Trapped-ion Quantum Computing For Collective Neutrino Oscillations

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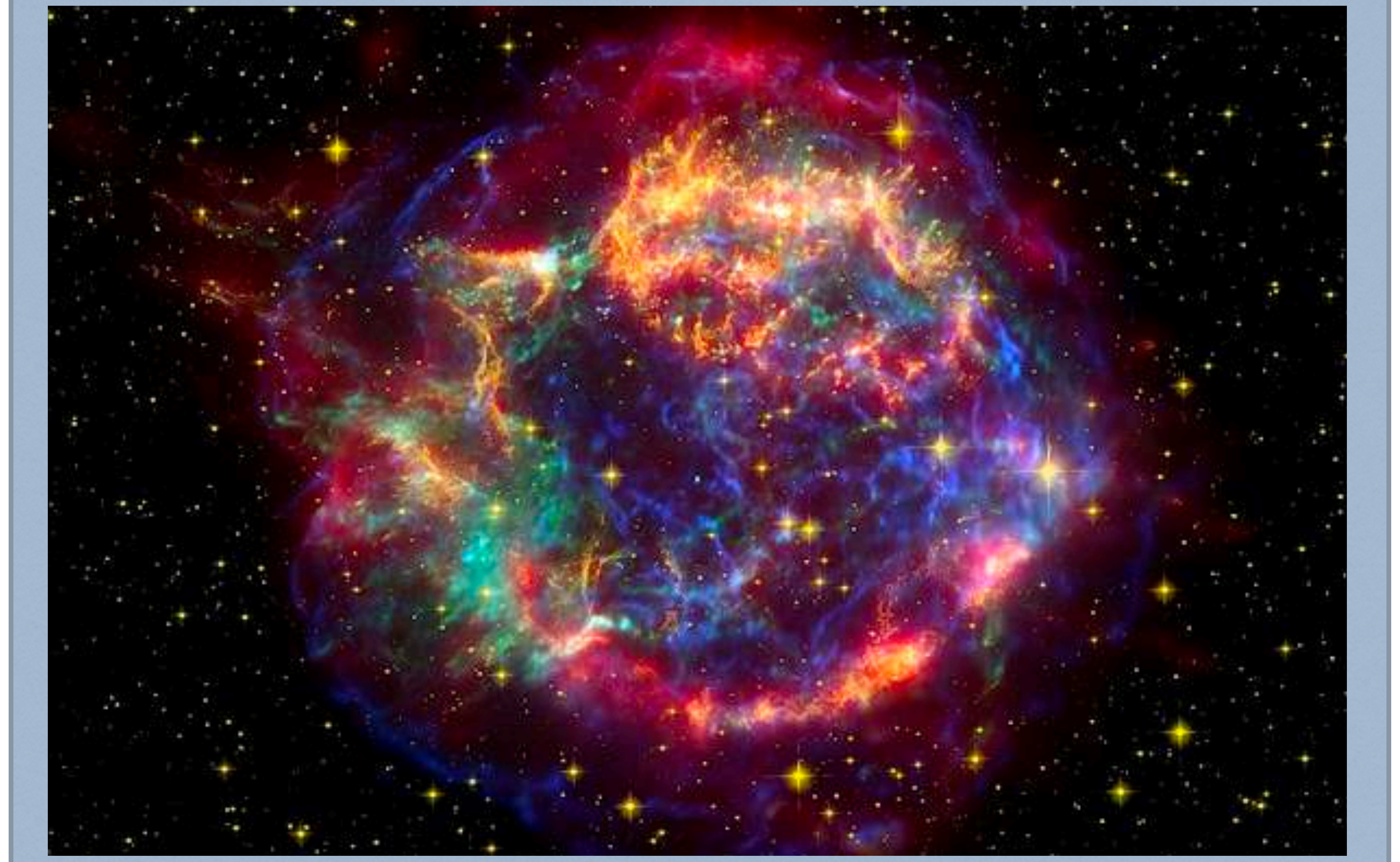
Quantum Computing @INFN Workshop

Bologna, 14-15 November 2022

Outline

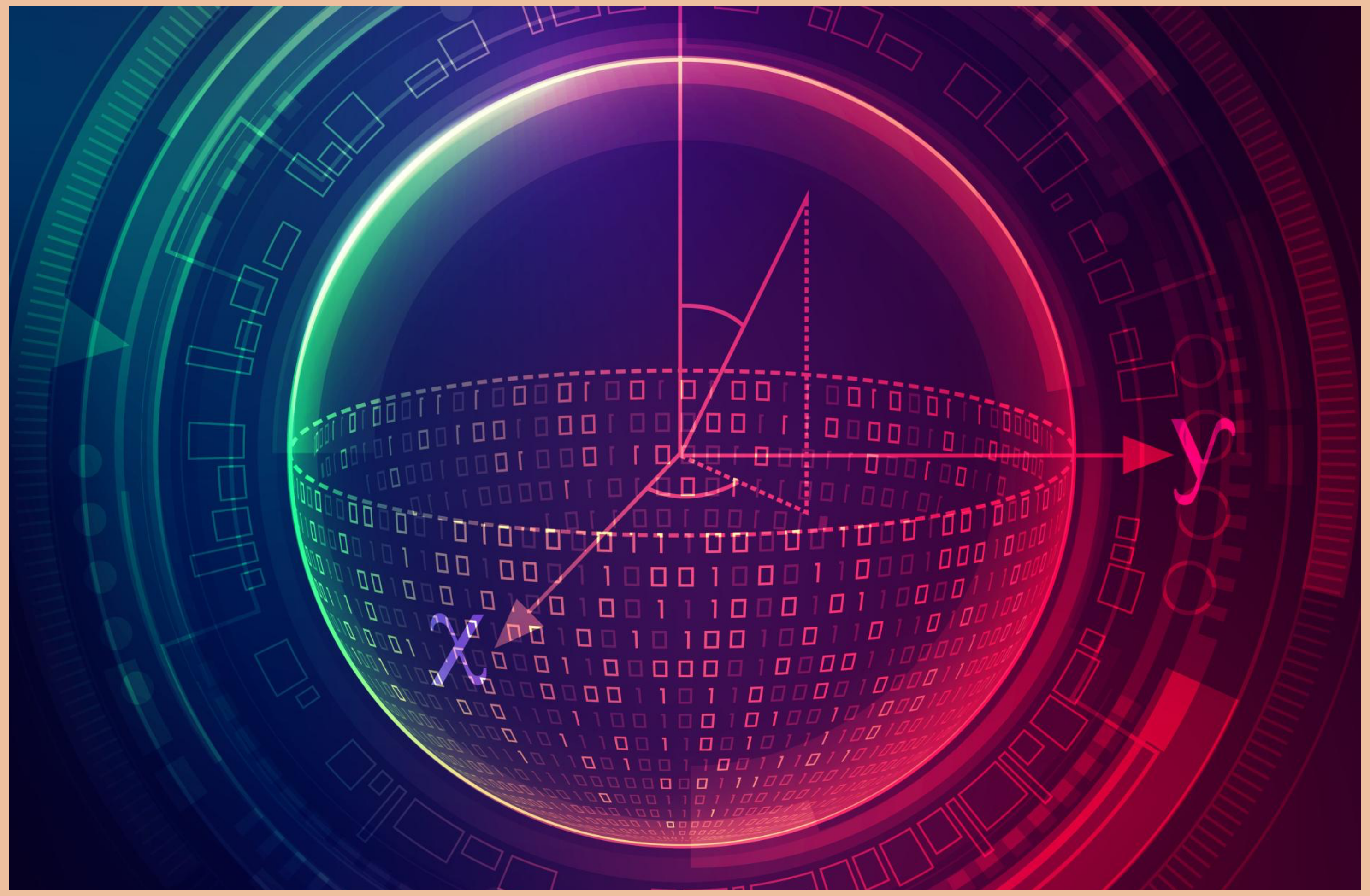
Introduction

- Motivation
- Physical description of the many-neutrino system in high density environment



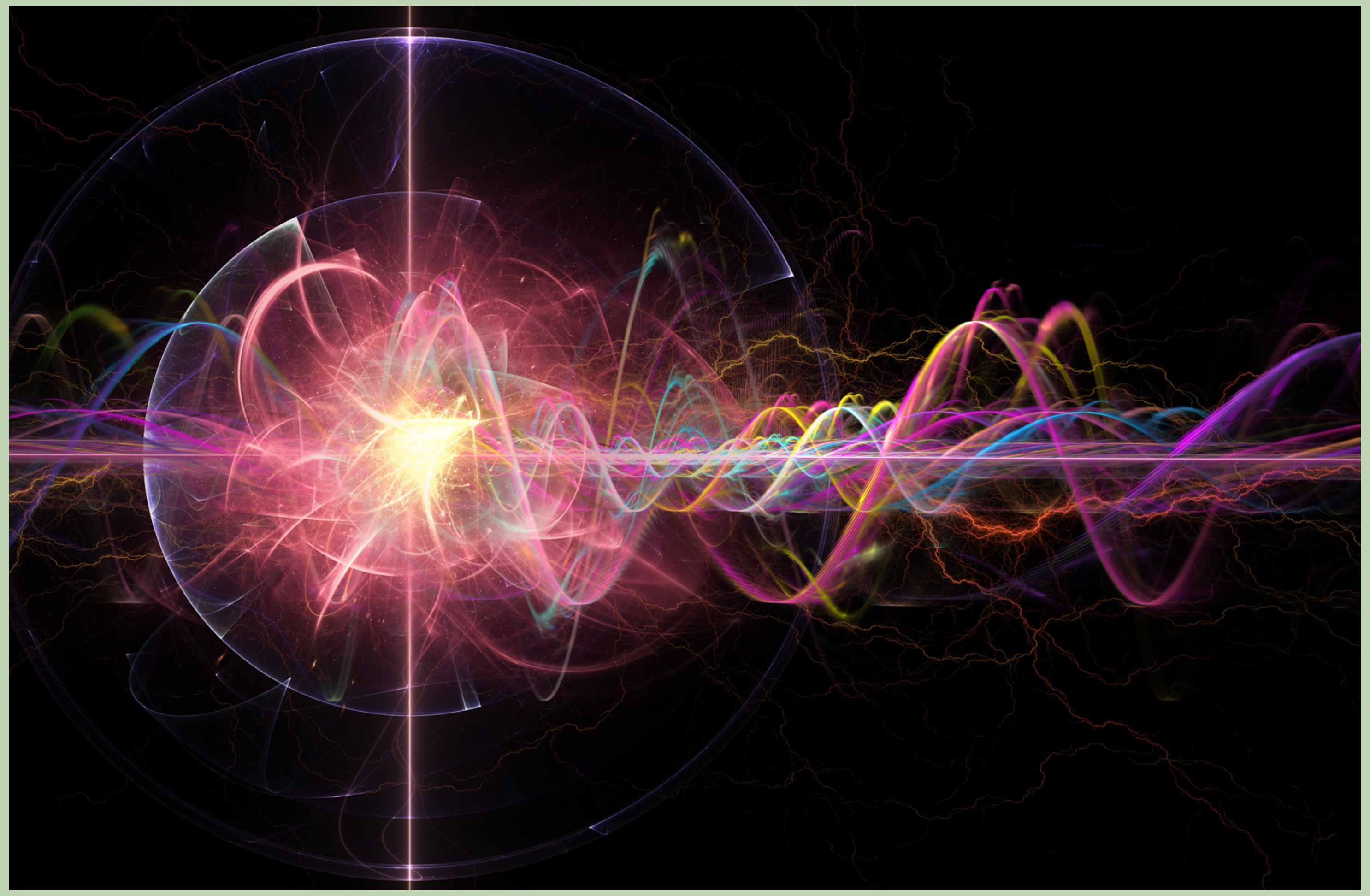
QC simulation

- Hamiltonian simulation
- The quantum algorithm implementation

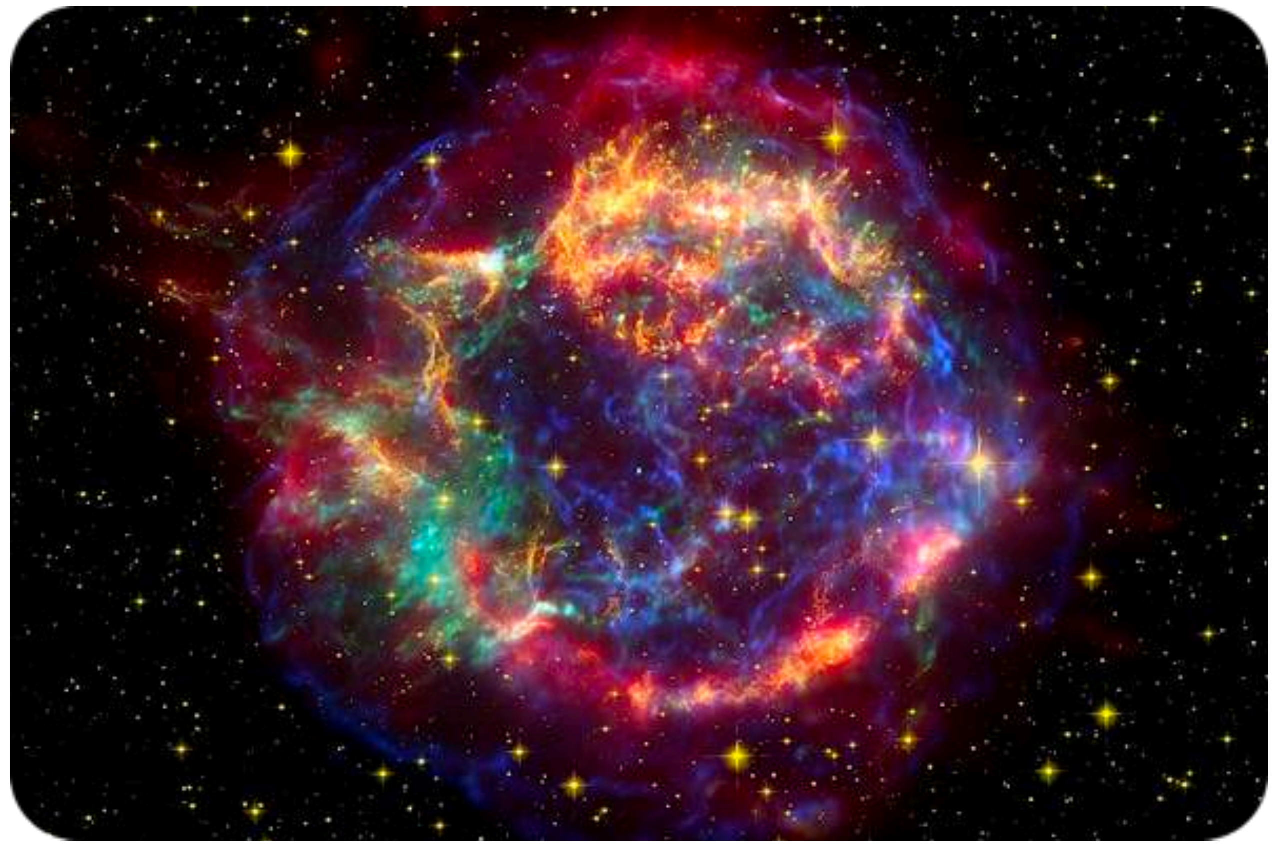


Results

- Data from the real trapped-ion quantum machine: Quantinuum System Model



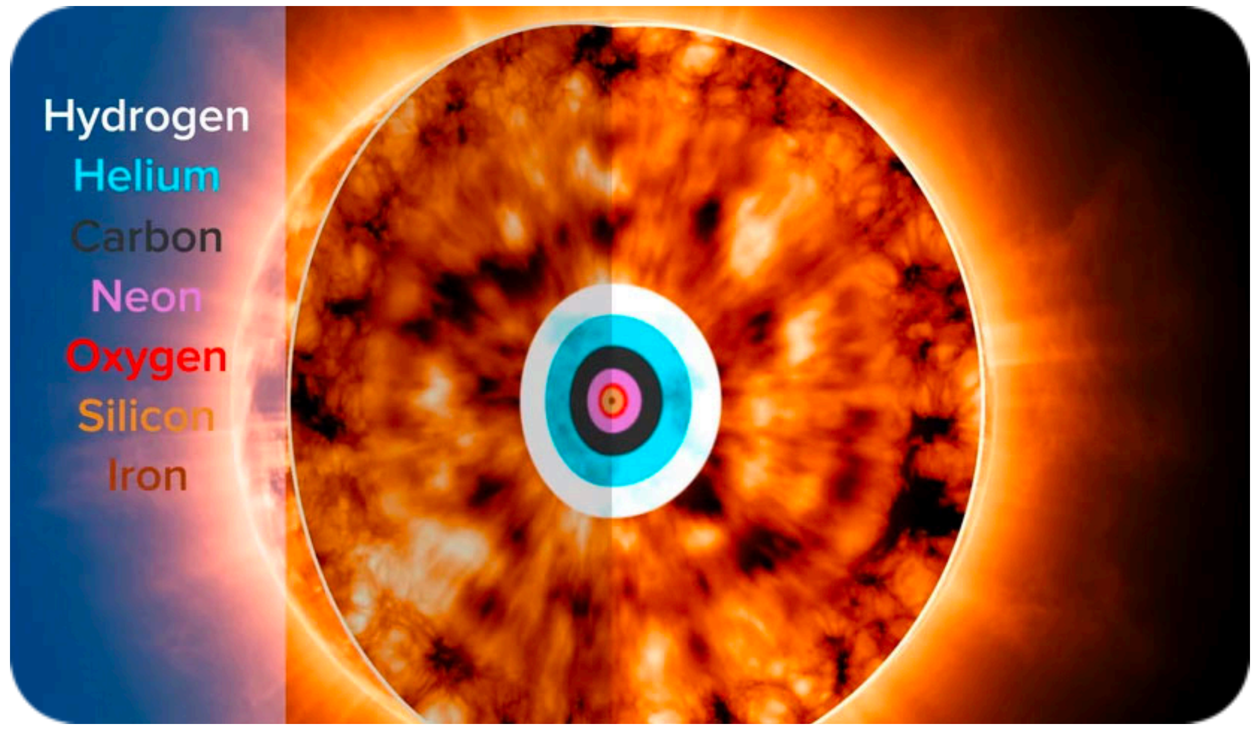
Why we care about neutrinos



Core-collapse supernovae

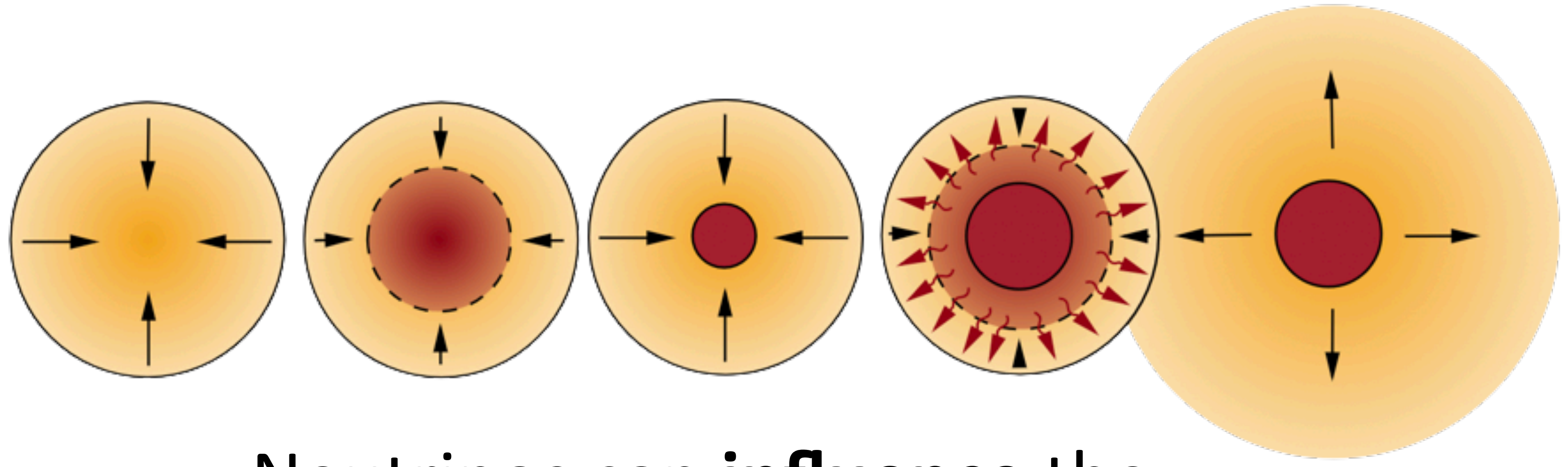
Neutrinos are **messengers of information** of physics under extreme conditions

Massive star mergers



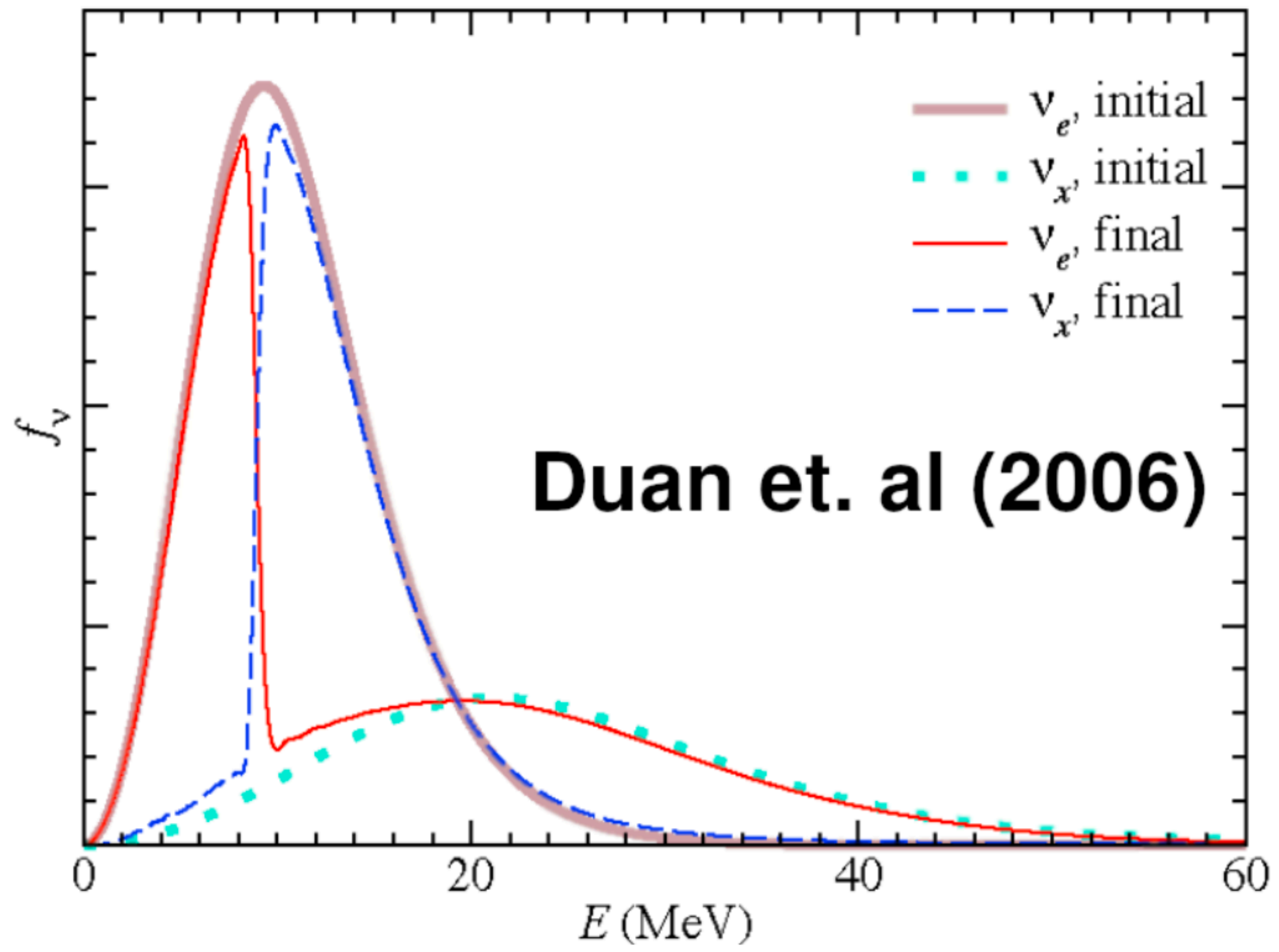
Nucleosynthesis and in general weak interaction is **flavor-dependent**

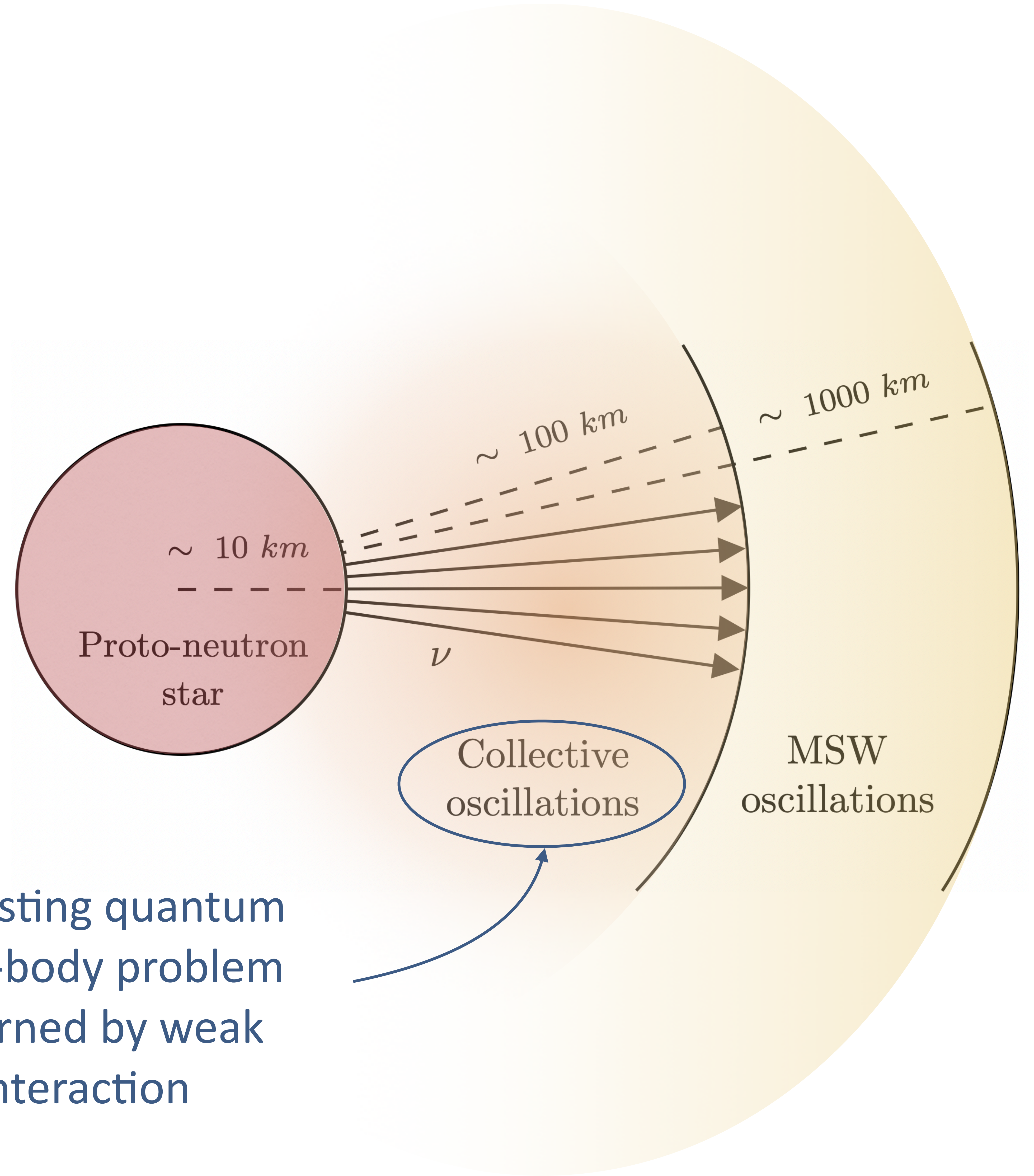
$$\nu_e + n \longleftrightarrow p + e^-$$
$$n \longleftrightarrow p + e^- + \bar{\nu}_e$$
$$\bar{\nu}_e + p \longleftrightarrow n + e^+$$



Neutrinos can **influence** the supernovae **explosion**

Spectral splits can happen at some distance from the emission sphere





Interesting quantum many-body problem governed by weak interaction

- Massive stars $M \geq 8\, M_{\odot}$ explode releasing a huge amount of energy and neutrinos $\sim 10^{58}$
- Flavor Hamiltonian of many-neutrino system

$$H = H_{vac} + H_{\nu e} + H_{\nu\nu}$$

Vacuum: Mass eigenstates $\neq$ flavor eigenstates

MSW: Scattering with matter

$\nu\nu$-interaction: Forward scattering

Two-flavor Hamiltonian (SU(2) model)

In the two-flavor approximation the flavor state of a neutrino is a **flavor isospin.**

The Hamiltonian can be written using the SU(2) algebra thanks to Pauli matrices:

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1-body term

$$H_{vac} = \sum_{i=1}^{N} \vec{b} \cdot \vec{\sigma}_i = \frac{\delta m^2}{4E} \sum_{i=1}^{N} \left(\sin(2\theta_\nu) X_i - \cos(2\theta_\nu) Z_i \right)$$

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2-body term

$$H_{\nu\nu} = \frac{\mu}{N} \sum_{i<j}^{N} J_{ij} \vec{\sigma}_i \cdot \vec{\sigma}_j = \frac{\mu}{N} \sum_{i<j}^{N} J_{ij} \left(X_i \otimes X_j + Y_i \otimes Y_j + Z_i \otimes Z_j \right)$$

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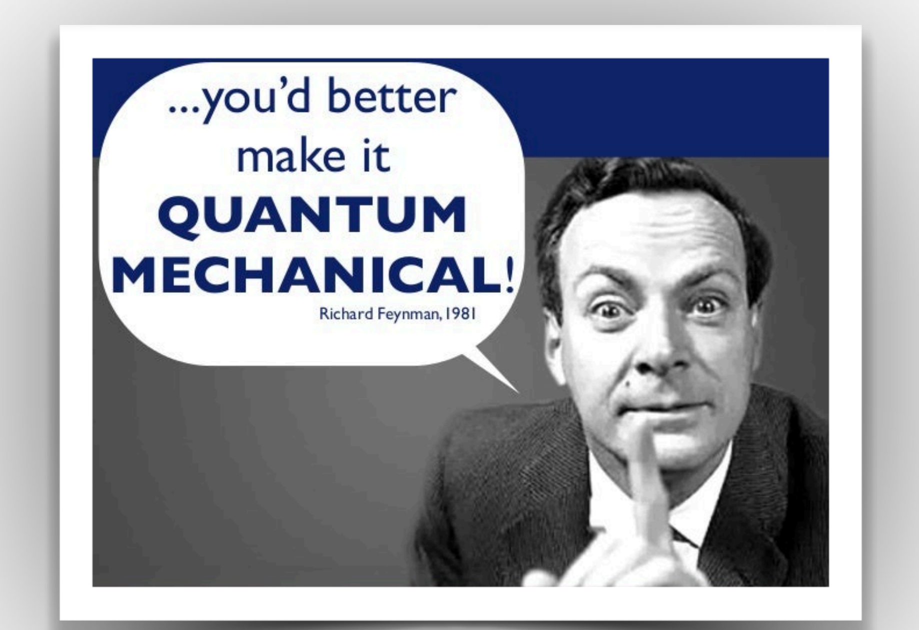
Simulating the full dynamics is impossible using classical resource



Ingredients for Hamiltonian simulation

1° ingredient:
Encoding map

2° ingredient:
Implementation of the unitary $U = e^{-iHt}$

$|\nu_e\rangle = |0\rangle$

$|\nu_x\rangle = |1\rangle$

- Two-flavor approximation $|\nu\rangle = \alpha|\nu_e\rangle + \beta|\nu_x\rangle$
- N neutrinos encoded into N qubits

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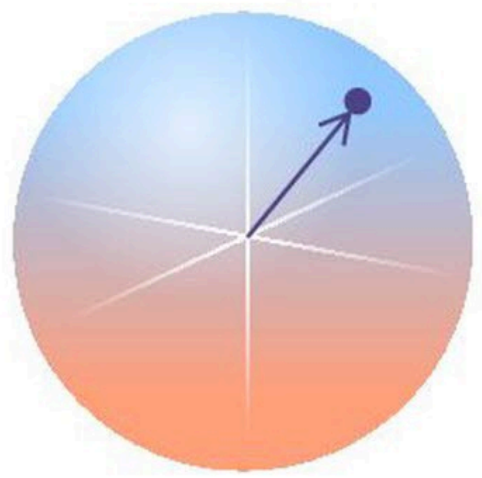

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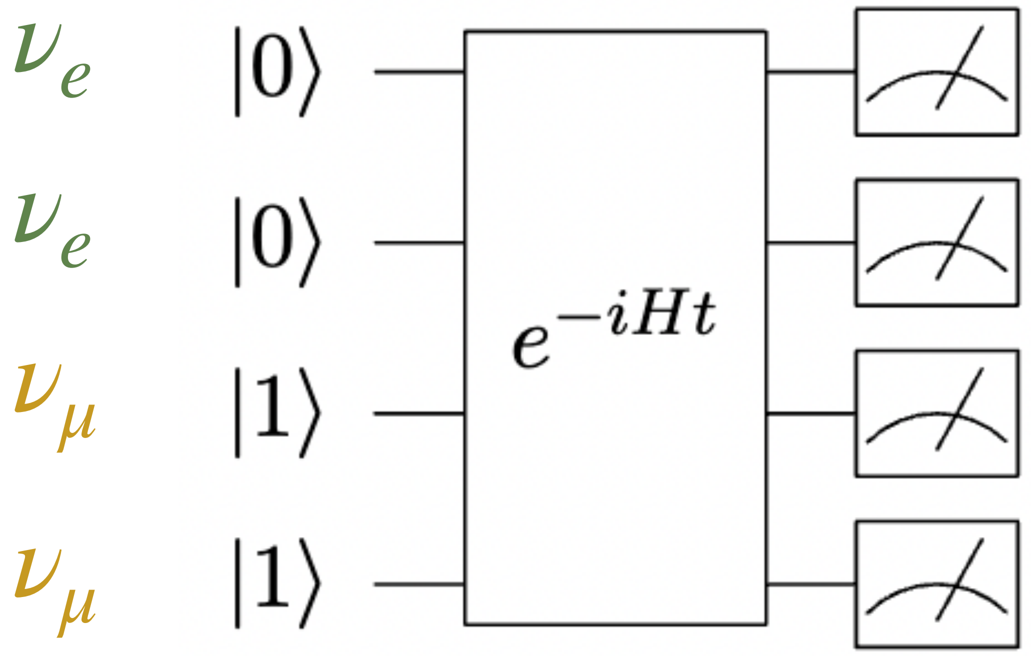
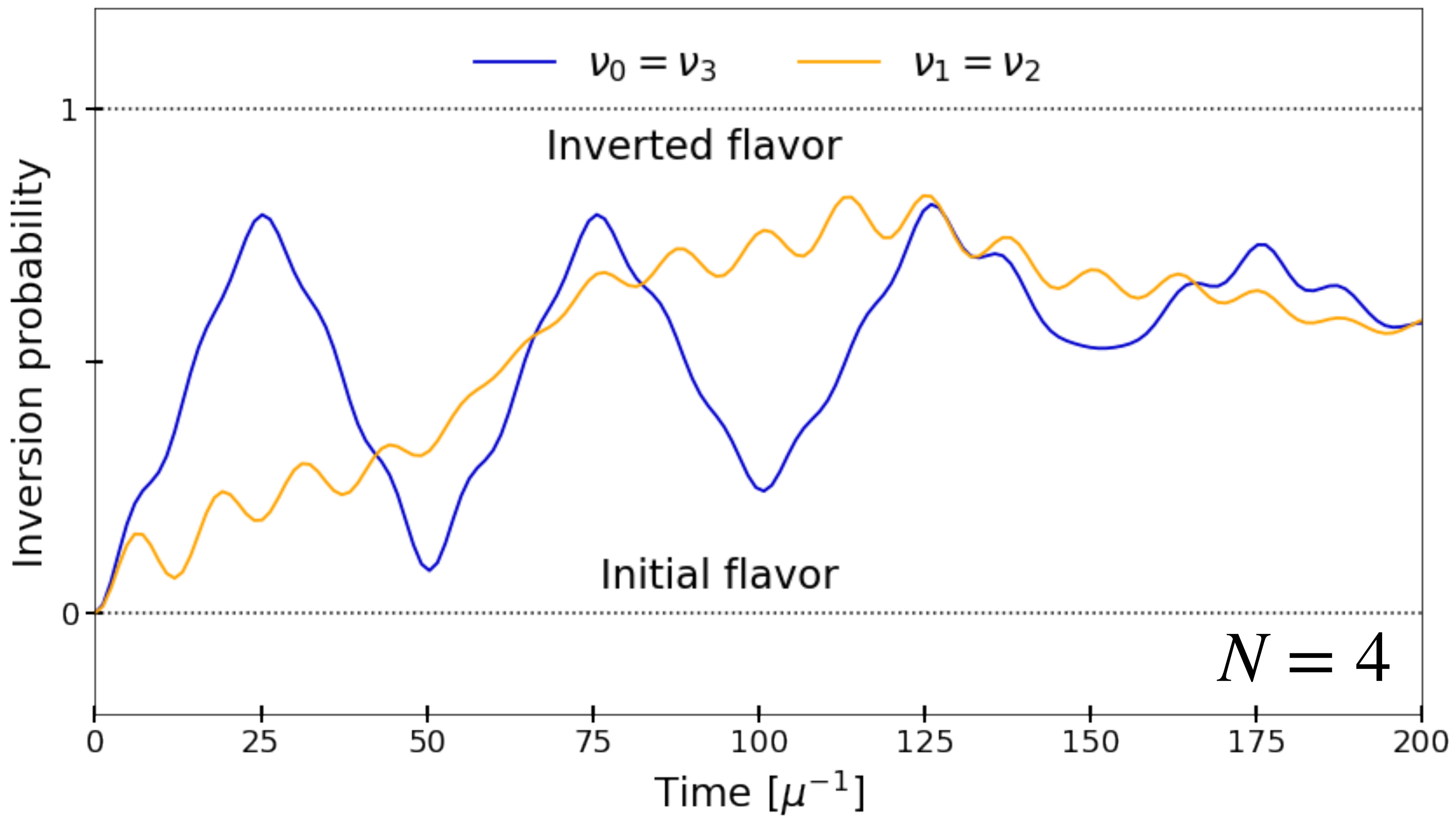


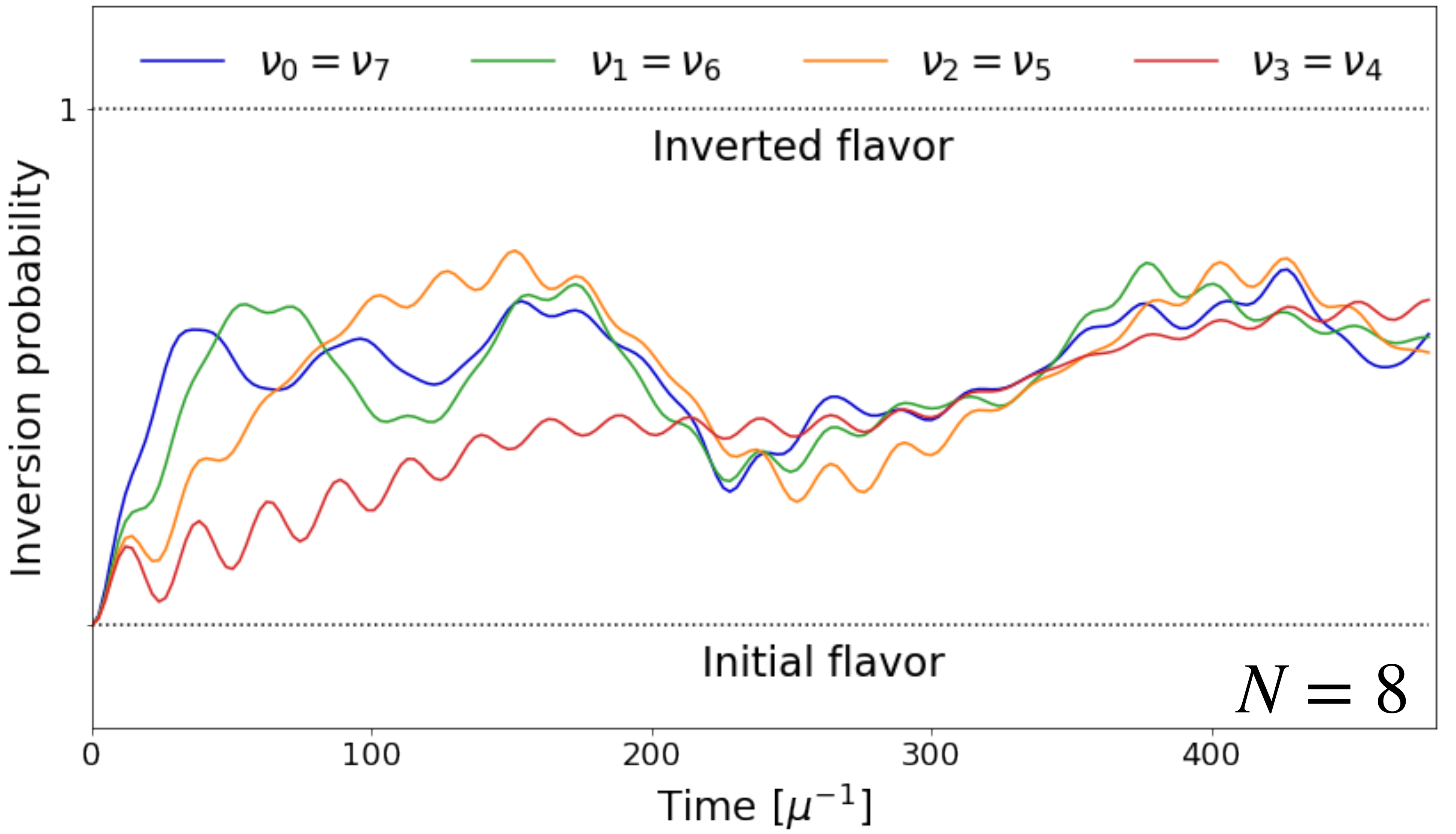
- Flavor evolution of N neutrinos generated by the Hamiltonian $H = \sum_i \vec{b} \cdot \vec{\sigma}_i + \sum_{i<j} J_{ij} \vec{\sigma}_i \cdot \vec{\sigma}_j$
- Implement the propagator $U(t) = e^{-iHt}$

2° ingredient:
Implementation of the unitary $U = e^{-iHt}$

We want to simulate the flavor evolution

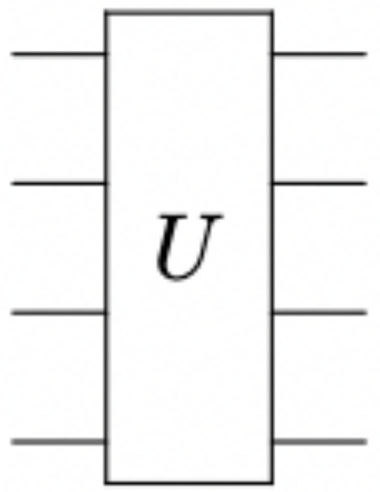
- Initial state $|\Psi_0\rangle$
  - $N = 4$ initial state $|\Psi_0\rangle = |0011\rangle$
  - $N = 8$ initial state $|\Psi_0\rangle = |00001111\rangle$
- Evolved state $|\Psi(t)\rangle = e^{-iHt}|\Psi_0\rangle$
- Measure the probability to be in the inverted flavor as a function of time $P_{inv}(t)$
- Note the symmetry under particle exchange



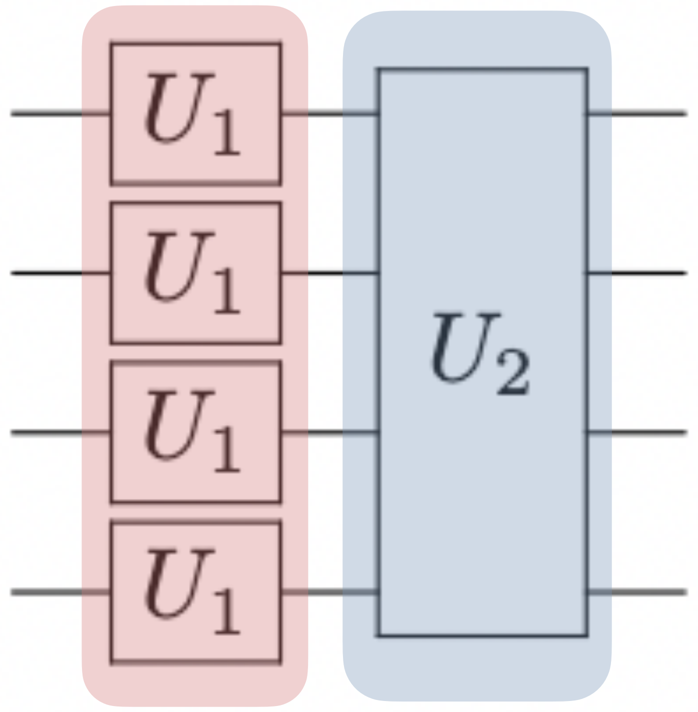


The Unitary Implementation: gate decomposition

$$U(dt) = e^{-i(H_{vac}+H_{\nu\nu})dt}$$



$$U(dt) = U_2(dt)U_1(dt)$$



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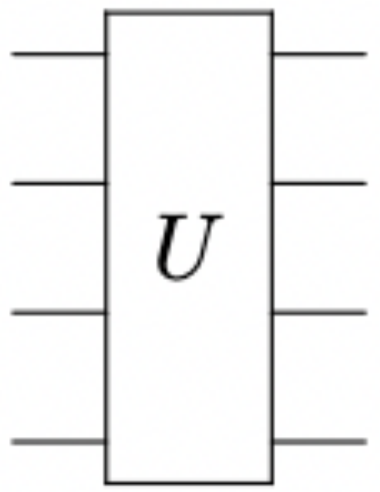

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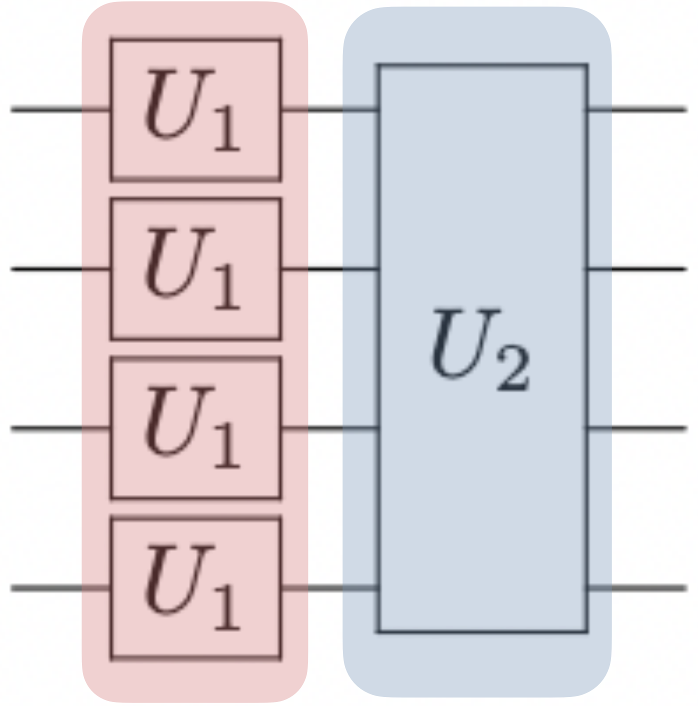
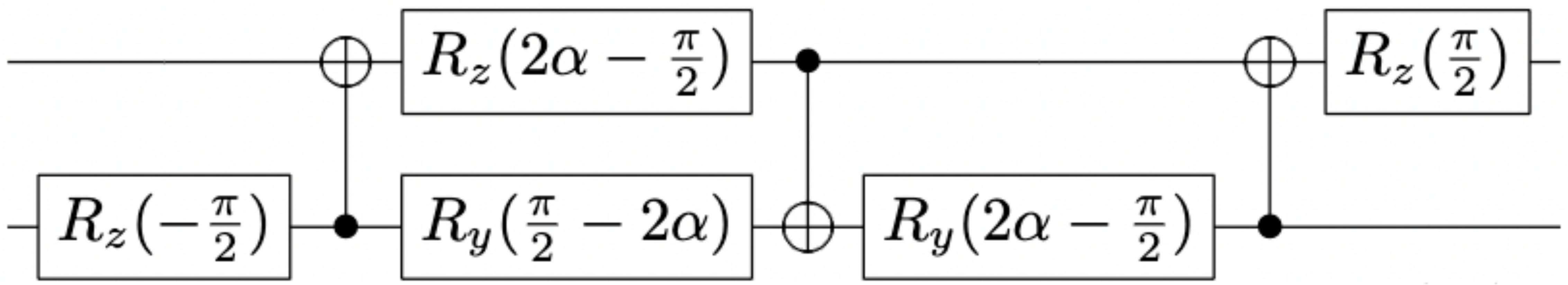


- **1-body part:** simple $U_1(t) = e^{-iH_{vac}t} = e^{-i\sum_i h_i t} = \prod_i e^{-ih_i t}$
- **2-body part:** tricky $U_2(t) = e^{-iH_{\nu\nu}t} = e^{-i\sum_{i<j} h_{ij}t}$
  - Different terms don't commute $[h_{ij}, h_{ik}] \neq 0$
  - Approximation $U_2(t) \approx \prod_{i<j} e^{-ih_{ij}t}$
  - It introduces an error $\sim \mathcal{O}(t^2)$
  - Simple implementation of $e^{-ih_{ij}t} = e^{-iJ_{ij}(X_i \otimes X_j + Y_i \otimes Y_j + Z_i \otimes Z_j)t}$



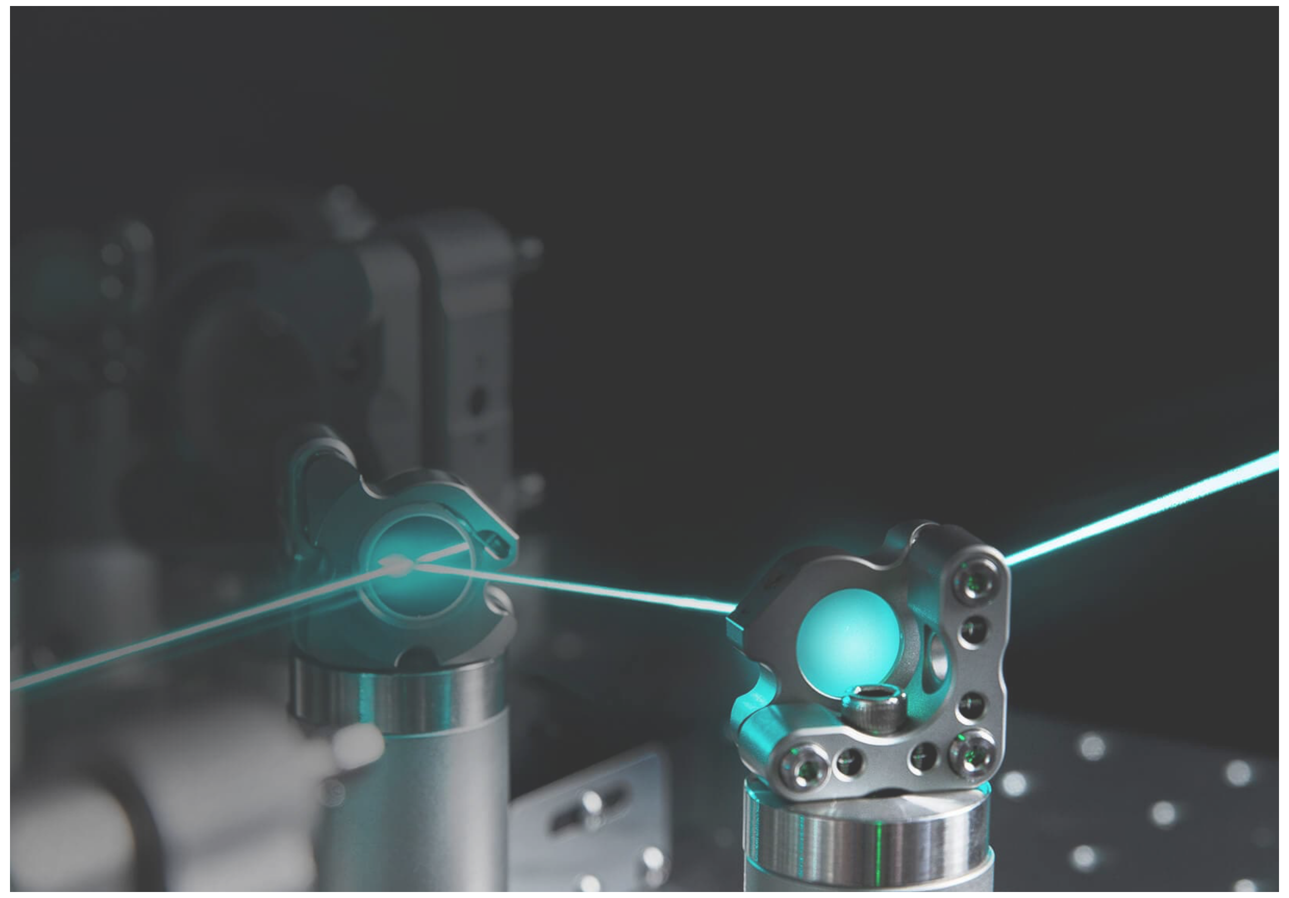
F. Vatan and C. Williams (2004)

$\alpha = -dtJ_{ij}$

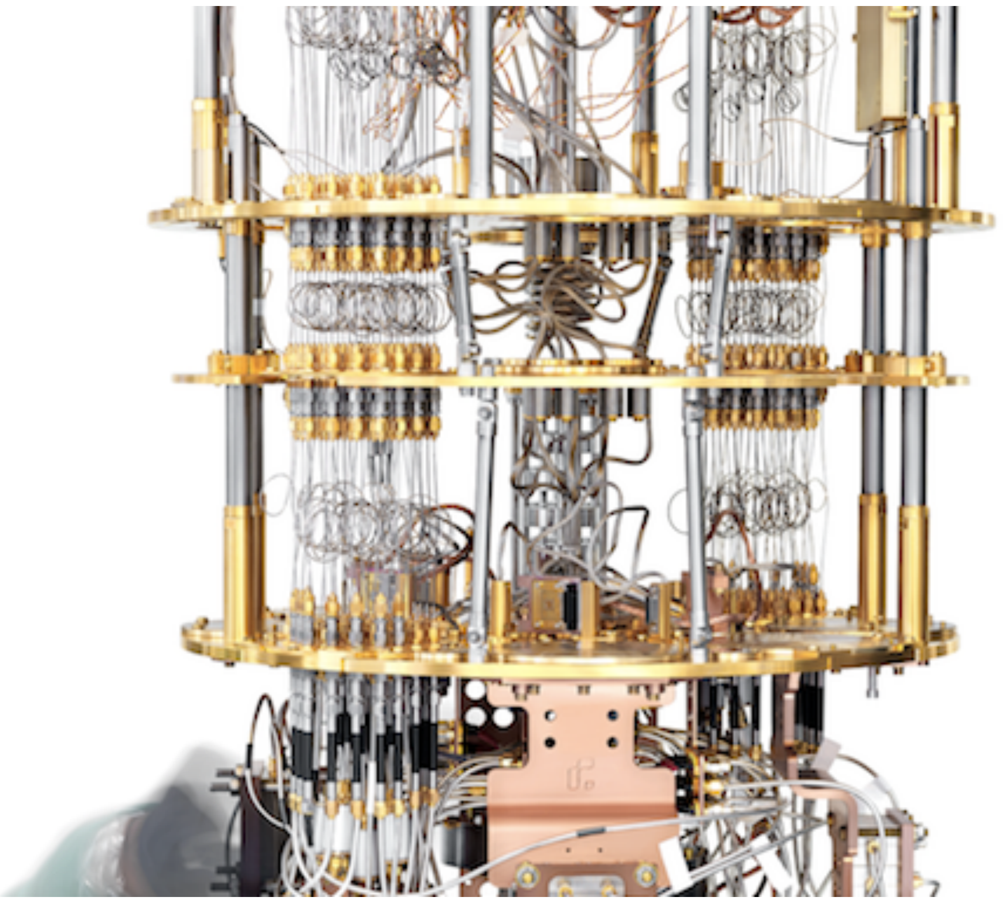
The Unitary Implementation: Machine aware compilation

- Different qubit
  - Superconductive circuit
  - Trapped ions
- Different qubit connectivity
  - Linear
  - All - to - all
  - Etc…
- Different universal gate set
  - Circuit optimization
  - More control on what we are running

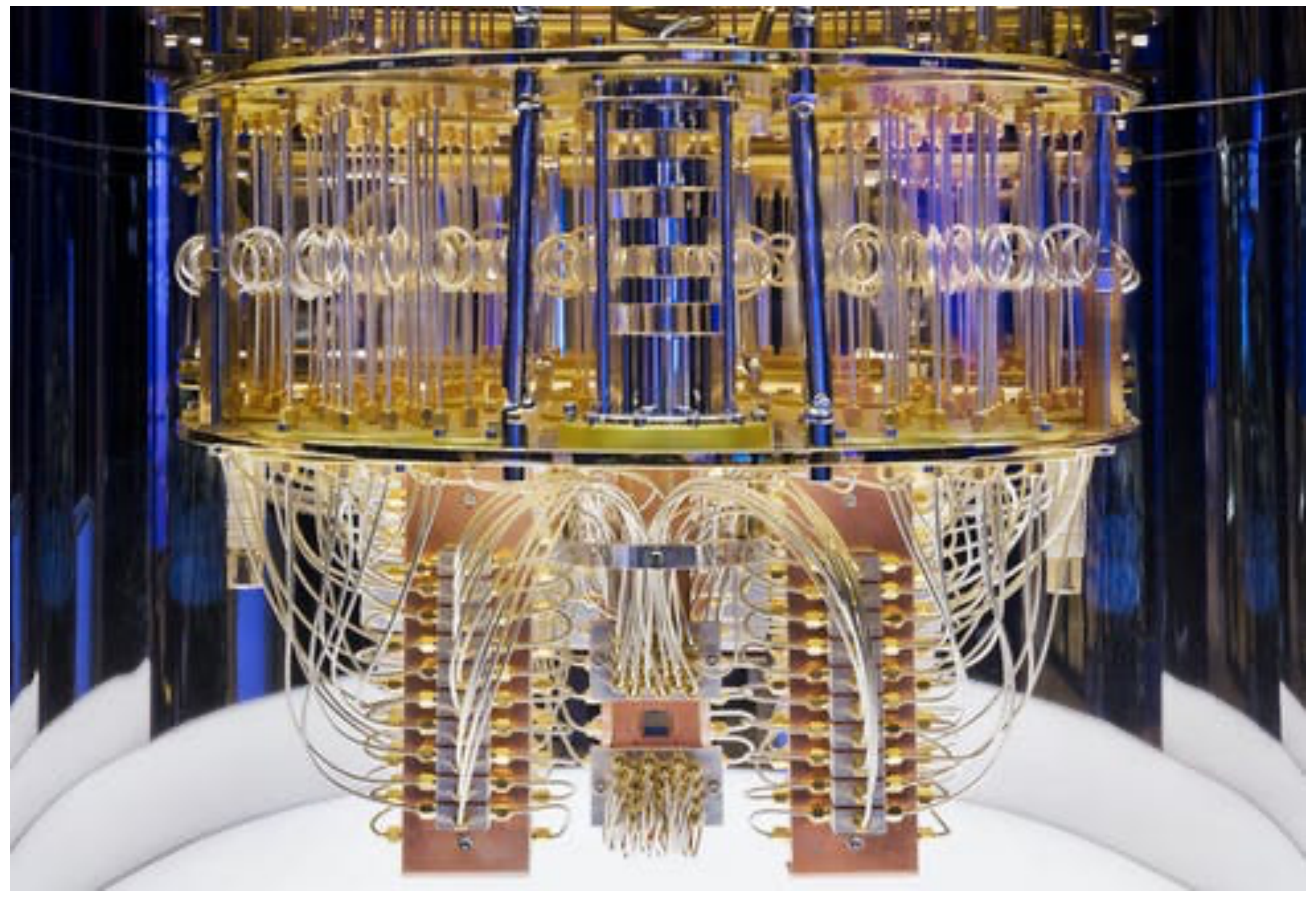
Honeywell Quantum



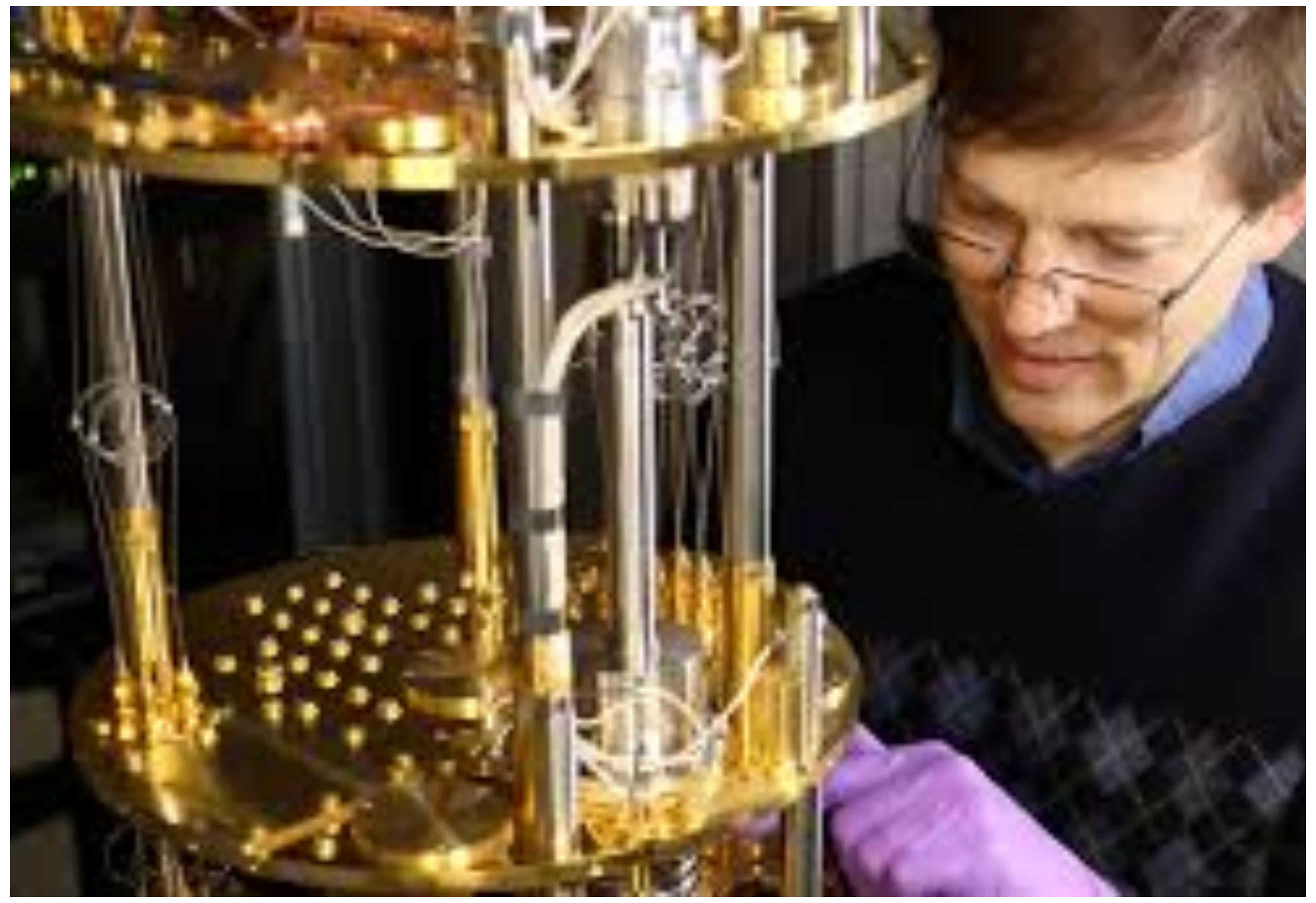
Rigetti Quantum



IBM Quantum



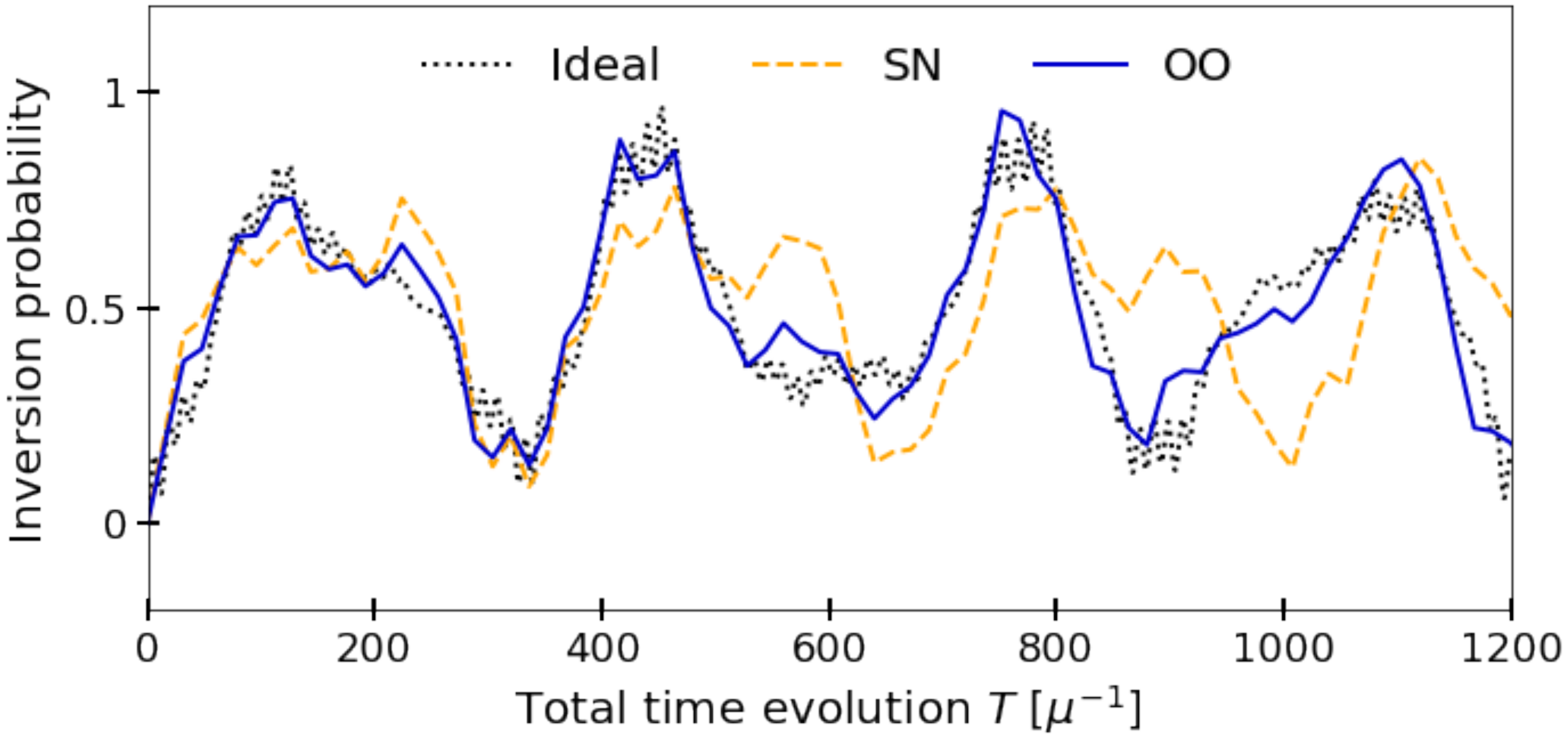
LLNL testbed



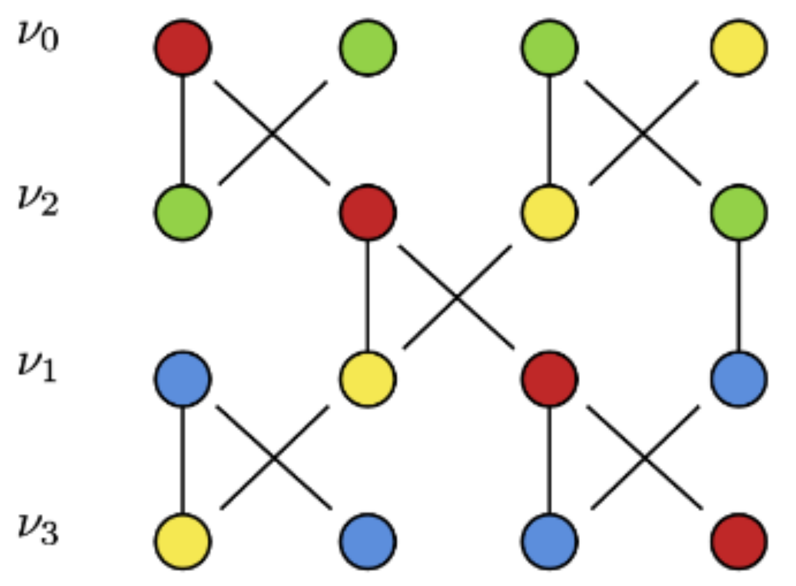
Long evolution of $N = 4$ neutrinos using $dt = 4N\mu^{-1}$

- The $\nu\nu$-term is an **all-to-all interaction:**
  - all qubits need to interact with all the others one time

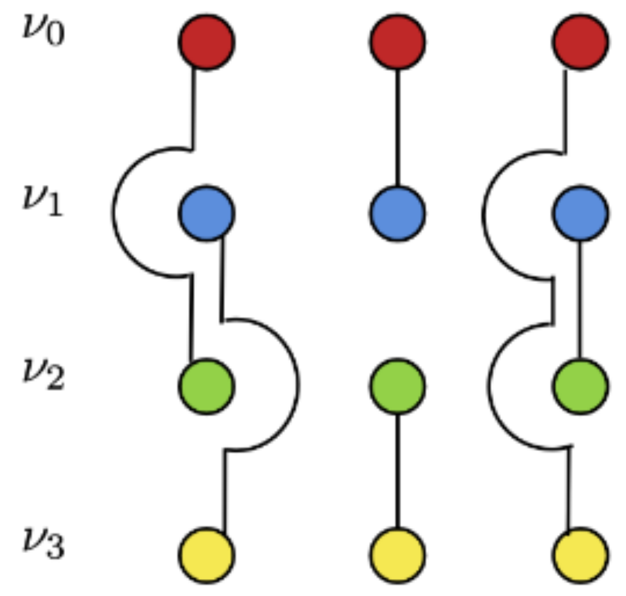
- The qubit connectivity introduces constraints in the quantum gate decomposition
  - **Swap network** algorithm for linear connectivity (B. Hall, A. Roggero et. al (2021))
  - Advantage of **full connectivity** of trapped ions
    - Less complexity
    - Error minimization



SN

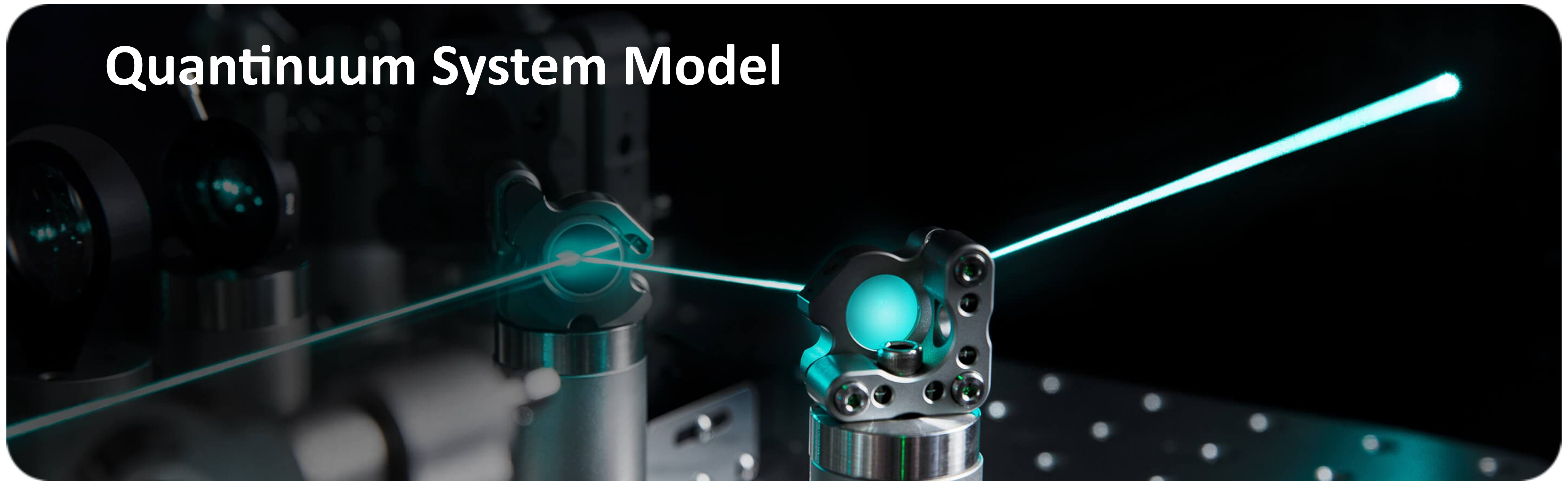


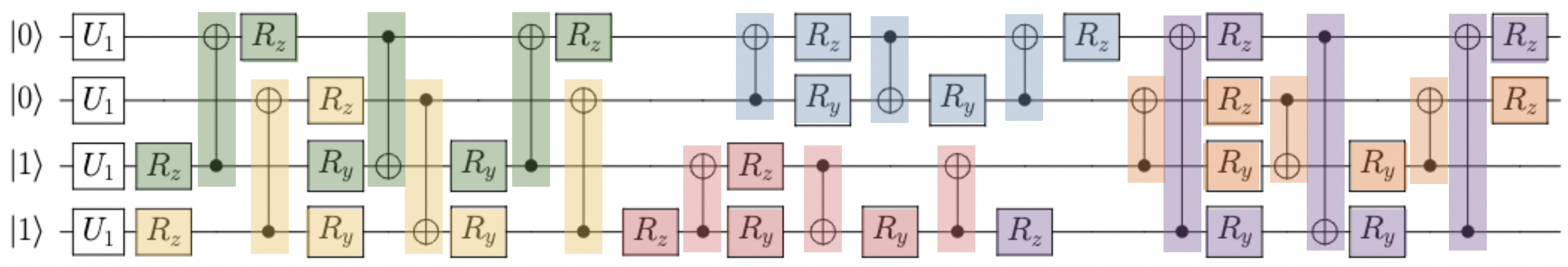
OO



Quantinuum System Model (QSM) H1-2

- Trapped-ion device
- Full-connected qubits
- High fidelity
  - $\varepsilon_q \sim 10^{-4}$
  - $\varepsilon_{qq} \sim 10^{-3}$

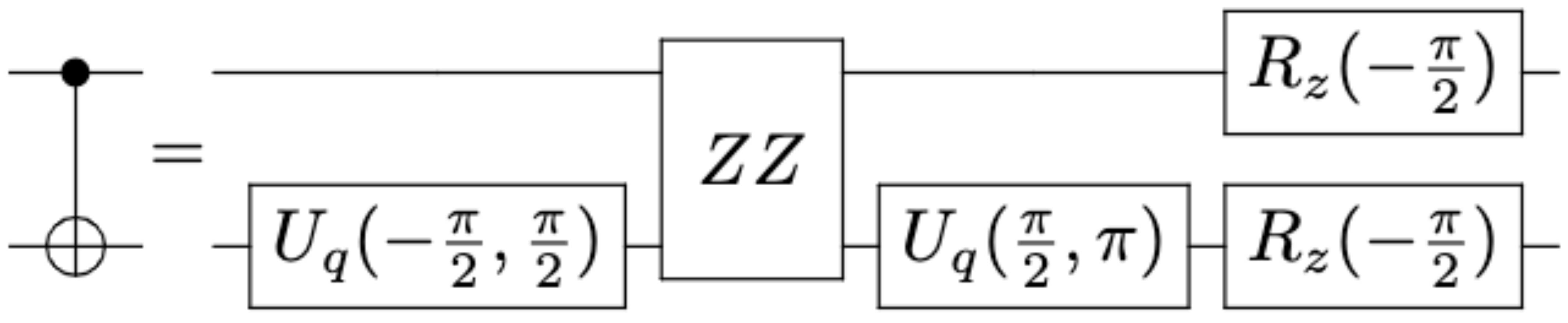




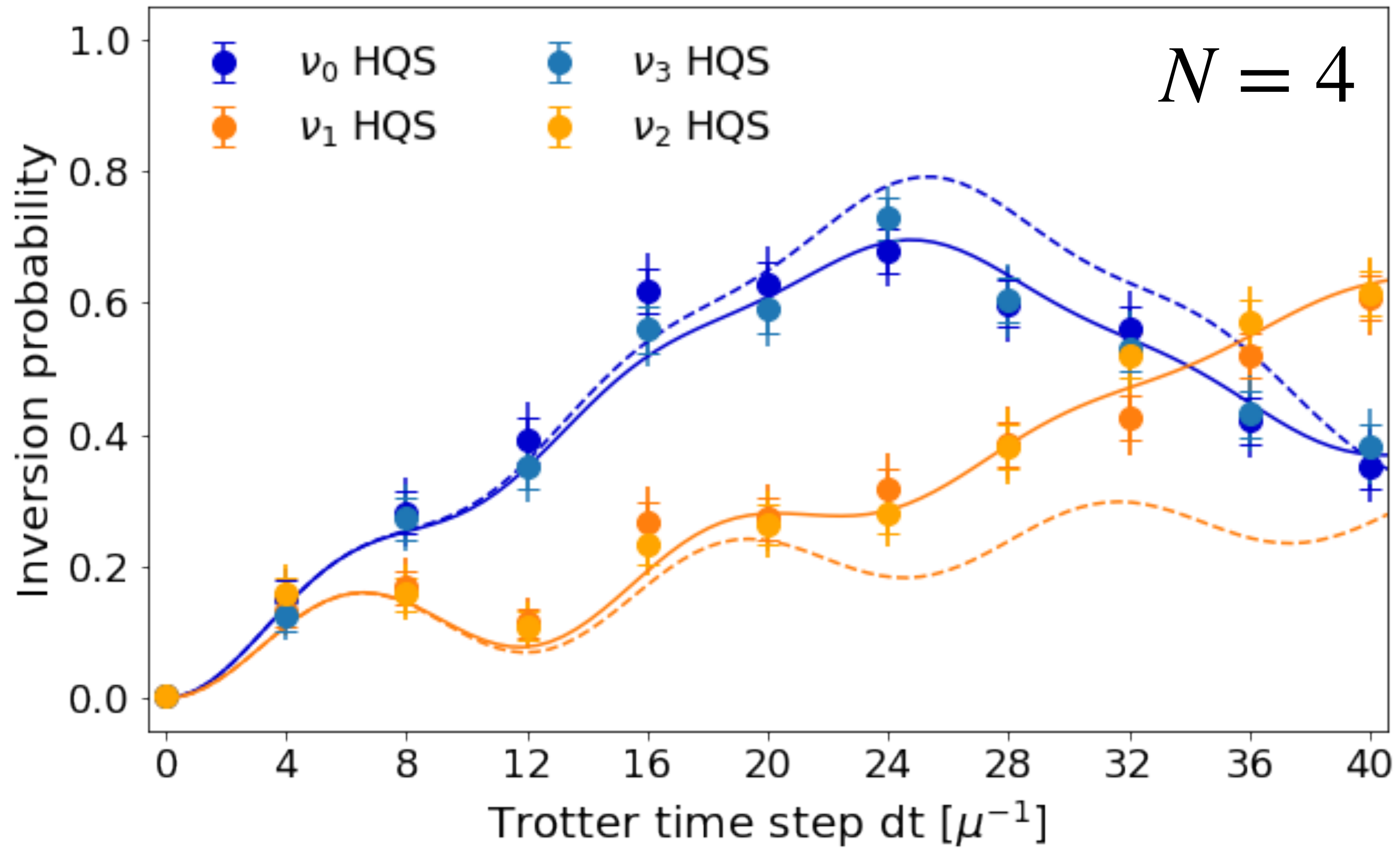
Machine aware implementation of the unitary propagator $U(t) = e^{-iHt}$:

- Qubit topology
- Quantum gate set
- Complexity scaling analysis

ZZ-based CNOT gate

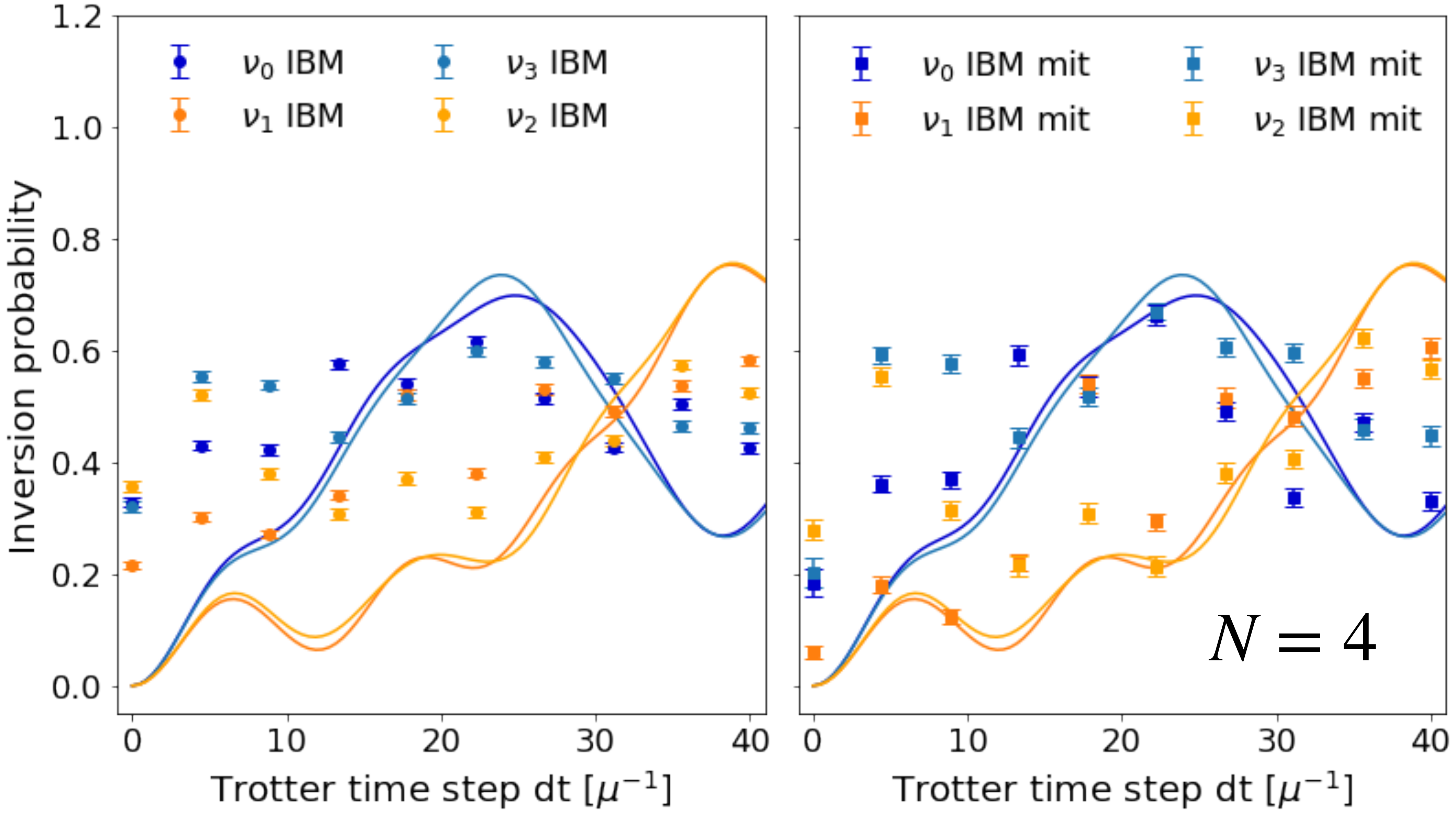


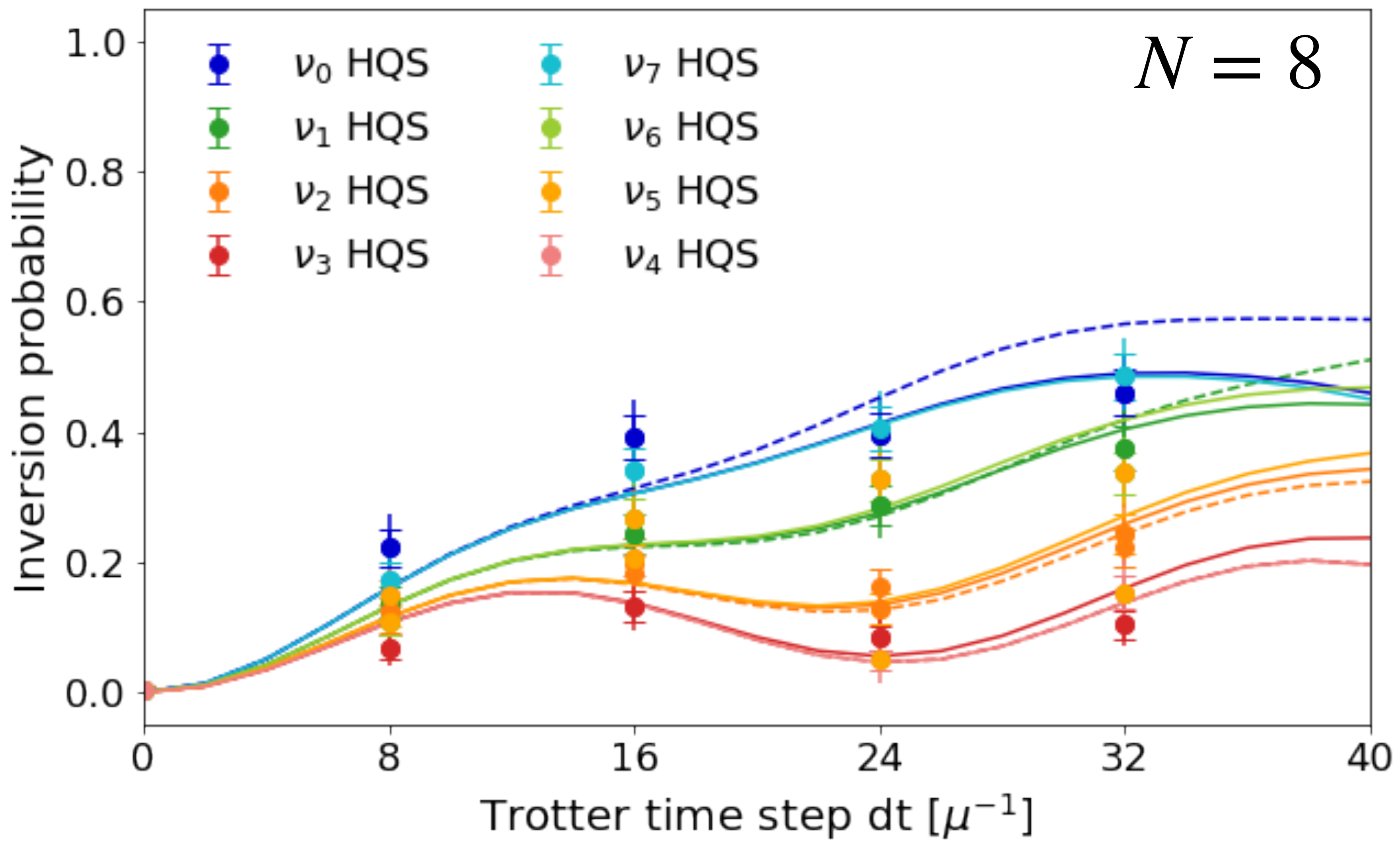
RESULTS: SINGLE TROTTER STEP



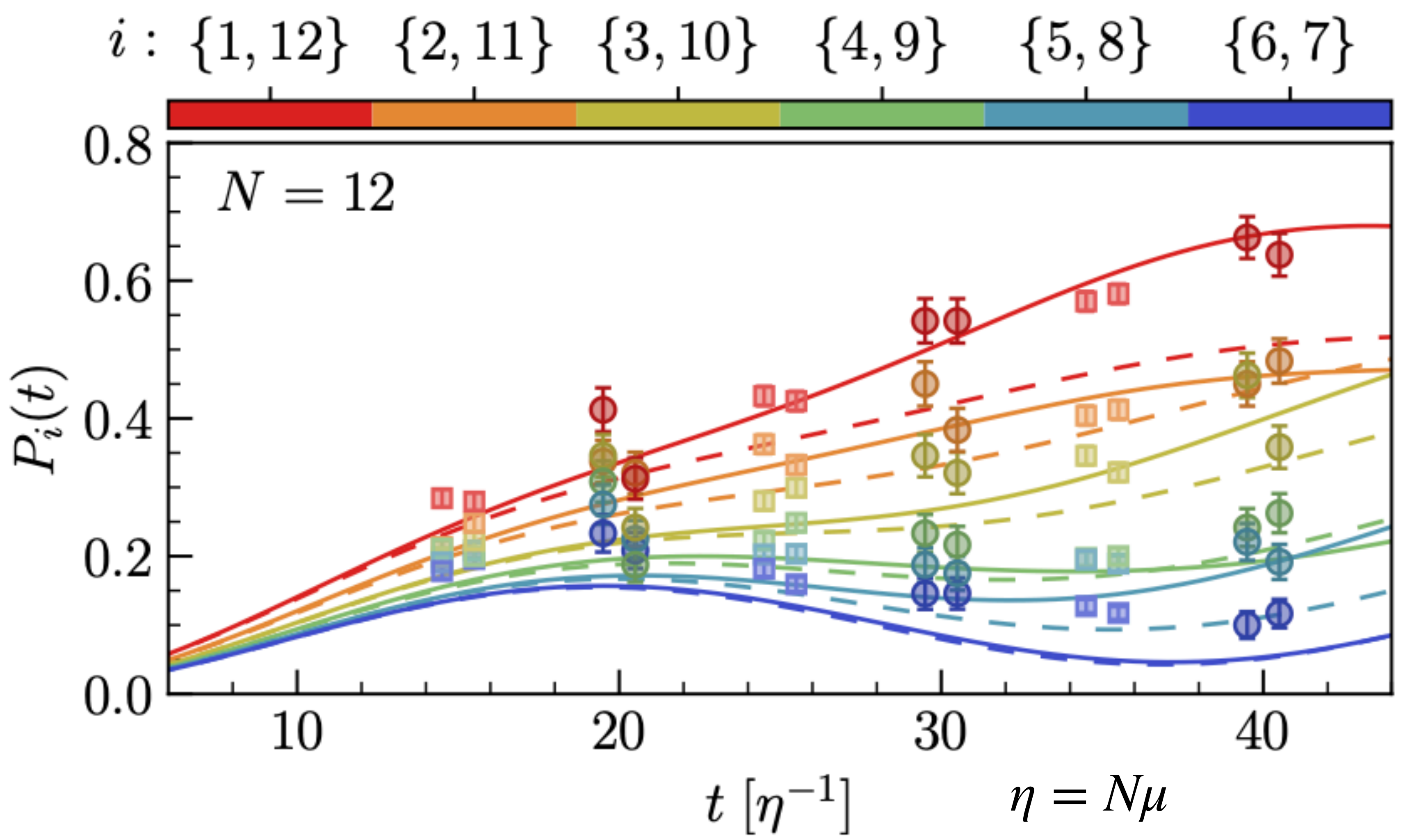
V. Amitrano et. al. (2022)
[arXiv:2207.03189](https://arxiv.org/abs/2207.03189)

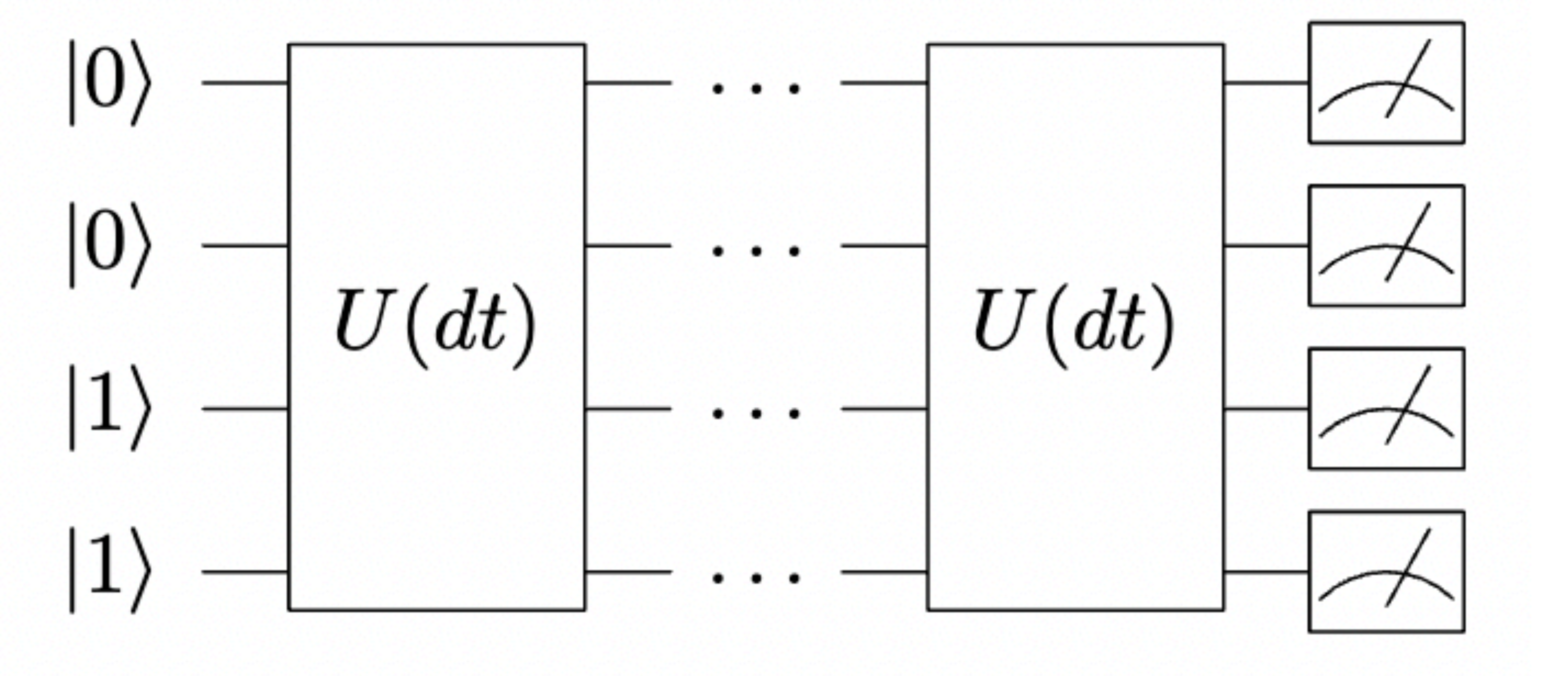
B. Hall, A. Roggero et. al (2021)
[arXiv:2102.12556](https://arxiv.org/abs/2102.12556)



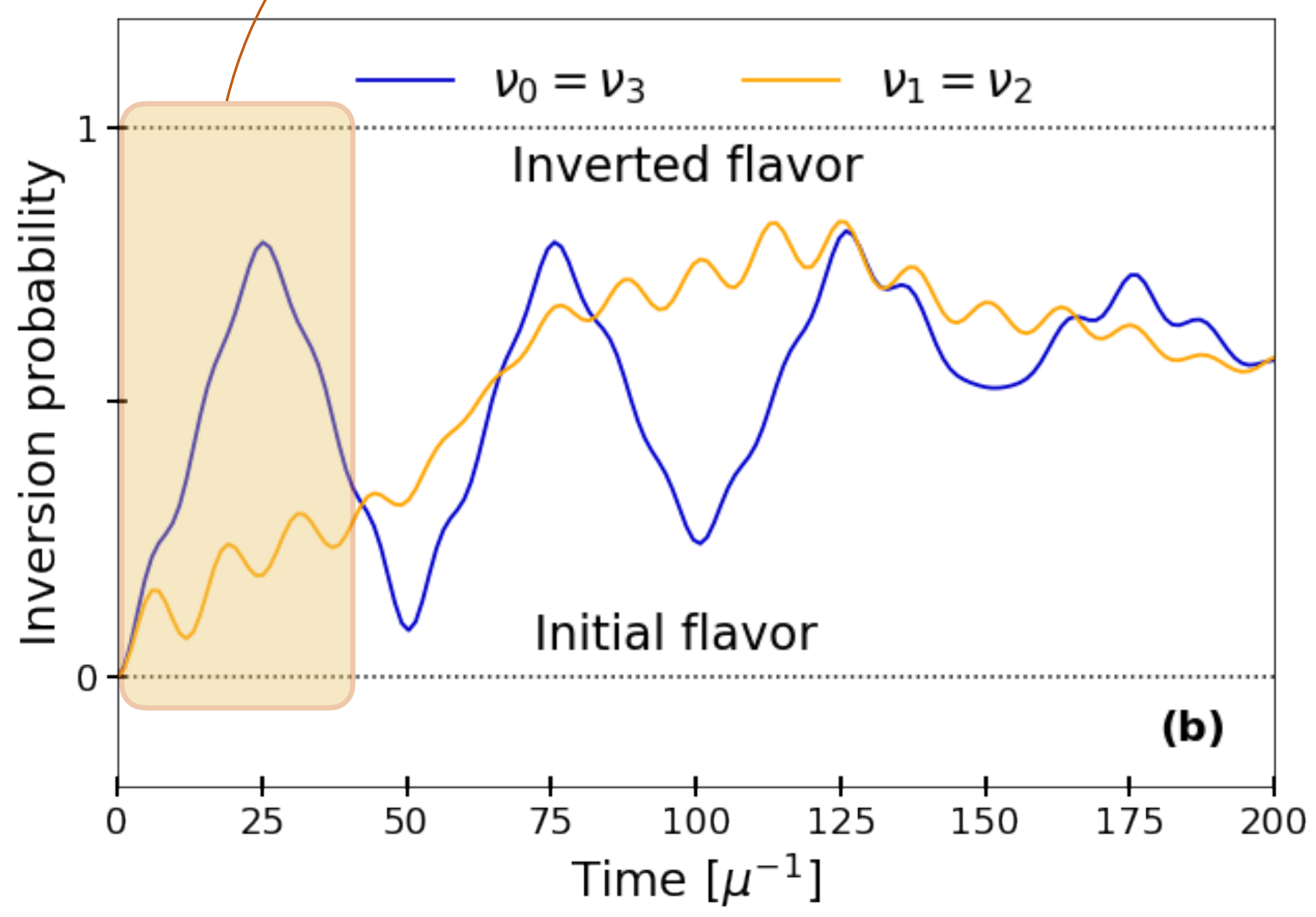


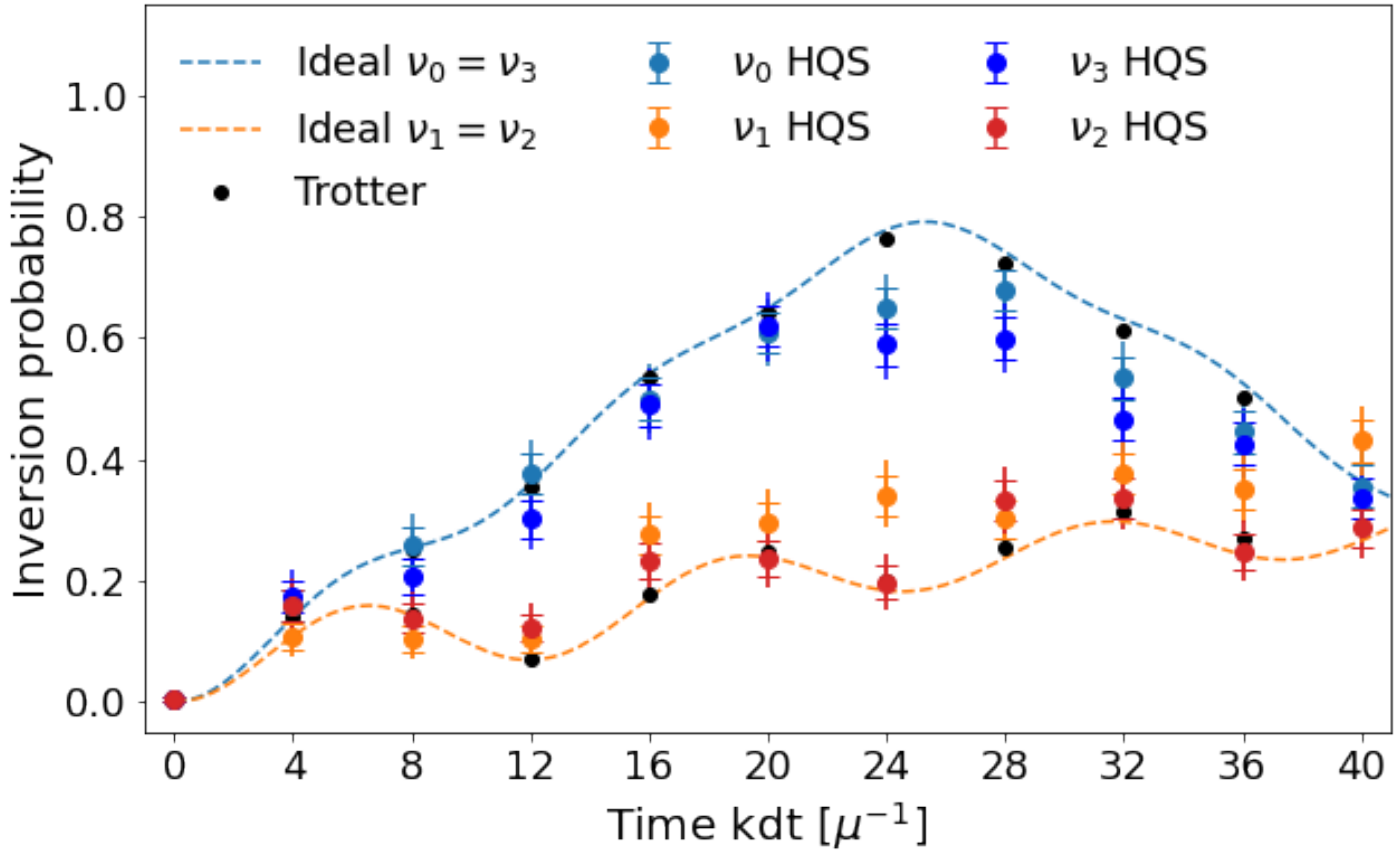
M. Illa and M. J. Savage (2022)
[arXiv:2210.08656](https://arxiv.org/abs/2210.08656)





- Short time-step $dt = 4\mu^{-1}$: small error decomposition $\sim \mathcal{O}(kdt^2)$
- Ideal ≈ trotterized evolution
- Very long quantum circuits (noise)





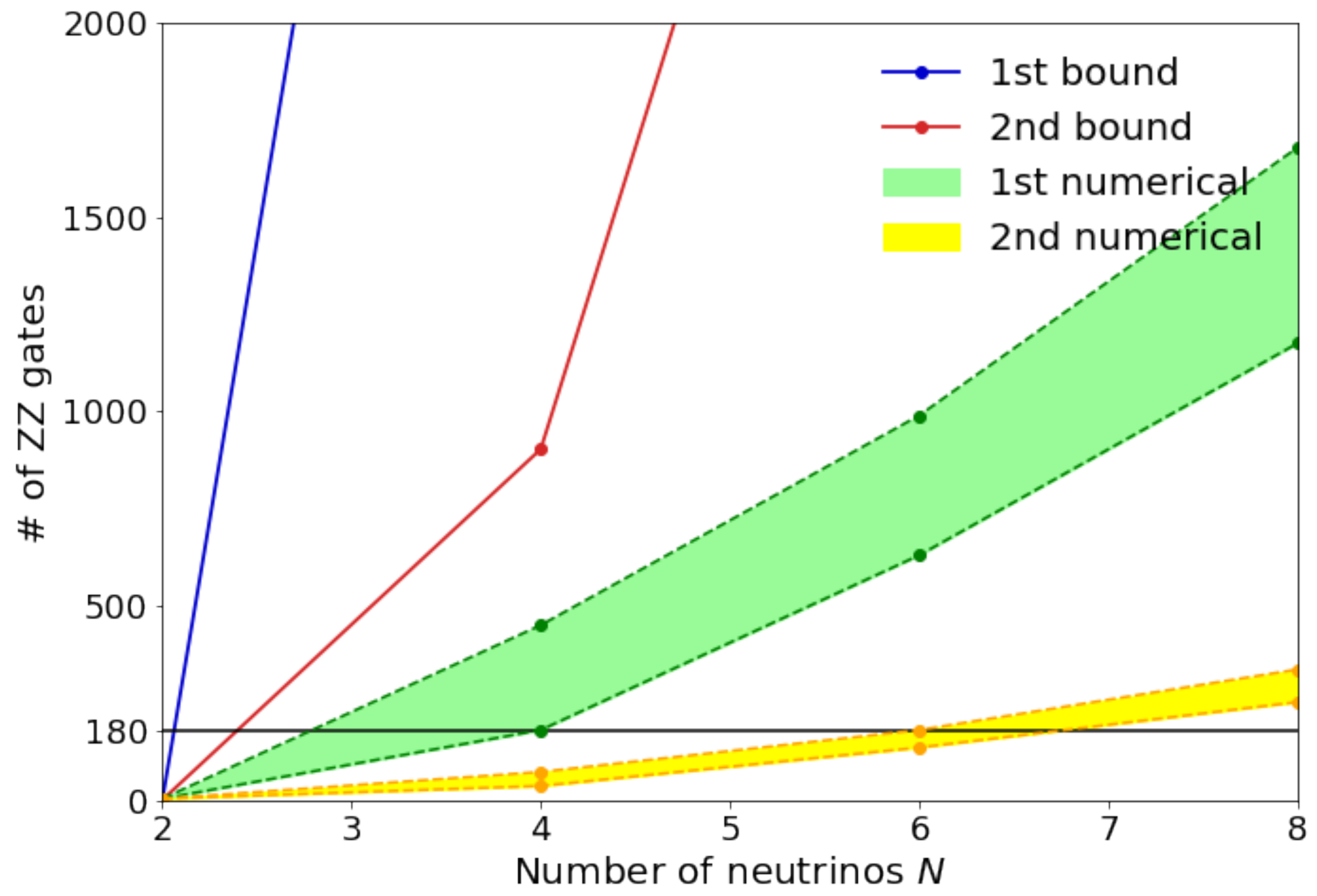
Steps	1	2	3	4	5	6	7	8	9	10
# ZZ	18	36	54	72	90	108	126	144	162	180
# SU(2)	36	68	100	132	164	196	228	260	292	324

Complexity scaling of the algorithm

We are interested in systems in which we fix $n_\nu = N/V$ and we look at the scaling with N

Complexity as the number of 2-qubit gates to evolve the system up to T keeping the error $< \epsilon$

- First order Trotter $\mathscr{C}_1 = \mathcal{O}\left(\frac{T^2\mu^2N^3}{\epsilon}\right)$
- Second order Trotter $\mathscr{C}_2 \leq \mathcal{O}\left(\frac{(T\mu)^{3/2}}{\sqrt{\epsilon}}N^{5/2}\right)$
- Higher order Trotter $\sim N^{2+\delta}$

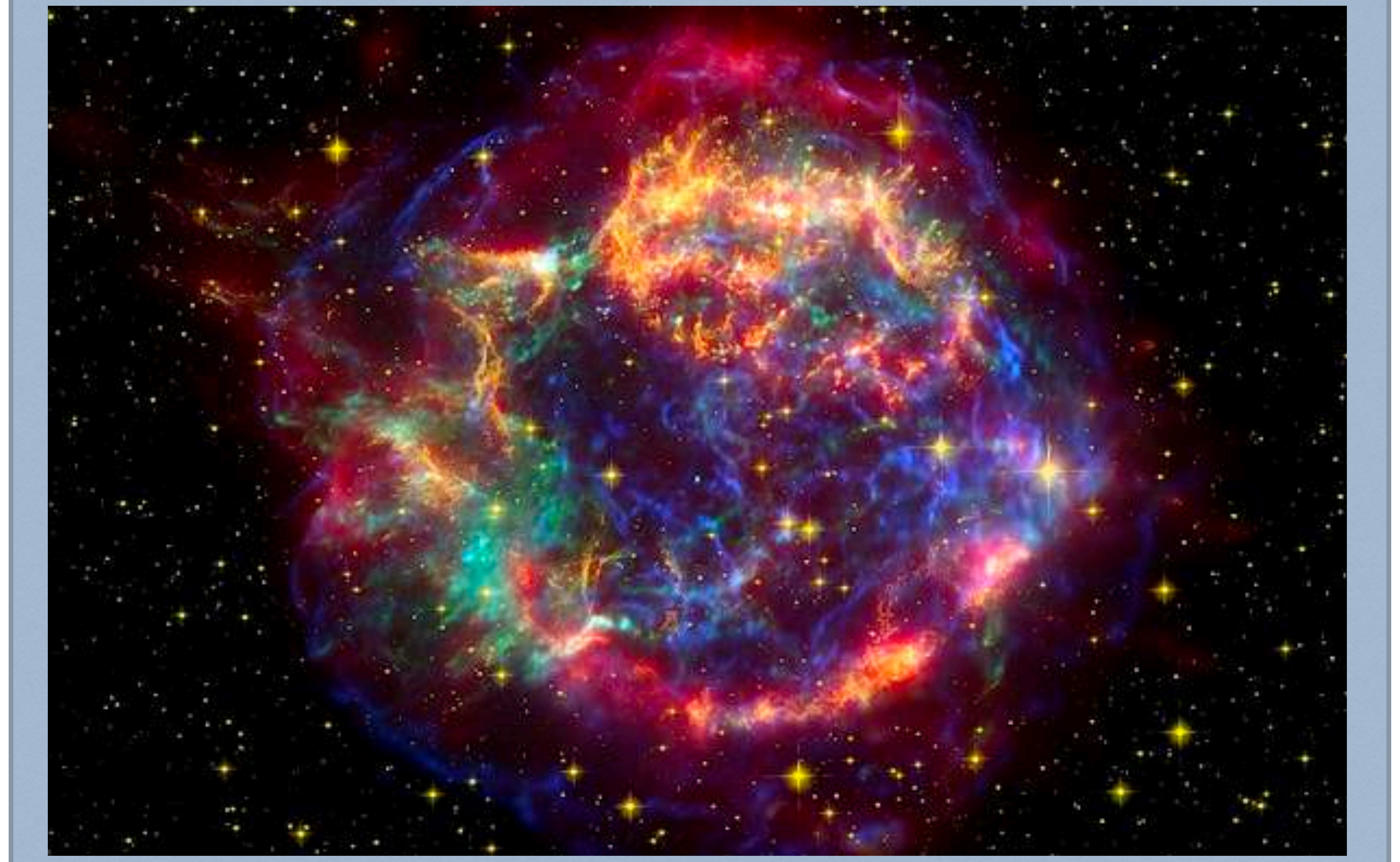


Real cost estimated by calculating the number of steps such that we evolve up to $T = 40\mu^{-1}$ with an error ≤ 0.15

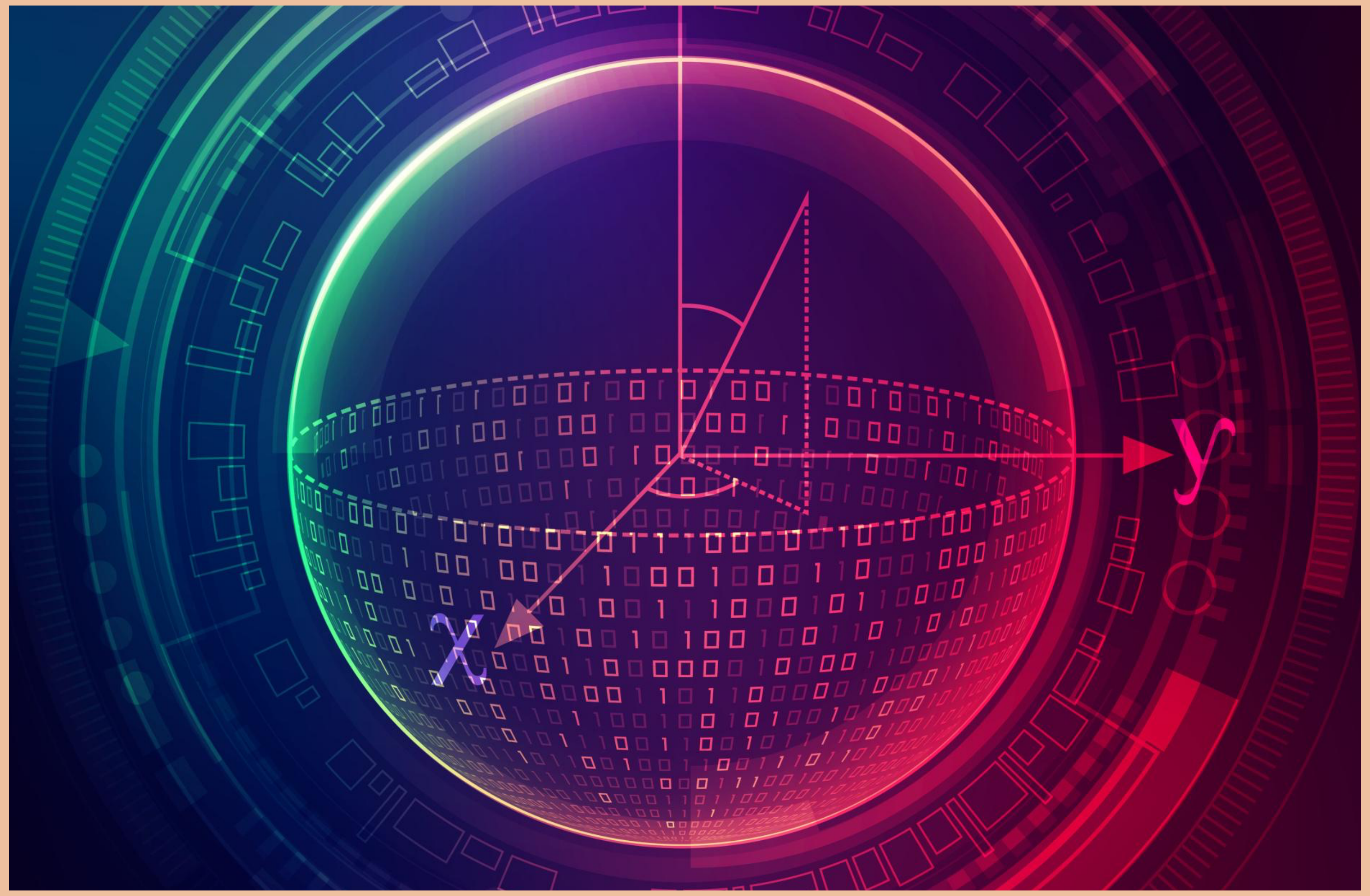
$$\varepsilon(dt) = \|U_{approx}(dt) - U_{exact}(dt)\|_\infty$$

Conclusions

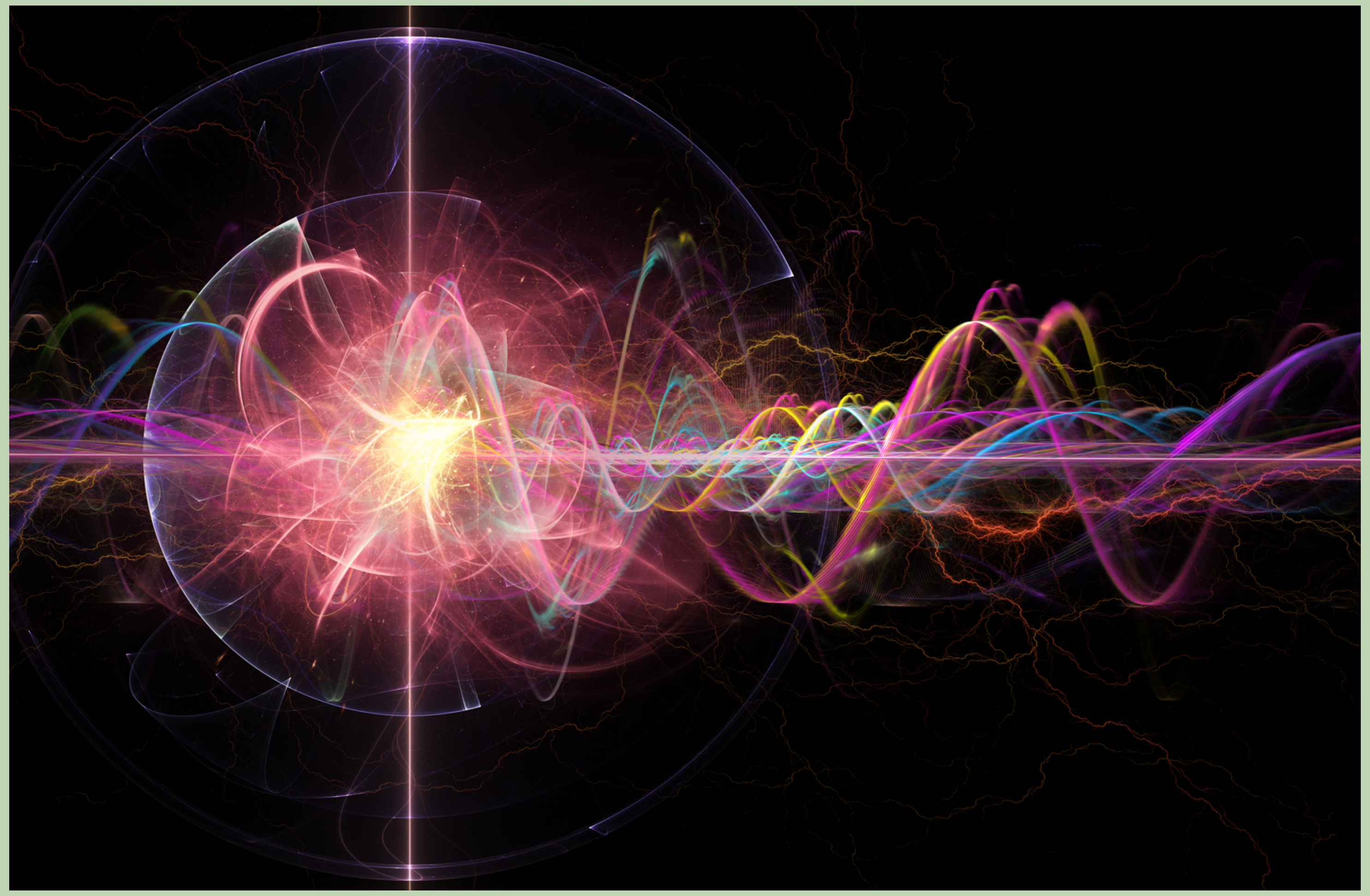
- Flavor dynamics is crucial to describe many effects in core-collapse supernovae
- Collective neutrino oscillations make the problem non linear and interesting to test quantum computing



- QC necessary for full dynamics simulation
- The gate decomposition must be machine aware and circuit optimization is crucial
- Full qubit connectivity allows for more freedom in gate decomposition



- Results are very promising
- We can increase the number of simulated neutrinos
- The algorithm scales polynomially



Thank you for your attention

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Quantum Computing @INFN Workshop

Bologna, 14-15 November 2022

Supplementary material

Two-flavor Hamiltonian Model

- Vacuum mixing (1-body term)

$$H_{vac} = \sum_{i=1}^{N} \vec{b} \cdot \vec{\sigma}_i = \frac{\delta m^2}{4E} \sum_{i=1}^{N} \left(\sin(2\theta_\nu) X_i - \cos(2\theta_\nu) Z_i \right)$$

- $\nu\nu$ - interaction (2-body term)

$$H_{\nu\nu} = \frac{\mu}{N} \sum_{i<j}^{N} J_{ij} \vec{\sigma}_i \cdot \vec{\sigma}_j = \frac{\mu}{N} \sum_{i<j}^{N} J_{ij} \left(X_i \otimes X_j + Y_i \otimes Y_j + Z_i \otimes Z_j \right)$$

! **$H_{\nu\nu}$ is an all-to-all interaction that makes the problem non-linear**

The model:

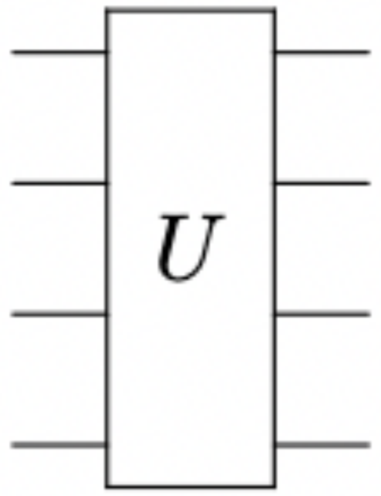
- $\theta_\nu = 0.195$ mixing angle
- Monochromatic flux $E \neq E_i$
- $\vec{b} = \frac{\delta m^2}{4E}(\sin(2\theta_\nu), 0, -\cos(2\theta_\nu))$
- $J_{ij} = 1 - \cos(\theta_{ij})$
- $\theta_{ij} = \arccos(0.9) \frac{|i-j|}{N-1}$
- Energy scale $\mu = \sqrt{2} G_F n_\nu$
- $X_2 = I \otimes I \otimes X \otimes I$
- $X_0 \otimes X_2 = X \otimes I \otimes X \otimes I$
- H is a $2^N \times 2^N$ Hermitian matrix

The Unitary Implementation

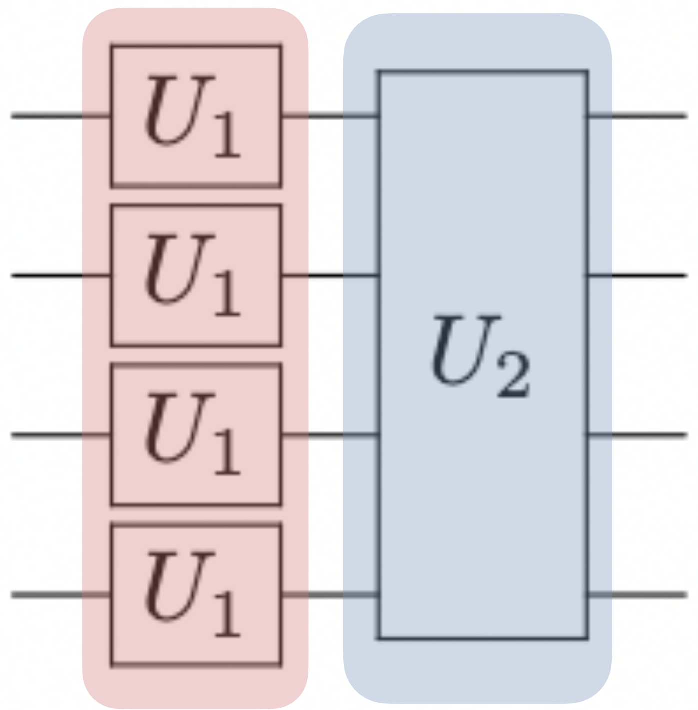
Single Trotter step

$$U(dt) = e^{-iHdt}$$

$$H = H_{vac} + H_{\nu\nu}$$

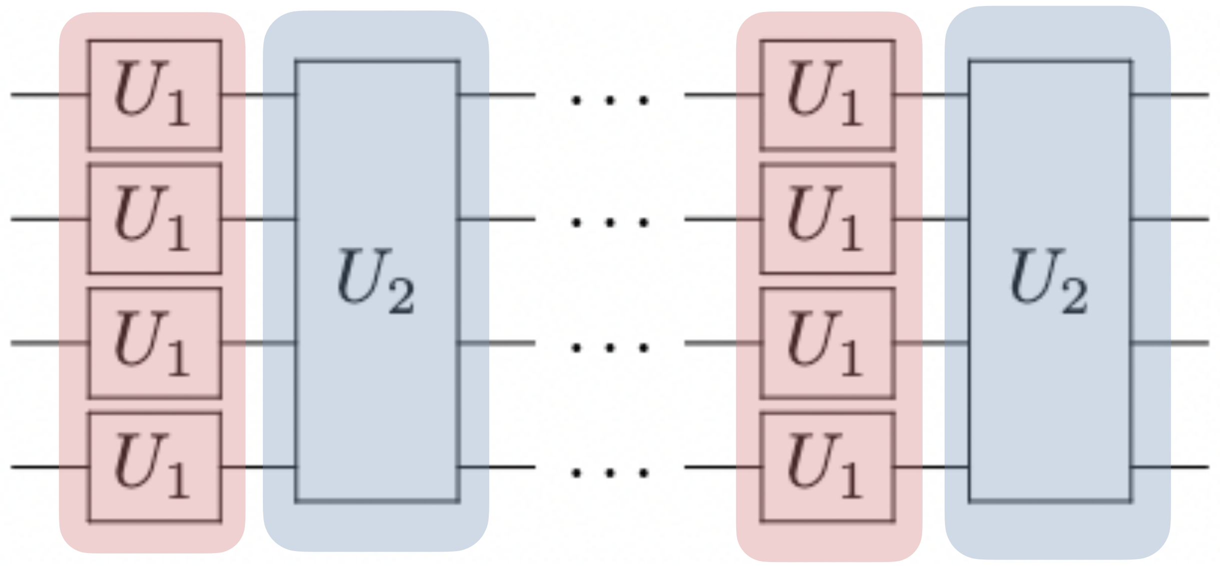


$$U(dt) = U_2(dt)U_1(dt)$$

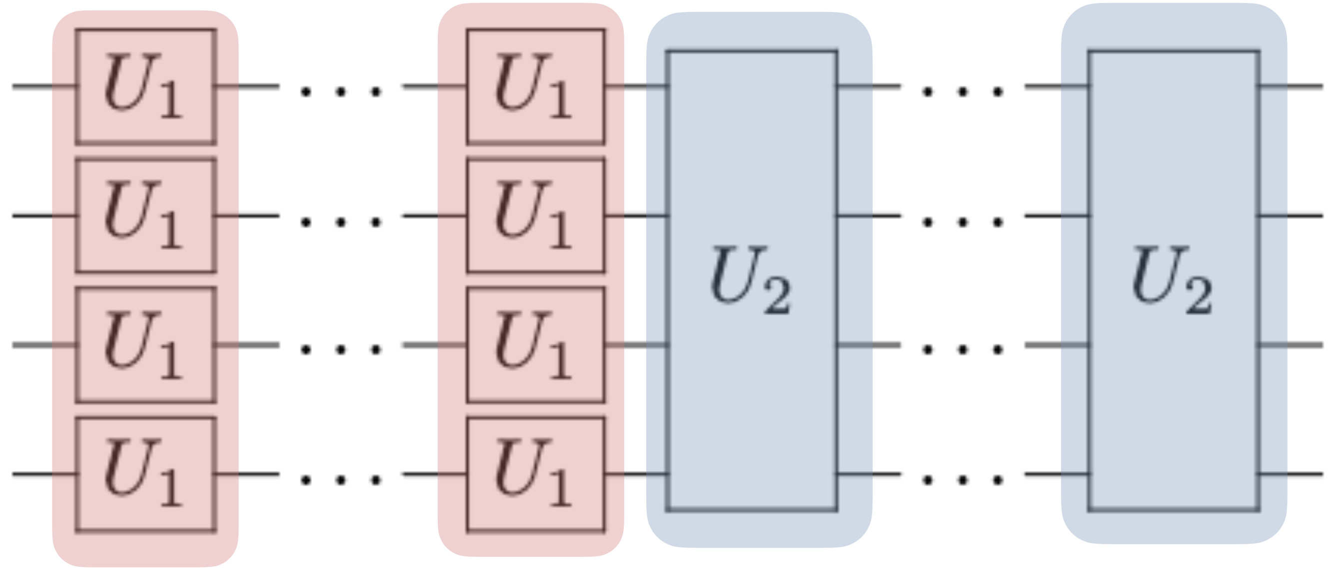


Evolution using multiple Trotter steps

$$U(T) = \prod_{k=1}^{r} U_2\left(\frac{T}{r}\right) U_1\left(\frac{T}{r}\right)$$



$$U(T) = \prod_{k=1}^{r} U_2\left(\frac{T}{r}\right) \prod_{k=1}^{r} U_1\left(\frac{T}{r}\right)$$

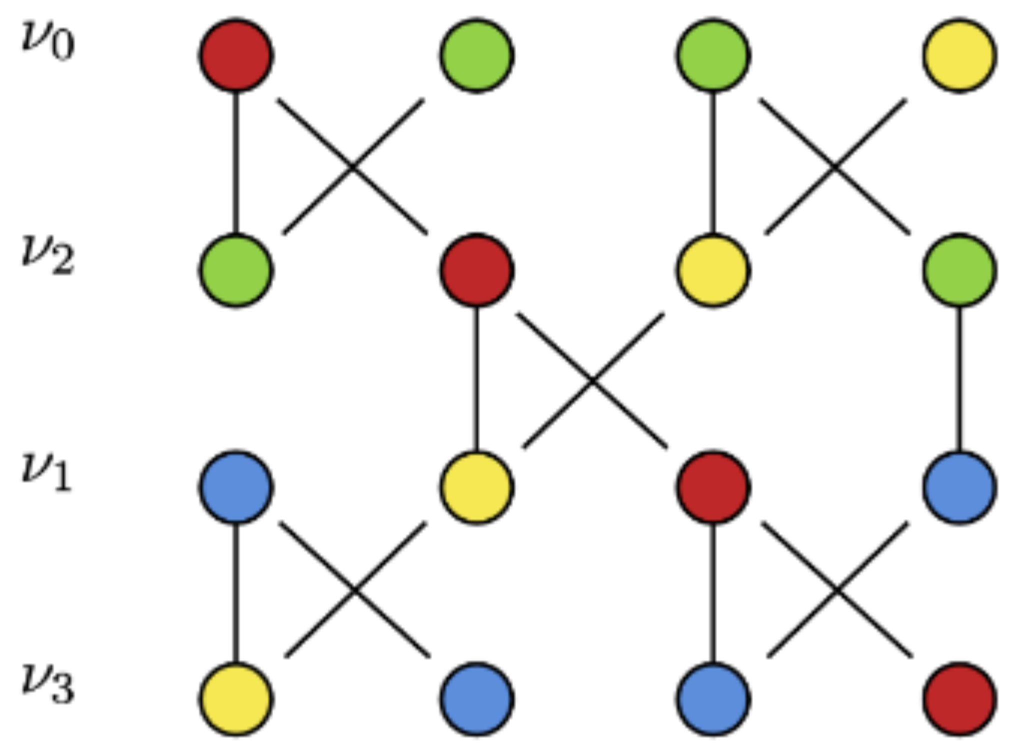


$$U_2(t) \approx \prod_{i<j}^{N} e^{-ih_{ij}t}$$

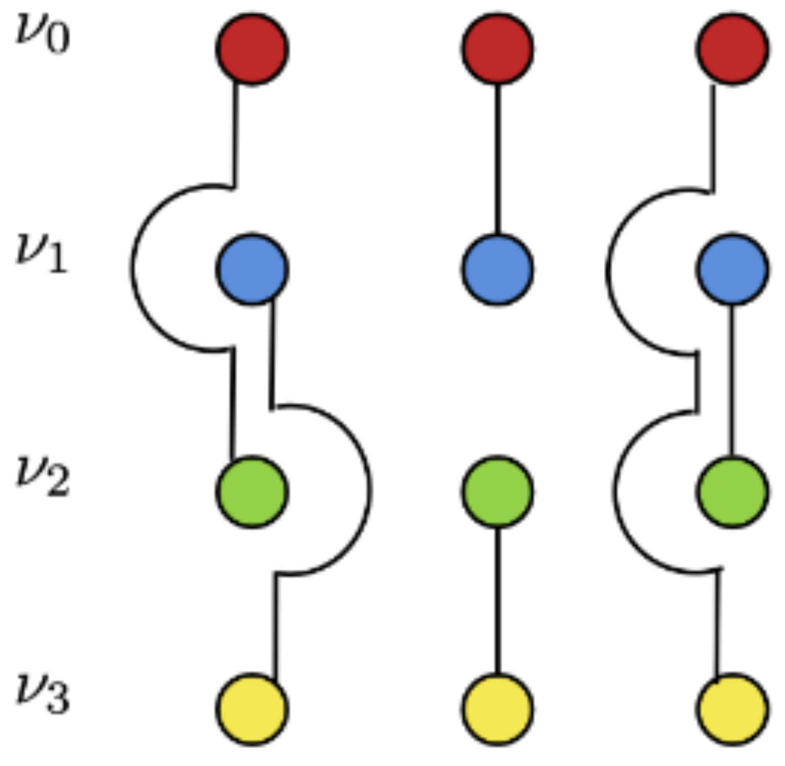
→ $\frac{N(N-1)}{2}$ terms → **All-to-all interaction means that each qubit interacts with all the others**

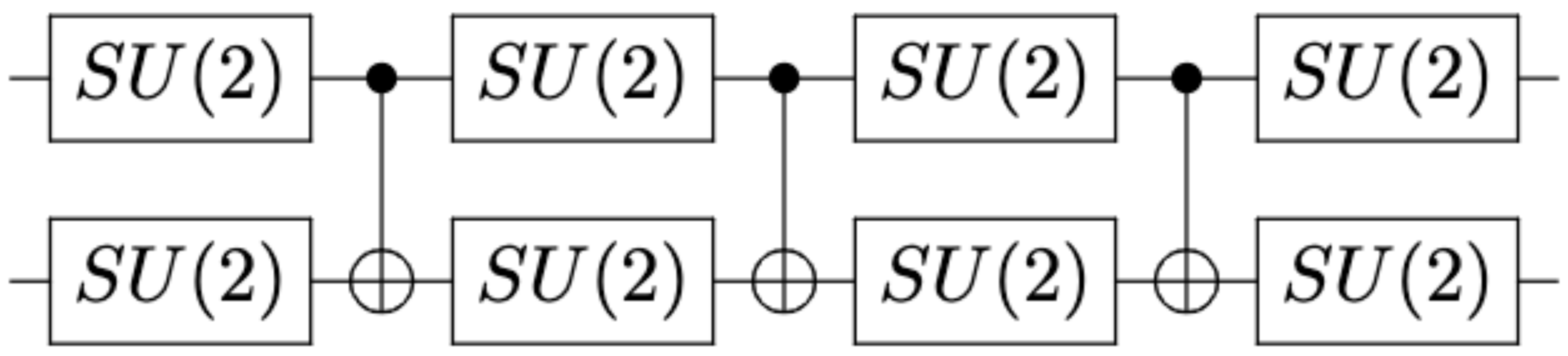
B. Hall, A. Roggero et. al (2021)

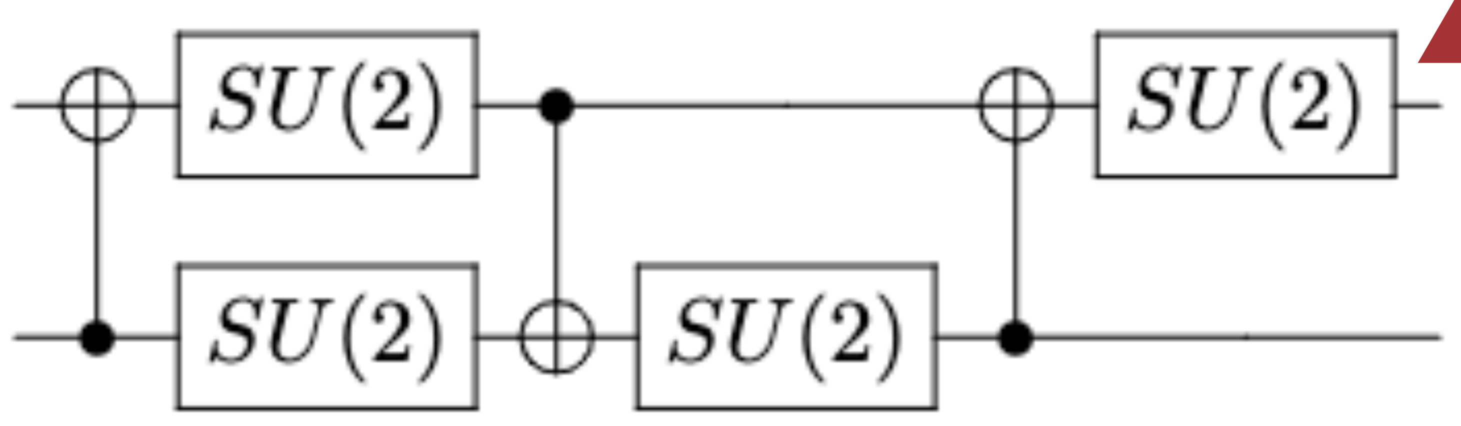
Best swap network



Optimal ordering





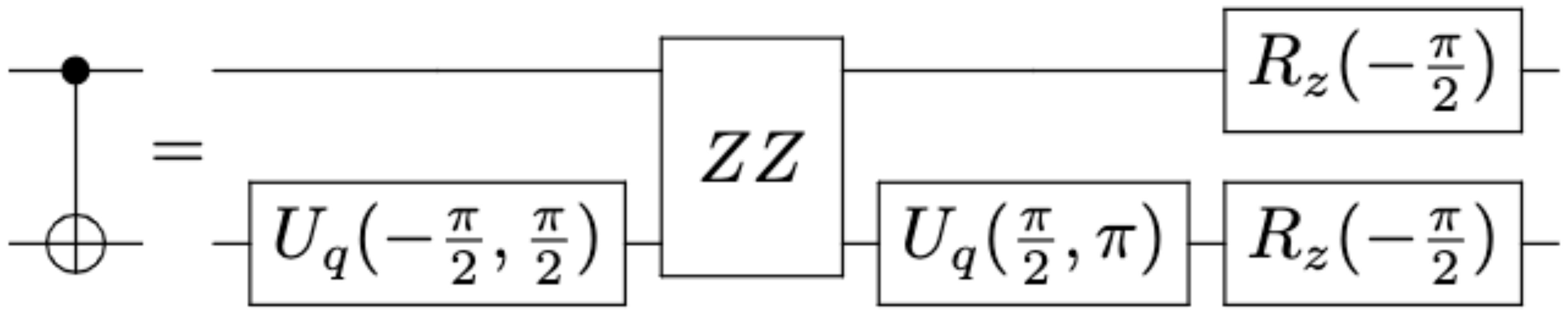


! **Less rotations and easier angles**

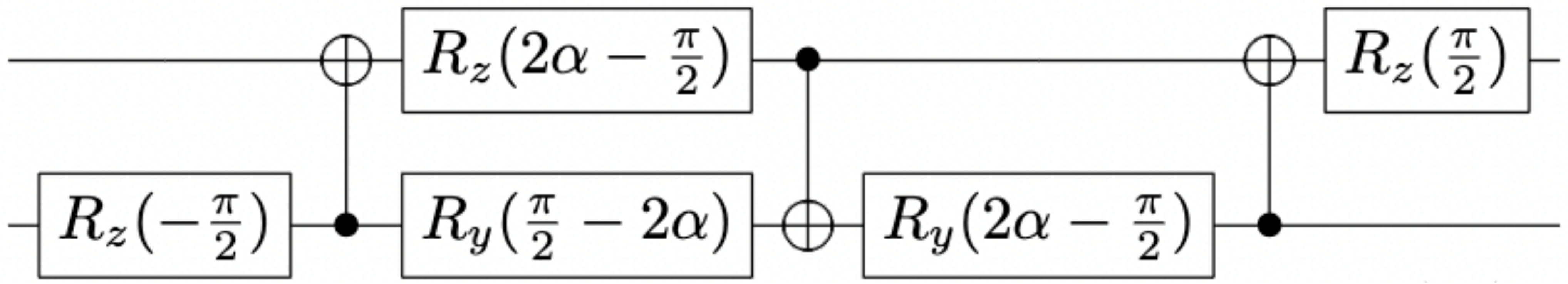
Native gate

- $R_z(\lambda) = \begin{pmatrix} e^{-i\lambda/2} & 0 \\ 0 & e^{i\lambda/2} \end{pmatrix}$
- $U_q(\theta, \varphi) = \begin{pmatrix} \cos\theta/2 & -ie^{-i\varphi}\sin\theta/2 \\ -ie^{i\varphi}\sin\theta/2 & \cos\theta/2 \end{pmatrix}$
- $ZZ = e^{-i\frac{\pi}{4}Z\otimes Z} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & i & 0 & 0 \\ 0 & 0 & i & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$

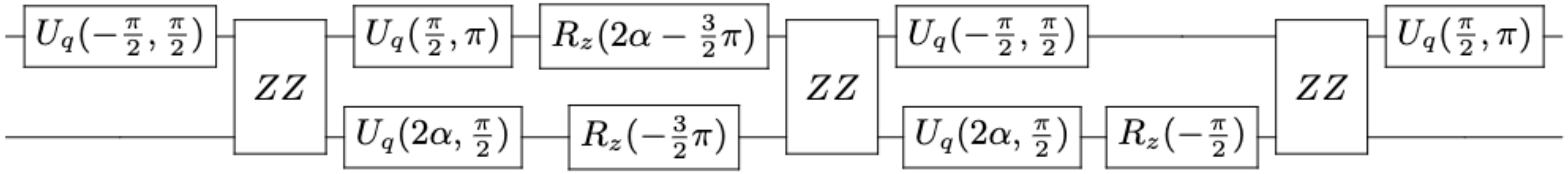
ZZ-based CNOT gate



CNOT-based u_{ij}



ZZ-based u_{ij}



Trotter error scaling for the approximation of $U(dt) = e^{-iHdt}$

First order Trotter decomposition $U(dt) \approx \mathscr{L}_1(dt) = \prod_{K=1}^{\Gamma} e^{-ih_{ij}dt}$

$$\varepsilon(dt) \leq \frac{dt^2}{2} \sum_{K=1}^{\Gamma} \left\| \sum_{L=K+1}^{\Gamma} [h_K, h_L] \right\|$$

$$\varepsilon(dt) \leq 12 dt^2 \mu^2 \frac{\Theta^2}{N^2} \binom{N}{3} = \mathcal{O}\left(dt^2 \mu^2 N\right)$$

$$\varepsilon(T) \leq r\varepsilon(dt) \quad T = rdt$$

$$r \leq 12 \frac{T^2 \mu^2 \Theta^2}{\epsilon N^2} \binom{N}{3} = \mathcal{O}\left(\frac{T^2 \mu^2 N}{\epsilon}\right)$$

$$\mathscr{C} \leq \binom{N}{2} r$$

$$\mathscr{C}_1 = \mathcal{O}\left(\frac{T^2 \mu^2 N^3}{\epsilon}\right)$$

Trotter error scaling for the approximation of $U(dt) = e^{-iHdt}$

Second order Trotter decomposition $U(dt) \approx \mathscr{L}_2(dt) = \mathscr{L}_1\left(\frac{dt}{2}\right)\mathscr{L}_1^\dagger\left(-\frac{dt}{2}\right)$

$$\varepsilon(dt) \leq \frac{dt^3}{12}\sum_K^\Gamma \| \sum_{L>K}^\Gamma \sum_{M>K}^\Gamma [h_L,[h_M,h_K]] \| + \frac{dt^3}{24}\sum_K^\Gamma \| \sum_{L>K}^\Gamma [h_K,[h_K,h_L]] \|$$

$$\varepsilon(dt) \leq dt^3 \frac{\mu^3\Theta^3}{N^3}\left[20\binom{N}{3} + 56\binom{N}{4}\right] = \mathcal{O}\left(dt^3\mu^3 N\right)$$

$$\varepsilon(T) \leq r\varepsilon(dt) \quad T = rdt$$

$$r \leq \frac{(T\mu\Theta)^{3/2}}{\sqrt{\epsilon}N^{3/2}}\sqrt{20\binom{N}{3} + 56\binom{N}{4}} = \mathcal{O}\left(\frac{T^{3/2}\mu^{3/2}\sqrt{N}}{\sqrt{\epsilon}}\right)$$

$$\mathscr{C} \leq \left(2\binom{N}{2} - \frac{N}{2}\right)r$$

$$\mathscr{C}_2 \leq \left(2\binom{N}{2} - \frac{N}{2}\right)r = \mathcal{O}\left(\frac{(T\mu)^{3/2}}{\sqrt{\epsilon}}N^{5/2}\right)$$

Statistical error analysis

- Number of repetitions $M = 200$
- Bayesian approach
    - Probability distribution of obtaining m times $|q\rangle$: $\mathscr{P}_b(m|p) = \binom{M}{m} p^m (1-p)^{M-m}$
    - Bayes theorem: $\mathscr{P}(p|m) = \dfrac{\mathscr{P}(m|p)\mathscr{P}(p)}{\mathscr{P}(m)}$

A priori distribution: Beta distribution

$$\mathscr{B}(\alpha,\beta) = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} p^{\alpha-1}(1-p)^{\beta-1}$$

Likelihood distribution: Binomial distribution

Posterior distribution

- Prior conjugate $\mathscr{B}(\alpha',\beta') = = \dfrac{\mathscr{P}_b(m|p)\mathscr{B}(\alpha,\beta)}{\int dq \mathscr{P}_b(m|p)\mathscr{B}(\alpha,\beta)}$ where $\alpha' = \alpha + m$ and $\beta' = \beta + M - m$
- $\alpha = 1$ and $\beta = 1$. We used $\mathscr{B}(\alpha',\beta')$ as posterior distribution and look for:
    - $\mathscr{P}(p_{min} < p < p_{max}) = 0.68$