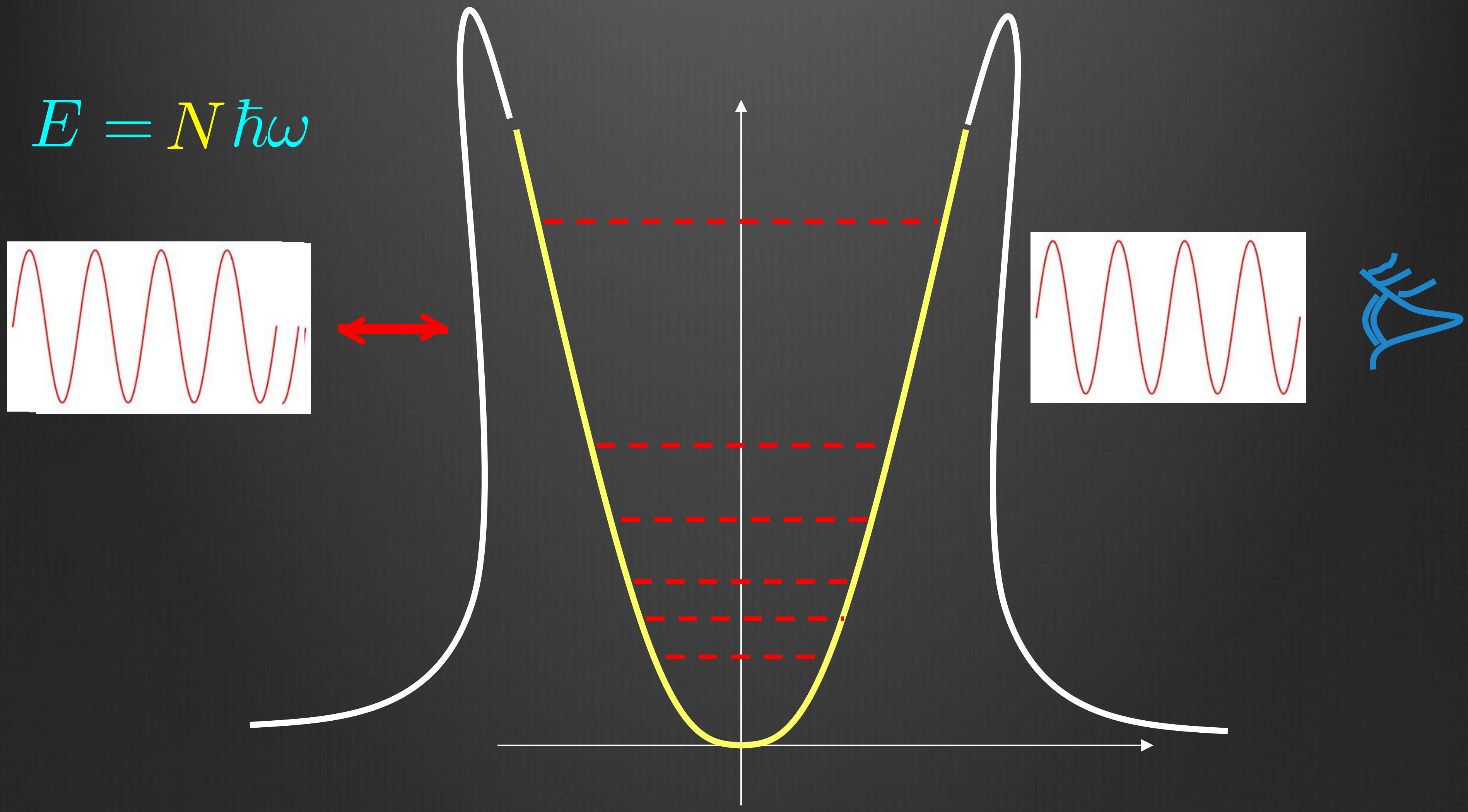
Given an admissible sequence of number S_n,

how to find the quantum potential V(x) ?

- Semi-classical method
- Dressing method (solitonic equations)

Primality test

$$E = N\hbar\omega$$



Factorization

$$N = p_1 \times p_2 \times \ldots \times p_k$$

The quantum abacus allows to crack the problem with the **MINIMUM** number of operations, k, i.e. the number of terms!

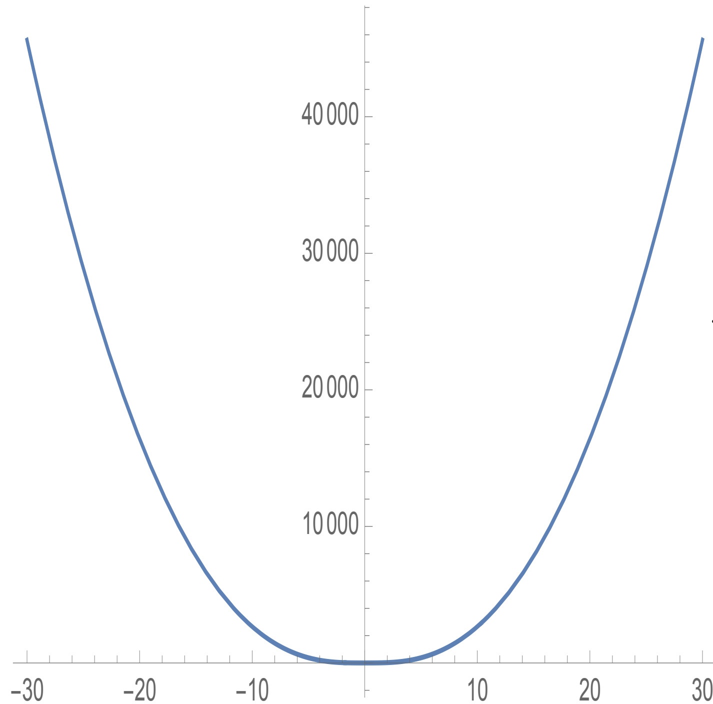
Semi-classical potential

GM, (1995)

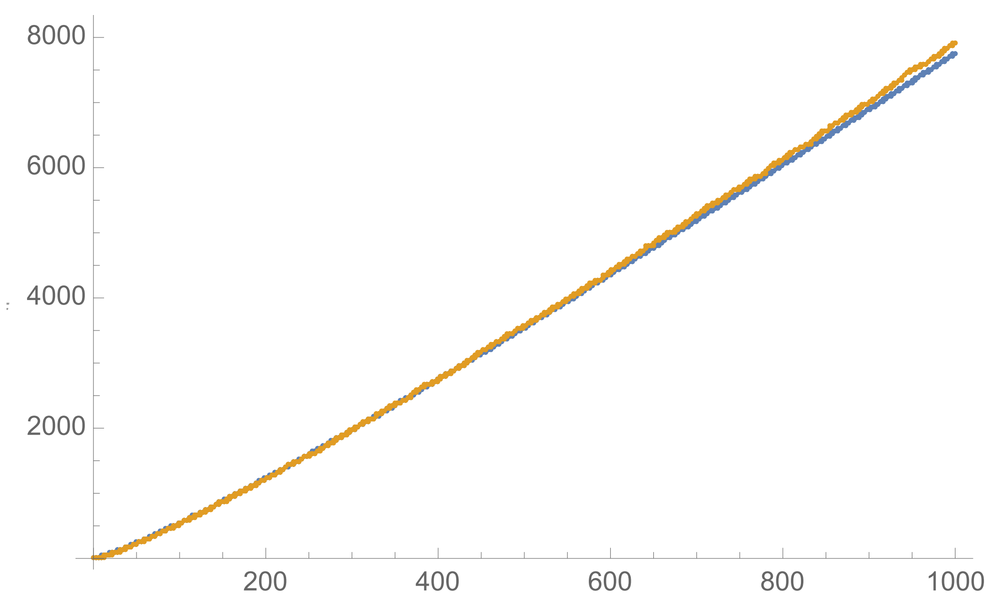
$$x(V) = \frac{\hbar}{\sqrt{2m}} \int_{E_0}^{V} \frac{dE}{\omega(E)\sqrt{V - E}} \,,$$

For prime numbers…

$$\omega^{-1}(E) = \sum_{n=1}^{\infty} \frac{\mu(n)}{n} \, \frac{\mathcal{E}^{\frac{1-n}{n}}}{\ln \mathcal{E}}$$



$$V(x) \simeq x^2 \log x$$



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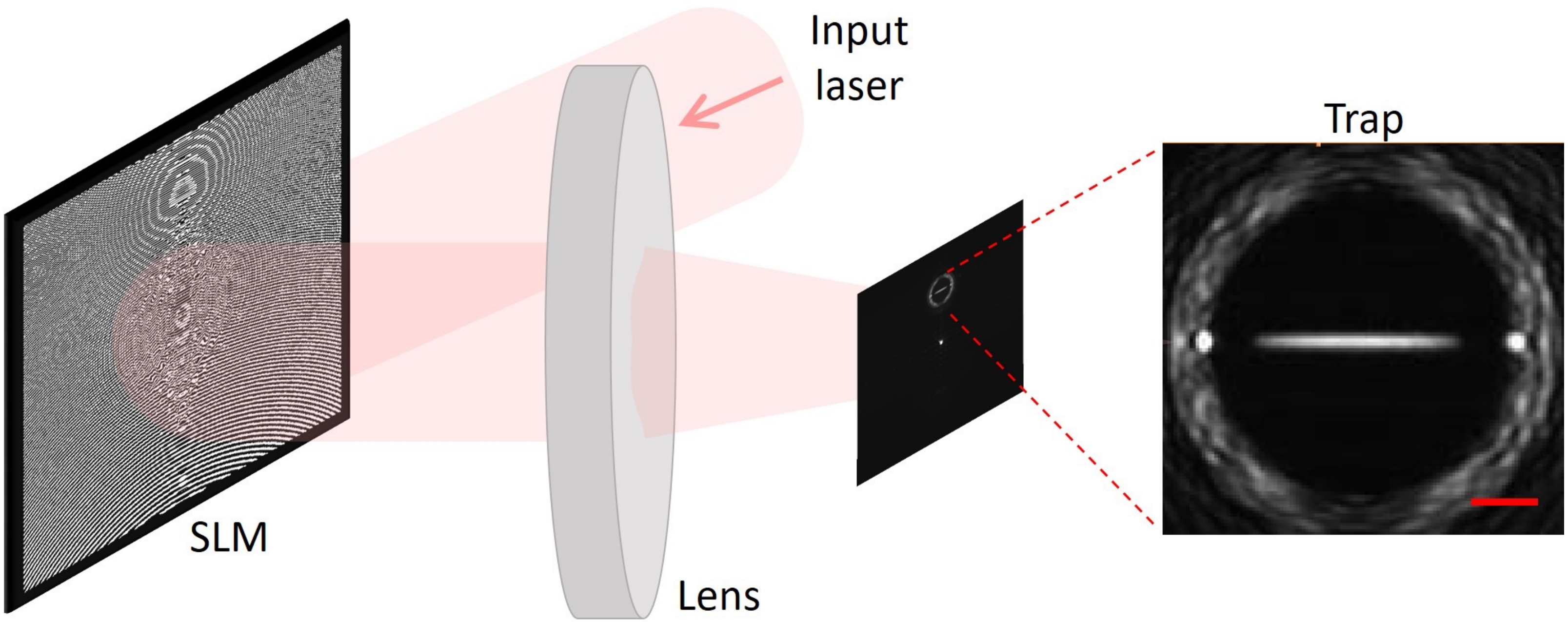
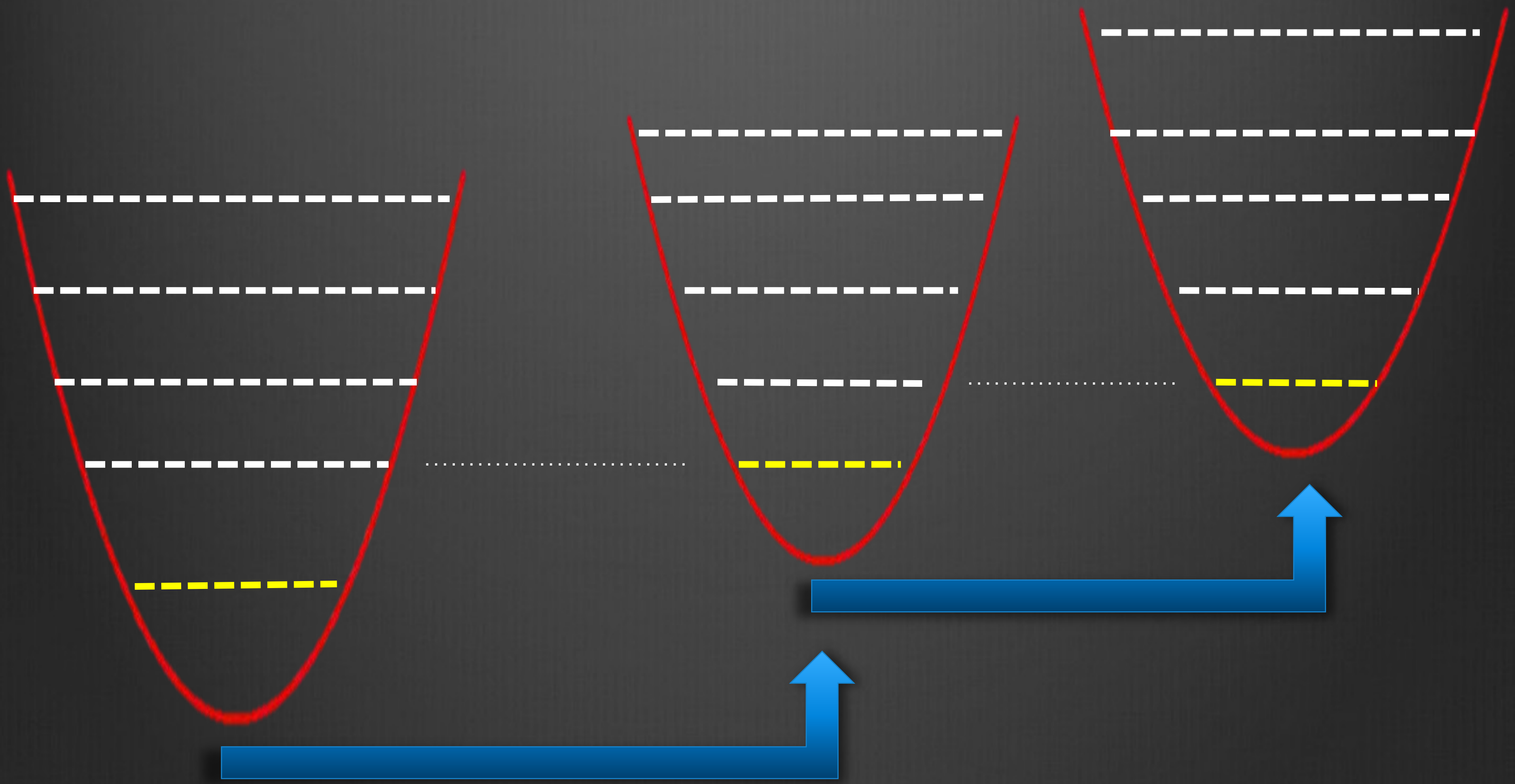
To appear in PNAS Nexus



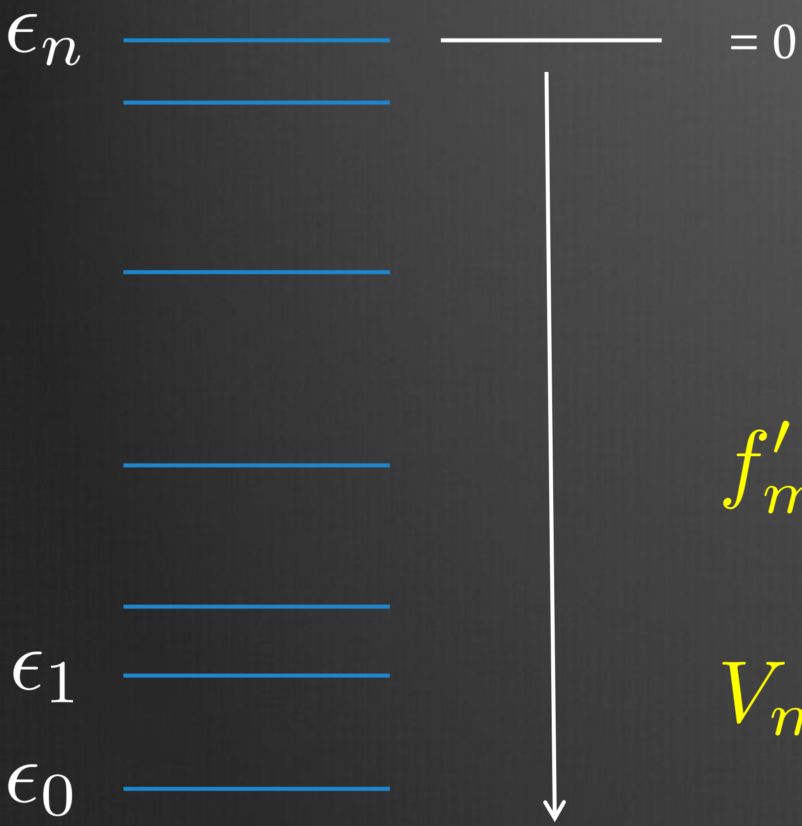
SUSY Quantum Mechanics



$$V_1(x) \to V_2(x) \to V_3(x) \to \cdots$$

$$E = \{E_1, E_2, E_3, \ldots, E_n\}$$

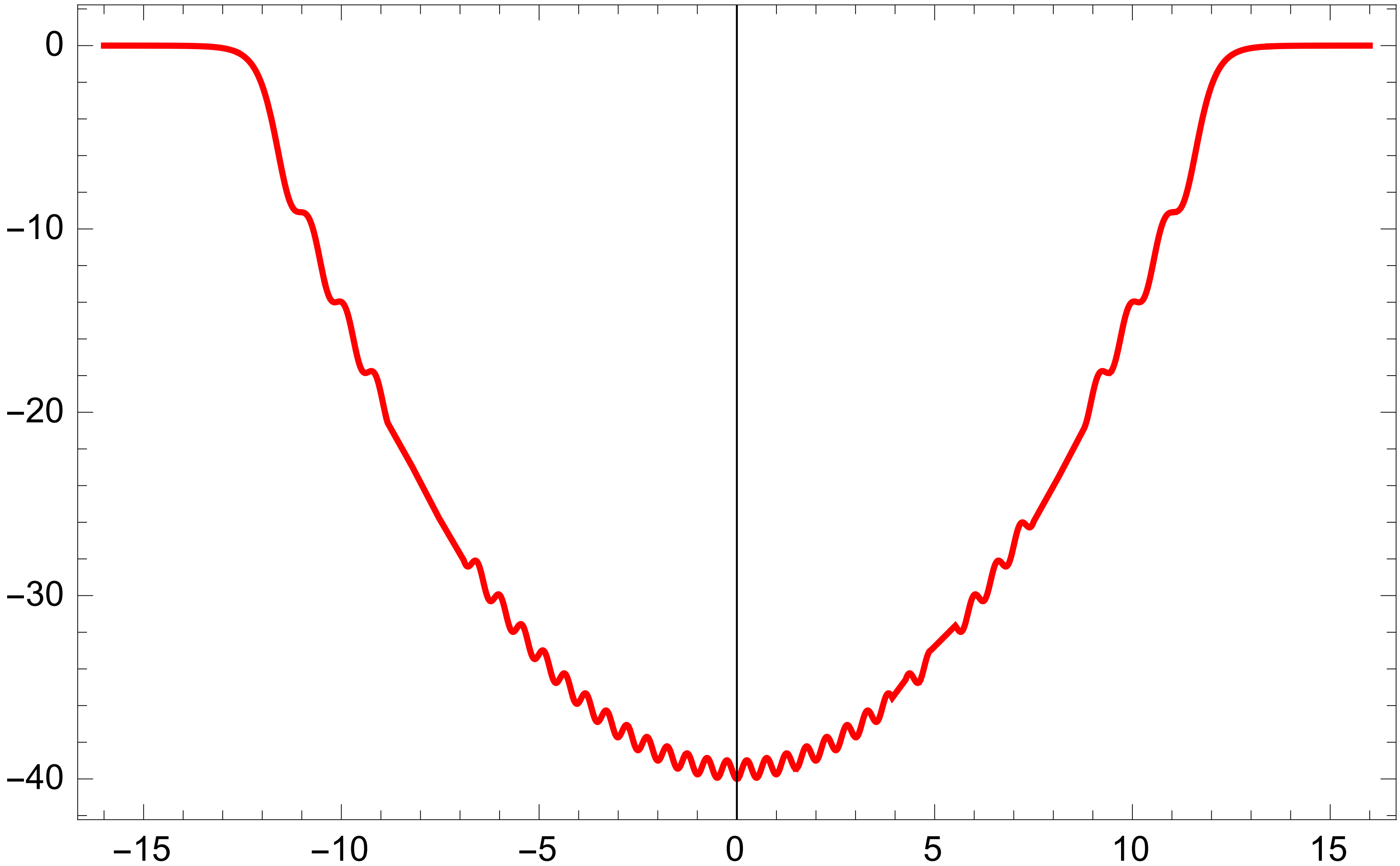
SUSY Potentials: top-down method



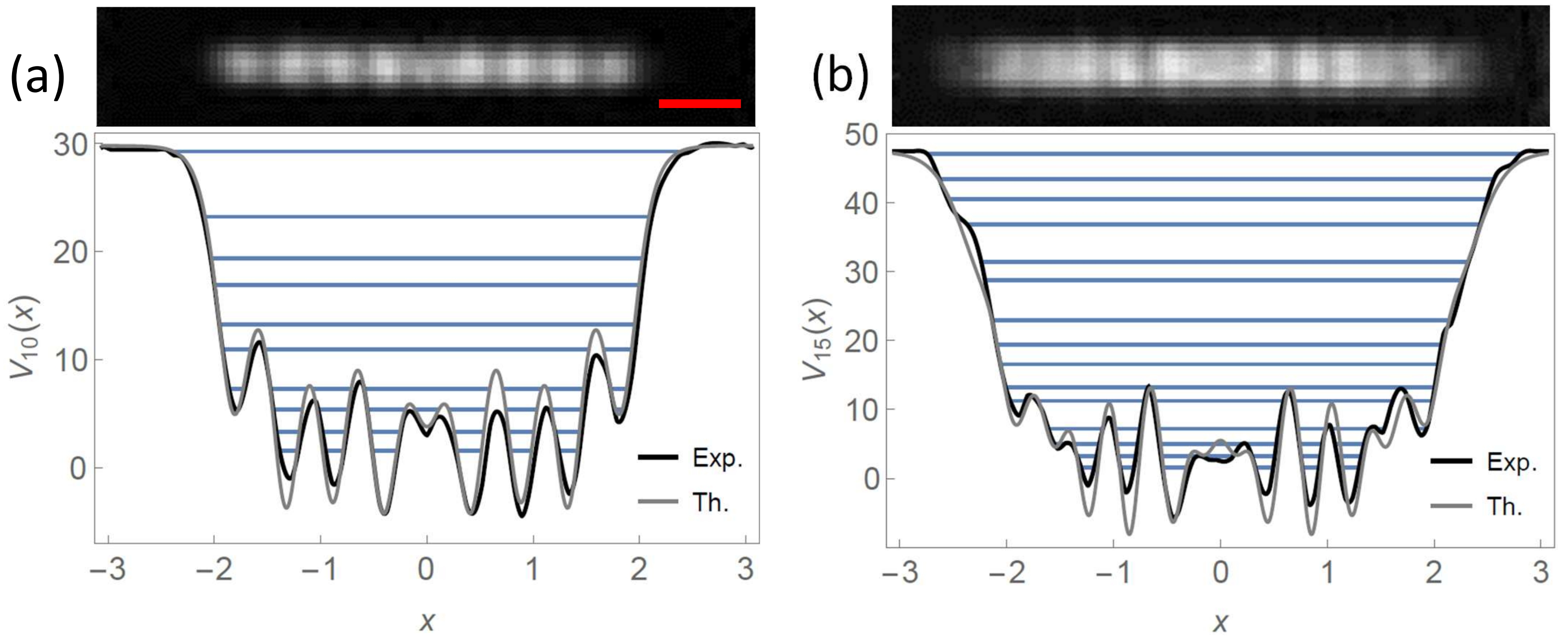
$$f'_m(x) - f^2_m(x) + V_m(x) = \epsilon_m$$

$$V_{m-1}(x) = 2\epsilon_m + 2f^2_m - V_m(x)$$

Potential with the first 40 levels of the harmonic oscillator



Holographic Quantum Potential with the first primes



(c)

Prime numbers	2	3	5	7	11	13	17	19	23	29	31	37	41	43	47
e_n for $V_{10}(x)$	1.58	3.31	5.40	7.33	10.9	13.2	16.9	19.4	23.2	29.3					
e_n for $V_{15}(x)$	1.58	3.21	5.00	7.22	11.3	13.2	16.6	19.4	22.9	28.8	31.4	36.9	40.6	43.4	47.1

Generalization to arbitrary sequences

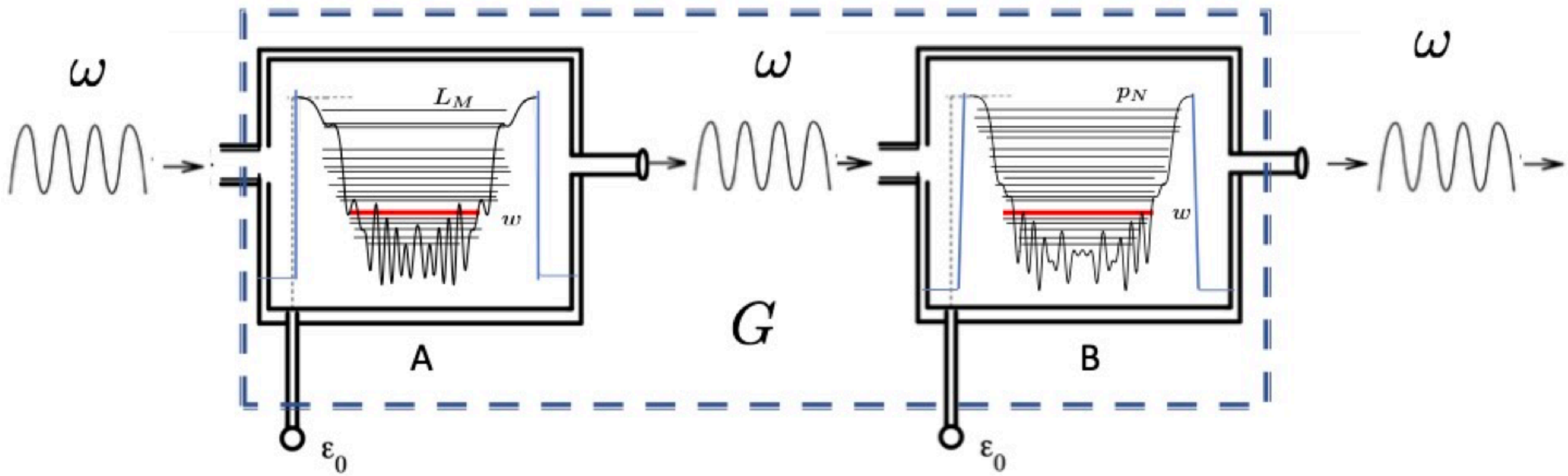
- Sequence of the logaritms of the primes or natural numbers

(this permits to address the factorization of integers, with analogical algorithm alternative to the Shor algorithm)

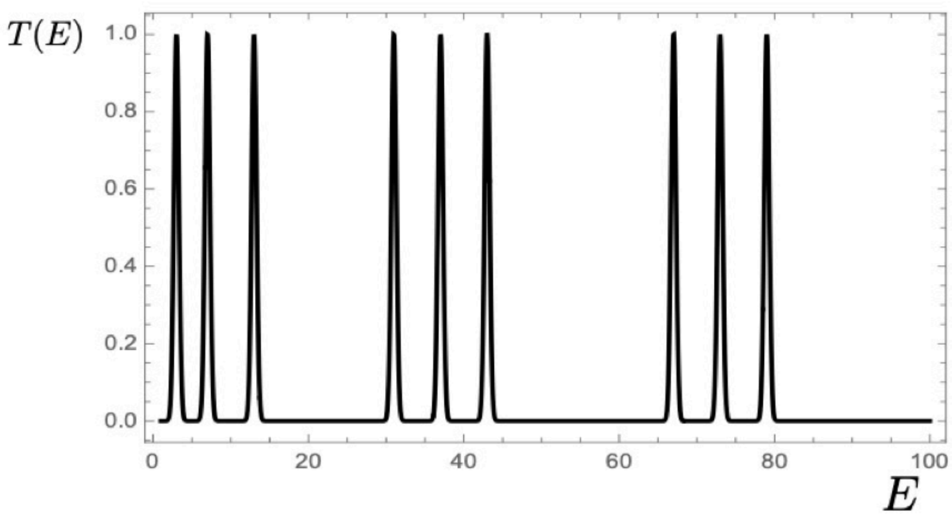
- Sequence of the lucky numbers

$$\{L_n\} = 1, 3, 7, 9, 13, 15, 21, 31, 33, \cdots$$

(this permits to address genuine questions in Number Theory such as the existence of Riemann Hypothesis for other classes)


ω
L_M
w
ω
p_N
w
ω
G
A
B
ε_0
ε_0

(a)


$T(E)$
1.0
0.8
0.6
0.4
0.2
0.0
0
20
40
60
80
100
E

(b)


Urbi et Orbi