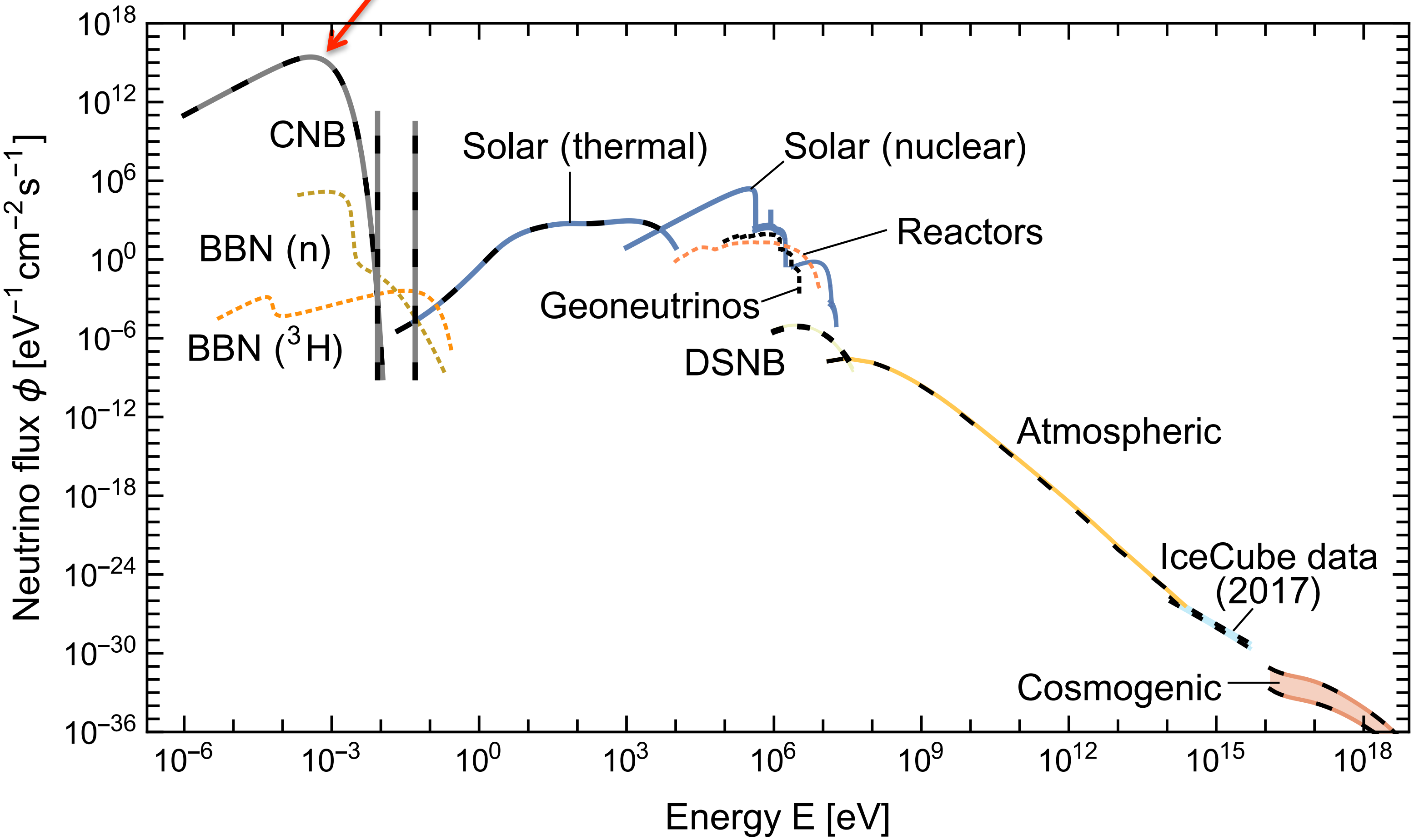
Inflation and CNB

- Effects of neutrinos on gravitational waves
- Inflation and neutrinos: further connections
- For the future (futuristic future??): measuring CNB and its spatial anisotropies
- Inflation & neutrinos: isocurvature modes

The Cosmic Neutrino Background (CNB)



Grand Unified Neutrino Spectrum (GUNS) at Earth, integrated over directions and summe1d over flavors. Solid lines are for neutrinos, dashed or dotted lines for antineutrinos. The CNB spectrum corresponds to the masses (m2= 0, m3 = 8.6, m = 50) meV, resulting in a blackbody spectrum plus two monochromatic lines of nonrelativistic neutrinos with energies corresponding to neutrinos with masses m2 and m3. Figure fromE. Vitagliano, I. Tamborra, and G. Raffelt Rev. Mod. Phys. 92 (2020).

The Cosmic Neutrino Background (CNB)

- **A sea of low energy neutrinos which decoupled from the primordial plasma when the Universe was ~ 1 s old (at $z\sim10^{10}$ at $T\sim1$ MeV)**

- After that they just free-stream → they carry information about the conditions of the Universe when it was 1s old and about neutrino properties.

- The **temperature of the CNB today** is expected to be

$T_\nu = (4/11)^{1/3} T_\gamma$ relative to CMB photons → $T_\nu = 1.946\ \mathrm{K}$

- This in turns implies (under the approximation of massless neutrinos)

$$\frac{n_{\nu_\alpha} + n_{\bar{\nu}_\alpha}}{n_\gamma} = \frac{3}{11} \longrightarrow n_\nu = \sum_\alpha n_{\nu_\alpha} \approx 336\ \mathrm{cm}^{-3}$$

N.B.: **remarkably large flux** with respect to other astrophysical neutrino sources (e.g. solar neutrinos). Unfortunately **at a very low energy.**

- Actually we know, based on the measured neutrino mass differences that at least 2 neutrinos are non-relativistic today, and one can be still relativistic.

The Cosmic Neutrino Background (CNB)

Let us (briefly) review how to obtain the previous numbers:
decoupling temperature

- At T>> 1 MeV neutrinos are kept in thermal equilibrium because of their wek interactions with charged leptons, baryons and photons (e.g. $\nu_\alpha \bar{\nu}_\alpha \leftrightarrow e^+ e^-$)

$$\longrightarrow \quad f^{(\mathrm{eq})}_{\nu_\alpha, \bar{\nu}_\alpha} = \left[\exp\left(\frac{p \mp \mu}{T} \right) + 1 \right]^{-1}$$

- To have a rough estimate of the decoupling temperature T_d

$$\Gamma\left(T\right) \sim H\left(T\right)$$

where Γ is the neutrinos interaction rate

The Cosmic Neutrino Background (CNB)

Let us (briefly) review how to obtain the previous numbers:

decoupling temperature

- $\Gamma = n_e \langle \sigma v \rangle \simeq G_F^2 T^5$ (for T << M_Z and M_W)

  $n \sim T^3 \quad \langle \sigma v \rangle \simeq G_F^2 T^2$

- $H^2 = \frac{8\pi G}{3} \rho_{\rm rad}\,, \qquad \rho_{\rm rad} = \frac{\pi^2}{30} g_*(T) T^4 \longrightarrow$

$$H(T) = \left(\frac{4\pi^3}{45}\right)^{1/2} g_*^{1/2}(T) \frac{T^2}{M_{\rm pl}}$$

$$\Longrightarrow \quad \boxed{T_{\nu,{\rm d}} \simeq \left(\frac{\sqrt{g_*}}{G_F^2 M_{\rm pl}}\right)^{1/3} \sim g_*^{1/6}\ {\rm MeV}}$$

The Cosmic Neutrino Background (CNB)

- N.B: decoupling however is not istananous. More refined computations require to solve the Boltzmann equations (e.g. Dolgov 2002)

$$\frac{\partial f}{\partial t} - Hp\frac{\partial f}{\partial p} = \mathbb{C}_{\text{tot}}\left[f\right]$$

<u>where in a FRW Universe</u>: $f\left(x^i, p_j, t\right) = f\left(p, t\right)$

- $$xH\frac{\partial f}{\partial x} = -\frac{80G_{\rm F}^2\left(g_{\rm L}^2 + g_{\rm R}^2\right)}{3\pi^3 x^5}yf \qquad x = am_0 \text{ and } y_j = ap_j$$

$$g_L^2 + g_R^2 = \begin{cases} \sin^4\theta_W + (\frac{1}{2} + \sin^2\theta_W)^2 & \text{for } \nu_e \\ \sin^4\theta_W + (-\frac{1}{2} + \sin^2\theta_W)^2 & \text{for } \nu_{\mu,\tau} \end{cases}$$

(couplings to left and right-handed currents)

$\Rightarrow$ Decoupling temperatures $\begin{cases} T_{\nu_e} = 2.4\ \text{MeV} \\ T_{\nu_{\mu,\tau}} = 3.7\ \text{MeV} \end{cases}$

The Cosmic Neutrino Background (CNB)

Let us (briefly) review how to obtain the previous numbers:
CNB Temperature today and abundance

- Recall that during the standard radiation dominated epoch the entropy is conserved

$$S = s\,a^3 = const. \quad \text{with} \quad s = \frac{2\pi^2}{45} g_{*s} T^3 \longrightarrow \boxed{T \propto a^{-1} g_{*s}^{-1/3}}$$

$$g_{*S} = \sum_{i=bosons} g_i \left(\frac{T_i}{T}\right)^3 + \frac{7}{8} \sum_{i=fermions} g_i \left(\frac{T_i}{T}\right)^3$$

- Shortly after neutrino decoupling, electron and positron become non-relativistic and annihilate, tansferring entropy to the relativistic particles in thermal equilibrium (photons in this case) so that g_{*s} increases ***after e^+- e^- annihilation thus making the T of the photons to decreases slightly less slowly than a^{-1}***

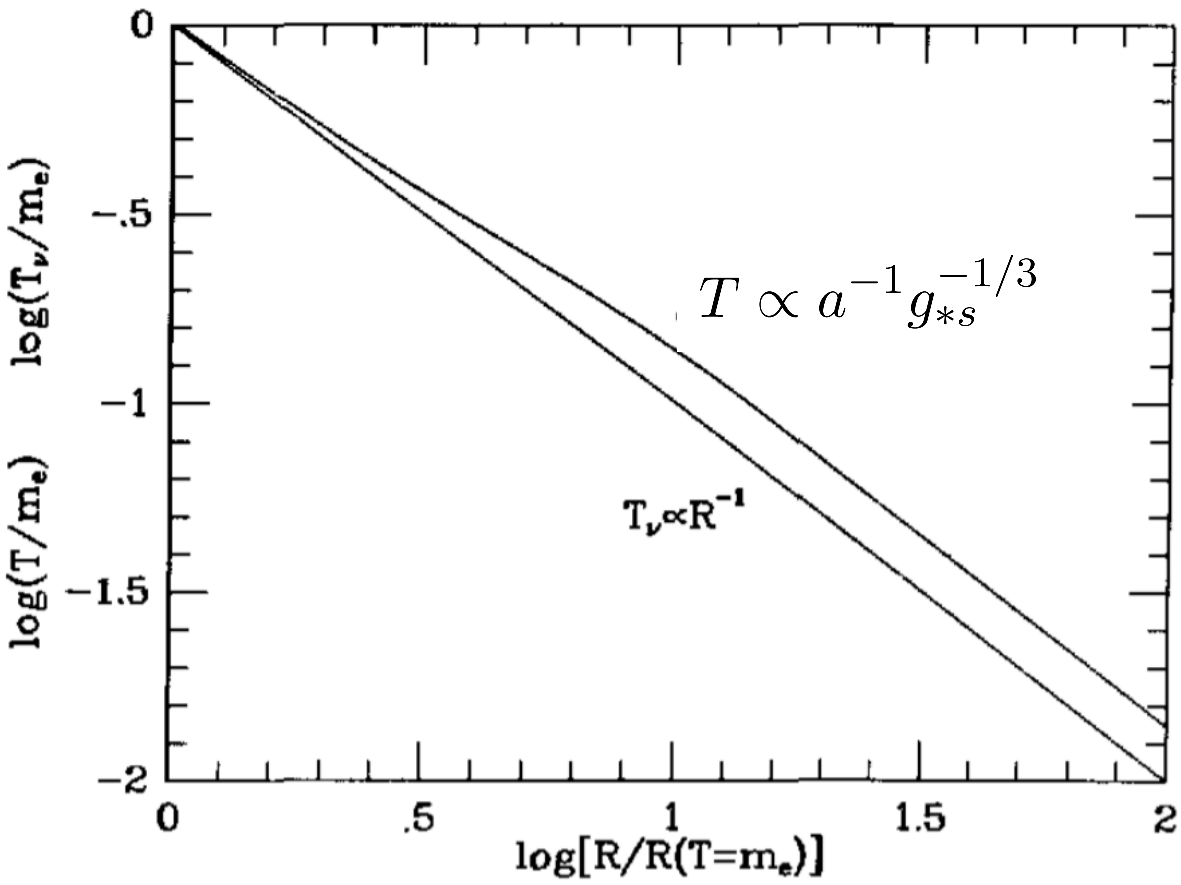
$$(sa)^3|_{\text{before}} = s(a^3)|_{\text{after}} \longrightarrow \boxed{\frac{T_{\text{after}}}{T_{\text{before}}} = \left(\frac{g_{*s\text{before}}}{g_{*s\text{after}}}\right)^{1/3} = \left(\frac{11}{4}\right)^{1/3}}$$

N.B. : **before *e^+- e^- annihilation*** *the relativistic particles in thermal equilibrium are: photons (g=2),* **e^+- e^-** *(g=4) – for a value of* ***$g_{*s\ before}$ = 11/2;***
after e^+- e^- annihilation *only photons giving* ***$g_{*s\ after}$ = 2***

The Cosmic Neutrino Background (CNB)

Let us (briefly) review how to obtain the previous numbers:

CNB Temperature today and abundance



The Cosmic Neutrino Background (CNB)

Let us (briefly) review how to obtain the previous numbers:
CNB Temperature today and abundance

- Being decoupled, neutrinos do not partecipate to such a transfer of entropy, and since their temperature (T_ν) continue to scale as a^{-1} we get today

$$T_\nu = (4/11)^{1/3} T_\gamma \rightarrow \quad T_\nu = 1.946 \text{ K}$$

- In the approximations of relativistc particles $n = \begin{cases} (\zeta(3)/\pi^2) g T^3 & \text{(BOSE)} \\ (3/4)(\zeta(3)/\pi^2) g T^3 & \text{(FERMI)} \end{cases}$

and therefore

$$\frac{n_{\nu_\alpha} + n_{\bar{\nu}_\alpha}}{n_\gamma} = \frac{3}{11} \longrightarrow \quad n_\nu = \sum_\alpha n_{\nu_\alpha} \approx 336 \text{ cm}^{-3}$$

Neutrino isocurvature modes

- Isocurvature (or entropic) perturbation modes are defined as (initial) difference in the relative perturbations of the different species

$$S_{XY} = -3\left(H\frac{\delta\rho_X}{\dot{\rho}_X} - H\frac{\delta\rho_Y}{\dot{\rho}_Y} \right)$$

- Therefore for N species, you will have N-1 isocurvature modes, and 1 adiabatic (or curvature) mode.
  Here X,Y= DM, Baryons, photons, and ***neutrinos***

- These different perturbation modes can be correlated (or uncorrelated

- Similar defintions hold for velocity isocurvature perturbations.

Neutrino isocurvature modes

- Equivalently: photons are taken as a reference species.
  Then, adiabaticity means that

$$\delta\left(\frac{n_i}{n_\gamma}\right)$$

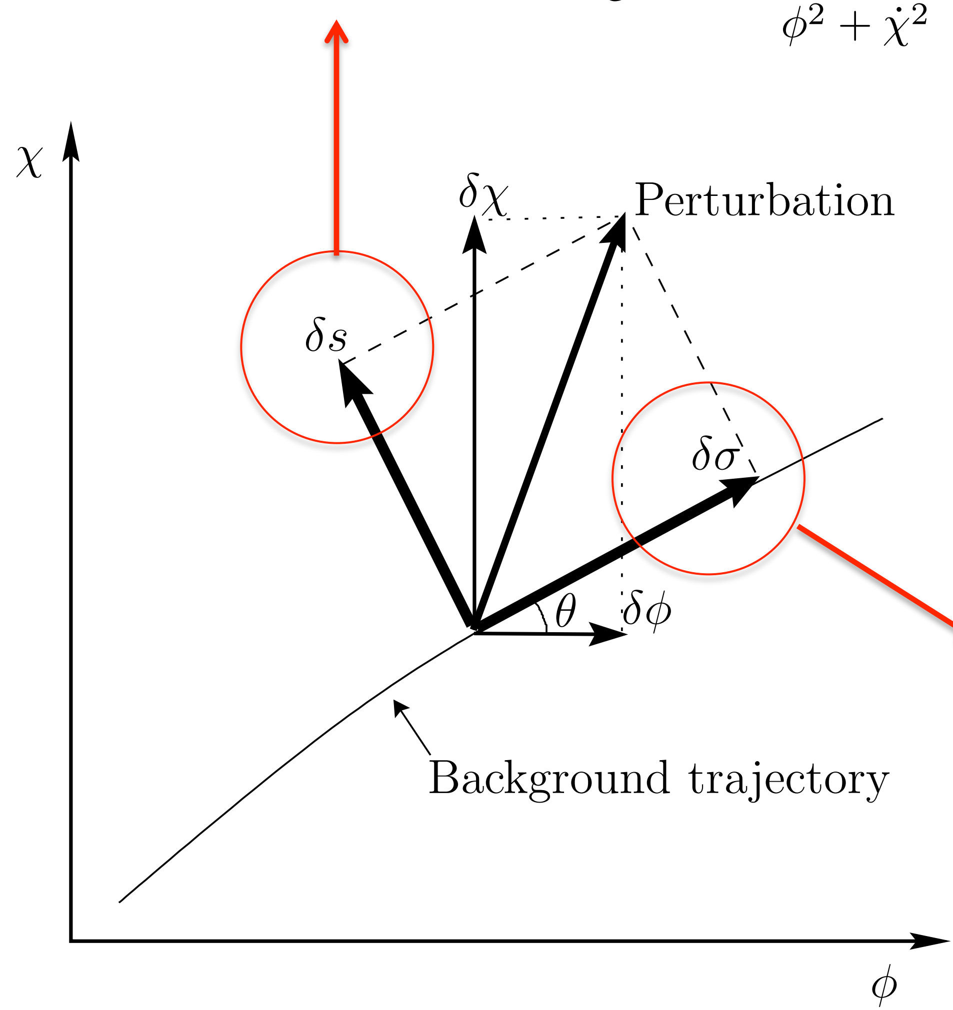
remains spatially constant, otherwise isocur. modes are switched on

- **Single-field models** of inflation (one single degree of freedom) can generate only super-horizon adiabatic perturbations

- **Multi-field models of inflation** can generate isocurvature modes (N.B.: can......they do not necessarily prdoduce them)

Isocurvature modes *during* inflation

Isocurvature mode $\mathcal{S} = \dfrac{H\left(\dot{\phi}\delta\chi - \dot{\chi}\delta\phi\right)}{\dot{\phi}^2 + \dot{\chi}^2}$



N.B.: if the trajectory in field space is a curved one (i.e. ϑ is not costant) then a cross correaltion between curvature and isocurvature pertbs. is generated: <R S> ≠0

Adiabatic mode $\mathcal{R} \simeq \dfrac{H\left(\dot{\phi}\delta\phi + \dot{\chi}\delta\chi\right)}{\dot{\phi}^2 + \dot{\chi}^2}$

Isocurvature modes *after* inflation

- The post-inflationary evolution determines how the isocurvature fluctuations generated during inflation transforms into specific isocurvature modes

On large (super-horizon) scales

$$\dot{\mathcal{R}} = \alpha H \mathcal{S}\,, \qquad \dot{\mathcal{S}} = \beta H \mathcal{S}$$

After inflation $\begin{pmatrix} \mathcal{R} \\ \mathcal{S} \end{pmatrix} = \begin{pmatrix} 1 & T_{\mathcal{RS}} \\ 0 & T_{\mathcal{SS}} \end{pmatrix} \begin{pmatrix} \mathcal{R} \\ \mathcal{S} \end{pmatrix}_*$ ***From inflation***

$$T_{\mathcal{RS}}(t_*, t) = \int_{t_*}^{t} \alpha(t') T_{\mathcal{SS}}(t_*, t') H(t') dt'\,,$$

Transfer functions

$$T_{\mathcal{SS}}(t_*, t) = \exp\left(\int_{t_*}^{t} \beta(t') H(t') dt' \right)\,.$$

An observational test of twofield inflation

- In particular for two field models of inflation

$$\boxed{\frac{\mathcal{P}_T}{\mathcal{P}_\mathcal{R}} = -8n_T(1-\cos^2\Delta)}$$

$$\begin{aligned}
\mathcal{P}_\mathcal{R} &= \left(1+T_{\mathcal{RS}}^2\right)\mathcal{P}_\mathcal{R}|_* \\
\mathcal{P}_\mathcal{S} &= T_{\mathcal{SS}}^2\mathcal{P}_\mathcal{R}|_*\,, \\
\mathcal{C}_\mathcal{RS} &= T_\mathcal{RS}T_\mathcal{SS}\mathcal{P}_\mathcal{R}|_*\,.
\end{aligned}$$

Cross-correlationadiabatic & socurvature modes $\quad \cos\Delta \equiv \frac{\mathcal{C}_\mathcal{RS}}{\mathcal{P}_\mathcal{R}^{1/2}\mathcal{P}_\mathcal{S}^{1/2}} \simeq \frac{T_\mathcal{RS}}{\sqrt{1+T_\mathcal{RS}^2}}$

See N.B., S. Matarrese & A. Riotto 2022; D. Wand, N. B., S . Maatarrese & A .Riotto 2003

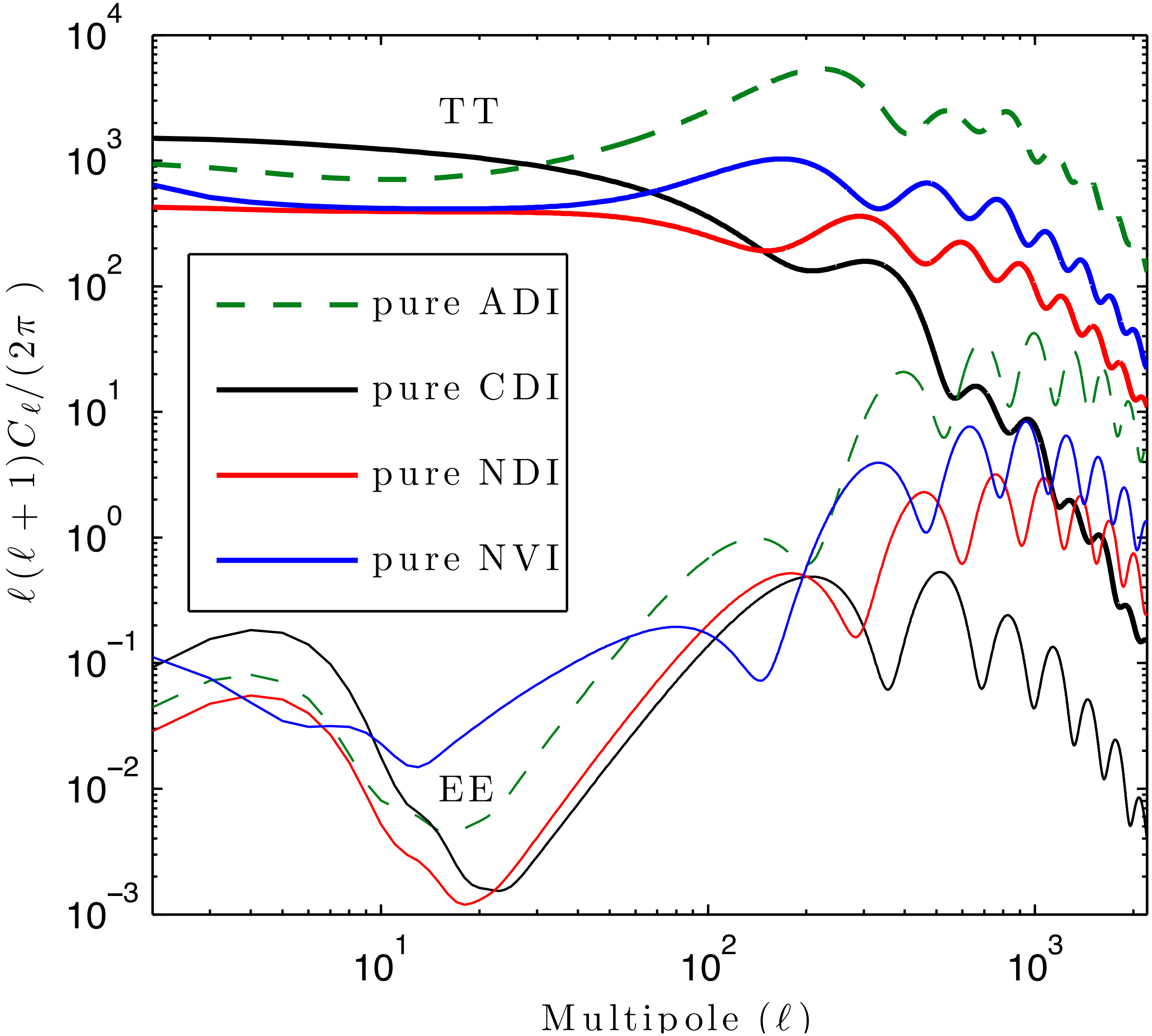
Neutrino isocurvature modes

- How these isocurvature modes can be set, initially, after inflation (at energies as high as 10^{15} GeV) is strongly model dependent:

  It depends on how inflation ends through the rehetaing phase: couplings of inflatons with other particles, inflaton decay channels.
  Thermalization processes in general tend to damp any isocurvature modes present during inflation

- ***Take-home message:***
  challenging but can provide a unique inside into couplings at energies never achievable in labs.

Neutrino isocurvature modes



Neutrino isocurvature modes

Setting the dictionary:

- one can assume simple **power-law power-spectra** (generally predicted by Inflation)

$$\mathcal{P}_{\mathcal{R}}(k) = A_{\mathcal{R}} \left(\frac{k}{k_0}\right)^{n_{\mathcal{R}}} \qquad \mathcal{P}_{\mathcal{I}}(k) = A_{\mathcal{I}} \left(\frac{k}{k_0}\right)^{n_{\mathcal{I}}}$$

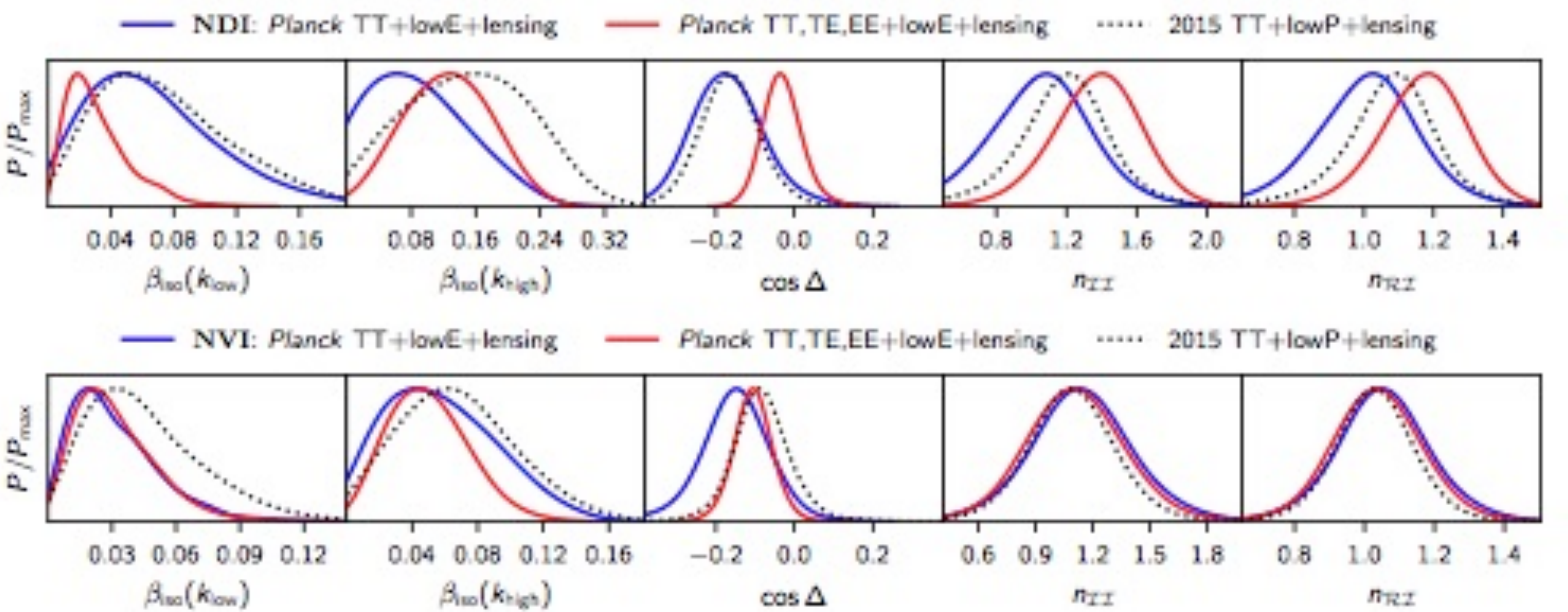
$$\langle \mathcal{R}(\mathbf{k}_1)\mathcal{S}(\mathbf{k}_2)\rangle = (2\pi)^3 \mathcal{P}_{\mathcal{R}\mathcal{I}}(k)\, \delta^{(3)}(\mathbf{k}_1 - \mathbf{k}_2) \qquad \mathcal{P}_{\mathcal{R}\mathcal{I}}(k) = A_{\mathcal{R}\mathcal{I}} \left(\frac{k}{k_0}\right)^{n_{\mathcal{R}\mathcal{I}}}$$

cross-correlation between adiabatic and isoc. perturbtions

- **cross-correlation coefficients and isocurvature fraction:**

$$\cos\Delta = \frac{\mathcal{P}_{\mathcal{R}\mathcal{I}}(k)}{\mathcal{P}_{\mathcal{R}}^{1/2}(k)\mathcal{P}_{\mathcal{I}}^{1/2}(k)} \qquad \beta_{\rm iso}(k) = \frac{\mathcal{P}_{\mathcal{I}}(k)}{\mathcal{P}_{\mathcal{R}}(k) + \mathcal{P}_{\mathcal{I}}(k)}$$

Neutrino isocurvature modes



At large scales β_{iso} < 7.4% for NDI and 6.8% for NVI

Overall the the non-adiabatic fraction is below 1.7 % with Planck TT,TE,EE+lowE+ lensing data for all three cases.

Neutrino isocurvature modes

- Interestingly, neutrino density isocurvature perturbations allow to probe the lepton number in the case of a non-vanishing chemical potential of neutrinos

$$S_{\nu\gamma} = -3H\left(\frac{\delta\rho_\nu}{\dot{\rho}_\nu} - \frac{\delta\rho_\gamma}{\dot{\rho}_\gamma}\right)$$

$$\Big\updownarrow$$

$$f_i(E) = [\exp(E/T_\nu \mp \xi_i)]^{-1}$$

Non-vanishing chemical potential

$$\Big\downarrow$$

$$n_{\rm L} = n_\nu - n_{\bar{\nu}} \neq 0$$

Neutrino isocurvature modes

- Interestingly, neutrino density iscurvature perturbations allow to probe the lepton number in the case of a non-vanishing chemical potential of neutrinos

$$f_i(E) = [\exp(E/T_\nu \mp \xi_i)]^{-1}$$

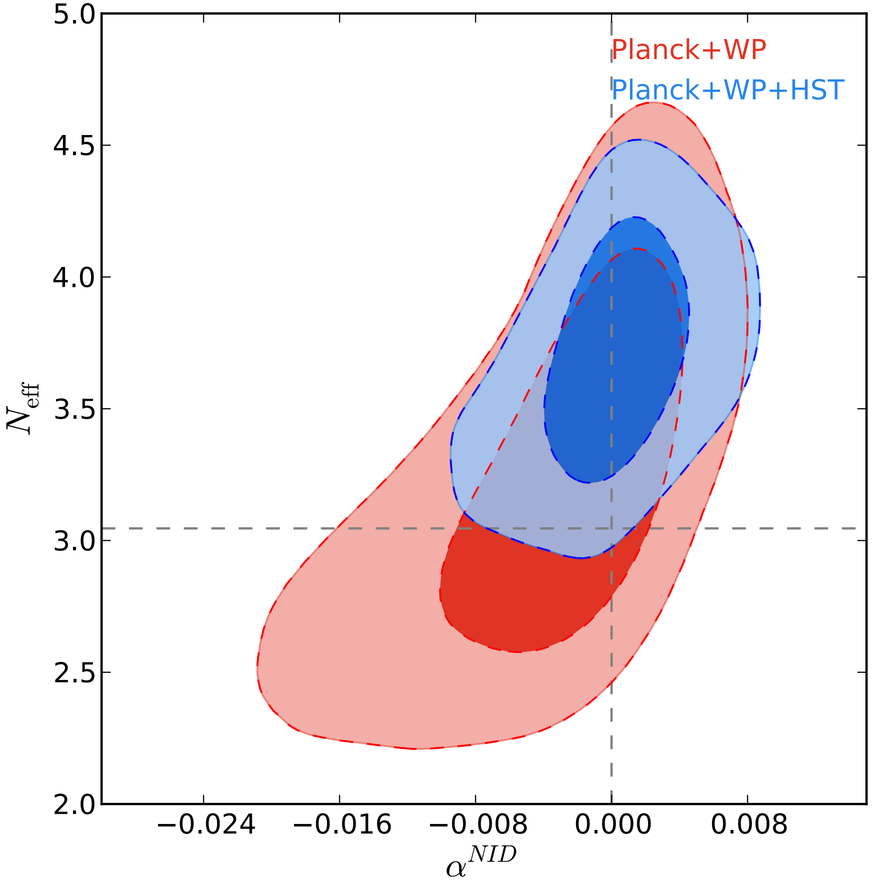
Non-vanishing chemical potential

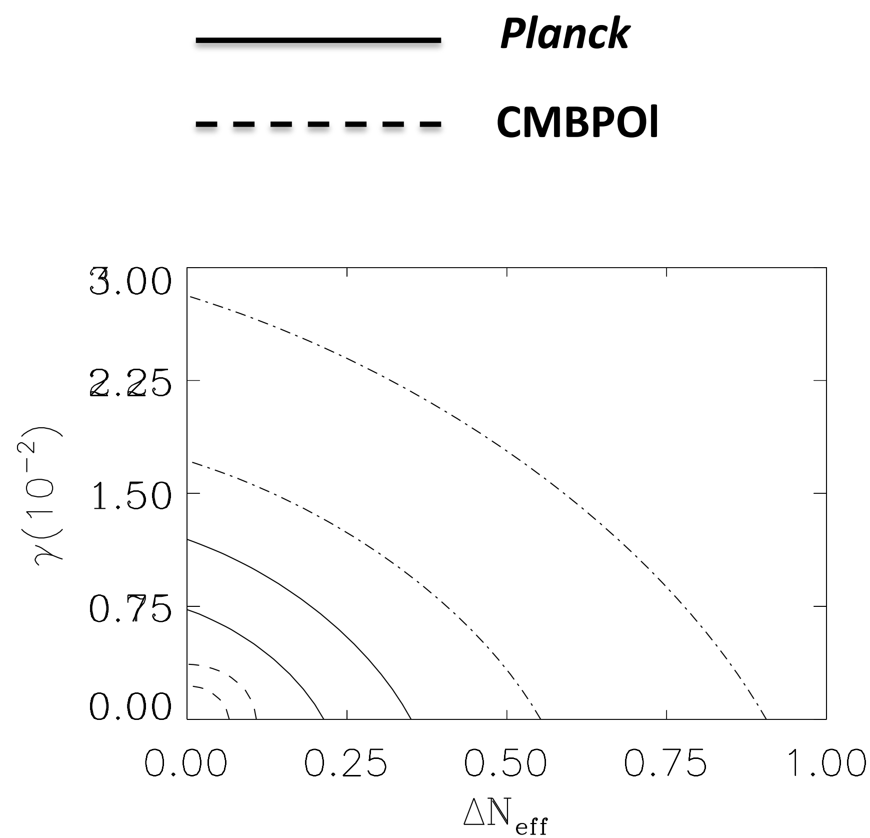
- $$\rho_{\nu_i} + \rho_{\bar{\nu}_i} = \frac{7\pi^2}{120} A_i T_\nu^4 = \frac{7}{8} A_i \left(\frac{T_\nu}{T_\gamma}\right)^4 \rho_\gamma \qquad A_i \equiv \left[1 + \frac{30}{7}\left(\frac{\xi_i}{\pi}\right)^2 + \frac{15}{7}\left(\frac{\xi_i}{\pi}\right)^4\right]$$

$$\rho_r = (1 + N_{eff}\frac{7}{8}\left(\frac{4}{11}\right)^{4/3})\rho_\gamma \Longrightarrow S_{\nu\gamma} \simeq \frac{\sum_i \delta N_{\mathrm{eff}}^{(i)}}{4N_{\mathrm{eff}}}$$

$$N_{\mathrm{eff}} = \sum_i A_i$$

Neutrino isocurvature modes





Here $\gamma = \sqrt{\beta_{ISO}}$

See, e.g. Di Valentino, Lattanzi, Mangano et al. 2012

Neutrinos and inflationary GWs

- You have already seen how neutrinos impact CMB and LSS
- Precision (data-driven) cosmology requires precise theoretical predictions
- Free-streaming neutrinos (and in general any decoupled relativistic species) do influence the propagation of cosmological gravitatonal waves, once the latter are produced from inflation.

Neutrinos and inflationary GWs

Take home messagge:

- *anisotropic stress of neutrinos damp the squared amplitude of primordial (inflationary) GWs by 35.6% for wavelengths that enter the horizon long-after neutrino decoupling during the radiation-dominated era*

- *the suppression is 9% for wavelengths that enter the horizon during the matter-dominated epoch*

Neutrinos and inflationary GWs

Main References:

- E.T. Vishniac, ApJ, 257, 456 (1982).
- M. Kasai and K. Tomita, Phys. Rev. D33, 1576 (1985).
- A.K. Rebhan and D.J. Schwarz, Phys. Rev. D50, 2541(1994)
- J.R. Bond, in Cosmology and Large-Scale structures, Les Houches Section IX, 1996
- **S. Weinberg PRD 69, 023503 (2004)**
- **Y. Watanabe, E, Komatsu PRD 73, 123515 (2006)**

The bold ones are the Refs. on which these notes are based on

See also:

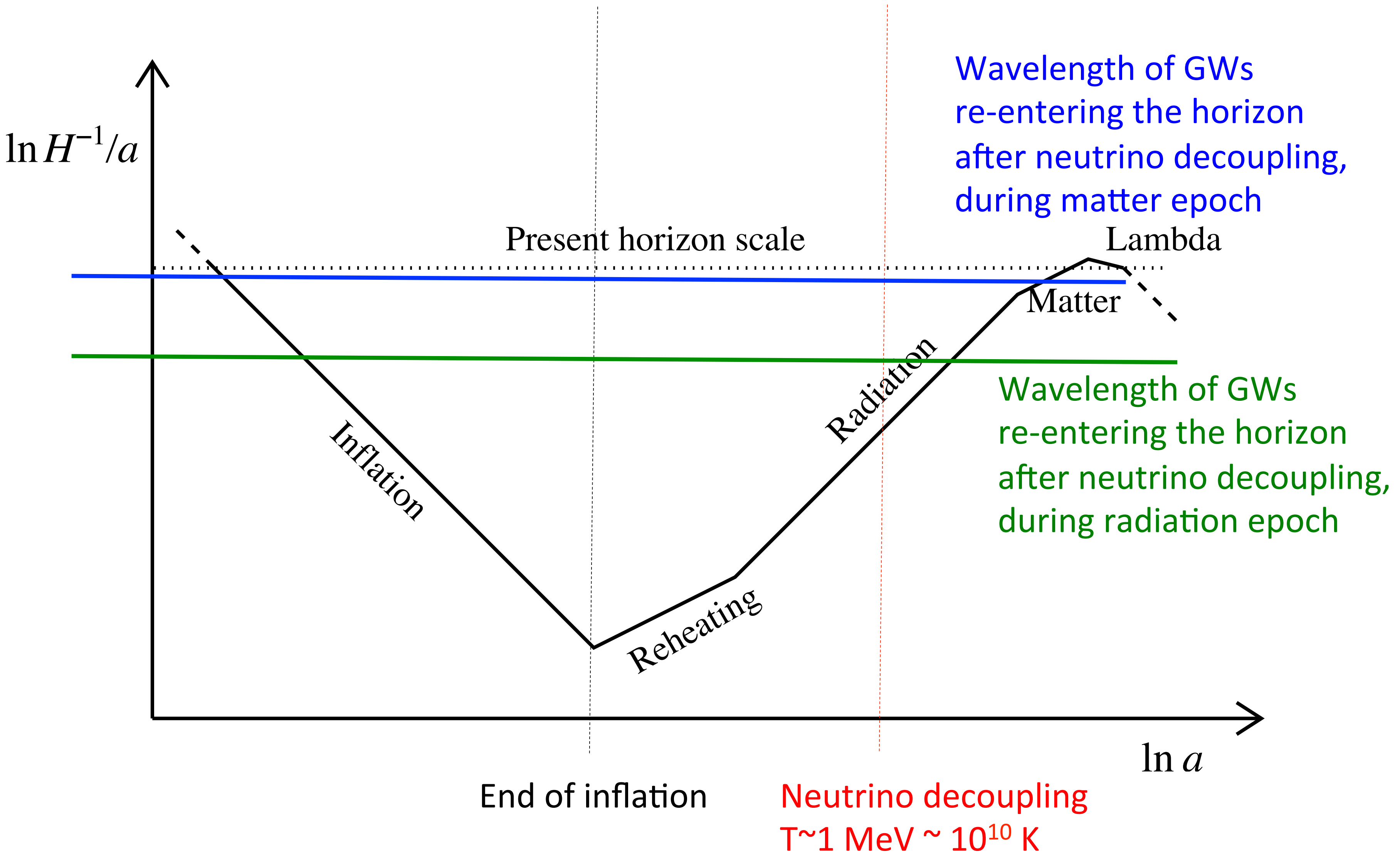
- N. Bartolo, A Mangilli, S. Matarrese, A Riotto,

for an extension to second-order in the perturbations

- M. Zarei, N. Bartolo et al.

for a derivation of the same results with a completely different way (using the quantum Boltzmann equation and for an extnsion to polarization GWs).

Neutrinos and inflationary GWs



Neutrinos and inflationary GWs

$$\ddot{h}_{ij} + \left(\frac{3\dot{a}}{a}\right)\dot{h}_{ij} - \left(\frac{\nabla^2}{a^2}\right)h_{ij} = 16\pi G \pi_{ij}$$

Transverse & traceless part of the (spatial part of) anisotropic-stress tensor of neutrinos

$\pi_{ij} = \pi_{ji} \quad \pi^i_{\ i} = 0 = \partial^i \pi_{ij}$

$$T_{\mu\nu} = \underbrace{\rho u_\mu u_\nu + p\mathcal{H}_{\mu\nu}}_{\text{Perfect fluid}} + \underbrace{\pi_{\mu\nu}}_{\text{Anisotropic stress}}$$

where: $\mathcal{H}_{\mu\nu} = g_{\mu\nu} + u_\mu u_\nu$ and the quadri-velocity of the fluid normalized as $u_\mu u^\mu = -1$

Neutrinos and inflationary GWs

Main reasoning

1. We know that the energy-momentum tensor $T_{\mu\nu}$ can be given in terms of the distribution function

$$T_{ij}^{(\nu)} = \frac{1}{\sqrt{-g}} \int \frac{d^3q}{q^0} q_i q_j F(q)$$

2. π_{ij} can be computed in terms of the distribution function of neutrinos

On its turn the distribution function of neutrinos depend on h_{ij}

3. On gets an integro-differential equation w.r.t. to time for h_{ij}

Neutrinos and inflationary GWs

A sketch of the main steps

- Boltzmann equation for *free-streaming* neutrinos (Vlasov equation):

$$\frac{dF(t,x^i,\gamma^i,P^0)}{dt} = \frac{\partial F}{\partial t} + \frac{dx^i}{dt}\frac{\partial F}{\partial x^i} + \frac{dP^0}{dt}\frac{\partial F}{\partial P^0} + \frac{d\gamma^i}{dt}\frac{\partial F}{\partial \gamma^i} = 0$$

- Perturb around a homogeneous and isotropic Fermi-Dirac distribution function

$$F = \bar{F} + \delta F(t,x^i,\gamma^i,P^0)$$

$$\bar{F}(P^0) = \frac{g_\nu}{e^{P^0/T}+1}$$

Neutrinos and inflationary GWs

Recall the various quantities: a technical note

- ✓ In the previous equations:

  $P^\mu = dx^\mu/d\lambda$ is the quadri-momentum of relativistic neutrinos with $g_{\mu\nu}P^\mu P^\nu = 0.$

- ✓ The metric is (since we are interested in linear tensor perturbations only here):

$$ds^2 \;=\; a^2(\tau)[-d\tau^2 + (\delta_{ij} + h_{ij})dx^i dx^j]$$

- ✓ Therefore $P^0 = \sqrt{g_{ij}P^i P^j}$

- ✓ γ^i is a unit directional vector ($\delta_{ij}\gamma^i\gamma^j = 1$) such that $P^i \equiv C\gamma^i P_0$

- ✓ Finally: $q^\mu = a(t)P^\mu$ with $q = q^0$ comoving momenta

Neutrinos and inflationary GWs

A sketch of the main steps

- Boltzmann equation for *free-streaming* neutrinos (Vlasov equation):

$$\frac{dF(t,x^i,\gamma^i,P^0)}{dt} = \frac{\partial F}{\partial t} + \frac{dx^i}{dt}\frac{\partial F}{\partial x^i} + \frac{dP^0}{dt}\frac{\partial F}{\partial P^0} + \cancel{\frac{d\gamma^i}{dt}\frac{\partial F}{\partial \gamma^i}} = 0$$

second-order in the perturbations

$$\frac{1}{P^0}\frac{dP^0}{dt} = -\frac{\dot{a}}{a} - \frac{1}{2}\frac{\partial h_{ij}}{\partial t}\gamma^i\gamma^j$$ (time-component of geodesic equation)

This equation describes the change in the neutrino energy as it propagates in a FRW universe with GWs.

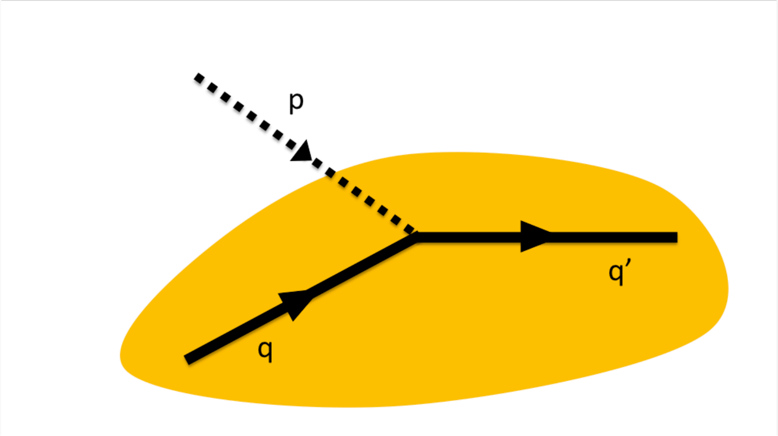
Case1: Neutrinos gain energy from GWs if $\frac{\partial h_{ij}}{\partial t} < 0$

Case2: Neutrinos loose energy if $\frac{\partial h_{ij}}{\partial t} > 0$

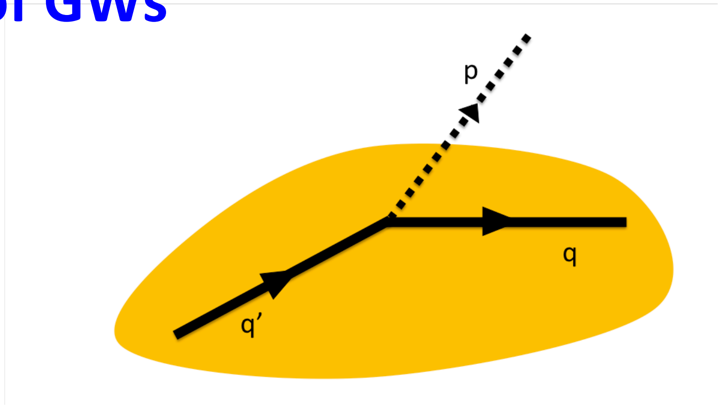
Neutrinos and inflationary GWs

- Therefore there is an energy flow from GWs to neutrinos (and viceversa) :

Case 1: Absorption of GWs (dotted lines) by a bath made of fermions → **Damping of GWs**



Case 2: Emission of GWs (dotted lines) by a bath made of fermions → **Amplification of GWs**



Neutrinos and inflationary GWs

- Going to Fourier space: $$\frac{\partial f_k}{\partial \tau} + ik\mu f_k = q\frac{\partial \bar{F}}{\partial q}\frac{1}{2}\frac{\partial h_k}{\partial \tau}$$

$$f_k(\tau, q, \mu) = e^{-i\mu k(\tau - \tau_{\nu\,\mathrm{dec}})} f_k(\tau_{\nu\mathrm{dec}}, q, \mu) + \frac{q}{2}\frac{\partial \bar{F}}{\partial q}\int_{\tau_{\nu\,\mathrm{dec}}}^{\tau} d\tau' h'_k(\tau') e^{-i\mu k(\tau - \tau')}$$

- Compute the anisotropic stress-tensor:

$$\Pi_k = -4\bar{\rho}_\nu(\tau)\int_{\tau_{\nu\,\mathrm{dec}}}^{\tau} d\tau' \left(\frac{j_2[k(\tau - \tau')]}{k^2(\tau - \tau')^2}\right) h'_k(\tau')$$

N.B.: here $q^\mu = a(\tau)\, P^\mu$ is the comving quadri-momentum

Neutrinos and inflationary GWs

A technical note to do the computations

✓ $$h_{ij}(t,\mathbf{x}) \;=\; \sum_{\lambda=+,\times}\int\frac{d^3k}{(2\pi)^3}h_{\lambda,k}(t)Q^{\lambda}_{ij}(\mathbf{x})$$

$$Q^{\lambda}_{ij|a}{}^{|a}(\mathbf{x}) + k^2 Q^{\lambda}_{ij}(\mathbf{x}) \;=\; 0$$

$$Q^{\lambda}_{ij} \;=\; Q^{\lambda}_{ji},\;\gamma^{ij}Q^{\lambda}_{ij} \;=\; Q^{\lambda}_{ij}{}^{|j} \;=\; 0$$

tensor harmonics: basis satisfying
the tensor Helmholtz equation

$$\gamma^{ij} \;\equiv\; a^2\bar{g}^{ij}$$

e.g.: in spatially flat geometry: $h_{ij}(\tau,\mathbf{x}) = \sum_{\lambda}\int\frac{d^3k}{(2\pi)^3}h_{\lambda}(\tau;\mathbf{k})e^{i\mathbf{k}\cdot\mathbf{x}}\epsilon^{\lambda}_{ij}$

✓ $$\delta F \;=\; \sum_{\lambda=+,\times}\int\frac{d^3k}{(2\pi)^3}f_{\lambda,k}(t,P^0,\mu)\gamma^i\gamma^j Q^{\lambda}_{ij}(\mathbf{x})$$

✓ $$\delta T^{(\nu)}_{ij} = a^2\sum_{\lambda=+,\times}\int\frac{d^3k}{(2\pi)^3}\Pi_{\lambda,k}Q^{\lambda}_{ij}(\mathbf{x})$$

on the other hand $\delta T^{(\nu)}_{ij} \;=\; a^{-4}\int\frac{d^3q}{q^0}\left[\bar{q}_i\bar{q}_j\delta F + (\delta q_i\bar{q}_j + \bar{q}_i\delta q_j)\bar{F}\right]$

$$\Rightarrow\quad \Pi_{\lambda,k}Q^{\lambda}_{ij}(\mathbf{x}) \;=\; a^{-4}\int\frac{d^3q}{q^0}q^2\gamma^i\gamma^j\gamma^l\gamma^m f_{\lambda,k}Q^{\lambda}_{lm}(\mathbf{x})$$

Neutrinos and inflationary GWs

- Obtain an integro-differential equation for h_{ij}:

$$h_k''(\tau) + \left[\frac{2a'(\tau)}{a(\tau)}\right] h_k'(\tau) + k^2 h_k(\tau) = -24 f_\nu(\tau) \left[\frac{a'(\tau)}{a(\tau)}\right]^2 \int_{\tau_{\nu\ \mathrm{dec}}}^{\tau} d\tau' \left[\frac{j_2[k(\tau-\tau')]}{k^2(\tau-\tau')^2}\right] h_k'(\tau')$$

$$f_\nu(\tau) \equiv \frac{\bar\rho_\nu(\tau)}{\bar\rho(\tau)} = \frac{\Omega_\nu (a_0/a)^4}{\Omega_M (a_0/a)^3 + (\Omega_\gamma + \Omega_\nu)(a_0/a)^4} = \frac{f_\nu(0)}{1 + a(\tau)/a_{EQ}}$$

Neutrino fraction

$$f_\nu(0) = \frac{\Omega_\nu}{\Omega_\gamma + \Omega_\nu} = 0.40523$$ (for 3 neutrinos flavours)

Neutrinos and inflationary GWs

- Integro-differential equation for h_{ij}:

$$h_k''(\tau) + \left[\frac{2a'(\tau)}{a(\tau)}\right] h_k'(\tau) + k^2 h_k(\tau) = -24 f_\nu(\tau) \left[\frac{a'(\tau)}{a(\tau)}\right]^2 \int_{\tau_{\nu\ \mathrm{dec}}}^{\tau} d\tau' \left[\frac{j_2[k(\tau-\tau')]}{k^2(\tau-\tau')^2}\right] h_k'(\tau')$$

- Solve for χ(u) where: $h_\lambda(u) \equiv h_\lambda(0)\chi(u)$

with initial conditions: $\chi(0) = 1, \quad \chi'(0) = 0$

Here u=k τ, τ is conformal time.

N.B.: on super-horizon scales (k τ <<1) h remains constant in time; and the r.h.s. vanishes for k τ <<1 → free-streaming neutrinos affects GWs only inside the horizon.

Neutrinos and inflationary GWs

- In absence of neutrinos the solution is

$$h_{ij}(u) = h_{ij}(0)\chi(u)$$

$$\chi(u) = j_0(u) = \frac{\sin(u)}{u} = \frac{\sin(k\tau)}{k\tau}$$

- Let us consider wavelengths that enter the horizon during the radiation era and long-after neutrino decoupling.
  Notice: very deep inside the horizon (k τ >> 1) the source term on the r.h.s. becomes negligible and therefore the solution approaches a homegneous solution

$$\chi(u) \to A\frac{\sin\left(u+\delta\right)}{u}$$

Neutrinos and inflationary GWs

- For wavelengths that enter the horizon during the radiation era and long-after neutrino decoupling

$$\chi(u) \to A\frac{\sin(u+\delta)}{u}$$

DAMPING EFFECT

with

$$A = 0.80313 \text{ and } \delta = 0$$

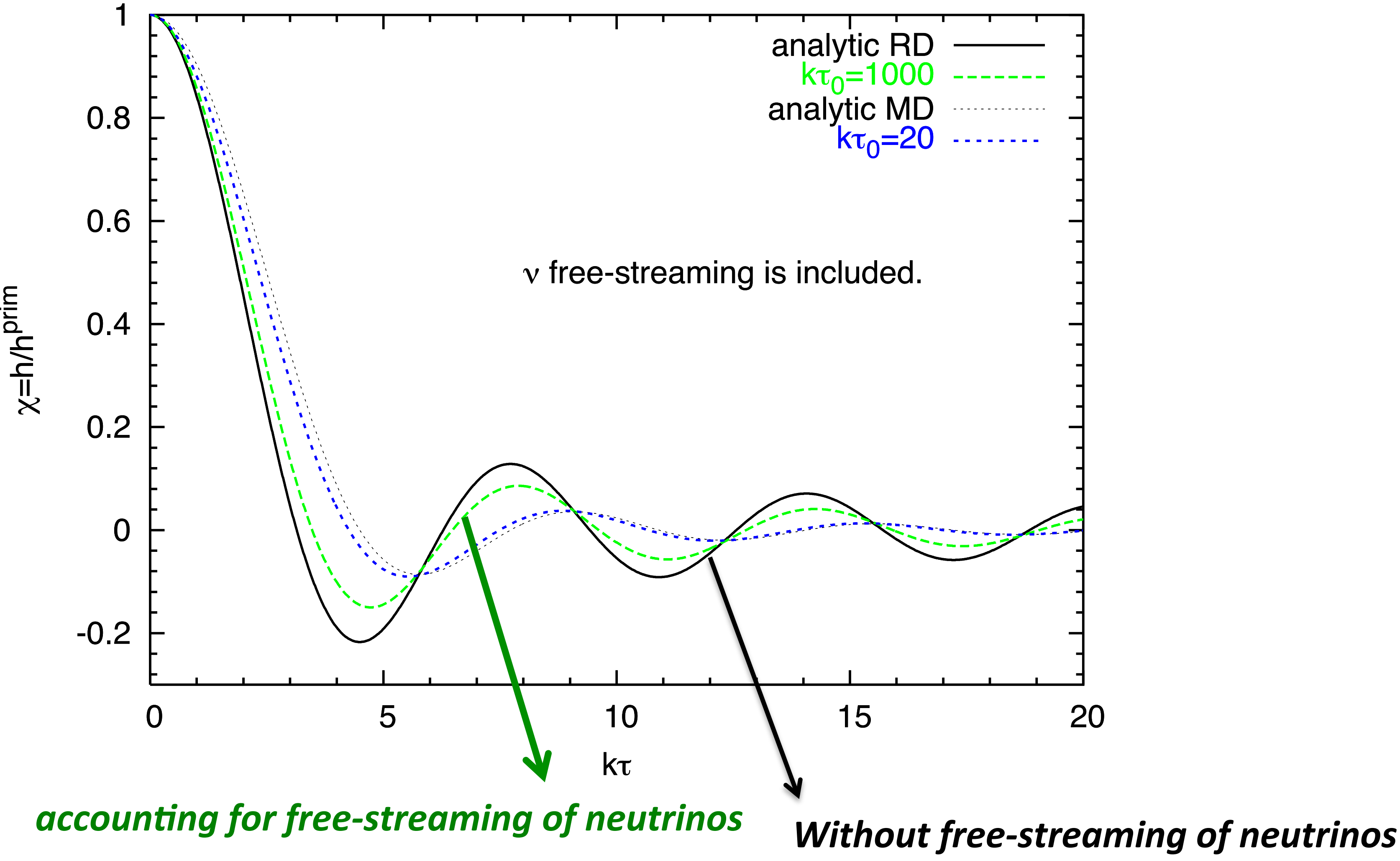
- (Heuristic) explanation:

  In absence of neutrinos the solution is $h_{ij}(u) = h_{ij}(0)\chi(u)$

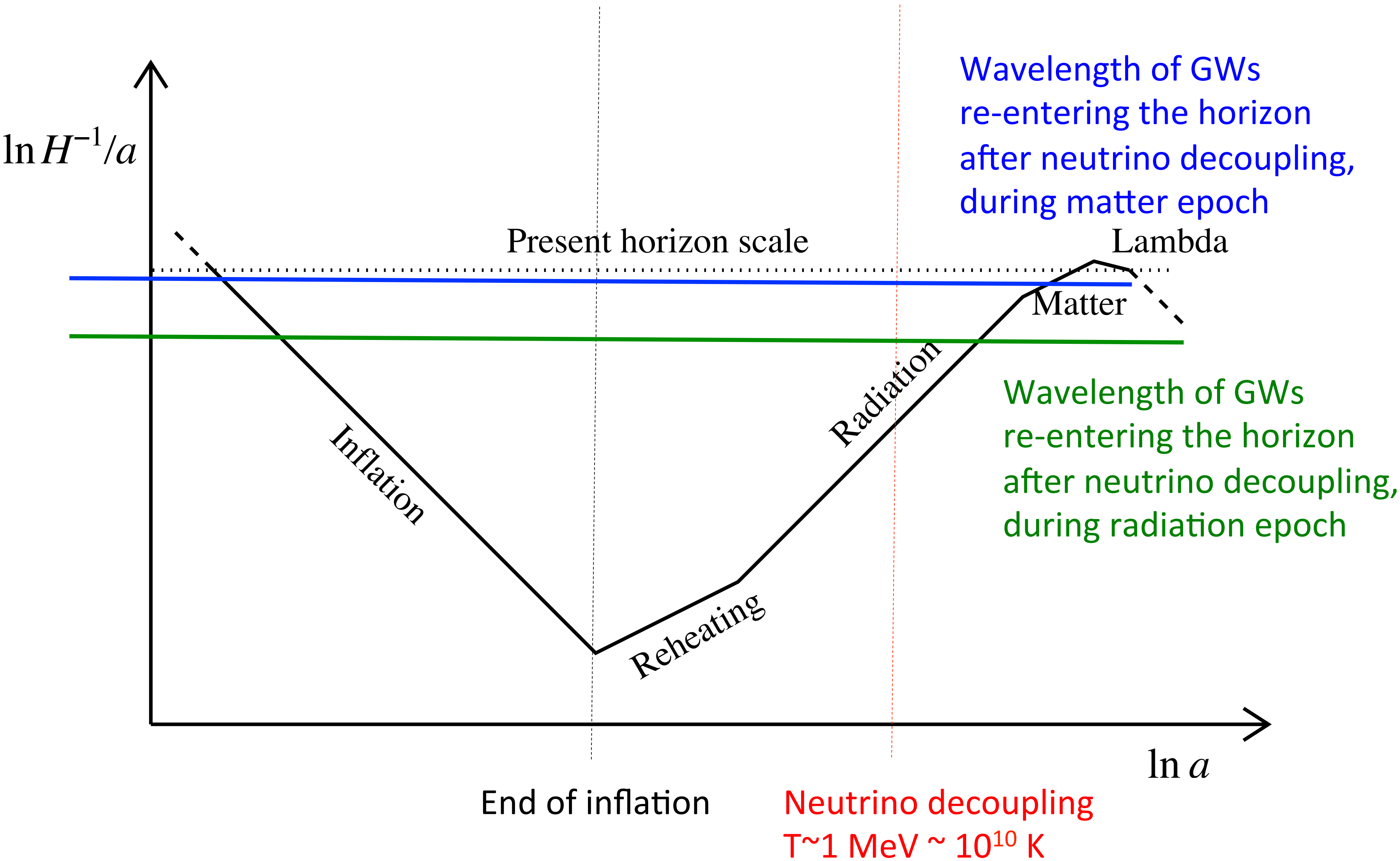
$$\chi(u) = j_0(u) = \frac{\sin(u)}{u} = \frac{\sin(k\tau)}{k\tau}$$

So on average h_{ij} (at leading-order) decreases in time→ neutrinos gain energy from GWs, and GWs are damped

Neutrinos and inflationary GWs

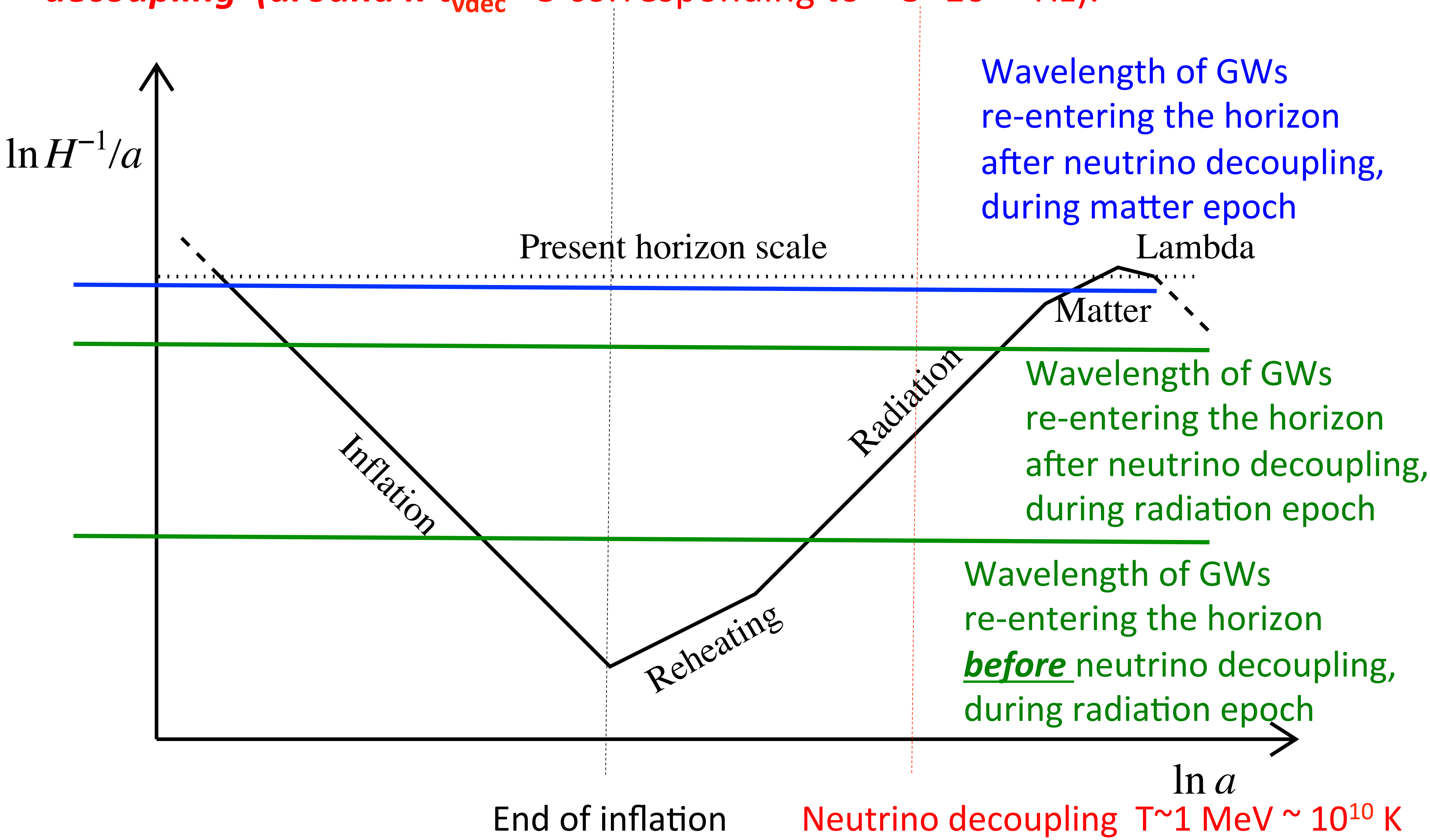


Neutrinos and inflationary GWs



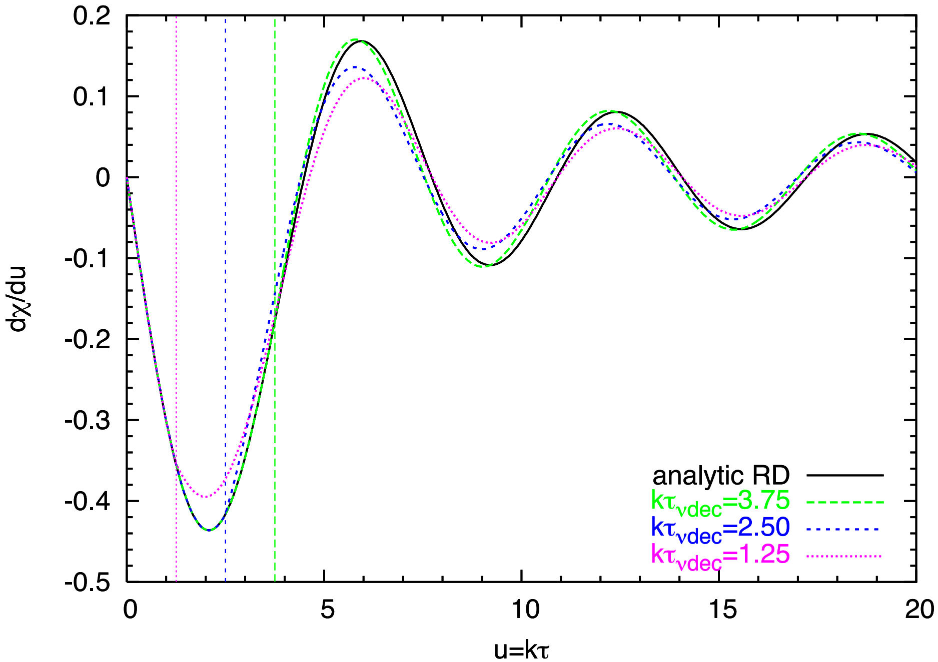
Neutrinos and inflationary GWs

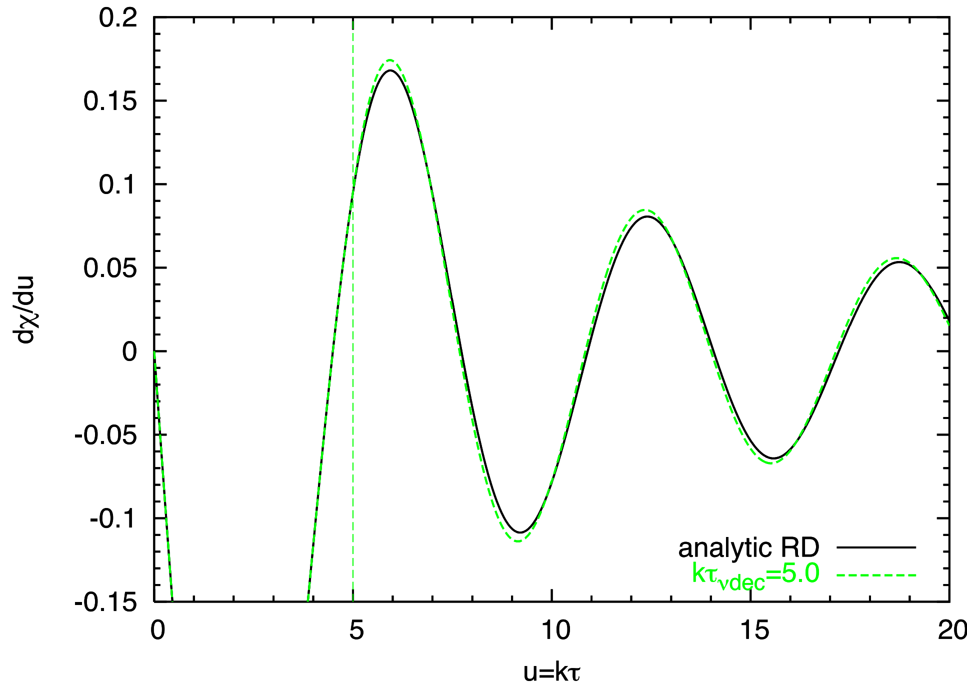
- However, there might be frequencies k for which GWs can be enhnaced. This happens for some wavelengths that enter the horizon ***before neutrinos decoupling (around k*** $\tau_{\nu dec}$ =5 corresponding to ~ 5×10^{-10} Hz).



Neutrinos and inflationary GWs

- However, there might be frequencies k for which GWs can be enhnaced. This happens for some wavelengths that enter the horizon ***before neutrinos decoupling*** ***(around*** $\boldsymbol{k\,\tau_{\nu dec}}$ =5 corresponding to ~ $5{\times}10^{-10}$ Hz).

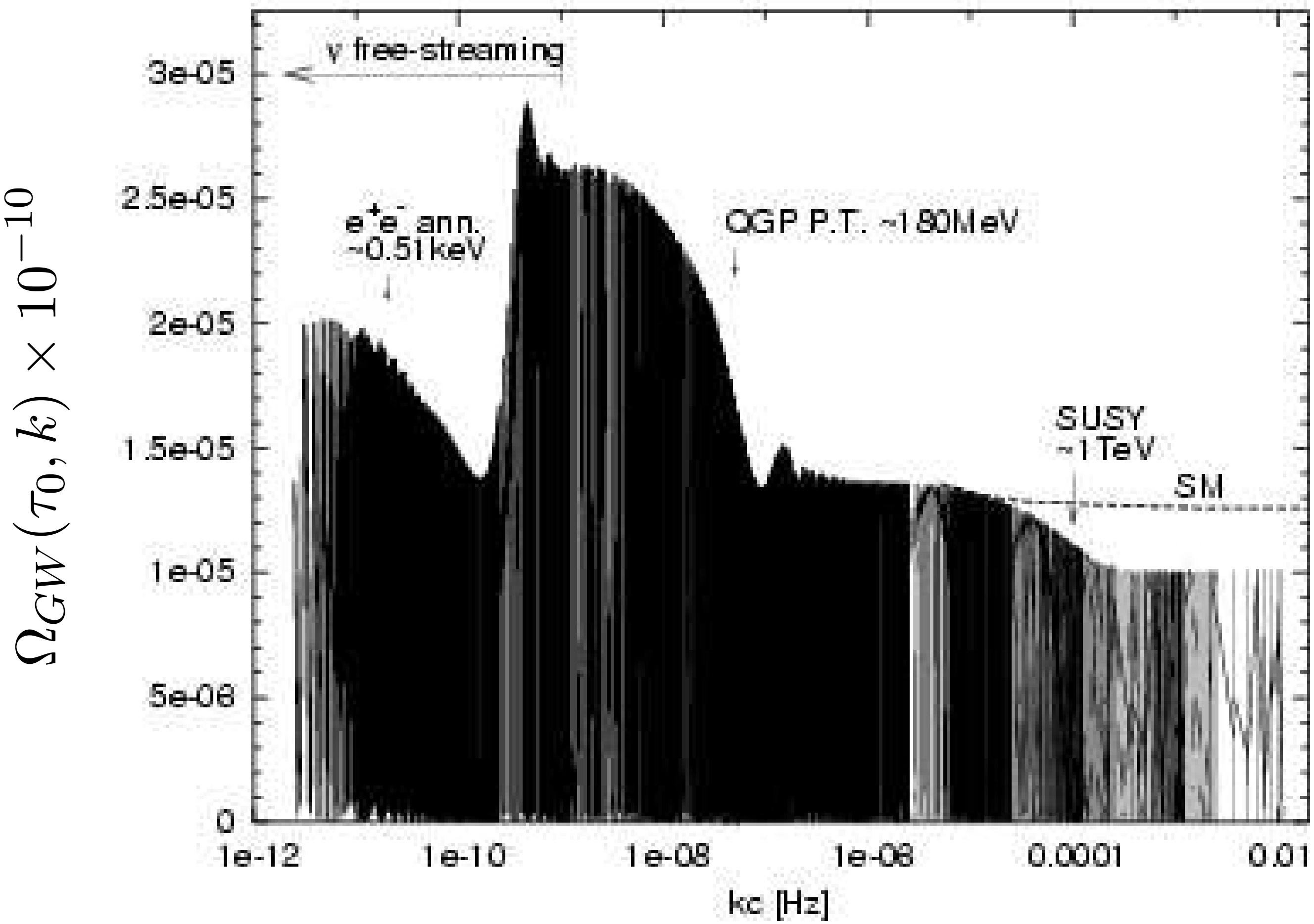




N.B.: For extremely short wavelength modes which have already been inside the horizon before neutrino decoupling, $k\tau_{\nu dec} \gg 1$ or $k > 10^{-9}$ Hz, the suppression becomes negligible: these modes are undamped as positive and negative contributions of χ' to the gravitational wave energy cancel out each other after several periods of χ'.

Neutrinos and inflationary GWs

- Therefore free-streaming neutrinos can damp or amplify the inflationary GWs depending on frequencies



Inflation and neutrinos: further connections

Impact of neutrinos on inflationary parameters

- Extended models w.r.t to the standard ΛCDM cosmological model can have also an impact on (the constraints of) primordial, inflationary parameters (i.e. A_s, n_s and the tensor-to-scalar perturbation ratio r).

Take home message

- ***In particular non-standard properties of neutrinos can in principle allow some inflationary models currently excluded by data to be still compatible with present CMB & LSS constraints.***

- This happens indeed because of some degeneracies between the neutrino-extended parameters and the inflationary parameters (e.g. between N_{eff} and n_s, see e.g. Gerbino et al. PRD 9, 2017).

Impact of neutrinos on inflationary parameters

- Another example has to do with ***self-interactions of massive neutrinos***

For example (Majoron models)**

$$\mathcal{L}_{\rm int} = g_{ij}\bar{\nu_i}\nu_j\Phi + h_{ij}\bar{\nu_i}\gamma_5\nu_j\Phi$$

$\downarrow$

If $m_\Phi > 1$ keV effective 4-fermion interactions $\nu\nu \to \nu\nu$

$$\Gamma \sim g^4 T_\nu^5/m_\phi^4 = G_{\rm eff}^2 T_\nu^5$$

$\downarrow$

Neutrinos decouple from the primordial plasma as usual (at T~1 MeV) but then they continue to scatter with themseleves (if $G_{\rm eff} > G_W$) until $\Gamma < H$. After they free-stream.

** e.g. Barenboim et al. '14; '19; Choudhury et al. '22

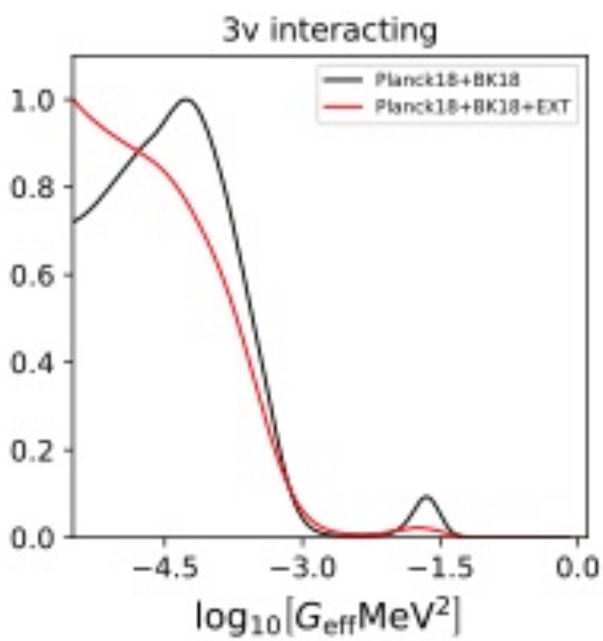
Impact of neutrinos on inflationary parameters

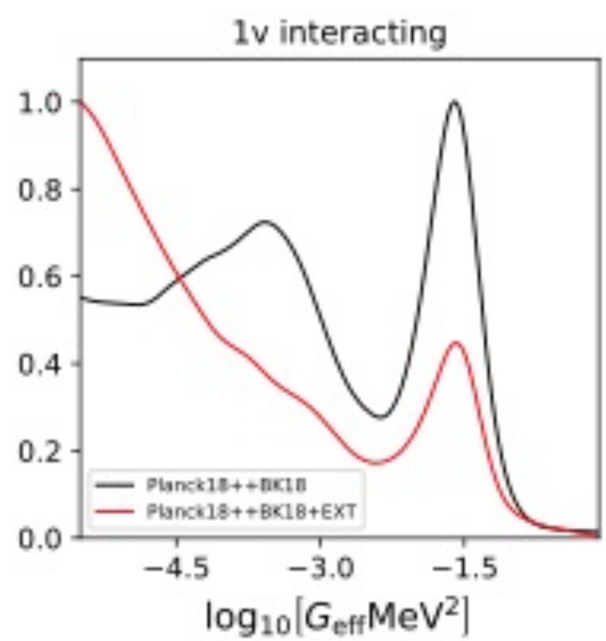
- In this specific scenario increasing G_{eff} brings to an increase of the CMB small-scale power, which can be compensated by decreasing n_s →

degeneracy between G_{eff} & n_s

Neutrino sector

Inflation sector

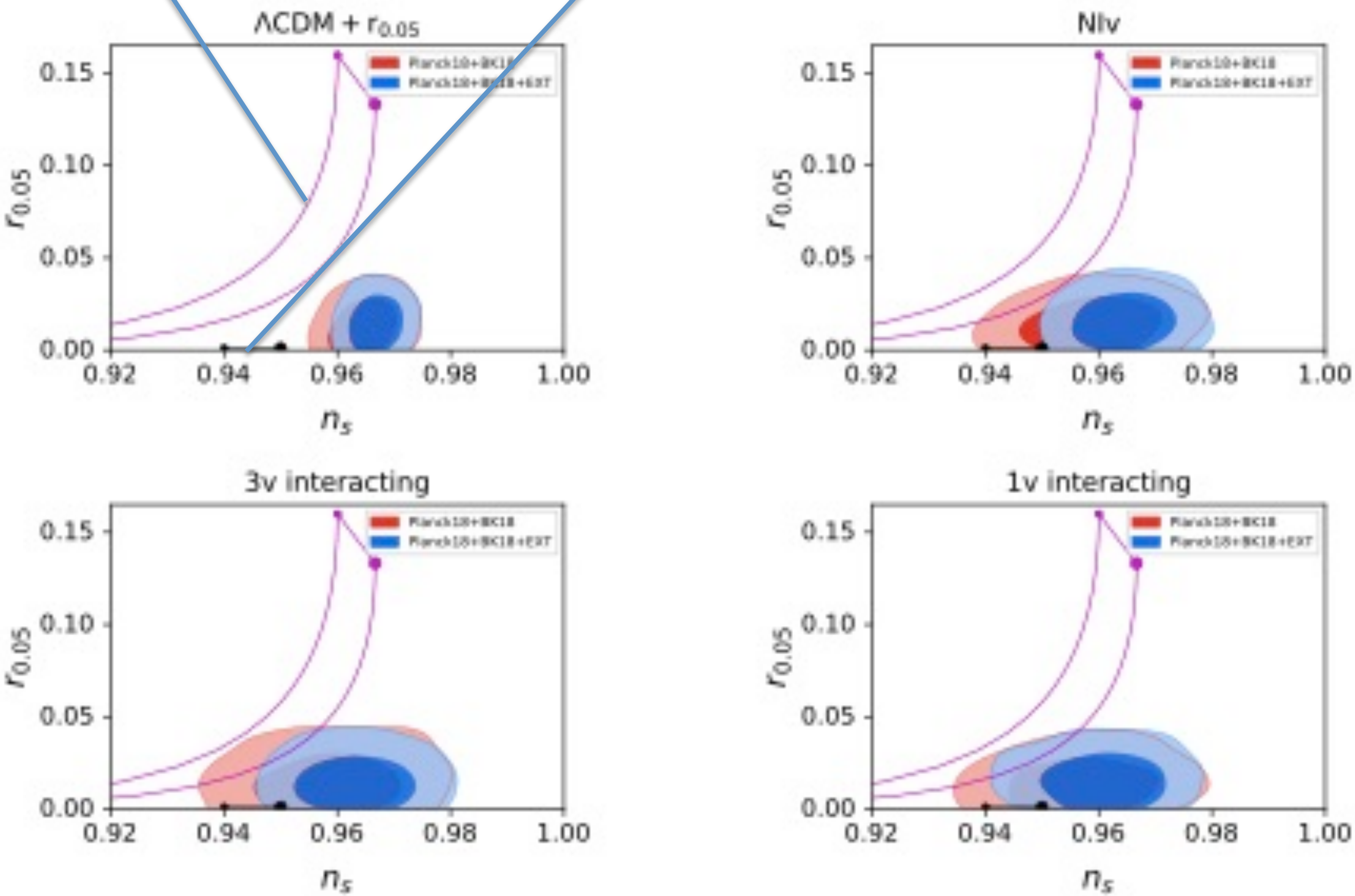




Choudhury et al. '22

Impact of neutrinos on inflationary parameters

$$V_{\rm NI}(\phi) = \lambda^4 \left(1 + \cos\left(\frac{\phi}{g}\right)\right) \qquad V_{\rm CWI}(\phi) = A\phi^4 \left[\ln\left(\frac{\phi}{f}\right) - \frac{1}{4}\right] + \frac{Af^4}{4}$$



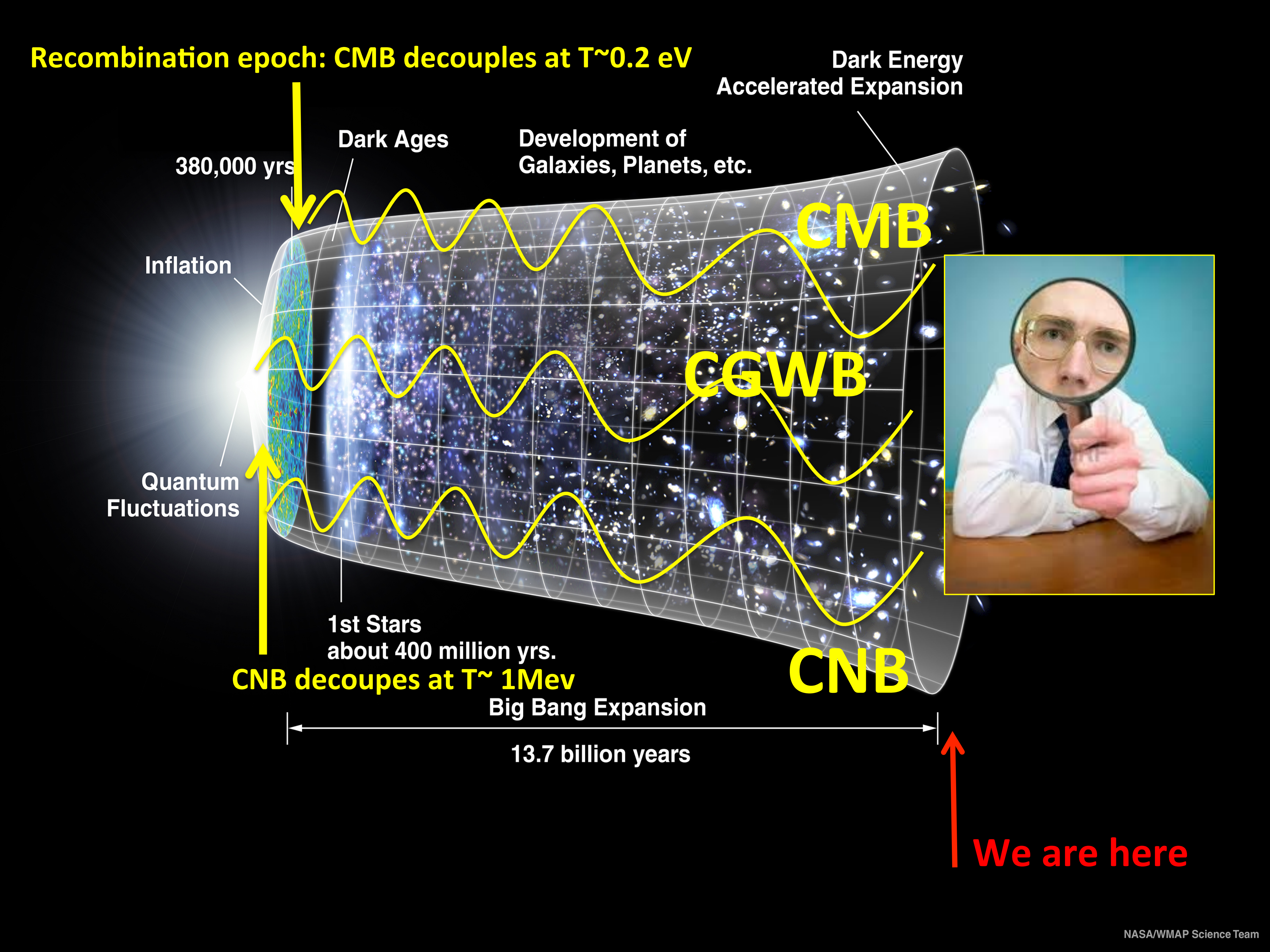
Choudhury et al. '22

The Cosmic Neutrino Background (CNB)

- ***Perhaps one of the most important not yet directly probed prediction of the standard cosmological model***

- Direct detection is extremely challenging because of: the feebleness of the weak interactions and the smallness of relic neutrino energies, diluted by cosmic expansion

- This is why various *indirect* cosmological signatures of neutrinos are central to (cosmologically) constrain their properties (e.g. their effects on CMB and Large-Scale Structures) and most of the more cosmologically-oriented lectures have been dealing with them


Recombination epoch: CMB decouples at T~0.2 eV
Dark Energy
Accelerated Expansion
Dark Ages
Development of
Galaxies, Planets, etc.
380,000 yrs
Inflation
CMB
CGWB
Quantum
Fluctuations
1st Stars
about 400 million yrs.
CNB decoupes at T~ 1Mev
CNB
Big Bang Expansion
13.7 billion years
We are here
NASA/WMAP Science Team

Even more science fiction?

***If yesterady Douglas discussed nicely about science-fiction…..
just wait for what I am going to discuss now…….***

Here I'll focus on a future (better…..futuristic) signature, related to an eventual direct detection of neutrinos: CNB spatial anisotropies.

The CNB: direct detection?

- First let me recall a possibility for direct detection of the CNB:

$$\nu_e + {}^3\mathrm{H} \rightarrow {}^3\mathrm{He} + e^-$$

  proposed by Weinberg in 1962

- See the PTOLEMY proposal for a direct detection experiment
  (Princeton Tritium Observatory for Light, Early-Universe, Massive-Neutrino Yield)

- Very challenging because of a number of reasons:
  - backgrounds to be resolved
  - energy resolution
  - precise computations of neutrino captures
  - quantity of tritium necessary to obtain the detection is an issue

Anisotropies of the CNB

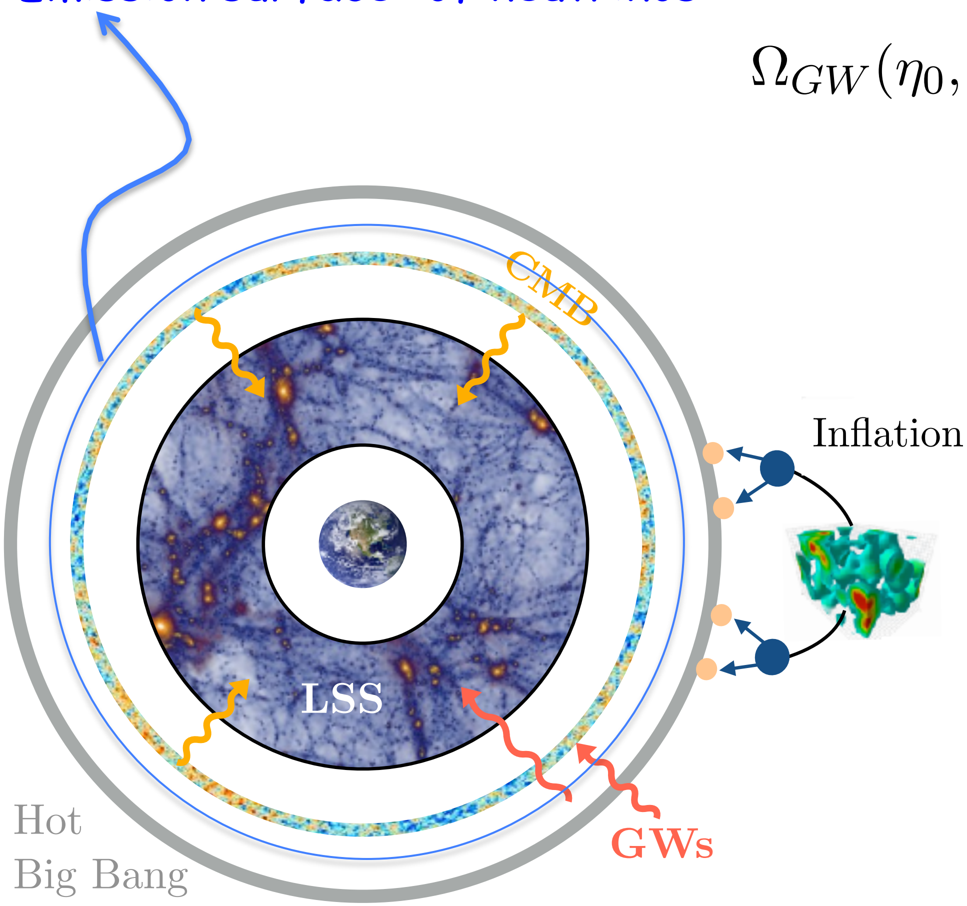
- But le me go to an even more futuristic signature:
  CNB spatial anisotropies
- The idea is very simple: as CMB photons, also the CNB has spatial anisotropies imprinted both at the production time and by its propagation through cosmic inhomogeneities when neutrinos free-stream towards us
- Many studies, for what we said, focused on measuring anisotropies of cosmic neutrinos through their (indirect) effects on CMB and LSS
- Here the approach is different, we are talking about a direct measurement of the CNB temperature anisotropies, in the exactly same way as for CMB temperature anisotropies.

After all, CNB anisotropies are out there........

N.B.: notice by the way that similar analyses hold for a stochastic background of GWs, see., e.g, *N.B., Bertacca, Matarrese, Peloso, Ricciardone, Riotto, Tasinato '19, '20; Valbusa, Ricciardone, N.B. et al. '21; Ricciardone, Valbusa, N.B. et al. PRL '21; Schulze, Valbusa, Lesgourgues, Ricciardone, N.B. et al '23; for previoius works: Alba & Maldacena '16; Contaldi '17.*

Anisotropies of CNB

"Emission surface" of neutrinos:



$$\Omega_{GW}(\eta_0, k, \hat{n}) = \overline{\Omega}_{GW}(\eta_0, k) + \delta\Omega_{GW}(\eta_0, k, \hat{n})$$

Isotropic background

Fluctuations (direction dependent)

Adopt a Boltzmann approach as for CMB:

Two contributions to nutrino anisotropies

1. At production (model dependent)
2. CNB propagation to the observer (model independent)

Anisotropies of the CNB

- Use the ***perturbed*** Boltzmann equations

- There are however some differences w.r.t. to CMB photons
  - neutrinos are massive particles
  - characterized by weak interactions (→ e.g. no equivalent of reionization as for CMB photons)
  - the last scattering surface is mass and momentum dependent
  - neutrinos' oscillations

- Despite these differences the overall pictue is clear:
  CNB anisotropies feature similar effects such as CMB anisotropies, as Sashs-Wolfe and Integrated Sachs-Wolfe effects.

- Pioneering works: W. Hu, D. Scott (one of your lecturers!!), N. Sugiyama, M . White '95; Michney & Caldwell 2006
  More recent works: Hannestad & Brandbyge 2009; Tully & Zhang '21

Anisotropies of the CNB

- Use the ***perturbed*** Boltzmann equations

$$\frac{df_\nu}{d\eta} = \frac{\partial f_\nu}{\partial \eta} + \frac{d\mathbf{x}}{d\eta}\frac{\partial f_\nu}{\partial \mathbf{x}} + \frac{d\hat{n}}{d\eta}\frac{\partial f_\nu}{\partial \hat{n}} + \frac{\partial \mathcal{E}}{\partial \eta}\frac{\partial f_\nu}{\partial \mathcal{E}}$$

- Perturb around a homogenous and isotropic Fermi-Dirac distirbution

$$f_\nu = \left[\exp\left(\frac{\mathcal{E}}{T(1+\mathcal{N})}\right)+1\right]^{-1} = f_\nu^{(0)} + \left.\frac{\partial f_\nu}{\partial \mathcal{N}}\right|_{\mathcal{N}=0} \mathcal{N}$$

fractional neutrino temperature perturbation

Anisotropies of the CNB

- Use the ***perturbed*** Boltzmann equations

$$\underbrace{\mathcal{N}' + ik\mu\frac{p}{\mathcal{E}}\mathcal{N}}_{\text{Free streaming}} + \underbrace{\frac{p^2}{\mathcal{E}^2}\left(i\frac{\mathcal{E}}{p}k\mu\Phi - \Psi' + \frac{1}{2}\chi'_{ij}n^i n^j\right)}_{\text{Gravitational effects that imprint anisotropies during propagation}} = 0$$

- Solution (along the line of sight, as CMB photons)

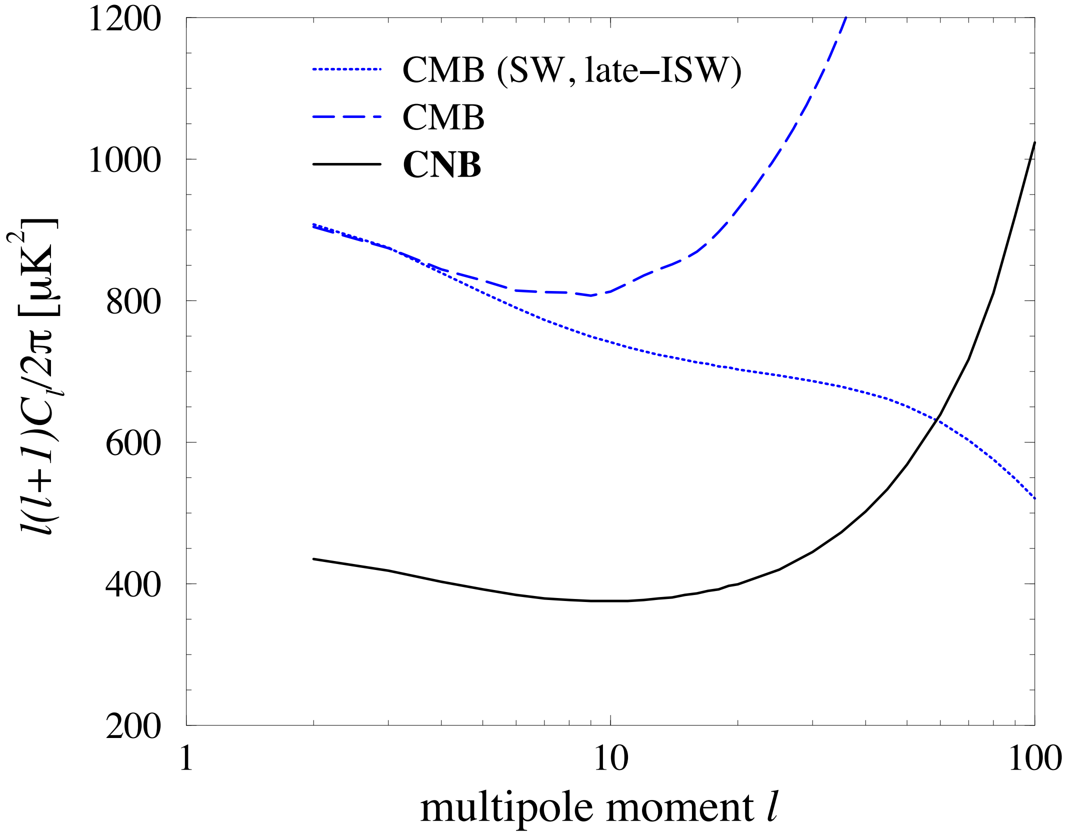
$$\mathcal{N}(\eta, \mathbf{k}, \hat{n}) = \int_{\eta_{in}}^{\eta} d\eta' e^{-ik\mu\chi(\eta',\eta)} \left\{ \boxed{\left[\mathcal{N}(\eta') + \Phi(\eta')\right]\delta(\eta' - \eta_{in})} + \boxed{\frac{\partial\left(\Phi(\eta',\mathbf{k}) + \frac{p^2}{\mathcal{E}^2}\Psi(\eta',\mathbf{k})\right)}{d\eta'}_{j}} \right\}$$

Sach-Wolfe effect **Integrated Sach-Wolfe effect**

$$\boxed{\chi(\eta_{in}, \eta) \equiv \int_{\eta_{in}}^{\eta} \frac{p}{\mathcal{E}} d\eta' = \int_{a_{in}}^{a} \frac{p}{\mathcal{E}} \frac{da'}{a\mathcal{H}}}$$

Comoving distance travelled by neutrinos

CNB angular power spectrum



$$\Delta T|_{CNB,SW} = \underbrace{\frac{3}{2}\times\frac{10}{9}}\times\left(\frac{4}{11}\right)^{1/3}\times\Delta T|_{CMB,SW}$$

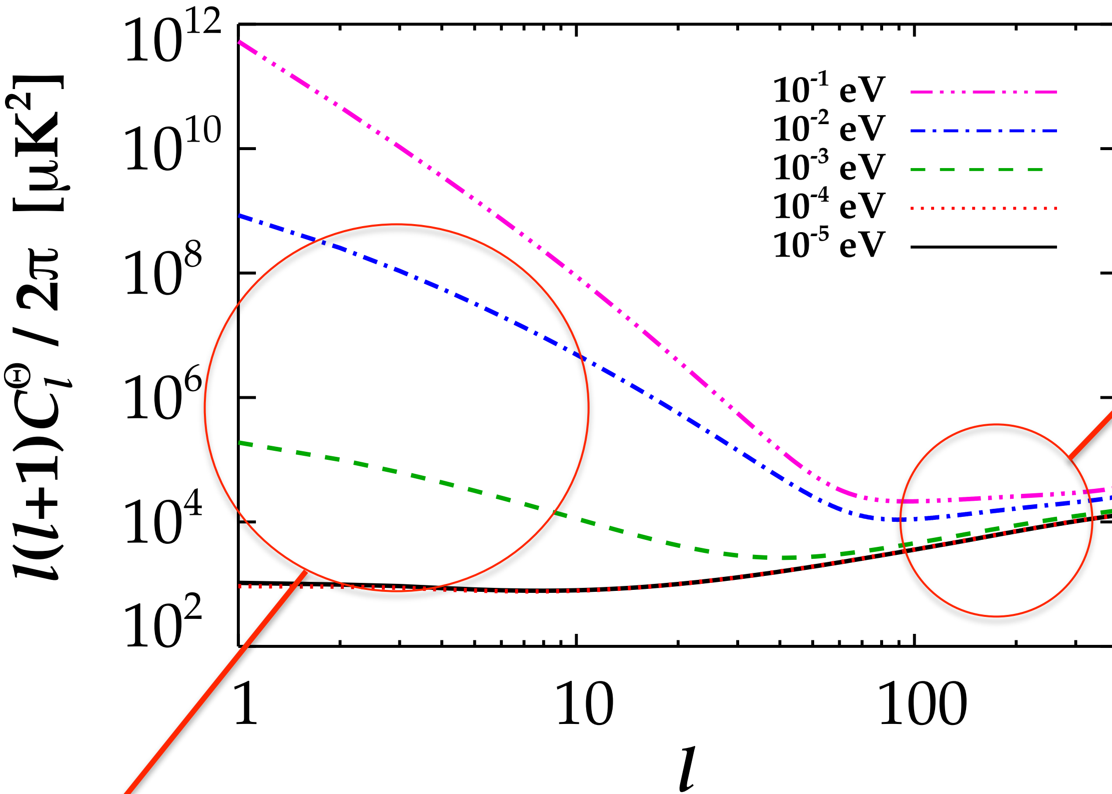
Rescaling of temperatures.

Due to generation of CNB taking place during radiation-domination

NEW PROBE OF LARGE SCALE ANISOTROPIES (like CMB photons)

CNB angular power spectrum

From Hannestad and Brandbyge 2009



At high l (small scales): weak dependence on m_ν. Why?
At small scales neutrinos are dominated by free-streaming, say $k>k_{FS}$ and in terms of multipoles $l_{FS} \sim k_{FS}\, \chi_*$
χ_* being the comoving distance from neutrino's last-scattering surface.
But $k_{FS} \sim m_\nu$, while $\chi_* \sim 1/m_\nu$,

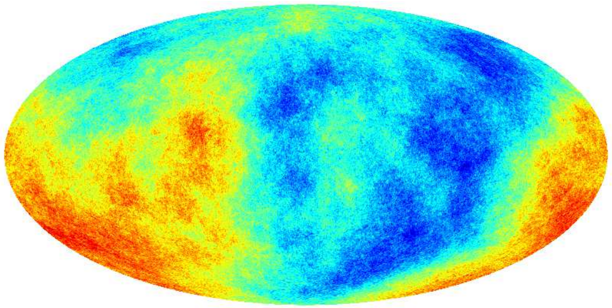
$$
\begin{aligned}
\dot{\Psi}_0 &= -k\frac{q}{\varepsilon}\Psi_1 + \dot{\phi}\frac{d\log f_0}{d\log q} && \text{if } l = 0 \\
\dot{\Psi}_1 &= k\frac{q}{3\varepsilon}\left(\Psi_0 - 2\Psi_2\right) - \frac{\varepsilon k}{3q}\psi\frac{d\log f_0}{d\log q} && \text{if } l = 1 \\
\dot{\Psi}_l &= k\frac{q}{(2l+1)\,\varepsilon}\left(l\Psi_{l-1} - (l+1)\,\Psi_{l+1}\right) && \text{if } l \geq 2
\end{aligned}
$$

At low l, the anisotropy increases with increasing neutrino masses

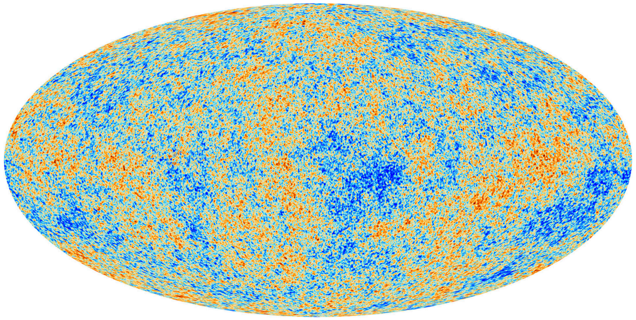
Cross-correlation of CNB with CMB

- ***Within GR, the anisotropies of the CNB and CMB share the same origin and the same perturbed geodesics***

- ***a cross-correlation among the two backgrounds naturally arises***



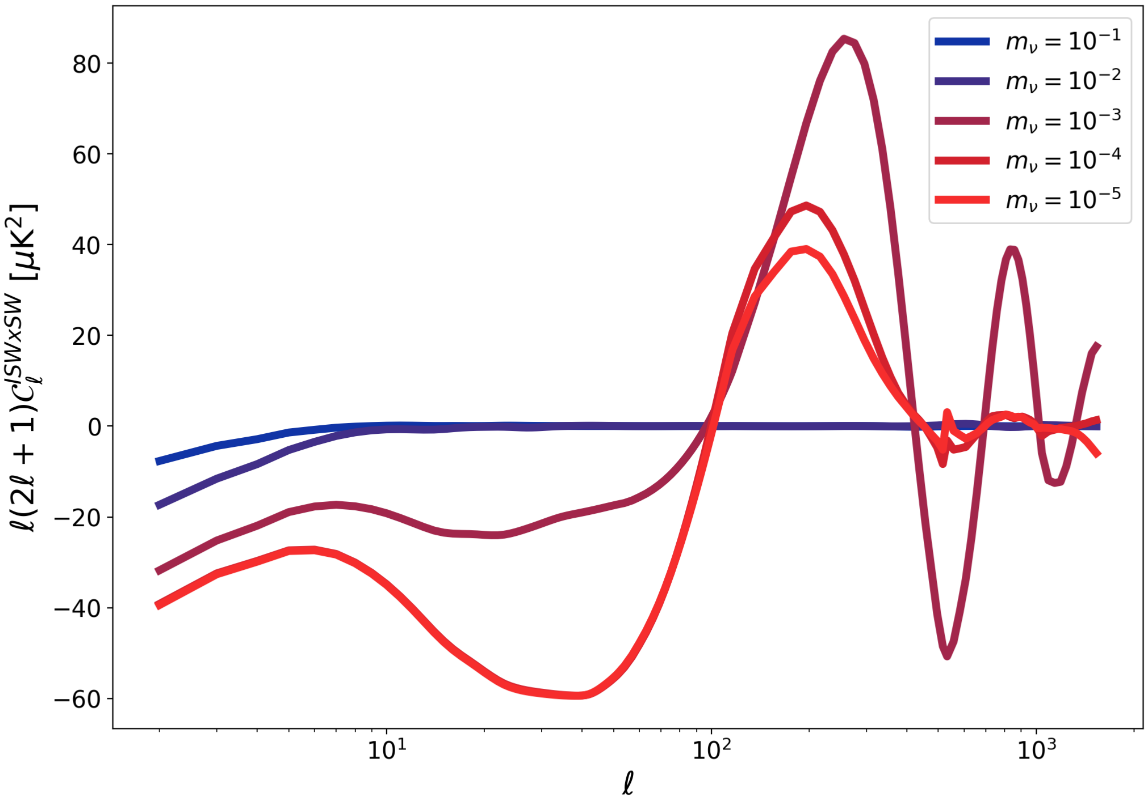
X



A. Raffelli, N.B, and J. Lesgourgues, in preparation

Cross-correlation of CNB with CMB

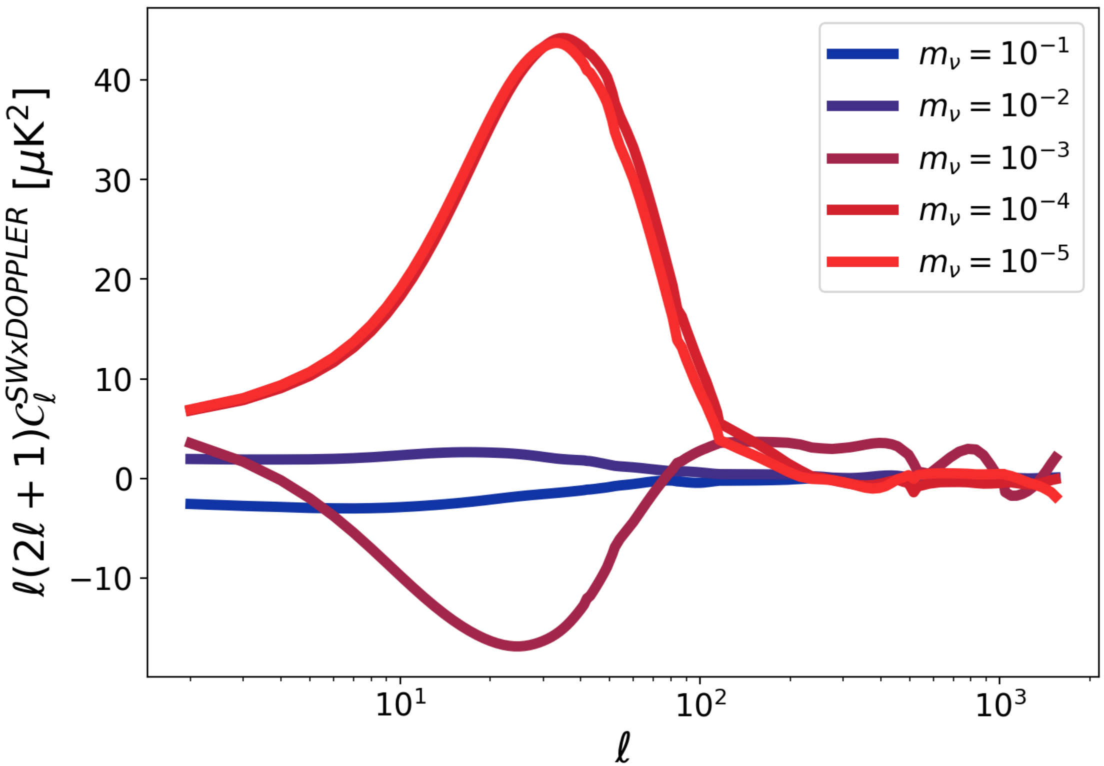
ISW (CNB) X SW (CMB)



A. Raffelli, N.B, and J. Lesgourgues, in preparation

Cross-correlation of CNB with CMB

SW (CNB) X Doppler (CMB)



A. Raffelli, N.B, and J. Lesgourgues, in preparation

CNB angular power spectrum

Other possible signals

- Notice that indeed an anisotropic signal is induced by the relative motion of the observer through the CMB (dipole contributon)

- Locally, the lowest energetic neutrinos can be gravitationally attracted by galaxies.

CNB angular power spectrum

- I know all this is futuristic, but it might resemble the history of CMB discovery, when some first theoretical predictions were given (by Peebles & Yu and Doroshkevich, Zel'Dovich and Syunyaev in the 70's) about CMB temperature anisotropies, even though at that time there was no hope to detect them

TABLE 1

RESIDUAL PERTURBATION $10^5 \times \langle|\delta'_0|^2\rangle^*$ TO THE MICROWAVE BACKGROUND

$\Delta\psi$†	Open General-Relativity Model	Flat General-Relativity Model	Closed General-Relativity Model	Flat Scalar-Tensor Model
0	73.0	9.0	4.40	1.55
3	16.0	7.8	4.00	1.09
6	7.4	5.4	3.20	0.44
9	4.1	3.5	2.45	0.22
12	...	2.3	1.80	0.13
15	...	1.6	1.30	0.084
$1+Z_m=5$‡	0.61	0.25	0.23	0.20
$1+Z_m=2$	0.41	0.040	0.026	0.023

From Peebles and Yu "Primeval Adiabatic Perturbation in an Expanding Universe", ApJ 162, 1970

CNB angular power spectrum

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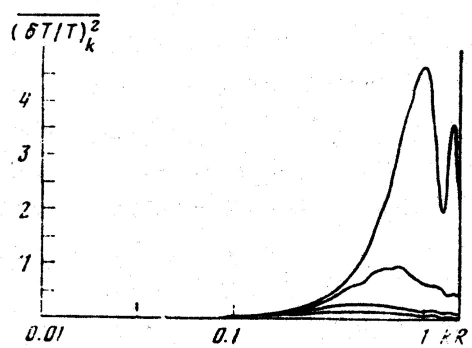
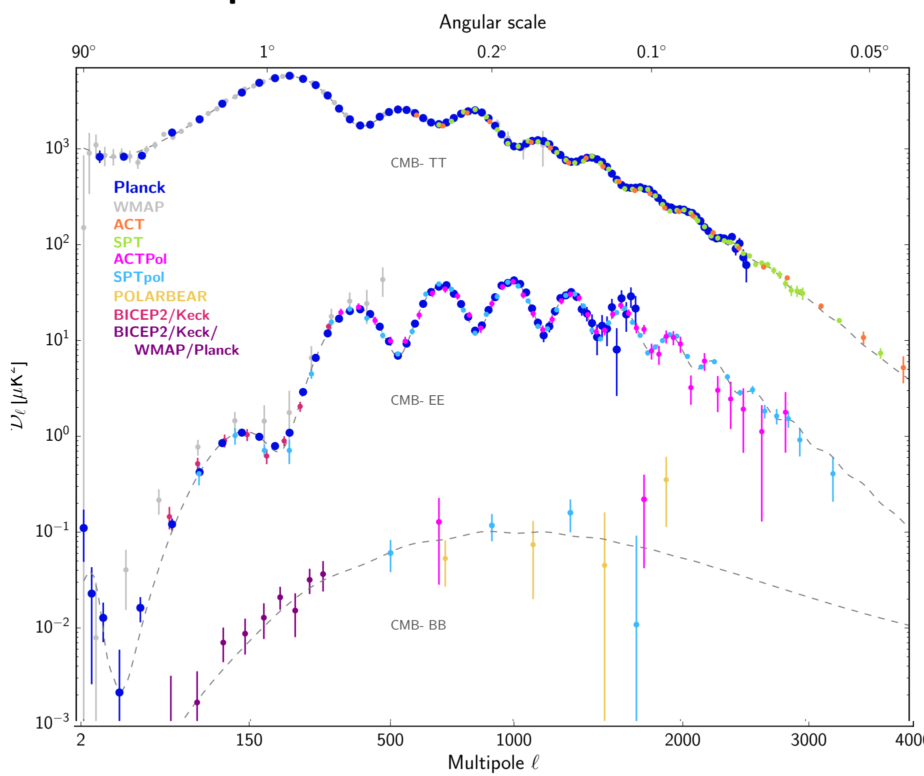


FIG. 4. Dependence of the function $\overline{(\delta T/T)_k^2} = (4\pi^2)^{-1} k^3 \int_{-1}^{1} d\mu\, \overline{|A(\mu, k)|^2}$ $e^{-k^2 r_\theta{}^2 \theta_0{}^2 (1-\mu^2)}$ on the wave number k for adiabatic perturbations with $\overline{|a(k)|^2} \propto k^2$, $\Omega = 0.1$, h = 1, and beamwidths $\theta_0 = 0, 5', 10', 15'$; $R = R_C/\beta_C$.

From Doroshkevich, Zel'Dovich & Syunyaev, Astron. Zh., 1978



Planck 2018 results. I. Overview and the cosmological legacy of Planck

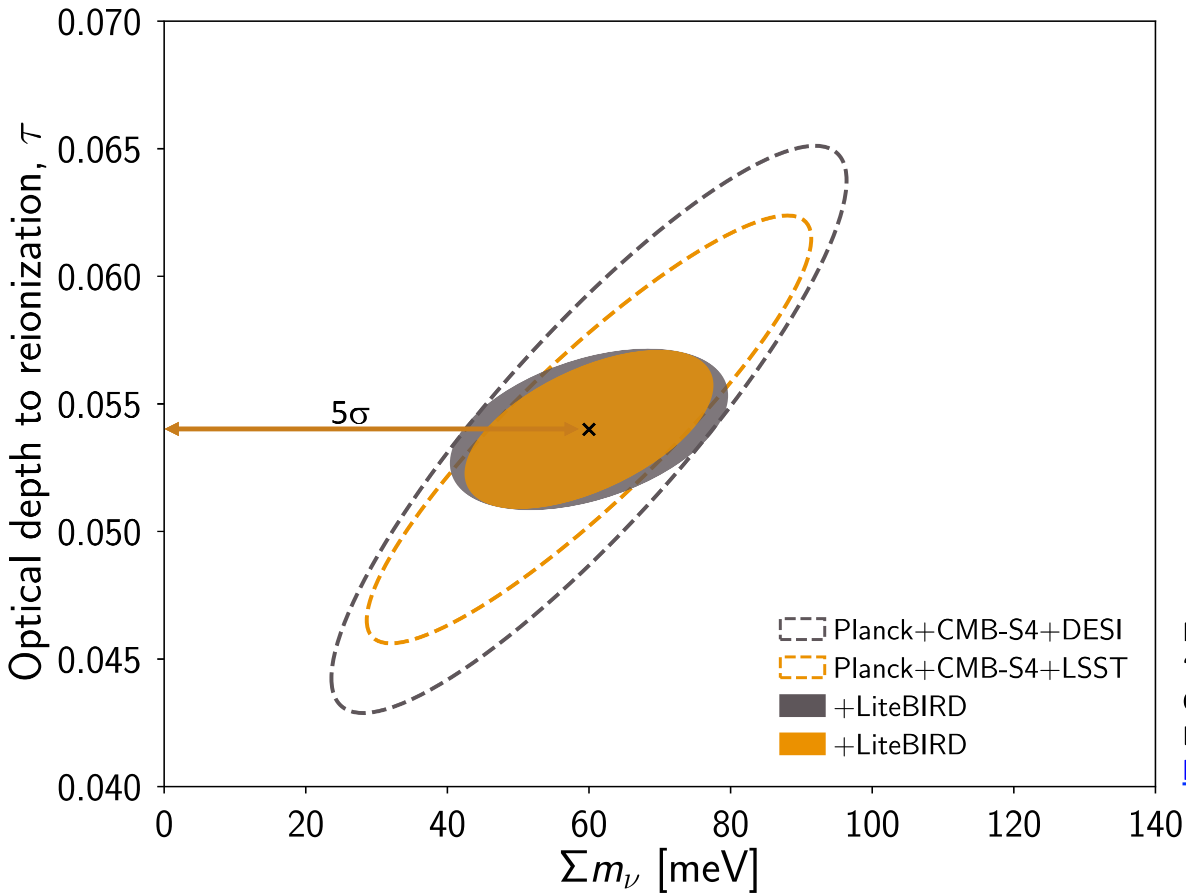
CNB angular power spectrum

Let us recall Wolfgang Pauli's remark shortly after conceiving of the neutrino in 1930: "I've done a terrible thing today, something which no theoretical physicist should ever do. I have suggested something that can never be verified experimentally." Hopefully, a discussion of CNB properties will someday cease to seem as exclusively theoretical as it does today, just as the neutrino itself once did to Pauli.

From Michney & Caldwell 2006

BACK UP SLIDES FOR SOME OF THE TOPICS WE MENTIONED IN PREVIOUS SLIDES

CMB & neutrinos properties: an example



From
"Probing Cosmic Inflation with the LiteBIRD Cosmic Microwave Background Polarization Survey"
LiteBIRD collaboration,
https://arxiv.org/pdf/2202.02773.pdf

- Determination of optical depth τ impact the determination of the sum of neutrinos masses: Massive neutrinos slow down structure formation and consequently, one can measure the neutrino mass by comparing the amplitude of fluctuations in the low-redshift Universe with that at the last-scattering surface (i.e., A_s). However, we cannot determine A_s unless we know τ.

The Cosmic Neutrino Background (CNB)

- N.B: More refined computations for neutrino decoupling also means to accout for finest details such as **QED finite-temperature corrections and neutrino oscillations**

e.g. QED finite-temperature corrections lead to a renormalization of the electron, positron and photon masses. This in its turn affect a series of relevant quantities via the energy of these particles in their distribution functions, e.g.:
- equation of state of the Erlay Universe
- the energy density and eherefore
- the expansion rate H

	$(aT_\gamma)_{\text{fin.}}$	$\delta\rho_{\nu_e}/\rho^0_\nu$	$\delta\rho_{\nu_\mu}/\rho^0_\nu$	$\delta\rho_{\nu_\tau}/\rho^0_\nu$	N_{eff}
no-QED	1.39910	0.946%	0.398%	0.398%	3.0340
FT-QED	1.39844	0.935%	0.390%	0.390%	3.0395
osc.$(\theta_{13} = 0)$ + QED	1.3978	0.73%	0.52%	0.52%	3.046
osc.$(s^2_{13} = 0.047)$ + QED	1.3978	0.70%	0.56%	0.52%	3.045

N.B.

Table 2: Results of kinetic equations numerical integration for neutrino decoupling in various theoretical models. First column shows the asymptotic dimensionless photon temperature. Second, third and fourth ones report the relative energy gain of neutrinos, while the last row presents the effective number of neutrino species. Data taken from Dolgov et al. 1999, Mangano et al. 2002 and Mangano et al. 2005.

Neutrinos and inflationary GWs

- In absence of neutrinos the solution is

$$h_{ij}(u) = h_{ij}(0)\chi(u)$$

$$\chi(u) = j_0(u) = \frac{\sin(u)}{u} = \frac{\sin(k\tau)}{k\tau}$$

- For wavelengths that enter the horizon during the radiation era and long-after neutrino decoupling

$$\chi(u) \to A\frac{\sin(u+\delta)}{u}$$

with

DAMPING EFFECT

$$A = 0.80313 \text{ and } \delta = 0$$

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- ***Here I'll focus on a future (better…..futuristic) signature, related to an eventual direct detection of neutrinos: CNB spatial anisotropies.***