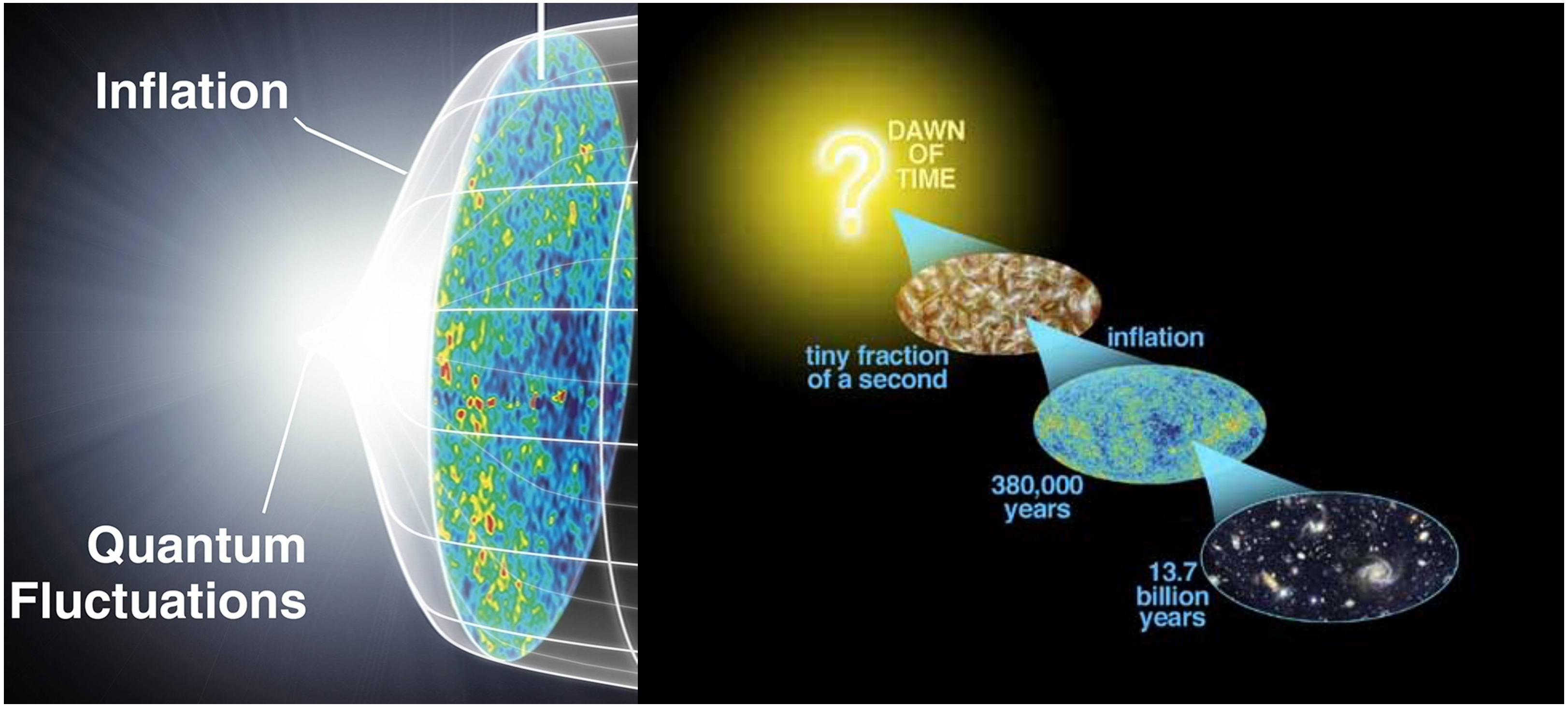
Very Early Universe & Neutrinos



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UNIVERSITÀ DEGLI STUDI DI PADOVA

ISAPP – SIF Varenna, 26 June – 6 July 2023

What are we going to learn??

- ***Lecture 1:***

  Some basics about inflation

  - dynamics of inflation: simplest models and slow-roll parameters

  - predictions (power-spectra of primordial density perturbations and gravitational waves)

  - contact with observations: present constraints on inflationary models

- ***Lecture 2:***

  Connections between Early Universe and neutrinos. A few examples:

  - *impact of neutrinos on inflationary gravitational waves*

  - *further connections with inflation*

  - *for the future: spatial anisotropies of the cosmic neutrinos background*

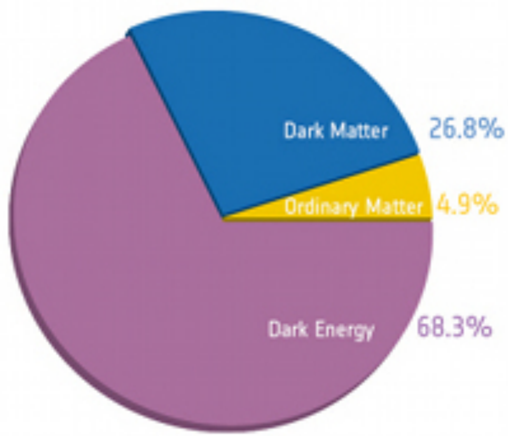
  - *neutrinos and isocurvature perturbations from inflation*

The big picture: precision cosmology

ΛCDM: The standard cosmological model

just 6 numbers…….
describe the Universe composition and evolution

Homogeneous background

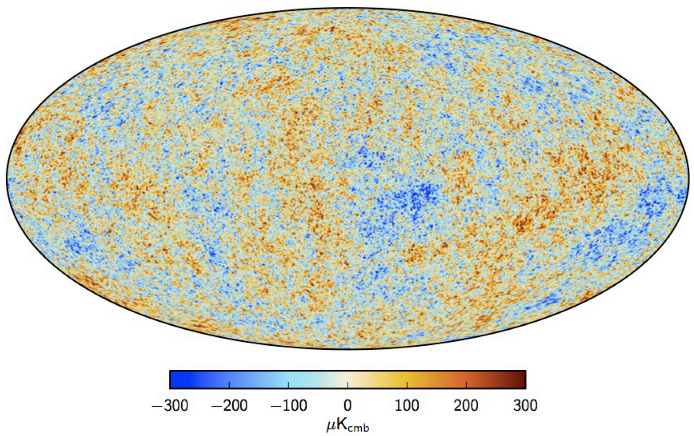


$\Omega_{b,}\ \Omega_{c,}\ \Omega_{\Lambda,} H_{0,}\ \tau$

- atoms 4%
- cold dark matter 23%
- dark energy 73%

Λ?? CDM??

Perturbations

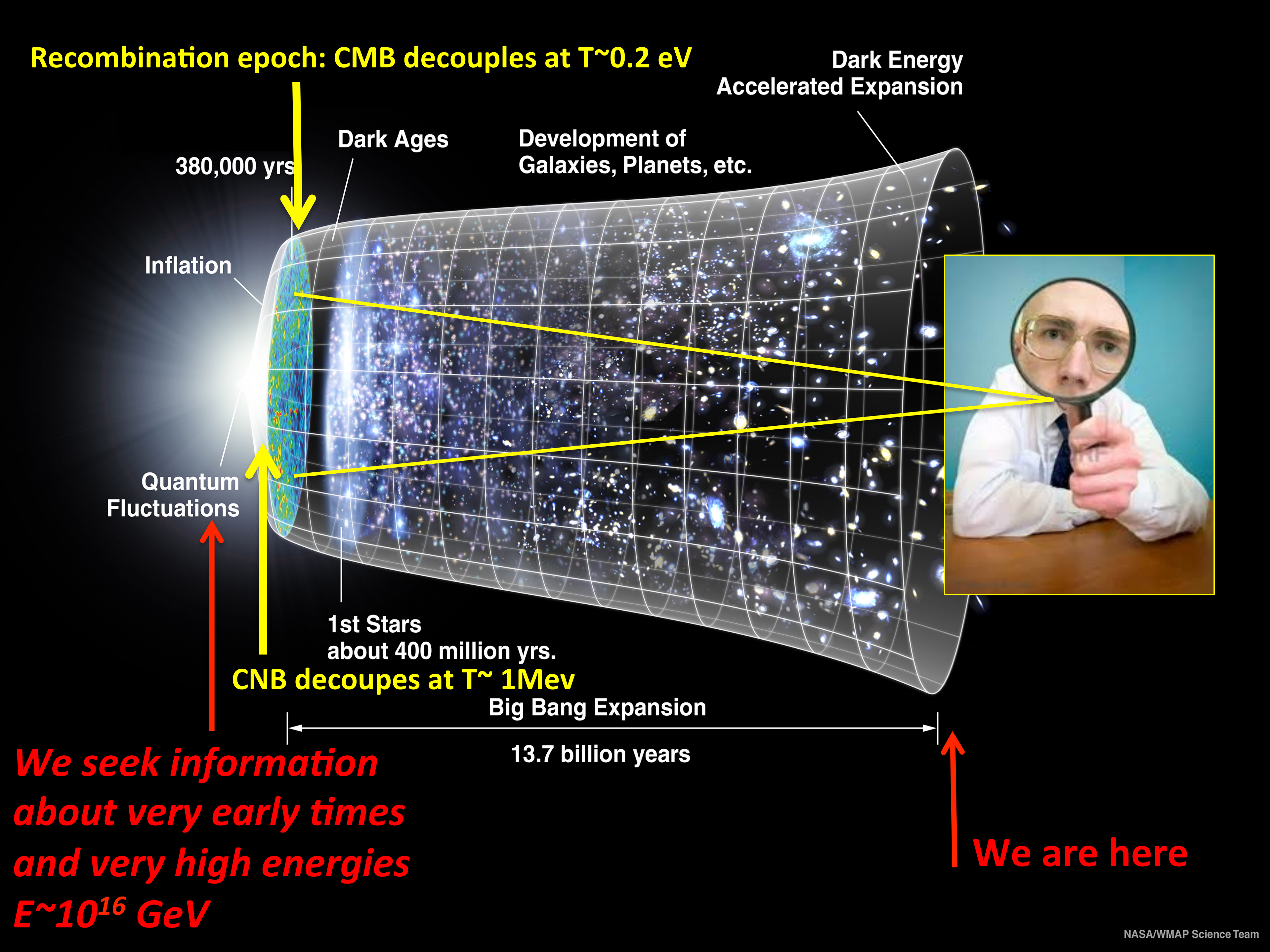


$A_{s,}\ n_{s,}\ r$

- nearly scale-invariant
- adiabatic
- (almost) Gaussian

ORIGIN???

Credit: L. Verde



Recombination epoch: CMB decouples at T~0.2 eV
Dark Energy
Accelerated Expansion
Dark Ages
Development of
Galaxies, Planets, etc.
380,000 yrs
Inflation
Quantum
Fluctuations
1st Stars
about 400 million yrs.
CNB decoupes at T~ 1Mev
Big Bang Expansion
13.7 billion years
We seek information
about very early times
and very high energies
E~10^16 GeV
We are here
NASA/WMAP Science Team

4 FACTS INFLATION CAN EXPLAIN

- The Universe is old
- The Universe is homogeneous and isotropic (on large scales)
- The Universe today is very close to be spatially flat
- **Most importantly**: Structures grew out of tiny, *nearly* scale invariant (*almost* Gaussian) perturbations

A step back in time

THE ASTROPHYSICAL JOURNAL, 162:815–836, December 1970

PRIMEVAL ADIABATIC PERTURBATION
IN AN EXPANDING UNIVERSE*

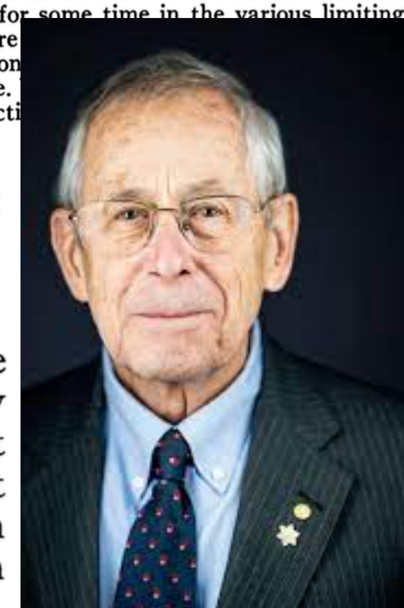
P. J. E. PEEBLES†
Joseph Henry Laboratories, Princeton University

AND

J. T. YU‡
Goddard Institute for Space Studies, NASA, New York
Received 1970 January 5; revised 1970 April 1

ABSTRACT

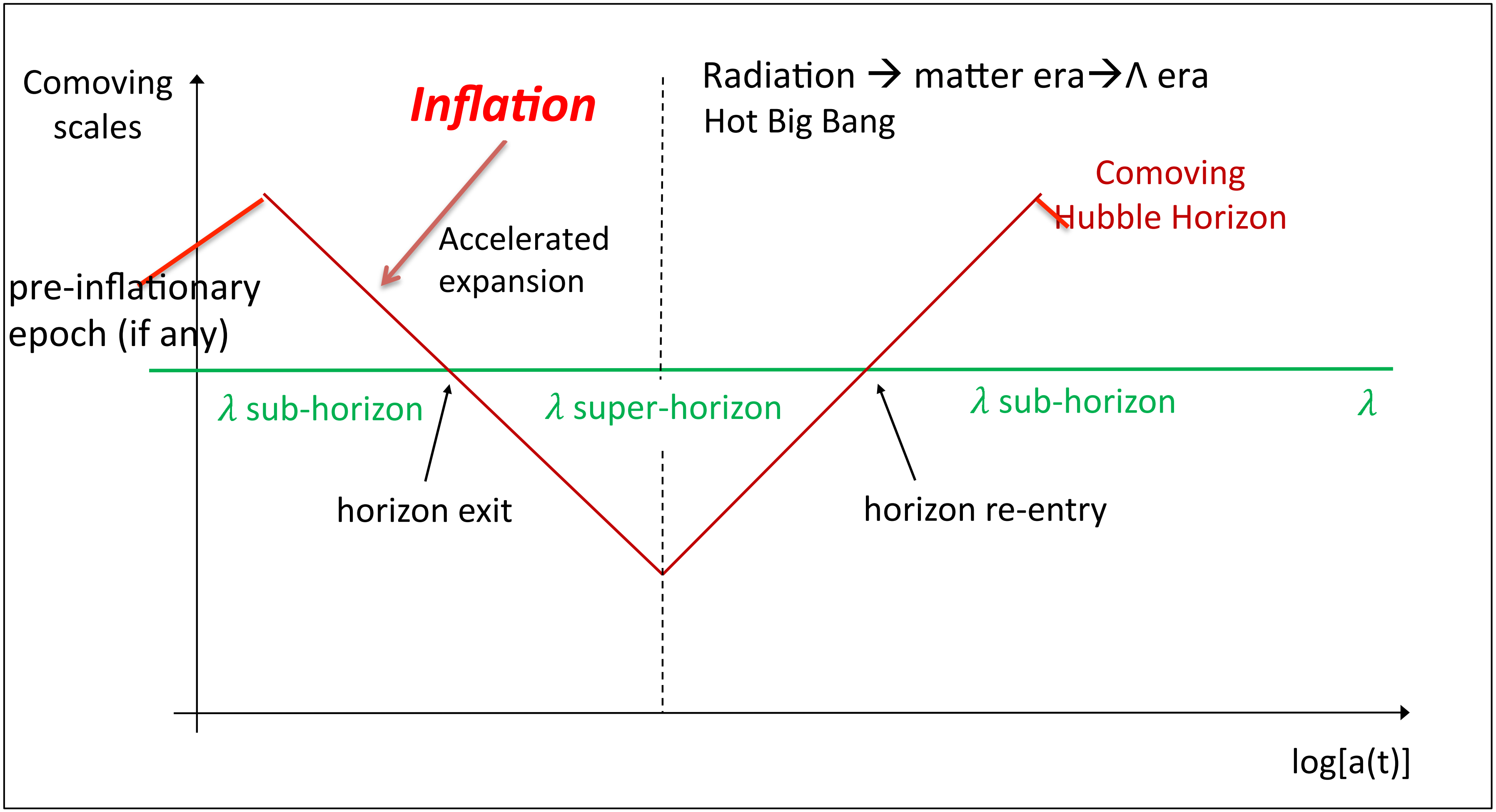
The general qualitative behavior of linear, first-order density perturbations in a Friedmann-Lemaître cosmological model with radiation and matter has been known for some time in the various limiting situations. An exact quantitative calculation which traces the entire
lacking because the usual approximations of a very short photon
combination, and a very long mean free path after, are inadequate.
integration of the collision equation of the photon distribution functi



b) Possible Significance

It is well to bear in mind that in this calculation the initial density fluctuations are invoked in an ad hoc manner because we do not have a believable theory of how they may have originated. Also, it is entirely possible that we have left out some relevant force, possibly that provided by a primeval magnetic field. Our calculation thus is at best exploratory; but we have remarked that one might consider the results of the exploration encouraging if, for example, the characteristic numbers one derives correspond to known phenomena.

The rise and fall … of the comoving Hubble horizon

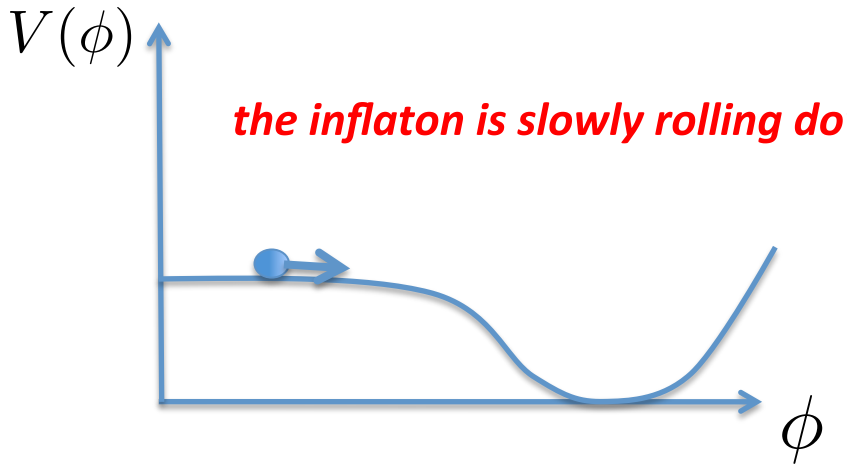


Inflation

A single real quantum scalar field with a canonical kinetic term on top of a rather ***flat potential***

$$\mathcal{L} = \frac{1}{2} M_{\rm Pl}^2 R - \frac{1}{2} g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - V(\phi)$$

(and minimally coupled to gravity; GR; Bunch-Davies vacuum)

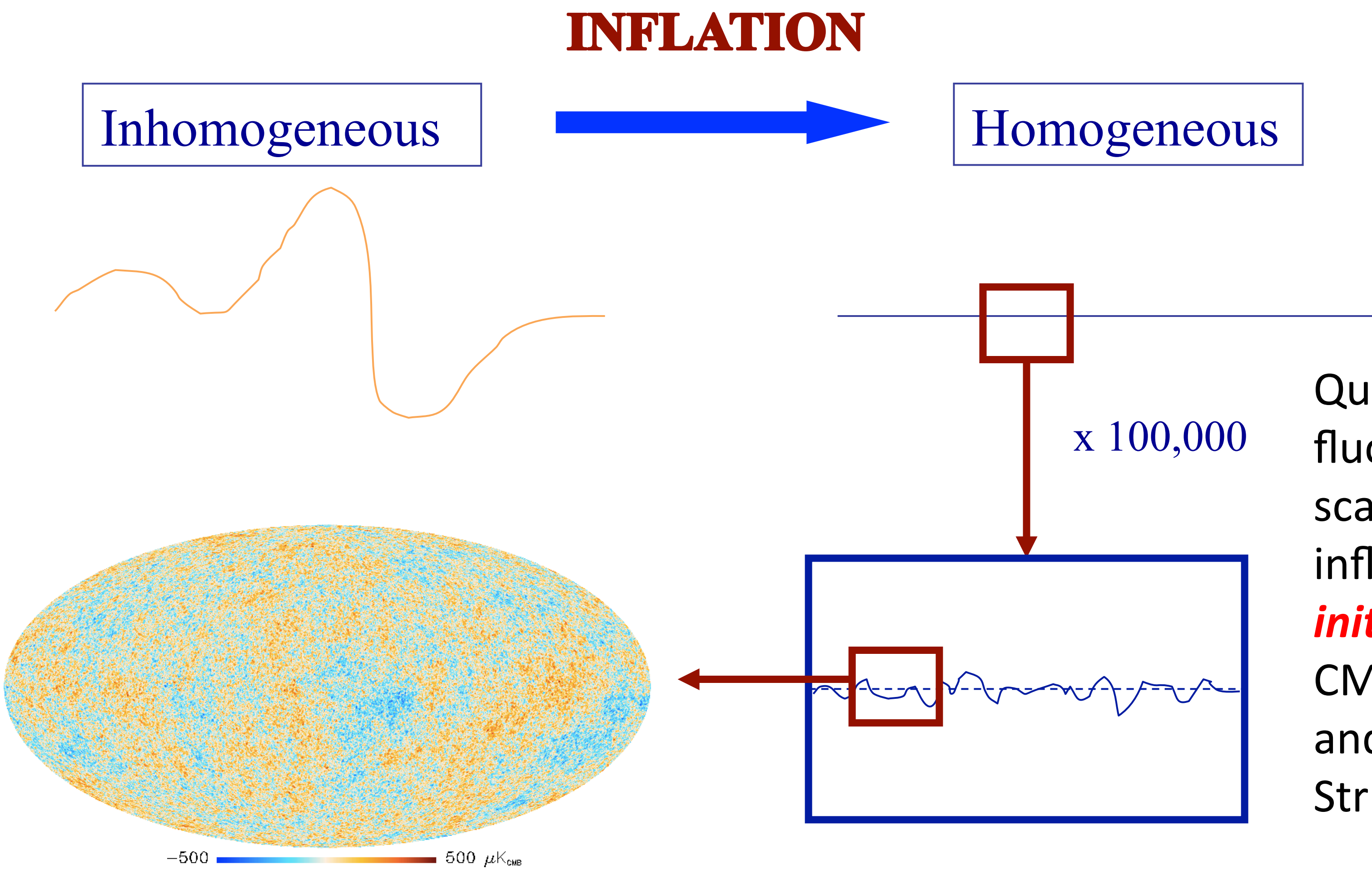


✓ $V(\phi) >> \frac{1}{2}\dot{\phi}^2 \longrightarrow H^2 = \frac{8\pi G}{3} V(\phi) \simeq const. \longrightarrow a(t) \simeq e^{Ht}$ ***Accelerated expansion***

$$\boxed{\varepsilon = \frac{M_{Pl}^2}{2}\left(\frac{V_\phi}{V}\right)^2 << 1}$$

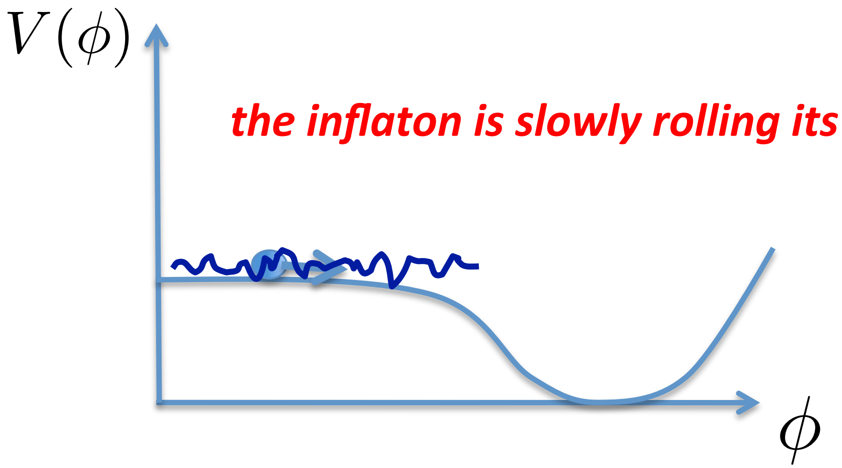
✓ ***To have long enough inflation*** $\boxed{\eta = M_{Pl}^2 \frac{V_{\phi\phi}}{V} << 1}$

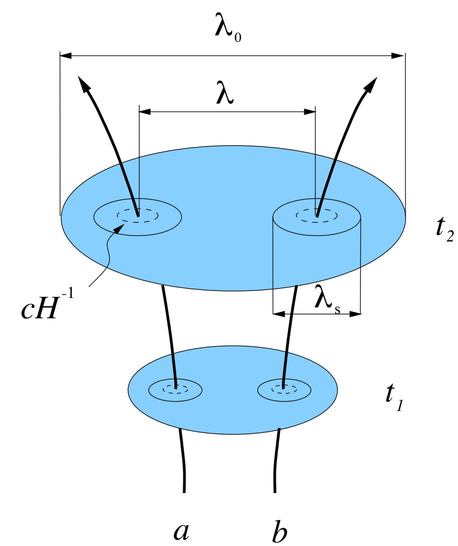
Initial conditions



Quantum fluctuations of a scalar field, the inflaton, set the ***initial conditions*** for CMB anisotropies and Large-Scale Structure formation

Inflation





- On large (super-horizon scales) each region in the universe goes through the same expansion history but at slightly different times:

$$\phi(\mathbf{x},t) = \phi_0(t - \delta t(\mathbf{x})) \longrightarrow \delta\phi(\mathbf{x},t) = -\delta t(\mathbf{x})\dot{\phi}_0(t)$$

- Fluctuations in the inflaton produce fluctuations in the universe expansion from place to place $H^2 \simeq \frac{8}{3}\pi G V(\phi) \longrightarrow$

$$\text{number of e-foldings } N = \ln(a_f/a_i) = \int_{t_i}^{t_f} H dt \longrightarrow \boxed{\zeta = H\delta t \quad = -H\frac{\delta\phi}{\dot{\phi}} \simeq -H\frac{\delta\rho}{\dot{\rho}}}$$

Additional expansion

Quantum fluctuations of a scalar field during inflation

$$\delta\ddot{\phi}_{\mathbf{k}} + 3H\delta\dot{\phi}_{\mathbf{k}} + \frac{k^2}{a^2}\delta\phi_{\mathbf{k}} = \cancel{-V_{,\phi\phi}}\,\delta\phi_{\mathbf{k}}$$

A massless or light scalar field: $m^2=V_{,\phi\phi} << H^2$ for slow-roll

✓ when the perturbation modes are ***within the horizon***:

λ << (comoving) Hubble radius = $(a\,H)^{-1}$

k>> (aH)

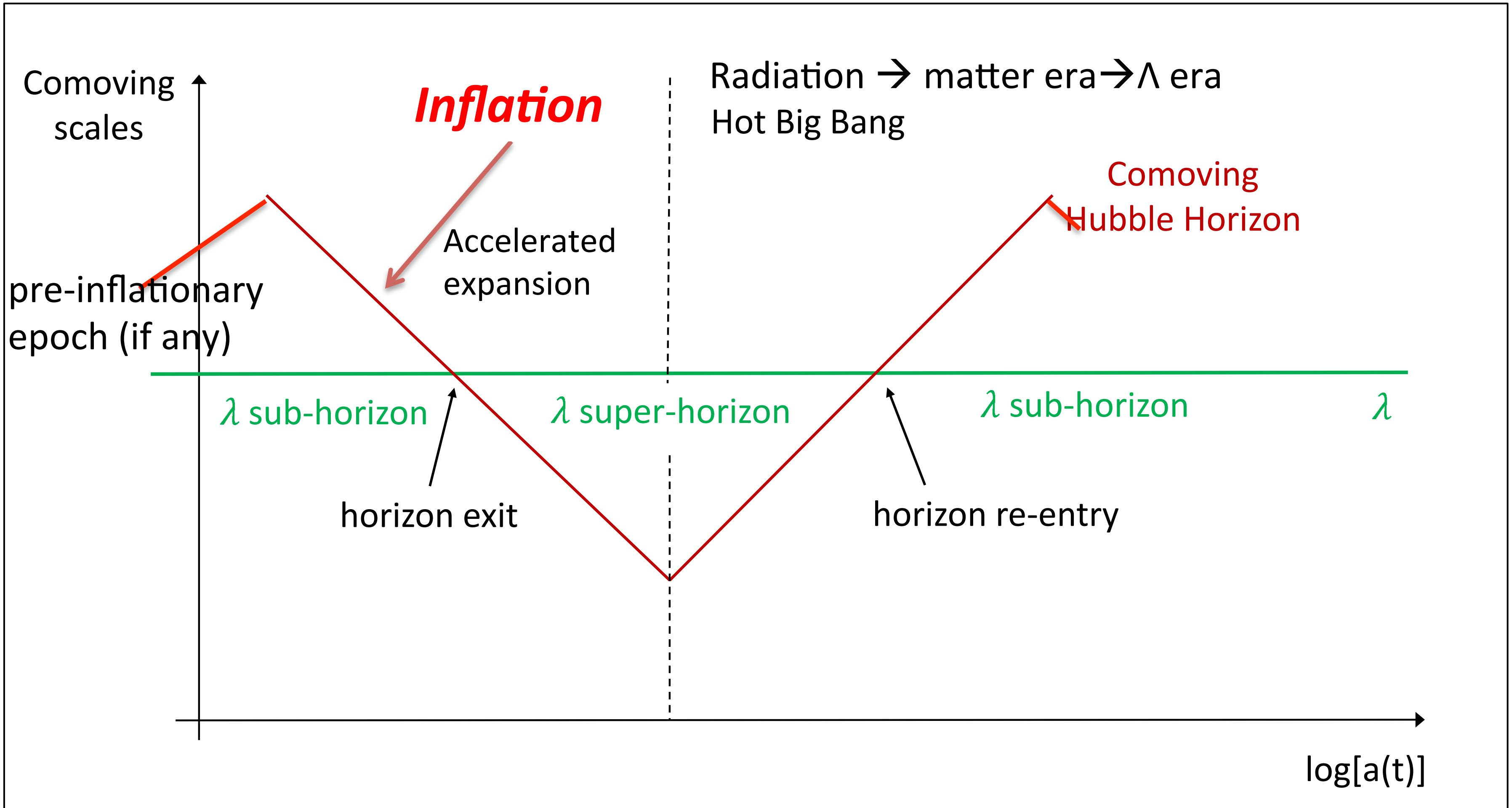
$$\delta\ddot{\phi}_{\mathbf{k}} + \frac{k^2}{a^2}\delta\phi_{\mathbf{k}} = 0 \ \rightarrow oscillations$$

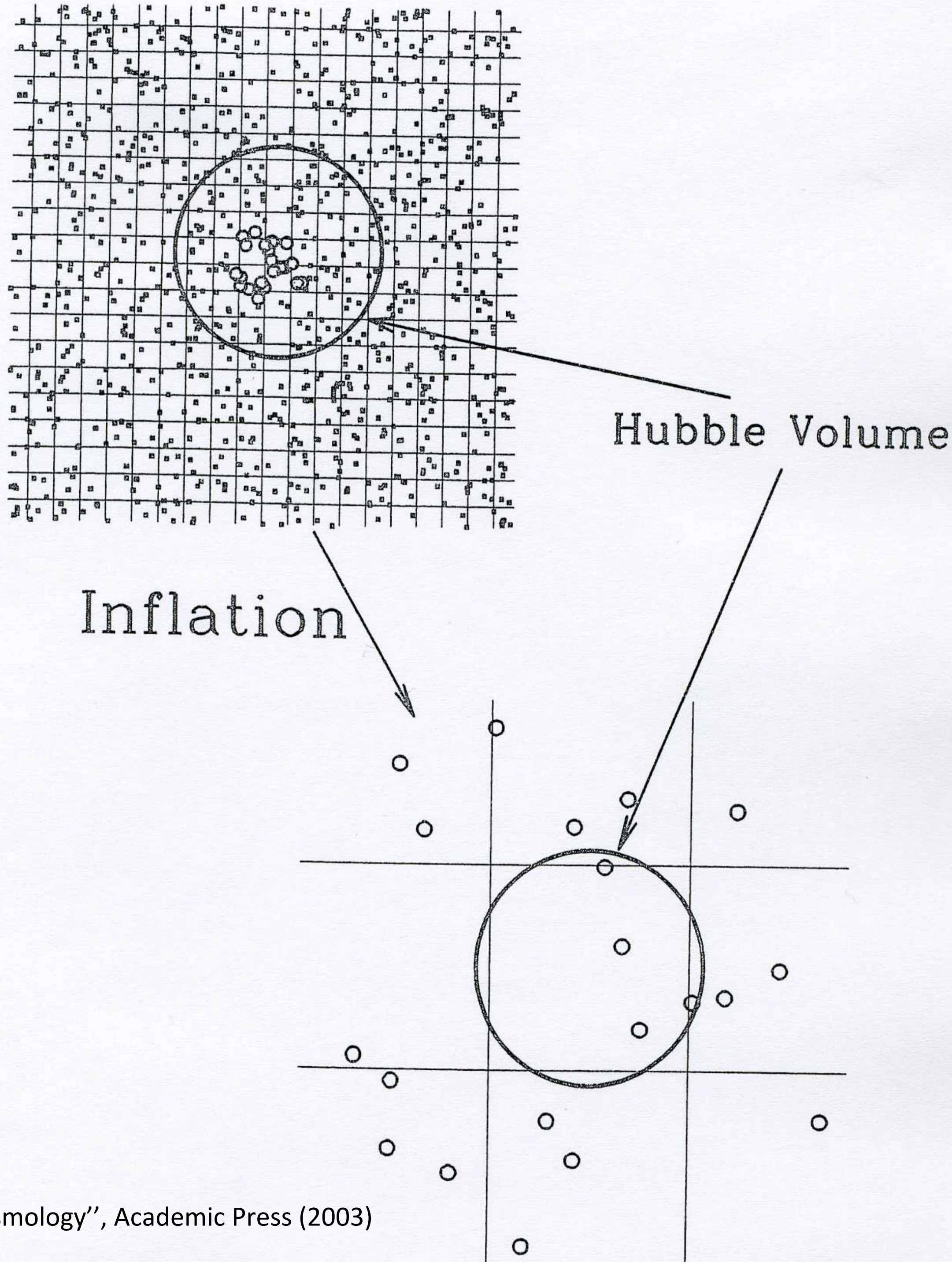
✓ when the perturbation modes are ***superhorizon scales***:

$$\delta\ddot{\phi}_{\mathbf{k}} + 3H\delta\dot{\phi}_{\mathbf{k}} = 0 \ \rightarrow \delta\phi_{\mathbf{k}} = cost. \quad \textbf{(k<<aH)}$$

Remember: H≈const.

The rise and fall ... of the comoving Hubble horizon





From S. Dodelson ``Modern Cosmology'', Academic Press (2003)

Primordial gravitational waves

GWs are tensor perturbations of the metric. Restricting ourselves to a flat FRW background (and disregarding scalar and vector modes)

$$\boxed{ds^2 = -\,dt^2 + a^2(t)\,(\delta_{ij} + h_{ij}(t,\underline{x}))\,dx^i\,dx^j}$$

where h_{ij} are tensor modes which have the following properties

$h_{ij} = h_{ji}$ (symmetric)

$h^i_{\ i} = 0$ (traceless)

$h^i_{\ j|i} = 0$ (transverse, i.e. divergence free)

and satisfy the equation of motion

$$\ddot{h}_{ij} + 3\frac{\dot{a}}{a}\dot{h}_{ij} - \frac{\nabla^2}{a^2}h_{ij} = 0$$

$$\boxed{\dot{} = d/dt}$$

i,j=1,2,3

Primordial gravitational waves

GWs have only (9→6-1-3=) 2 independent degrees of freedom, corresponding to the 2 polarization states of the graviton

$$h_{ij}(\vec{x},\tau) = \int \sum_{\lambda=+,\times} \frac{d^3k}{(2\pi)^3} e^{i\vec{k}\bullet\vec{x}} h_\lambda(\vec{k},\tau)\varepsilon_{ij}^\lambda(\vec{k})$$

polarization tensor

$$h_\lambda'' + 2\frac{a'}{a}h_\lambda' + k^2 h_\lambda = 0$$

free massless, minimally coupled scalar field

behaviour:

k « aH (outside the horizon) **h ≈ const + decaying mode**

k » aH (inside the horizon**)** **h ≈ $e^{\pm ik\tau}$/a** gravitational wave; it freely streams, experiencing redshift and dilution, like a free photon)

‘ = d/dτ
dτ=dt/a(t)

Observational predictions

- **Primordial density (scalar) perturbations**

$$\mathcal{P}_\zeta(k) = \frac{16}{9}\frac{V^2}{M_{\rm Pl}^4\dot{\phi}^2}\left(\frac{k}{k_0}\right)^{n-1}$$

amplitude

spectral index: $n - 1 = 2\eta - 6\epsilon$

describes deviations from scale invariance

$$\epsilon = \frac{M_{\rm Pl}^2}{16\pi}\left(\frac{V'}{V}\right)^2 \ll 1; \;\; \eta = \frac{M_{\rm Pl}^2}{8\pi}\left(\frac{V''}{V}\right) \ll 1$$

- **Primordial (tensor) gravitational waves:** ***a smoking gun for inflation***

$$\mathcal{P}_{\rm T}(k) = \frac{128}{3}\frac{V}{M_{\rm Pl}^4}\left(\frac{k}{k_0}\right)^{n_{\rm T}}$$

Tensor spectral index: $n_{\rm T} = -2\epsilon$

Energy scale of inflation

- **Tensor-to-scalar perturbation ratio**

$$r = \frac{\mathcal{P}_{\rm T}}{\mathcal{P}_\zeta} = 16\epsilon$$

- **Consistency relation** (valid for *all* single field models of slow-roll inflation):

$$r = -8n_T$$

Two simple but very important examples

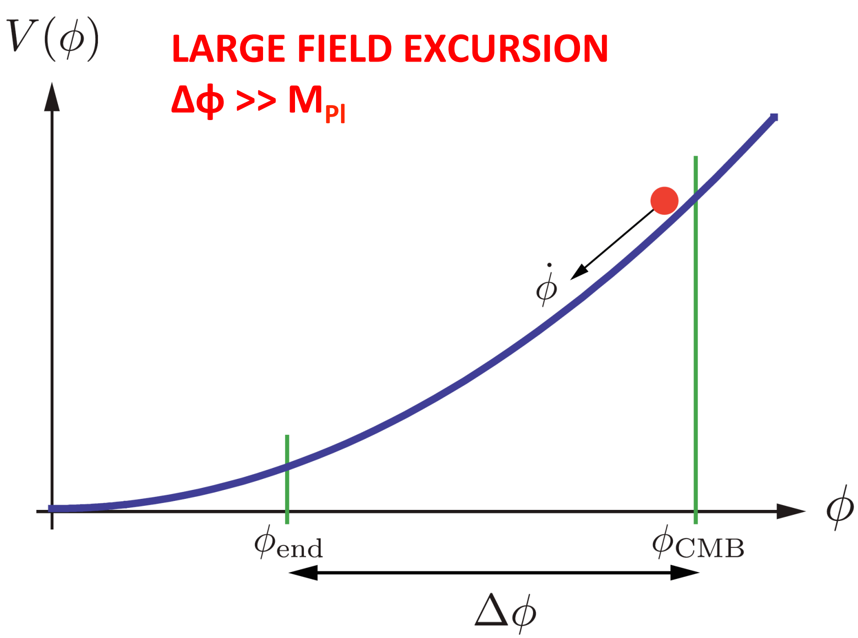
``Large field'' like potential

$$V(\phi) \propto \phi^{\alpha}$$

$$\epsilon \sim \frac{1}{\pi G}\left(\frac{V_{,\phi}}{V}\right)^2 \sim \alpha^2 \frac{1}{\pi G}\frac{1}{\phi^2} \sim \alpha^2 \frac{M_{\rm Pl}^2}{\phi^2}$$

$$\epsilon \ll 1 \Rightarrow \phi \gg M_{\rm Pl}$$

$$M_{Pl} = (\hbar c/G)^{1/2} \equiv G^{-1/2} \simeq 10^{19} GeV$$



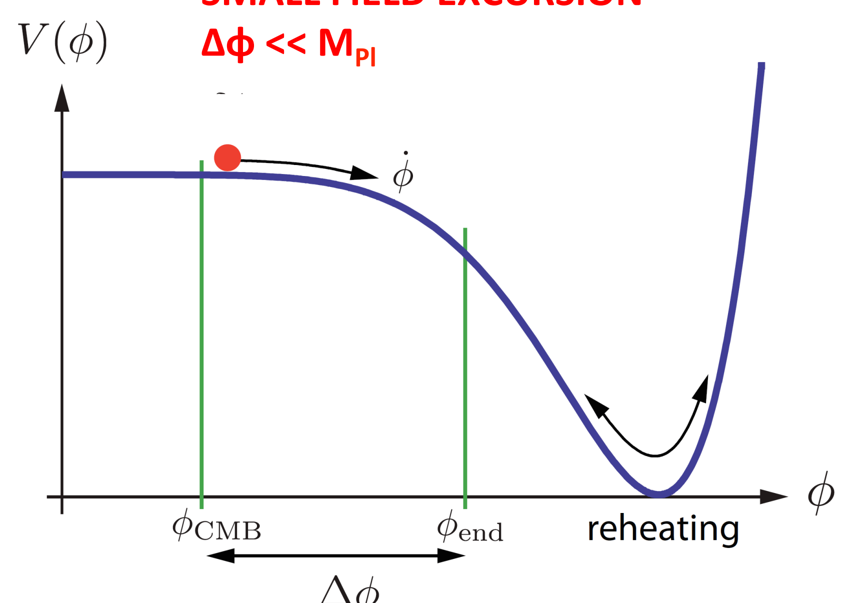
``Small field'' like potential

$$V(\phi) = V_0\left[1 - \left(\frac{\phi}{\mu}\right)^p\right] \qquad \phi < \mu < M_{\rm Pl}$$

p>2

$$\epsilon \sim p^2 \frac{\phi^{2p}}{\mu^{2p}}\frac{M_{\rm Pl}^2}{\phi^2}\left[1 - \left(\frac{\phi}{\mu}\right)^p\right]^{-1}$$

$\epsilon \rightarrow 0$ for $\phi \rightarrow 0$

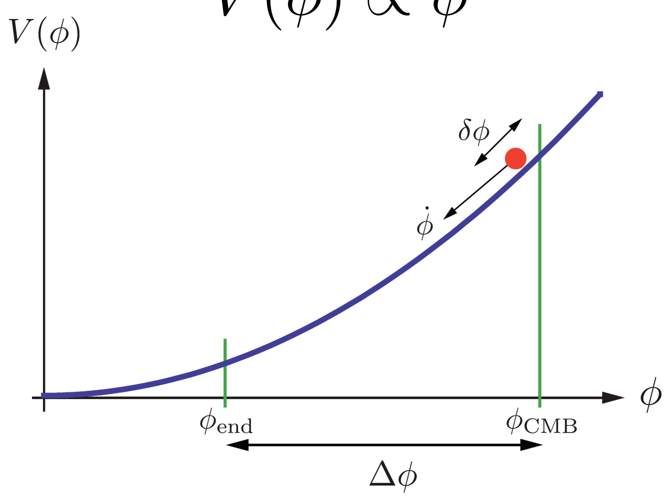


Inflaton dynamics and the level of gravity waves

Roughly speaking: ``Large field'' models can produce a high level of gravity waves;
``small field'' models produce a low level of gravity waves

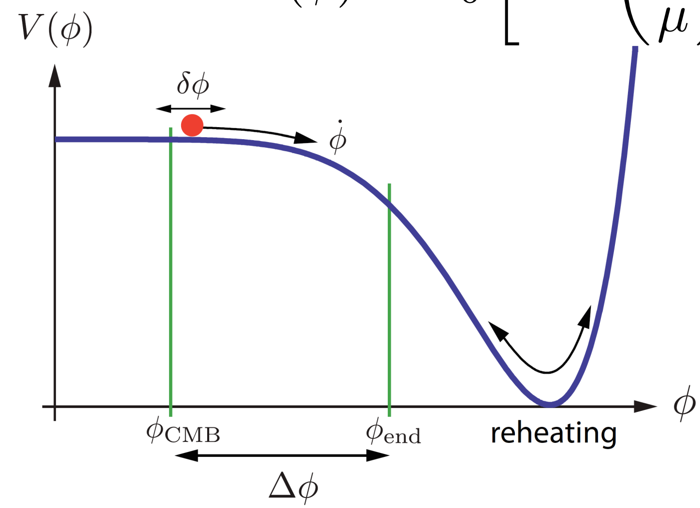
``Large field'' like potential

$$V(\phi) \propto \phi^{\alpha}$$



``Small field'' like potential

$$V(\phi) = V_0 \left[1 - \left(\frac{\phi}{\mu}\right)^p\right]$$



$$\boxed{\frac{\Delta\phi}{m_{\rm Pl}} \simeq \left(\frac{N}{30}\right) \times \left(\frac{r}{0.01}\right)^{1/2}}$$

$30 \lesssim N \lesssim 60$

Current observational status

Constraints from CMB data

- **Primordial density perturbations:** Amplitude $\ln(10^{10} A_s) = 3.044 \pm 0.014 \quad (68\%\,\mathrm{CL})$

$$n_s = 0.9649 \pm 0.0042 \quad (68\%\,\mathrm{CL})$$

n_s=1 (Harrison Zeld' ovich spectrum) excluded at 8.4 sigmas!!

Two fundamental observational constants of cosmology in addition to three very well known (Ω_b, Ω_{cdm}, Ω_Λ)

Latest constraints

- **Primordial gravitational waves:**

Using Planck data release 3 and 4 (including BB power spectrum data)
+ BICEP/Keck Array 2015 and 2018

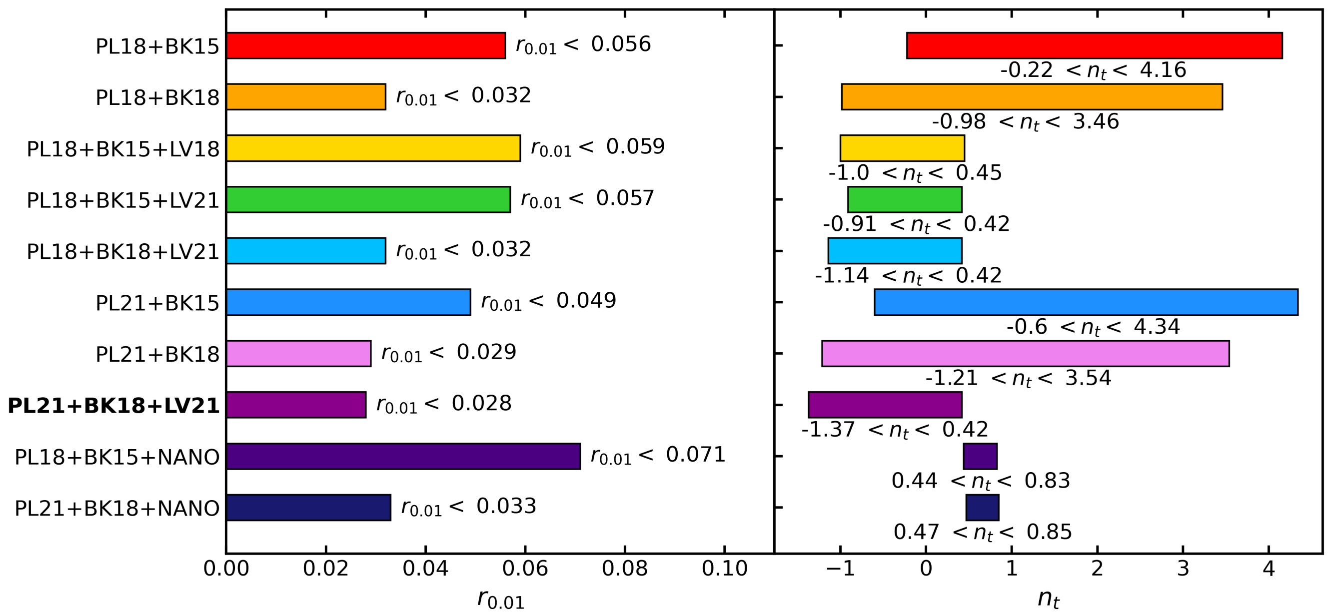
$$r < 0.028 \quad (95\%\ \mathrm{C.L.})$$

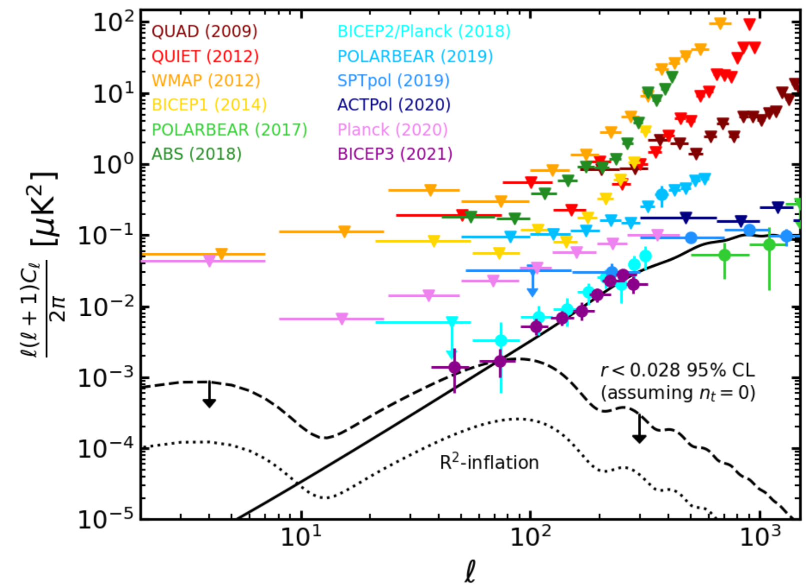
Galloni, N.B., Matarrese, Migliaccio, Ricciardone, Vittorio '23

Energy scale of inflation $V^{1/4} < 1.4 \times 10^{16}\ \ \mathrm{GeV}$

A new era (the CMB B-mode era) has started!
Target of future CMB experiments: r <10^{-3}

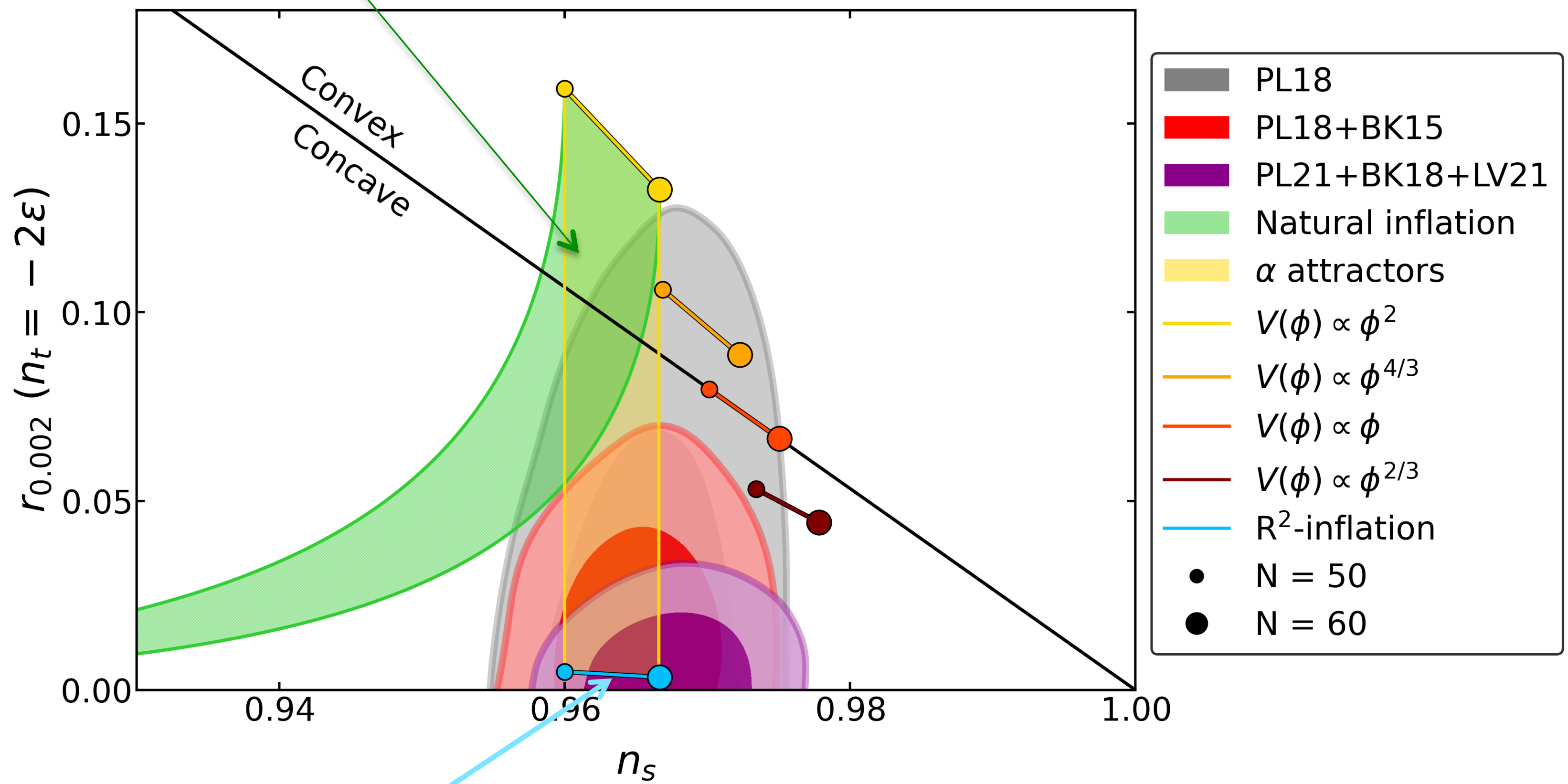
Constraints from CMB data





Implications for standard single-field models of slow-roll inflation

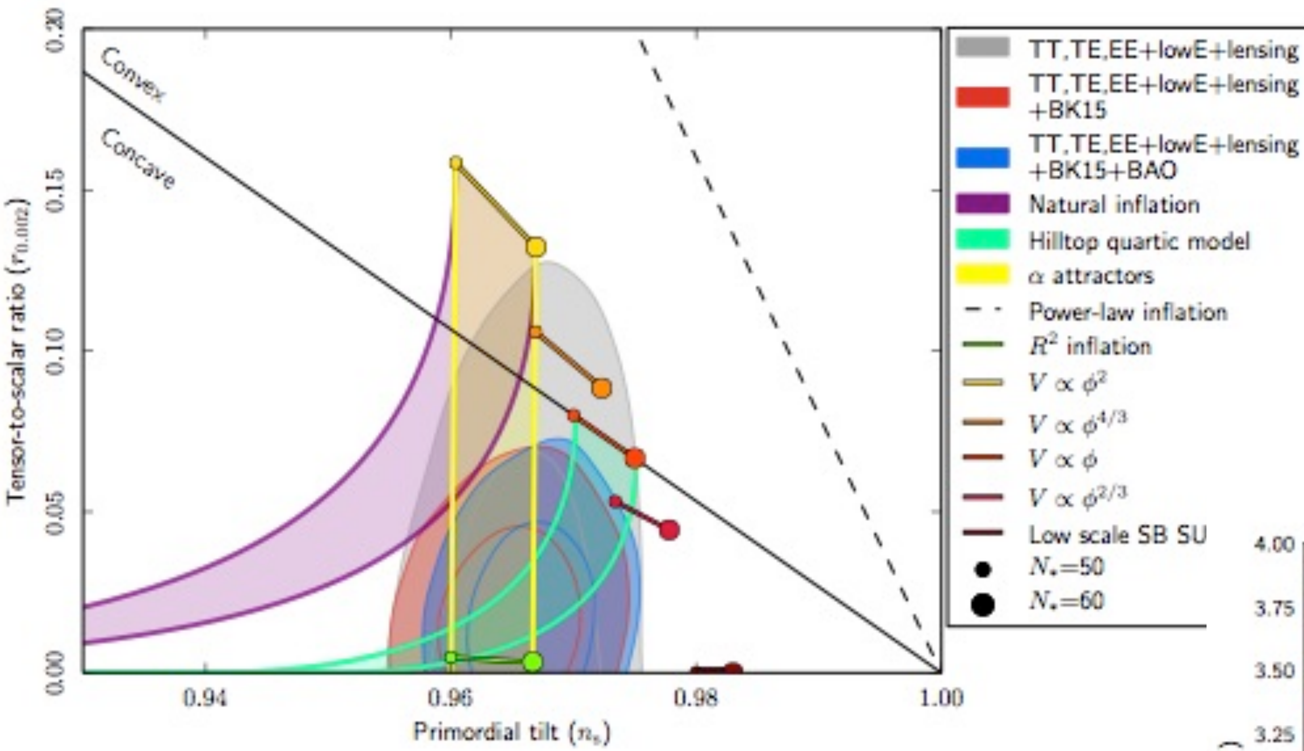
Natural inflation $V(\phi) \propto 1 - \cos(\phi / f)$



Starobinsky model $R + (R^2 / 6M^2)$

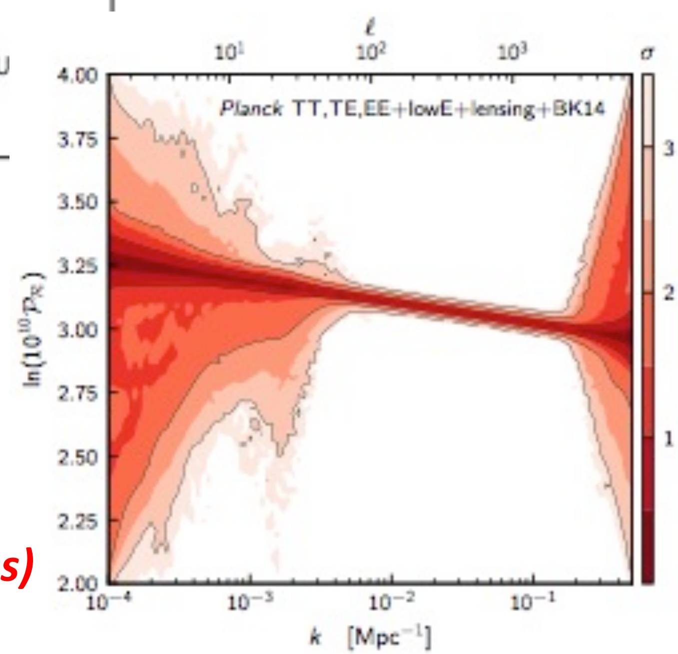
$\rightarrow V(\phi) \propto (1 - e^{-2\sqrt{2/3}\,\phi / M_{Pl}})^2$

Beyond the r-n_s plane



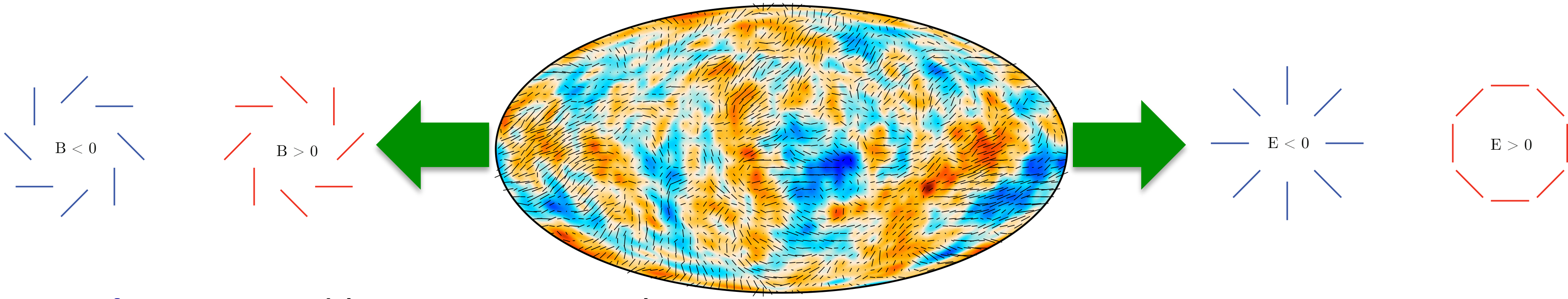
f_{NL}

Shape	Independent	Lensing subtracted
	SMICA $T+E$	
Local	4.1 ± 5.1	−0.9 ± 5.1
Equilateral	−25 ± 47	−26 ± 47
Orthogonal	−47 ± 24	−38 ± 24



No evidence of deviations from a featureless power-spectrum (for curvature perturbations on CMB scales)

Searching for inflationary GWs via CMB polarization

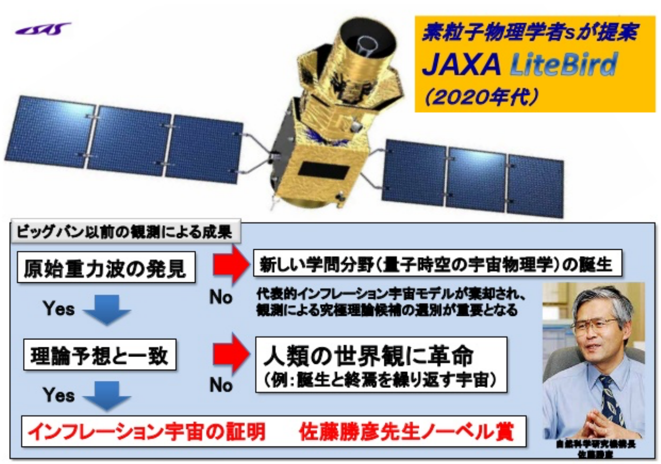


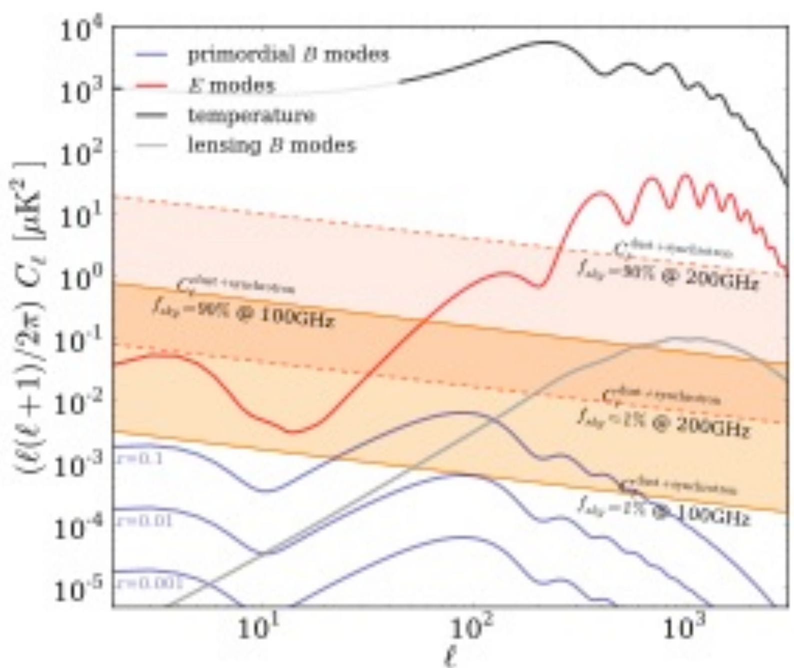
B-modes: Sourced by tensor perturbations but not by density perturbations

E-modes: from scalar and tensor pert.

$$P_{\rm T} \sim \left(\frac{V}{M_{\rm Pl}}\right)^4$$

Primary goal for future CMB surveys: δr<10^{-3}

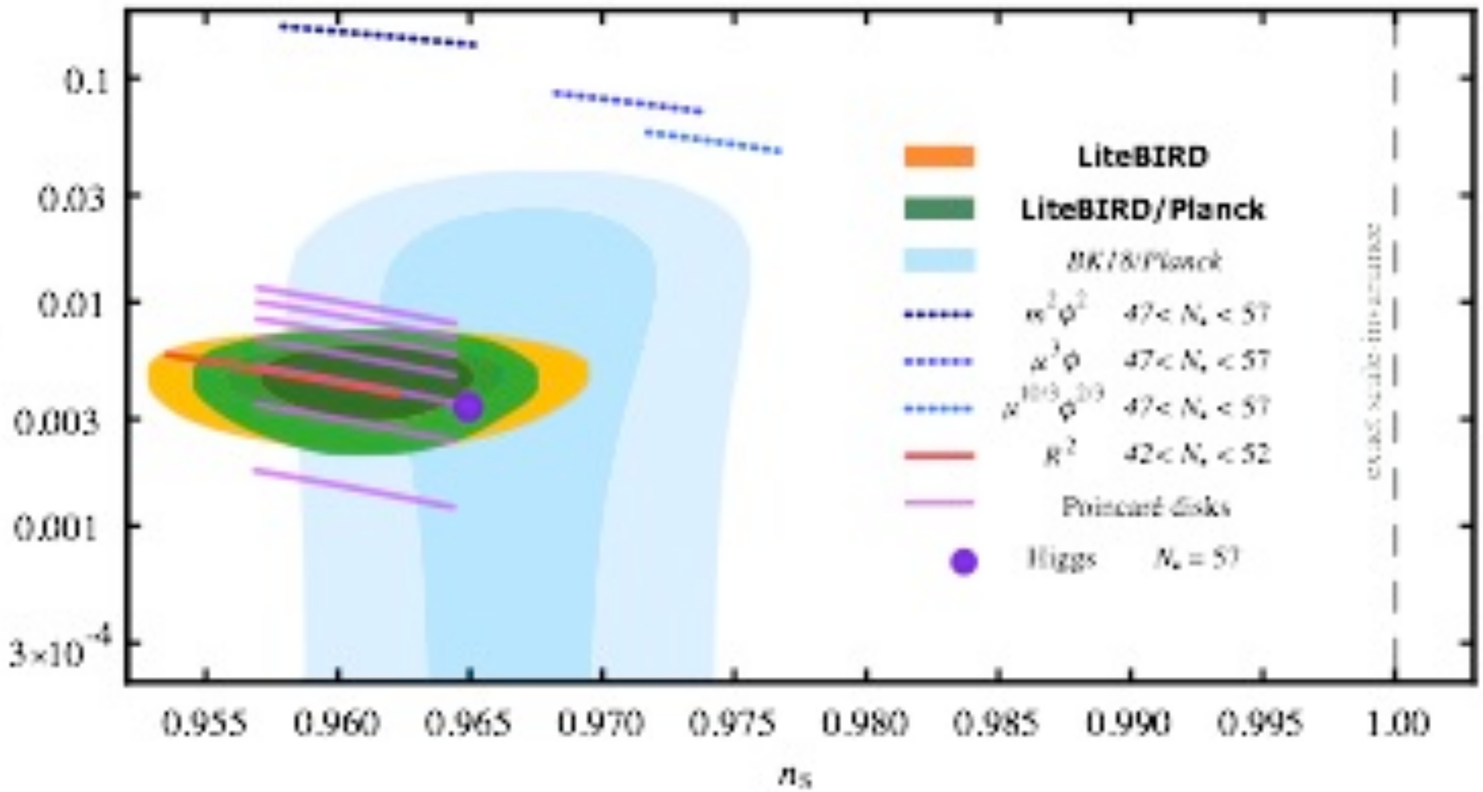




Forecasts for tensor-to-scalar ratio r

- **For future space CMB missions.**

Future constraints on inflationary models



Fro…

"Probing Cosmic Inflation with the LiteBIRD Cosmic Microwave Background Polarization Survey"

LiteBIRD collaboration, https://arxiv.org/pdf/2202.02773.pdf

When inflation ends??

Inflation ends when the inflaton field starts to ``feel'' the curvature of the potential

So this means that η starts to be $|\eta| \gtrsim 1$, and this in turn pushes in general $\epsilon \to 1$
So that inflation comes to an end

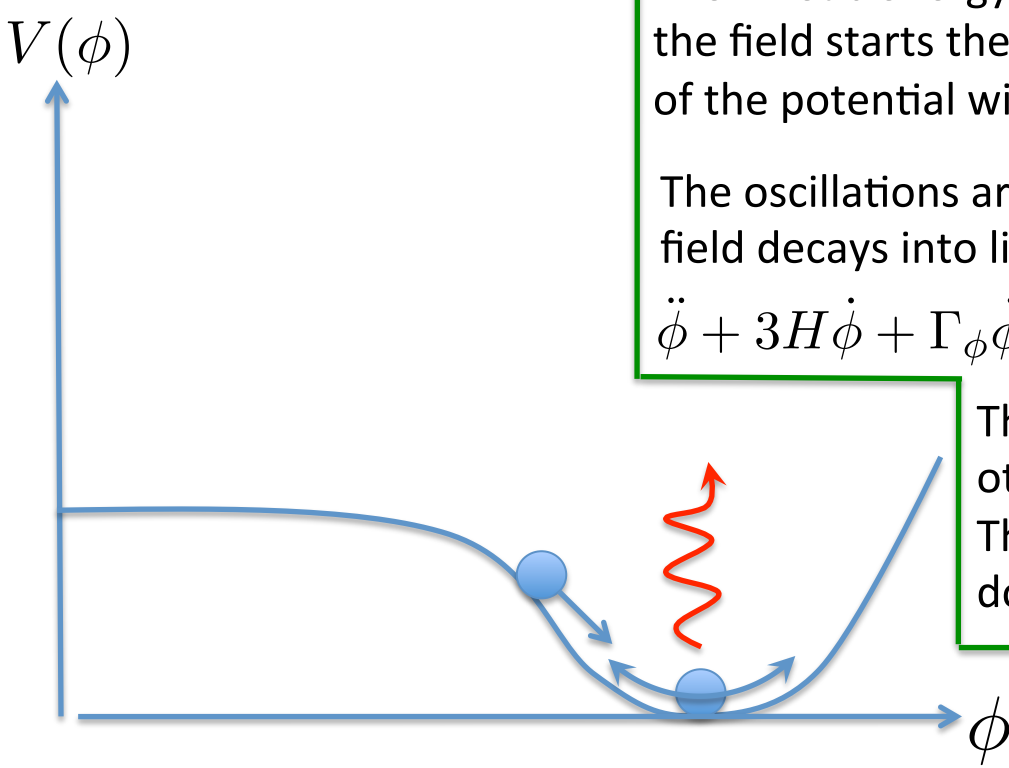
$$\frac{\ddot{a}}{a} = \dot{H} + H^2 = (1-\epsilon)H^2$$

REHEATING PHASE

The kinetic energy is not negligible anymore;
the field starts then to oscillate around the minimum
of the potential with frequency $V_{,\phi\phi} \gg H^2$

The oscillations are damped because in this regime the inflaton field decays into lighter particles with a decay rate Γ_ϕ

$$\ddot{\phi} + 3H\dot{\phi} + \Gamma_\phi \dot{\phi} = -V_{,\phi} \longrightarrow \ \dot{\rho}_\phi + 3H\rho_\phi = -\Gamma_\phi \rho_\phi$$



The energy of the inflaton is transferred to other lighter particles (release of latent heat). These particles thermalize and start to dominate→ ***the standard FRW universe starts***

How we arrive at these predcitions?

The quantum origin of cosmological perturbations: details

Inflation and the Inflaton

Consider a simple real scalar field:

$$S = S_{EH} + S_\phi = \int d^4x\sqrt{-g}\frac{1}{16\pi G}R + S_\phi$$

$$S_\phi = \int d^4x\sqrt{-g}\mathcal{L}_\phi[\phi, g_{\mu\nu}] = \int d^4x\sqrt{-g}\left[-\frac{1}{2}g^{\mu\nu}\phi_{,\mu}\phi_{,\nu} - V(\phi)\right]$$

3 ingredients:

- The scalar field (the so called inflaton field)
- the gravitational field (i.e., the metric)
- the ``rest of the world'': fermions, gauge bosons, other scalars.
  Usually, in the simplest models, these additional components turns out to be subdominant w.r.t. the inflaton field (because e.g., we know that for pressurless matter $\rho_m \sim a^{-3}$ while radiation $\rho_r \sim a^{-4}$, and so they decrease almosr exponentially during inflation).

Generating the primordial density perturbations

✓ first step: take a scalar field during an inflationary phase

$$\ddot{\phi} + 3H\dot{\phi} - \frac{\nabla^2 \phi}{a^2} = -\frac{\partial V}{\partial \phi}$$

split the scalar field into a ``classical'' background expectation value (on the vacuum state) and **quantum fluctuations** around the mean value

$$\phi(\mathbf{x}, t) = \phi(t) + \delta\phi(\mathbf{x}, t)$$

✓ ***Perturb linearly*** the equation of motion of the scalar field around its background value

$$\delta\ddot{\phi} + 3H\delta\dot{\phi} - \frac{\nabla^2 \delta\phi}{a^2} = -\frac{\partial^2 V}{\partial \phi^2}\,\delta\phi$$

Note:

It is convenient to go to Fourier space: at linear-order perturbations in Fourier space evolve independently (k-mode by k-mode)

Consider a random field f(t,**x**):

$$f_{\mathbf{k}}(t) = \int \frac{d^3x}{(2\pi)^{3/2}} e^{-i\mathbf{k}\cdot\mathbf{x}} f(t,\mathbf{x})$$

$$f(t,\mathbf{x}) = \int \frac{d^3\mathbf{k}}{(2\pi)^{3/2}} e^{i\mathbf{k}\cdot\mathbf{x}} f_{\mathbf{k}}(t)$$

N.B.: we are using a three-dimensional Fourier transform because in the equation of motion of the perturbations we have some term that depend on time. Moreover we are using plane-waves (OK if we can neglect the spatial curvature).

$$\delta\ddot{\phi} + 3H\delta\dot{\phi} - \frac{\nabla^2\delta\phi}{a^2} = -\frac{\partial^2 V}{\partial\phi^2}\delta\phi \longrightarrow \boxed{\delta\ddot{\phi}_{\mathbf{k}} + 3H\delta\dot{\phi}_{\mathbf{k}} + \frac{k^2}{a^2}\delta\phi_{\mathbf{k}} = -\frac{\partial^2 V}{\partial\phi^2}\delta\phi_{\mathbf{k}}}$$

Quantum fluctuations of a scalar field during inflation

$$\delta\ddot{\phi}_{\mathbf{k}} + 3H\delta\dot{\phi}_{\mathbf{k}} + \frac{k^2}{a^2}\delta\phi_{\mathbf{k}} = \cancel{-V_{,\phi\phi}}\,\delta\phi_{\mathbf{k}}$$

A massless or light scalar field: $m^2=V_{,\phi\phi} << H^2$ for slow-roll

✓ when the perturbation modes are ***within the horizon***:

λ << (comoving) Hubble radius = (a H)$^{-1}$

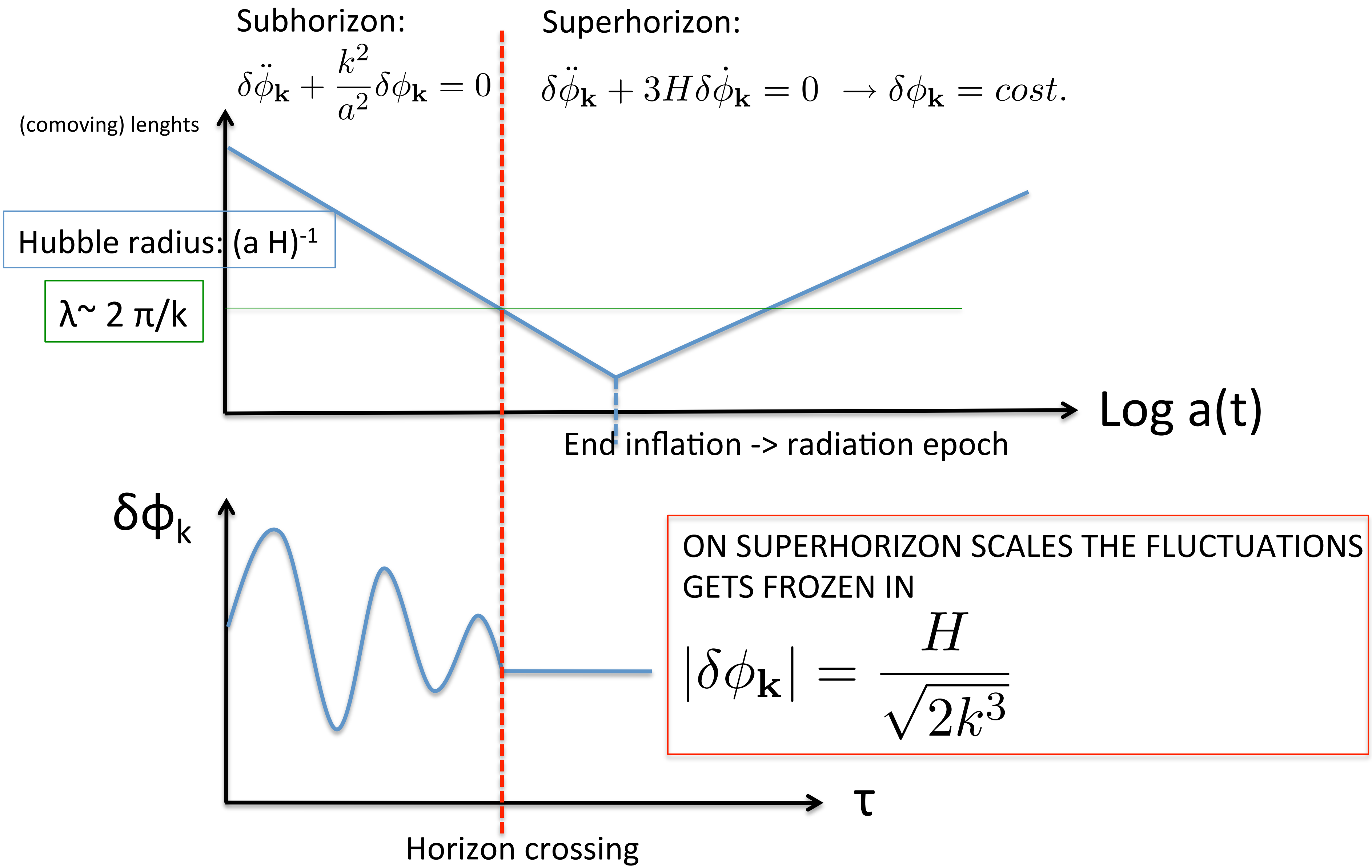
k>> (aH)

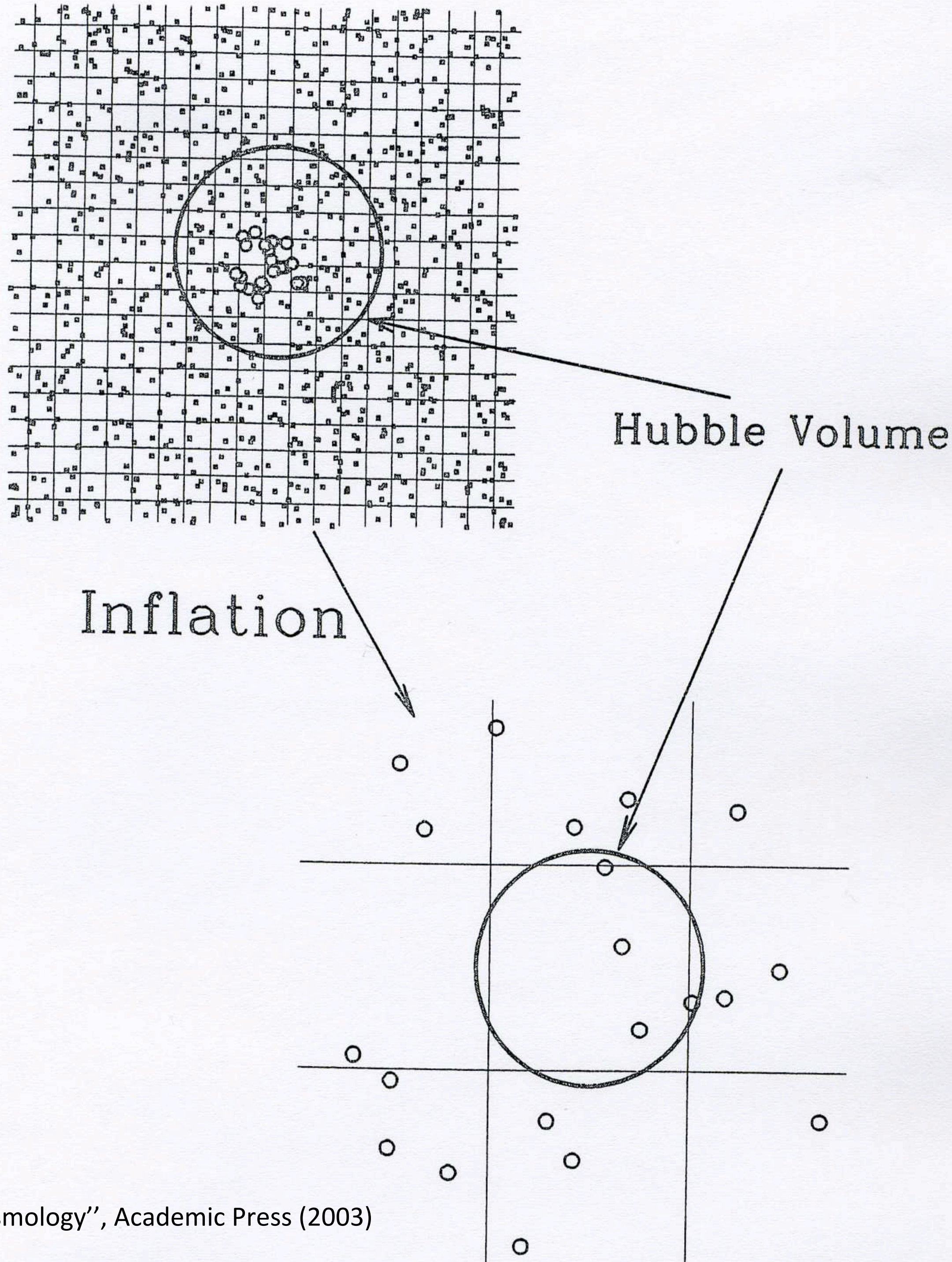
$$\delta\ddot{\phi}_{\mathbf{k}} + \frac{k^2}{a^2}\delta\phi_{\mathbf{k}} = 0 \ \rightarrow oscillations$$

✓ when the perturbation modes are ***superhorizon scales***:

$$\delta\ddot{\phi}_{\mathbf{k}} + 3H\delta\dot{\phi}_{\mathbf{k}} = 0 \ \rightarrow \delta\phi_{\mathbf{k}} = cost. \quad \textbf{(k<<aH)}$$

Remember: H≈const.


Subhorizon:
$\delta\ddot{\phi}_{\mathbf{k}} + \frac{k^2}{a^2}\delta\phi_{\mathbf{k}} = 0$
Superhorizon:
$\delta\ddot{\phi}_{\mathbf{k}} + 3H\delta\dot{\phi}_{\mathbf{k}} = 0 \;\rightarrow\; \delta\phi_{\mathbf{k}} = cost.$
(comoving) lenghts
Hubble radius: $(a\,H)^{-1}$
$\lambda \sim 2\,\pi/k$
Log a(t)
End inflation -> radiation epoch
$\delta\phi_k$
ON SUPERHORIZON SCALES THE FLUCTUATIONS GETS FROZEN IN
$|\delta\phi_{\mathbf{k}}| = \frac{H}{\sqrt{2k^3}}$
τ
Horizon crossing



From S. Dodelson ``Modern Cosmology'', Academic Press (2003)

So what's going on?

On microscopic scales (well inside the horizon) microphysics is at work: use quantum field theory. There are quantum fluctuations of the scalar field; if averaged over macroscopic intervals of time they vanish (quantum fluctuations of vacuum: particles are continuously created and destroyed).

However the space-time background is exponentially inflating so their physical wavelengths grow exponentially

$$\lambda_{phys} \propto a(t) \propto e^{Ht}$$

until they become greater then the horizon H^{-1} (which remains almost constant). On super-horizon scales the fluctuations get frozen (because of the friction term $3H\delta\dot{\phi}$). The fluctuations do not vanish if averaged on macroscopic time intervals: a classical fluctuation has been generated.

Said in other words: if on superhorion scales $\delta\phi \neq 0$ over macroscopic time intervals then the final result is a state with a net number of particles. This is a gravitational mechanism of amplification. The crucial point is the "in" and "out" (of the horizon) state of the fluctuations

Let me give you some details about the computation of

Quantum fluctuations of a scalar field during inflation*

Let us find an exact (meaning valid at every k) solution

Quantum fluctuations of a scalar field during inflation

$\widetilde{\delta\phi} = a(t)\delta\phi$; quantize the theory by promoting the scalar field to an operator

$$\widetilde{\delta\phi}(\tau, \mathbf{x}) = \int \frac{d^3\mathbf{k}}{(2\pi)^{3/2}} \left[u_k(\tau) a_{\mathbf{k}} e^{i\mathbf{k}\cdot\mathbf{x}} + u_k^*(\tau) a_{\mathbf{k}}^\dagger e^{-i\mathbf{k}\cdot\mathbf{x}} \right]$$

Classical function of time normalized as $u_k^* u_k' - u_k u_k^{*\prime} = -i$

$a_{\mathbf{k}}$ is the ***annihillation operator:*** $a_{\mathbf{k}}|0\rangle = 0$ for all **k**; $|0\rangle$ is the ***(free) vacuum state.***

$a_{\mathbf{k}}^\dagger$ is the ***creation operator:*** $\langle 0|a_{\mathbf{k}}^\dagger = 0$ for all **k**

(classically they would correspond to two constants of integration)

They obey the commutation relation for bosons

$$[a_{\mathbf{k}}, a_{\mathbf{k}'}] = 0, \quad [a_{\mathbf{k}}, a_{\mathbf{k}'}^\dagger] = \hbar\, \delta^{(3)}(\mathbf{k} - \mathbf{k}')$$

quantum mechanics!

Quantum fluctuations of a scalar field during inflation

$$\delta\ddot{\phi}_{\mathbf{k}} + 3H\delta\dot{\phi}_{\mathbf{k}} + \frac{k^2}{a^2}\delta\phi_{\mathbf{k}} = -V_{,\phi\phi}\,\delta\phi_{\mathbf{k}} \longrightarrow \quad u_k'' + \left(k^2 - \frac{a''}{a} + V_{,\phi\phi}\,a^2\right)u_k = 0$$

Primes denote a derivative w.r.t. conformal time

✓ In flat space-time, once the commutation relations are fixed, everything is fixed: the solution is a plane wave. In a curved space time there is some ambiguity in defining the vacuum state (it depends on the choice of $u_k(\tau)$).

So we require that at very short distances and at early times (when the expansion is negligible) the solutions reproduce the correct form of a flat space-time. So we require that when a given mode is well inside the horizon

$$u_k(\tau) \rightarrow \frac{1}{\sqrt{2k}}e^{-ik\tau} \quad \text{for} \quad \left(\frac{k}{aH}\right) \gg 1$$

This is the so called "Bunch-Davies vacuum choice"

Quantum fluctuations of a generic scalar field in quasi de-Sitter

- If we are not in de Sitter, and if the mass of the scalar field is small but not zero, then

$$\epsilon = -\frac{\dot{H}}{H^2} \qquad \eta_V = \frac{1}{3}\frac{V_{,\phi\phi}}{H^2}$$

and the equation to solve reads $$u_k'' + \left(k^2 - \frac{a''}{a} + M^2 a^2\right) u_k = 0$$

with $$\frac{M^2}{H^2} = 3\eta_V - 6\epsilon$$

$$a(\tau) = -\frac{1}{H\tau(1-\epsilon)} \longrightarrow a''/a \simeq \frac{2}{\tau^2}\left(1 + \frac{3}{2}\epsilon\right) \longrightarrow \boxed{u_k'' + \left(k^2 - \frac{\nu^2 - \frac{1}{4}}{\tau^2}\right) = 0}$$

same Bessel equation as before (and same solution up to first-order in the slow-roll parameters) but now with

$$\boxed{\nu \simeq \frac{3}{2} + 3\epsilon - \eta_V}$$

N.B.: actually we are also accounting for the metric perturbations that enter into the equation for the scalar field

Quantum fluctuations of a generic scalar field in quasi de-Sitter

✓ So the solution is $u_k(\tau) = \frac{\sqrt{\pi}}{2} e^{i\left(\nu+\frac{1}{2}\right)\frac{\pi}{2}} \sqrt{-\tau}\, H_\nu^{(1)}(-k\tau)$

We are interested in the value of the fluctuations ***on superhorizon scales* -k τ=k /(aH)<<1**

$$H_\nu^{(1)}(x \ll 1) \sim \sqrt{2/\pi}\, e^{-i\frac{\pi}{2}}\, 2^{\nu_\chi - \frac{3}{2}} \left(\Gamma(\nu\ )/\Gamma(3/2)\right) x^{-\nu} \quad \rightarrow$$

$$u_k(\tau) \simeq \frac{1}{\sqrt{2k}} e^{i\left(\nu-\frac{1}{2}\right)\frac{\pi}{2}} 2^{\nu-\frac{3}{2}} \left(\frac{\Gamma(\nu)}{\Gamma(\frac{3}{2})}\right) (-k\tau)^{\frac{1}{2}-\nu}$$

✓ At lowest-order in the slow roll parameter then one finds

(use τ= -1/(H a)) and $\delta\phi_k = u_k(\tau)/a$)

$$\boxed{|\delta\phi_k| = \frac{H}{\sqrt{2k^3}} \left(\frac{k}{aH}\right)^{\frac{3}{2}-\nu}}$$

ON SUPERHORIZON SCALES: -k τ=k /(aH)<<1

$$\boxed{\nu \simeq \frac{3}{2} + 3\epsilon - \eta_V}$$

The power spectrum of cosmological perturbations: a quick definition

Consider a random field f(t,**x**): $f(t,\mathbf{x}) = \int \frac{d^3\mathbf{k}}{(2\pi)^{3/2}}\, e^{i\mathbf{k}\cdot\mathbf{x}}\, f_{\mathbf{k}}(t)$

$$\langle f_{\mathbf{k_1}} f^*_{\mathbf{k_2}} \rangle = \frac{2\pi^2}{k^3}\mathcal{P}_f(k_1)\delta^{(3)}(\mathbf{k_1}-\mathbf{k_2})$$

(dimensionless) Power-spectrum

f(t,**x**) can be the fractional energy density perturbation δρ/ρ, or the scalar field perturbation, in which case the brackets denote the expectation value on the vacuum state and it can be computed using creation and annihiliaton operators.

N.B.: $f^*_{\mathbf{k}2} = f_{-\mathbf{k}2}$ if f is real and so we could also write $\langle f_{\mathbf{k_1}} f_{\mathbf{k_2}} \rangle = \frac{2\pi^2}{k^3}\mathcal{P}_f(k_1)\delta^{(3)}(\mathbf{k_1}+\mathbf{k_2})$

Spectral index of the power spectrum: definition

$$n_s - 1 = \frac{d \ln \mathcal{P}(k)}{d \ln k}$$

So, if n_s is a constant

$$\mathcal{P}(k) = \mathcal{P}(k_0) \left(\frac{k}{k_0} \right)^{n_s - 1}$$

- So the spectral index describes the shape of the power spectrum (i.e. its dependence with k~(2 π)/λ, or equivalently with the cosmological scales).

- If n_s=1 we have an exact scale-invariant power spectrum which is also called Harrison-Zel' dovich power-spectrum: the amplitude of the initial fluctuations is the same on all cosmological scales.

Let us compute the power spectrum for the scalar field fluctuations

$$\langle \delta\phi_{\mathbf{k_1}} \delta\phi^*_{\mathbf{k_2}} \rangle \equiv \langle 0 | \delta\phi_{\mathbf{k}_1} \delta\phi^*_{\mathbf{k}_2} | 0 \rangle$$

Use

$$\widetilde{\delta\phi}(\tau, \mathbf{x}) = \int \frac{d^3\mathbf{k}}{(2\pi)^{3/2}} \left[u_k(\tau) a_{\mathbf{k}} e^{i\mathbf{k}\cdot\mathbf{x}} + u^*_k(\tau) a^\dagger_{\mathbf{k}} e^{-i\mathbf{k}\cdot\mathbf{x}} \right]$$

$$a_{\mathbf{k}} |0\rangle = 0$$

$$[a_{\mathbf{k}}, a_{\mathbf{k}'}] = 0, \quad [a_{\mathbf{k}}, a^\dagger_{\mathbf{k}'}] = \hbar\, \delta^{(3)}(\mathbf{k} - \mathbf{k}')$$

Then the only non zero combinations are $\langle 0 | a a^\dagger | 0 \rangle \neq 0$

$$\langle \delta\phi_{\mathbf{k_1}} \delta\phi^*_{\mathbf{k_2}} \rangle = \frac{|u_k|^2}{a^2} \delta^{(3)}(\mathbf{k_1} - \mathbf{k_2}) \longrightarrow$$ (remember $\delta\phi_k = u_k(\tau)/a$)

$$\boxed{\mathcal{P}_{\delta\phi}(k) = \frac{k^3}{2\pi^2} |\delta\phi_k|^2}$$

$$\longrightarrow \quad \boxed{\mathcal{P}_{\delta\phi}(k) = \left(\frac{H}{2\pi}\right)^2 \left(\frac{k}{aH}\right)^{3-2\nu}}$$

ON SUPERHORIZON SCALES

$$3 - 2\nu = 2\eta_V - 6\epsilon$$

From quantum fluctuations to density perturbations

The inflaton field is special: it dominates the energy density of the universe during inflation with $\rho_\phi \simeq V(\phi)$

$$\delta\phi \rightarrow \delta\rho \simeq V'(\phi)\delta\phi \simeq -3H\dot{\phi}\,\delta\phi$$

Fluctuations in the inflaton produce fluctuations in the universe expansion from place to place, so that each region in the universe goes through the same expansion history but at slightly different times:

$\delta t = -\dfrac{\delta\phi}{\dot{\phi}}$; now remember that $\text{number of e-foldings}\, N = \ln(a_f/a_i) = \int_{t_i}^{t_f} H dt$

$\rightarrow \text{additional expansion}\, \zeta = H\delta t$

$$\boxed{\zeta = H\delta t \quad = -H\frac{\delta\phi}{\dot{\phi}} \simeq -H\frac{\delta\rho}{\dot{\rho}}}$$

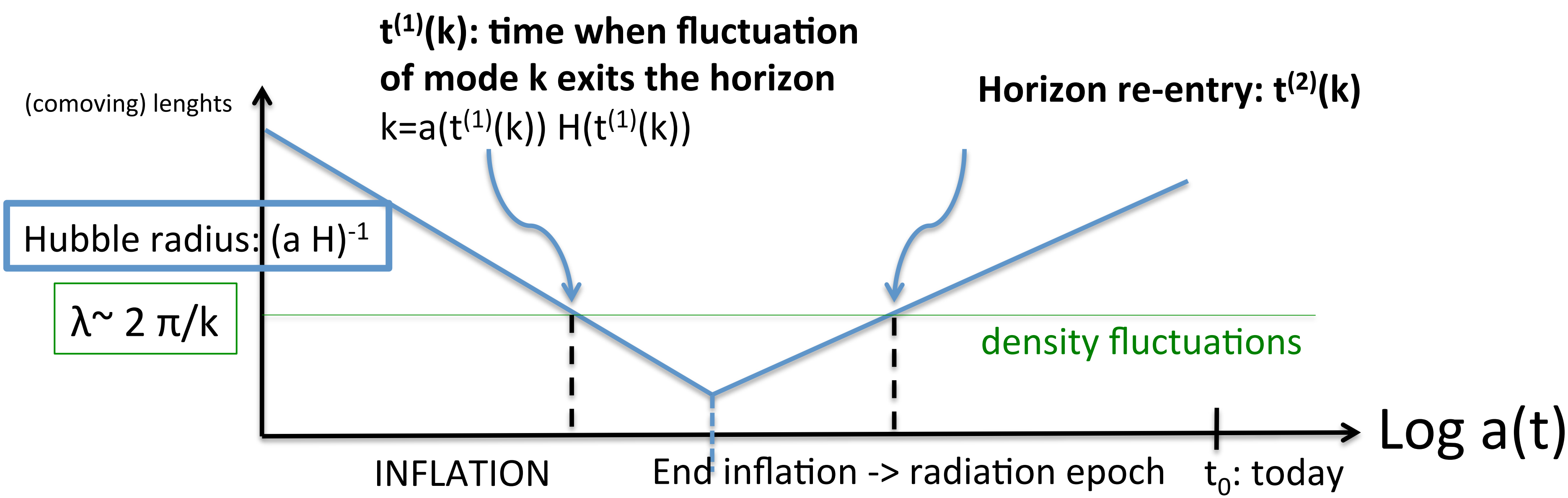
ζ remains constant on superhorizon scales (ζ is the uniform energy density curvature pert.)

N.B.: to obtain the last expression for ζ just use $\dot{\rho_\phi} = -3H(\rho_\phi + p_\phi) = -3H\dot{\phi}^2$

Generating the primordial density perturbations

N.B: ***on superhorizon scales ζ=constant***,
so one can easily relate density fluctuations after inflation with the quantum fluctuations of the inflaton field $\delta\phi \sim H/2\pi$

$$\zeta|_{t^{(2)}(k)} = -H\frac{\delta\rho}{\dot\rho}\,|_{t^{(2)}(k)} = \zeta|_{t^{(1)}(k)} \simeq -H\frac{\delta\phi}{\dot\phi}|_{t^{(1)}(k)}$$



Primordial power spectrum

$$-H\frac{\delta\rho}{\dot{\rho}}\ |_{t^{(2)}(k)} \simeq -H\frac{\delta\phi}{\dot{\phi}}|_{t^{(1)}(k)}$$

- Therefore the power-spectrum of density perturbations (i.e., their amplitude and dependence on the scale) will depend on the specific inflationary model, since, remember that

$$H^2 \simeq \frac{8\pi G}{3}V(\phi) \qquad \dot{\phi} \simeq -\frac{V_{,\phi}}{3H}$$

- The scale dependence comes from evaluating at the epoch of horizon crossing during inflation ($t^{(1)}(k)$). But we know that H and φ vary in time very slowly: the level of density fluctuations depends weakly on the cosmological scale $\lambda \sim 2\pi/k$ ***(if exact scale-invariance: Harrison-Zel' dovich spectrum).***

In fact H and φ vary a little bit: you expect a spectrum of density perturbations which is ***nearly*** **scale-invariant.**

So let us compute precisely (at first-order in the slow-roll parameters) the power spectrum of curvature perturbations and its spectral tilt (called scalar spectral titl)

$$\boxed{\mathcal{P}_\zeta = \frac{H^2}{\dot{\phi}^2}\mathcal{P}_{\delta\phi} = \left(\frac{H^2}{2\pi\dot{\phi}}\right)^2_{t^{(1)}(k)} = \left(\frac{H^2}{2\pi\dot{\phi}}\right)^2\left(\frac{k}{aH}\right)^{3-2\nu}}$$

(remember $\mathcal{P}_{\delta\phi} \sim H^2$)

Where the last equality is due to the fact that the curvature perturbation ζ remains constant on super-horizon scales

<u>To compute the spectral tilt</u>

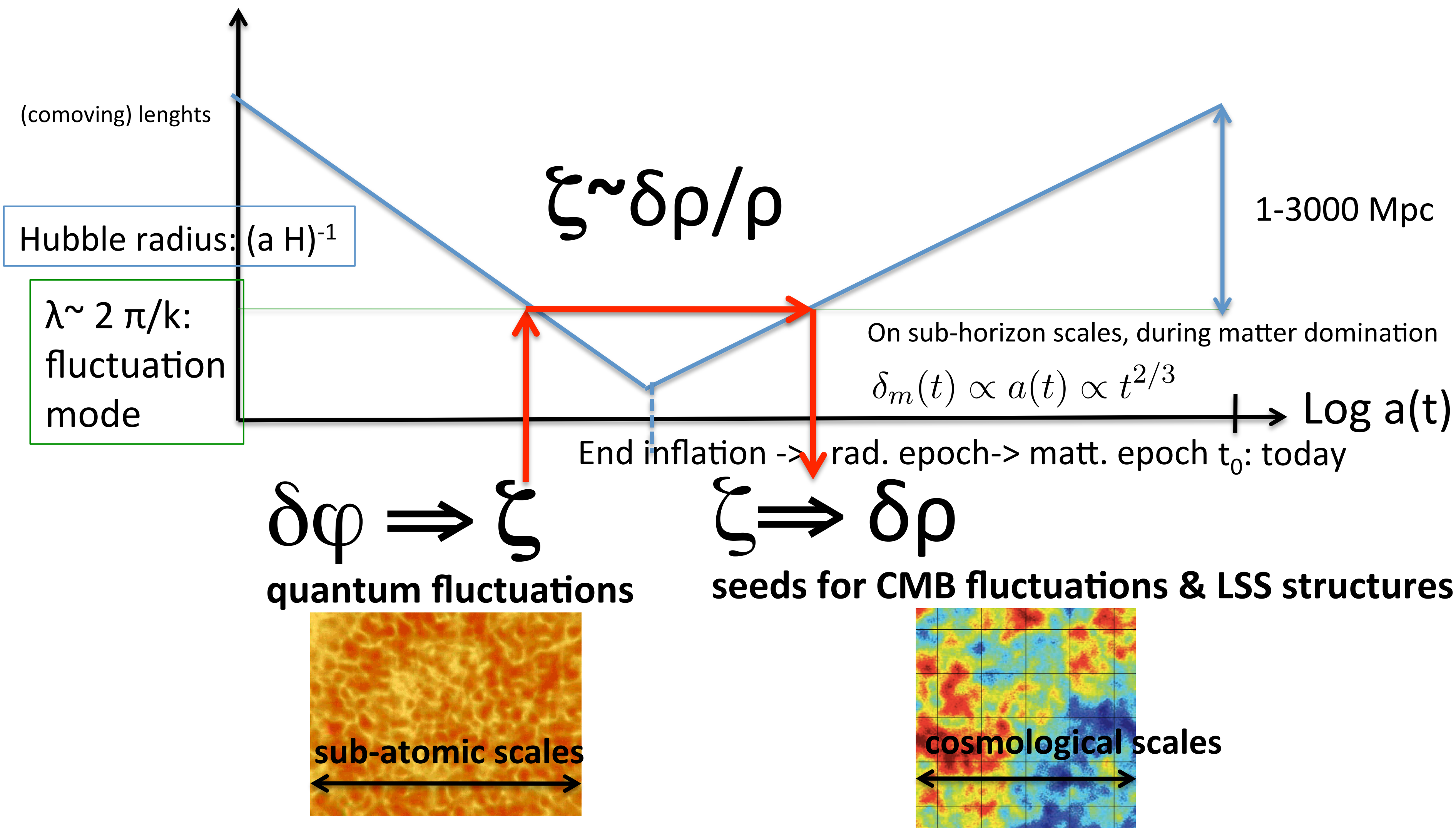
$$n_s - 1 = \frac{d\ln\mathcal{P}_\zeta}{d\ln k}$$

Then simply it follows from the last expression for in the power spectrum $\mathcal{P}_\zeta$

$$\boxed{n_s - 1 = 3 - 2\nu = 2\eta_V - 6\epsilon}$$

Structure formation within the inflationary scenario

Quantum fluctuations are streched from microscopic to cosmological scales



Primordial gravitational waves

GWs are tensor perturbations of the metric. Restricting ourselves to a flat FRW background (and disregarding scalar and vector modes)

$$\boxed{ds^2=a^2(\tau)[- d\tau^2 + (\delta_{ij} + h_{ij}(\underline{x},\tau))\, dx^i\, dx^j]}$$

where h_{ij} are tensor modes which have the following properties

$h_{ij} = h_{ji}$ (symmetric)

$h^i_i = 0$ (traceless)

$h^i_{j|i} = 0$ (transverse, i.e. divergence free)

and satisfy the equation of motion

$$h''_{ij} + 2\frac{a'}{a}h'_{ij} - \nabla^2 h_{ij} = 0$$

$\boxed{' = d/d\tau}$

i,j=1,2,3

Primordial gravitational waves

GWs have only (9→6-1-3=) 2 independent degrees of freedom, corresponding to the 2 polarization states of the graviton

$$h_{ij}(\vec{x},\tau) = \int \sum_{\lambda=+,\times} \frac{d^3k}{(2\pi)^3} e^{i\vec{k}\bullet\vec{x}} h_\lambda(\vec{k},\tau)\varepsilon_{ij}^\lambda(\vec{k})$$

$$h_\lambda '' + 2\frac{a'}{a} h_\lambda ' + k^2 h_\lambda = 0$$

polarization tensor

free massless, minimally coupled scalar field

behaviour:

k « aH (outside the horizon) **h ≈ const + decaying mode**

k » aH (inside the horizon**)** **h ≈ $e^{\pm ik\tau}/a$** gravitational wave; it freely streams, experiencing redshift and dilution, like a free photon)

Primordial gravitational waves

In a similar way one can compute the power spectrum of the gravitational waves

$$\ddot{h}_\lambda + 3H\dot{h}_\lambda + \frac{k^2}{a^2}h_\lambda = 0$$

We see that the 2 polarization states corresponds to 2 massless minimally coupled scalar fields. Then we have (a "*" here indicates evaluation at horizon crossing during inflation)

This equality holds because, on super-horizon scales, tensor fluctuations remain constant in time (see results for a massless scalar field) and so its value on those scales is fixed at horizon-crossing during inflation (similarly to what we did for the curvature perturbations)

$$\mathcal{P}_{h_{+,\times}} = 32\pi G\,\mathcal{P}_{\phi_{+,\times}} = \frac{4}{M_{\rm Pl}^2}\left(\frac{H_*}{2\pi}\right)^2 = \frac{4}{M_{\rm Pl}^2}\left(\frac{H}{2\pi}\right)^2\left(\frac{k}{aH}\right)^{-2\epsilon}$$

And hence, summing over the 2 polarization states:

$$\mathcal{P}_T = \frac{8}{M_{\rm Pl}^2}\left(\frac{H}{2\pi}\right)^2\left(\frac{k}{aH}\right)^{-2\epsilon}$$ with ***tensor spectral index*** $n_T = \dfrac{d\ln\mathcal{P}_T}{d\ln k} = -2\epsilon$

Current observational status

Constraints from CMB data

- **Primordial density perturbations:** Amplitude $\ln(10^{10}A_s) = 3.044 \pm 0.014 \quad (68\%\,\mathrm{CL})$

$$n_s = 0.9649 \pm 0.0042 \quad (68\%\,\mathrm{CL})$$

n_s=1 (Harrison Zeld' ovich spectrum) excluded at 8.4 sigmas!!

Two fundamental observational constants of cosmology in addition to three very well known (Ω_b, Ω_{cdm}, Ω_Λ)

Latest constraints

- **Primordial gravitational waves:**

Using Planck data release 3 and 4 (including BB power spectrum data)
+ BICEP/Keck Array 2015 and 2018

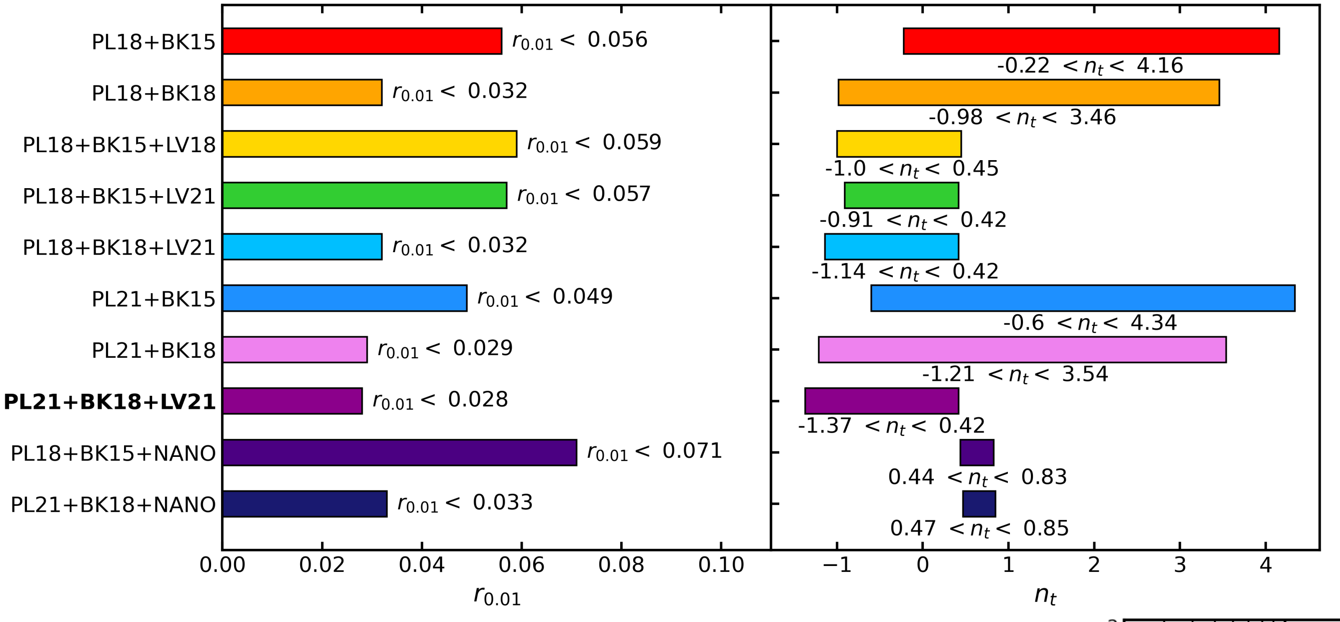
$$r < 0.028 \quad (95\%\ \mathrm{C.L.})$$

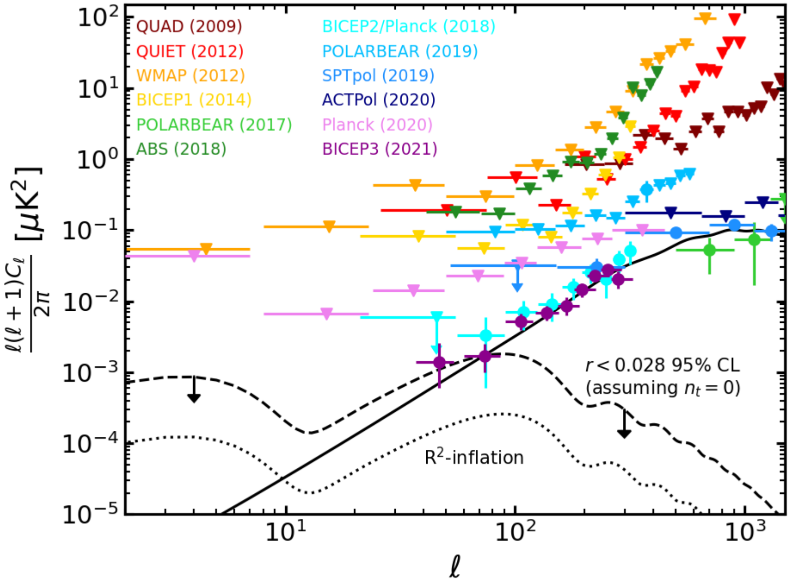
Galloni, N.B., Matarrese, Migliaccio, Ricciardone, Vittorio '23

Energy scale of inflation $V^{1/4} < 1.4 \times 10^{16}\ \ \mathrm{GeV}$

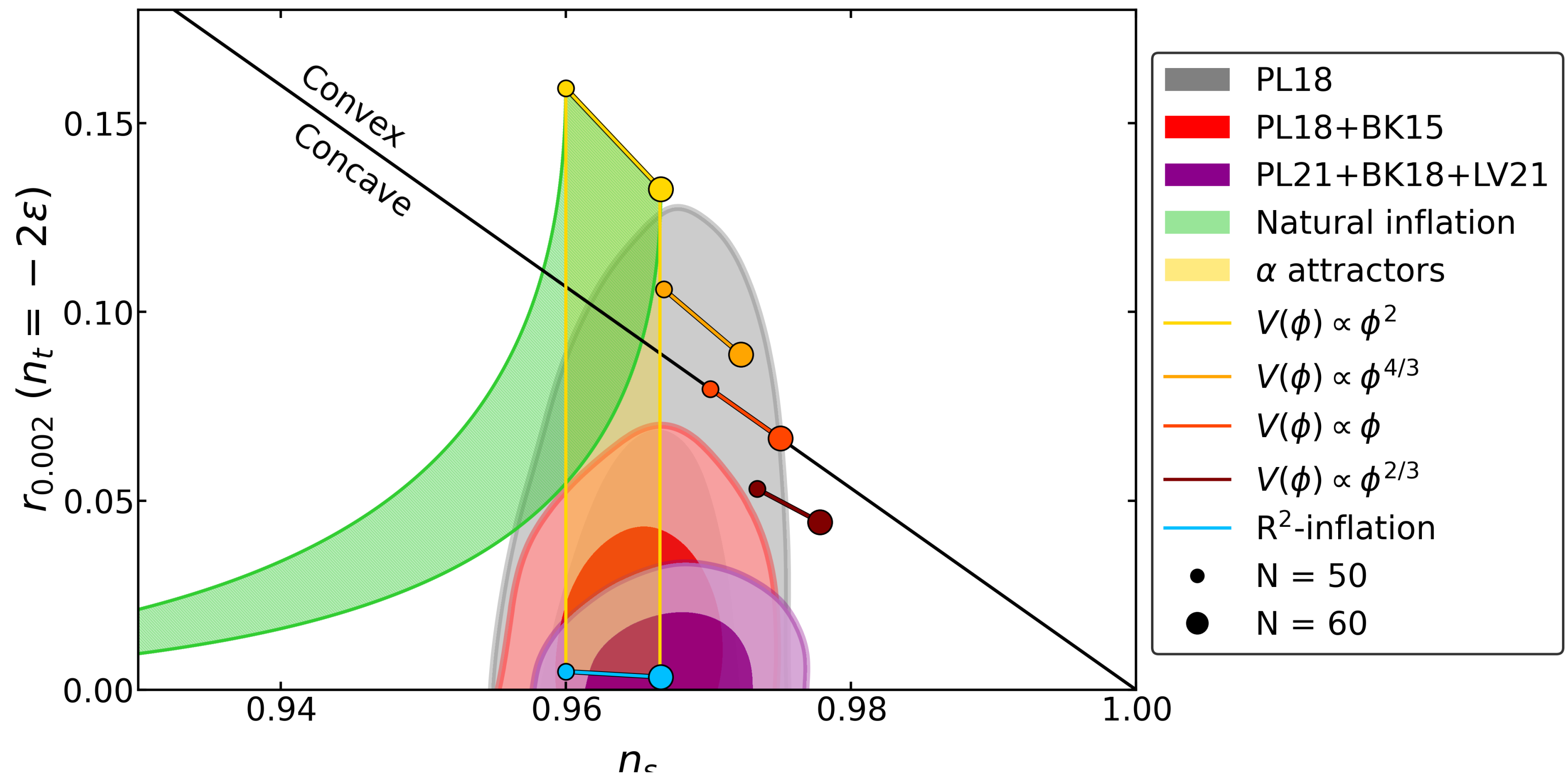
A new era (the CMB B-mode era) has started!
Target of future CMB experiments: $r < 10^{-3}$

Constraints from CMB data

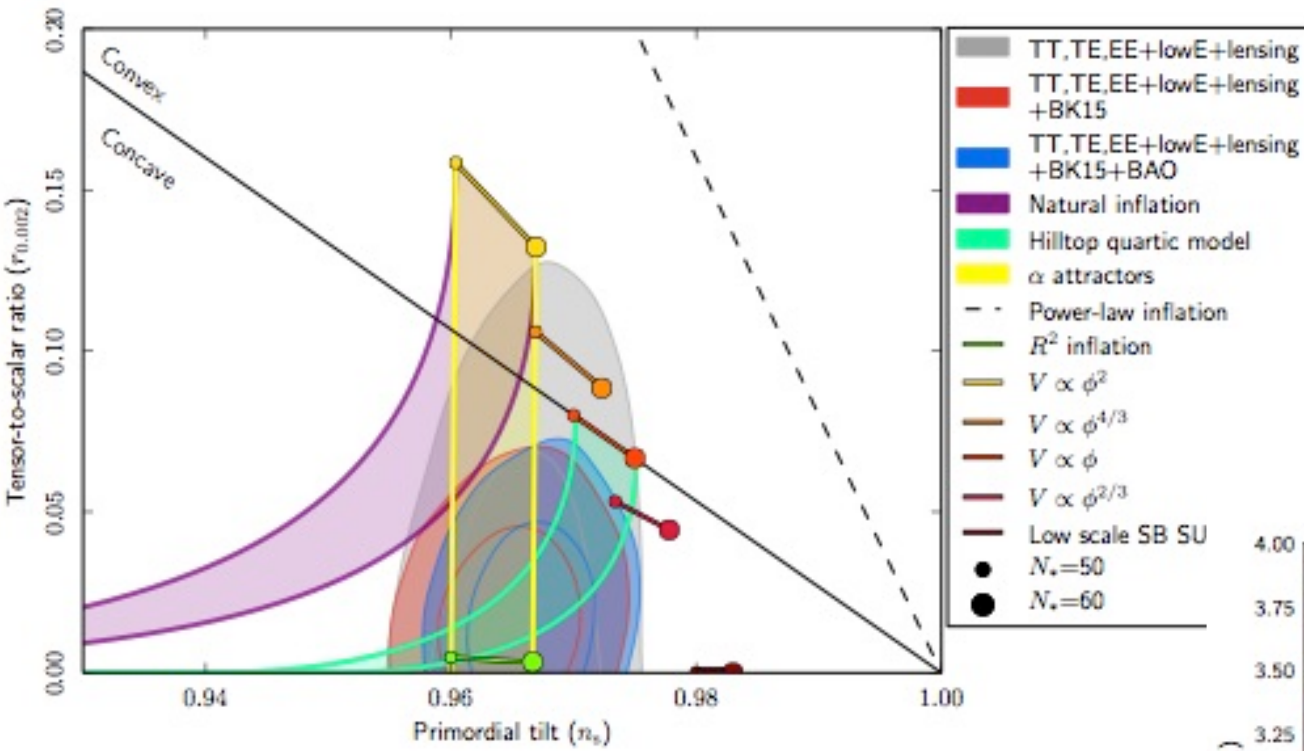




Implications for standard single-field models of slow-roll inflation

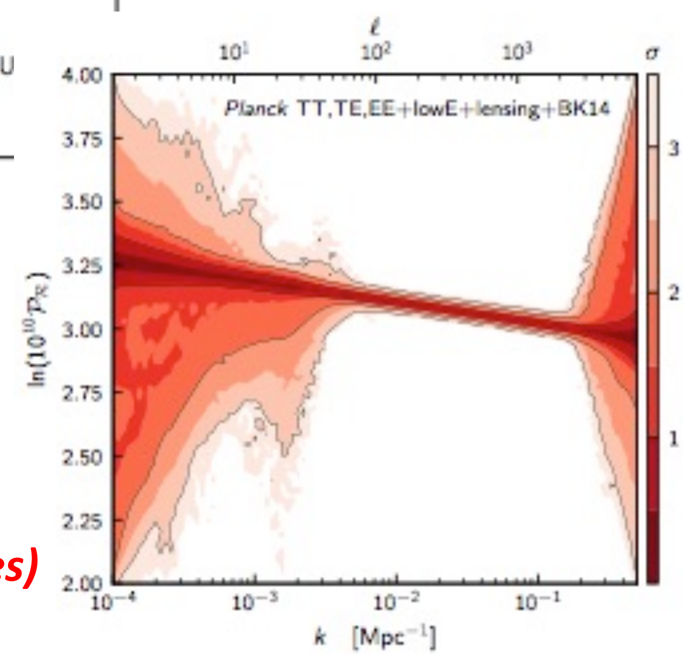


Beyond the r-n_s plane



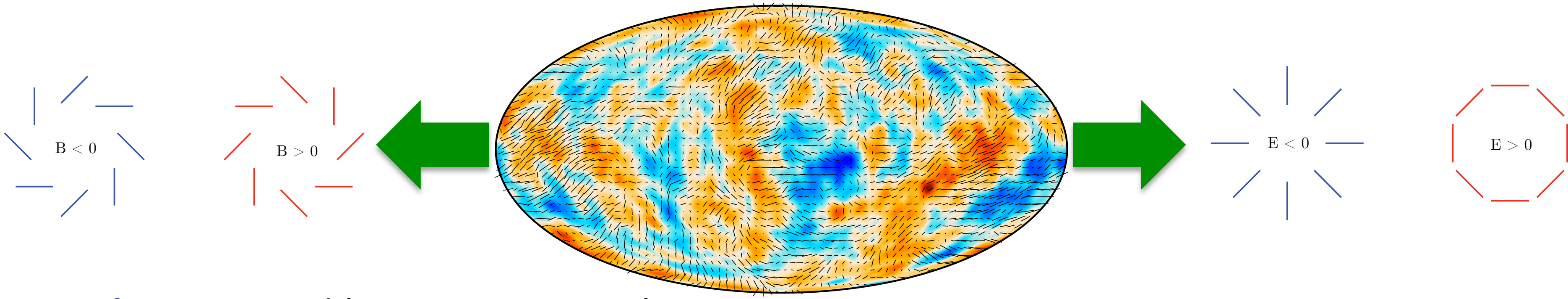
$f_{\rm NL}$

Shape	Independent	Lensing subtracted
	SMICA $T+E$	
Local	4.1 ± 5.1	-0.9 ± 5.1
Equilateral	-25 ± 47	-26 ± 47
Orthogonal	-47 ± 24	-38 ± 24



No evidence of deviations from a featureless power-spectrum (for curvature perturbations on CMB scales)

Searching for inflationary GWs via CMB polarization

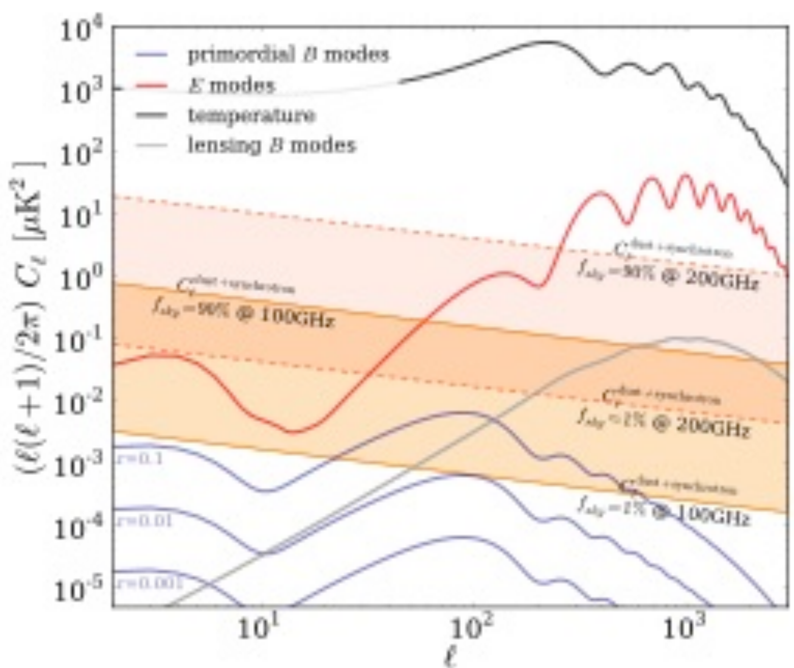


B-modes: Sourced by tensor perturbations but not by density perturbations

E-modes: from scalar and tensor pert.

$$P_{\mathrm{T}} \sim \left(\frac{V}{M_{\mathrm{Pl}}}\right)^4$$

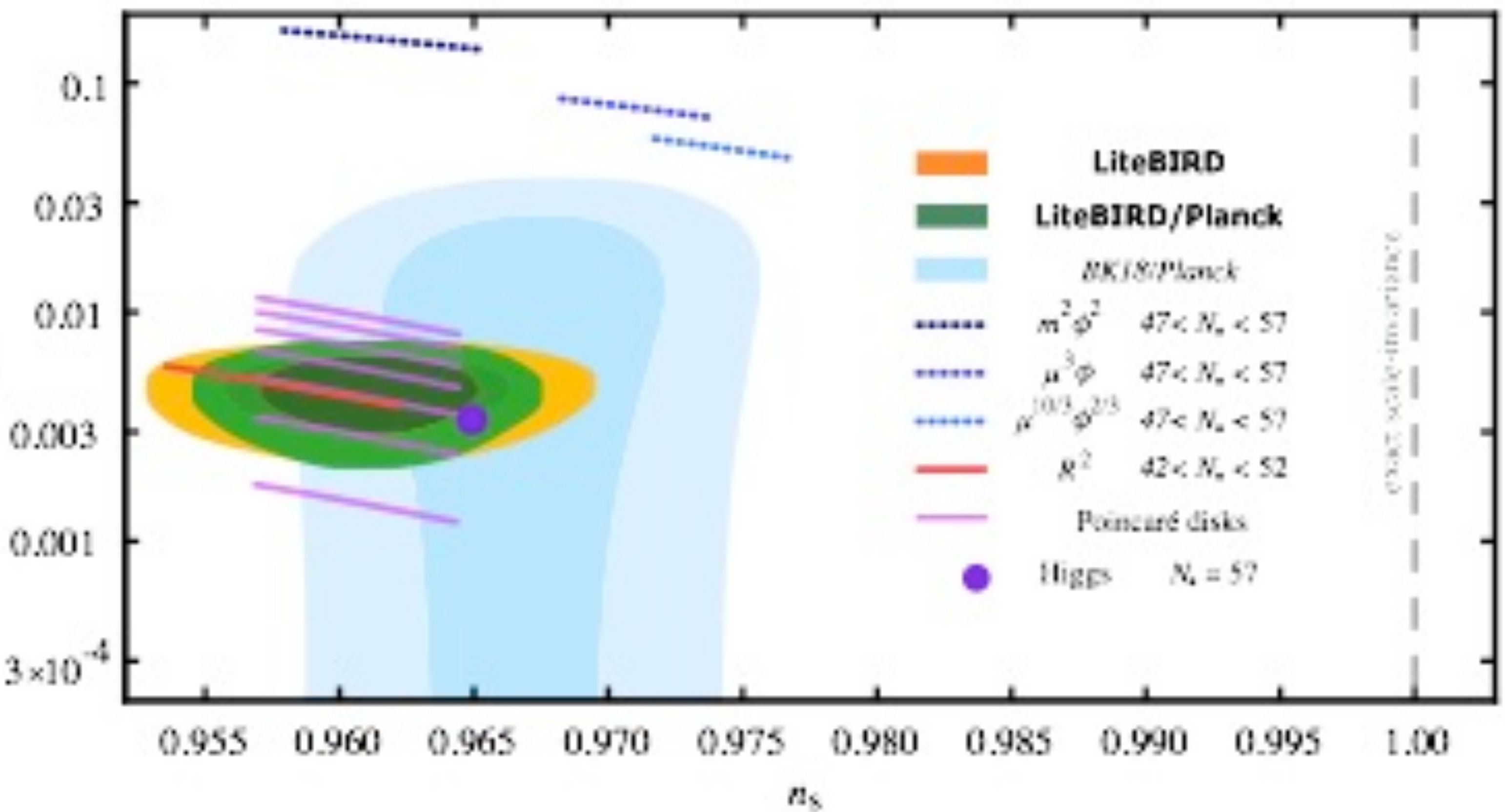
Primary goal for future CMB surveys: $\delta r < 10^{-3}$



Forecasts for tensor-to-scalar ratio r

- **For future space CMB missions.**

Future constraints on inflationary models



Fro...
"Probing Cosmic Inflation with the LiteBIRD Cosmic Microwave Background Polarization Survey"
LiteBIRD collaboration, https://arxiv.org/pdf/2202.02773.pdf

When inflation ends??

Inflation ends when the inflaton field starts to ``feel'' the curvature of the potential

So this means that η starts to be $|\eta| \gtrsim 1$, and this in turn pushes in general $\epsilon \to 1$
So that inflation comes to an end

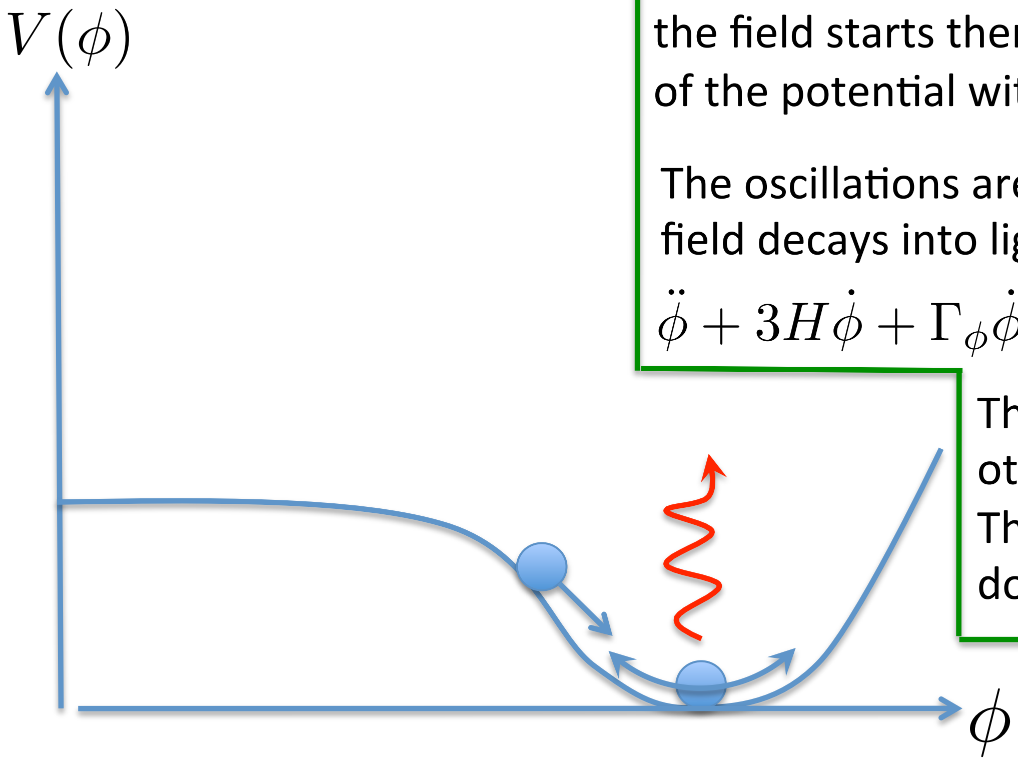
$$\frac{\ddot{a}}{a} = \dot{H} + H^2 = (1-\epsilon)H^2$$

REHEATING PHASE

The kinetic energy is not negligible anymore;
the field starts then to oscillate around the minimum of the potential with frequency $V_{,\phi\phi} \gg H^2$

The oscillations are damped because in this regime the inflaton field decays into lighter particles with a decay rate Γ_ϕ

$$\ddot{\phi} + 3H\dot{\phi} + \Gamma_\phi \dot{\phi} = -V_{,\phi} \longrightarrow \dot{\rho}_\phi + 3H\rho_\phi = -\Gamma_\phi \rho_\phi$$



The energy of the inflaton is transferred to other lighter particles (release of latent heat). These particles thermalize and start to dominate→ ***the standard FRW universe starts***

What is the inflation model……..??

$$S = \int d^4x \sqrt{-g} \left[????? \right]$$

Standard single-field

Features in the potential

Higher-order derivative interactions or non-canonical kinetic term

$\mathcal{L}(\phi, X)$ with $X = \frac{1}{2} g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi$

Modified gravitational sector

Multiple-fields

2 ways to reach the goal:

1. Gravitational waves
2. Primordial non-Gaussianity

Primordial non-Gaussianity

Primordial NG

ζ(**x**): primordial perturbations

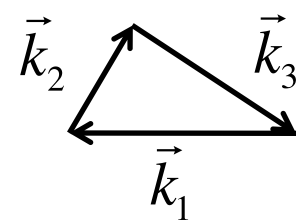
If the fluctuations are Gaussian distributed then their statistical properties are completely characterized by the two-point correlation function, <ζ($\mathbf{x}_1$)ζ($\mathbf{x}_2$)> or its Fourier transform, the power-spectrum.

Thus a non-vanishing ***three point function***, or its Fourier transform, the ***bispectrum is an indicator of non-Gaussianity***

$$\left\langle \zeta(\vec{k}_1)\zeta(\vec{k}_2)\zeta(\vec{k}_3) \right\rangle = (2\pi)^3 \delta^{(3)}(\vec{k}_1 + \vec{k}_2 + \vec{k}_3) f_{NL} F(k_1, k_2, k_3)$$

Amplitude

Shape



$$\rightarrow \left\langle \frac{\Delta T}{T}(n_1) \frac{\Delta T}{T}(n_2) \frac{\Delta T}{T}(n_3) \right\rangle$$

Why primordial NG is important?

Among many good reasons:

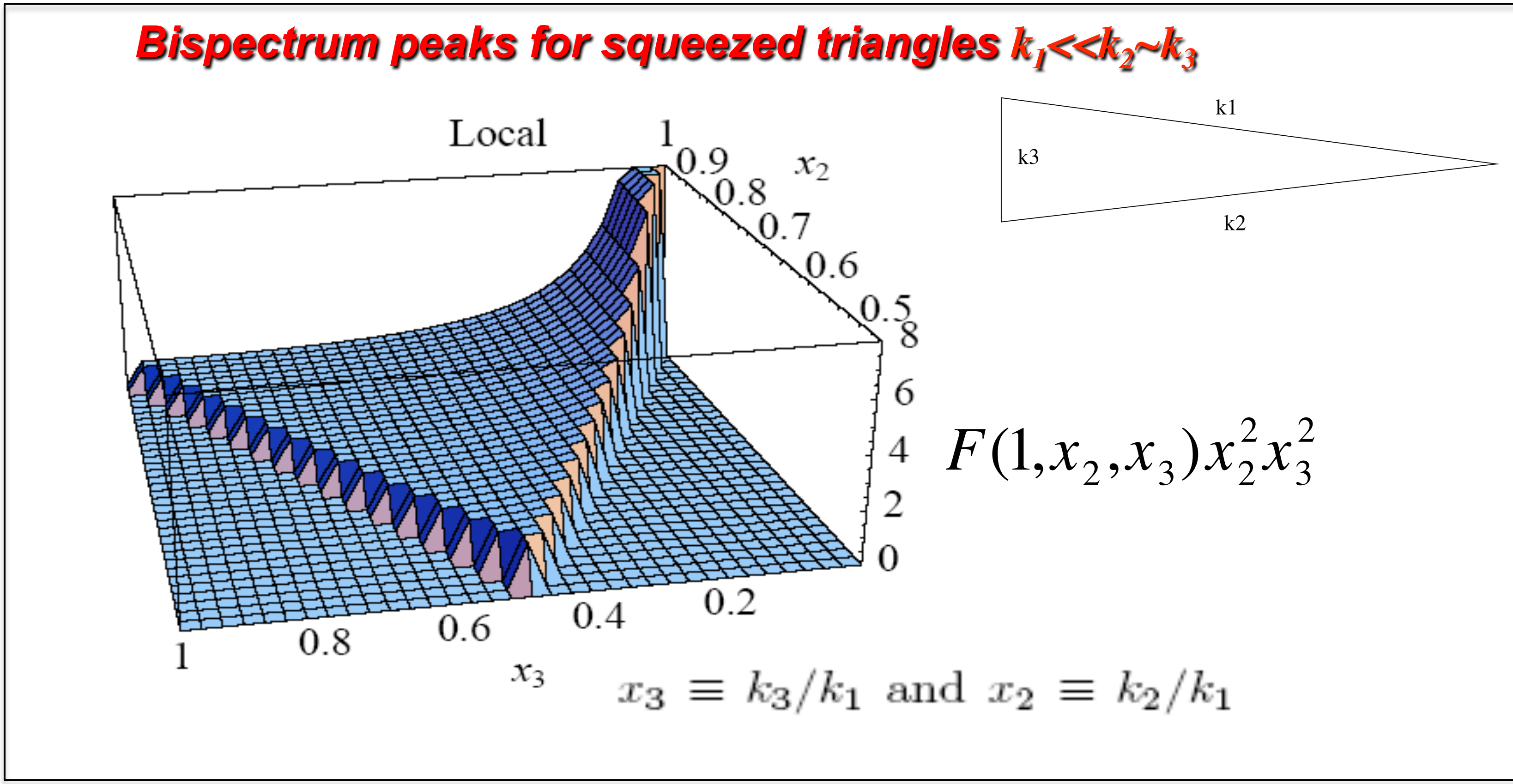
f_{NL} and shape are model dependent:
e.g.: standard single-field models of slow-roll inflation predict

$$f_{NL} \sim O(\varepsilon,\eta) \ll 1$$

(Acquaviva, Bartolo, Riotto, Matarrese 2002; Maldacena 2002)

A detection of a primordial $|f_{NL}|\sim 1$ would rule out *all* standard single-field models of slow-roll inflation

SHAPES OF NG:LOCAL NG



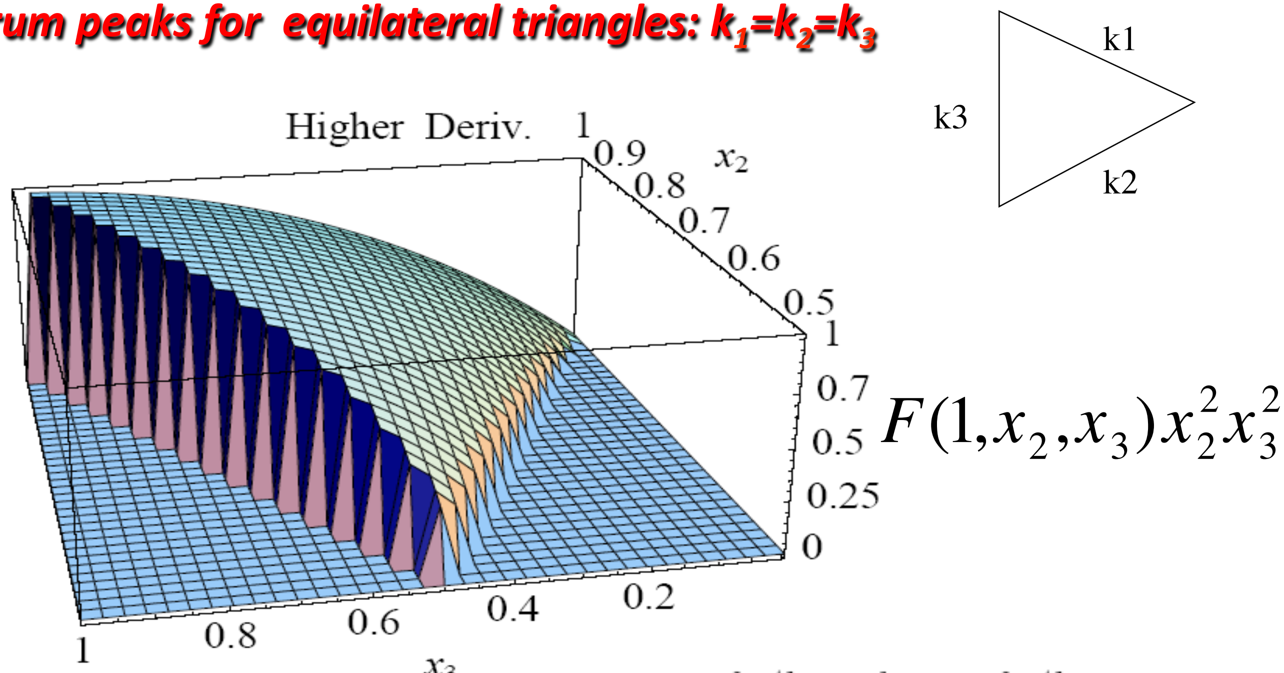
Babich et al. astro-ph/0405356

$$\Phi(\mathbf{x}) = \Phi_L(\mathbf{x}) + f_{\rm NL}\Phi_L^2(\mathbf{x})$$

Non-linearities develop outside the horizon during or immediately after inflation (e.g. ***multifield models of inflation***)

EQUILATERAL NG

Bispectrum peaks for equilateral triangles: $k_1=k_2=k_3$



$x_3 \equiv k_3/k_1$ and $x_2 \equiv k_2/k_1$

Babich et al. (2004)

Single field models of inflation with non-canonical kinetic term L=P(φ, X) where X=(∂ φ)^2 (DBI or K-inflation) where NG comes from higher derivative interactions of the inflaton field

Example: $\dot{\delta\phi}(\nabla\delta\phi)^2$

Observational limits set by Planck

	f_{NL}	
Shape	Independent	Lensing subtracted
	SMICA $T+E$	
Local	4.1 ± 5.1	-0.9 ± 5.1
Equilateral	-25 ± 47	-26 ± 47
Orthogonal	-47 ± 24	-38 ± 24

e.g. models with non-standard kinetic terms

e.g. multi-field models of inflation

Planck 2018 results. IX. Constraints on primordial non-Gaussianity.

Implications for inflation models

- The standard models of single-field slow-roll inflation has survived the most stringent tests of Gaussianity to-date: ***deviations from primordial Gaussianity are less than 0.01% level. This is a fantastic achievement, one of the most precise measurements in cosmology!***

$$\Phi(\mathbf{x}) = \Phi^{(1)}(\mathbf{x}) + f_{\mathrm{NL}} \left(\Phi^{(1)}(\mathbf{x}) \right)^2 + \ldots\ldots$$

$\sim 10^{-5}$ $\sim$few $\sim 10^{-10}$

- ***The NG constraints*** on different primordial bispectrum shapes ***severly limit/rule out specific key (inflationary) mechanisms alternative to the standard models of inflation***

Primordial non-Gaussianity allows to answer to some very simple, but fundamental questions you might have about inflation:

- ***What is the sound speed the inflaton fluctuations propagate with?***

- ***Are there other particles other than the inflaton?***

- ***What are their masses and spins?***

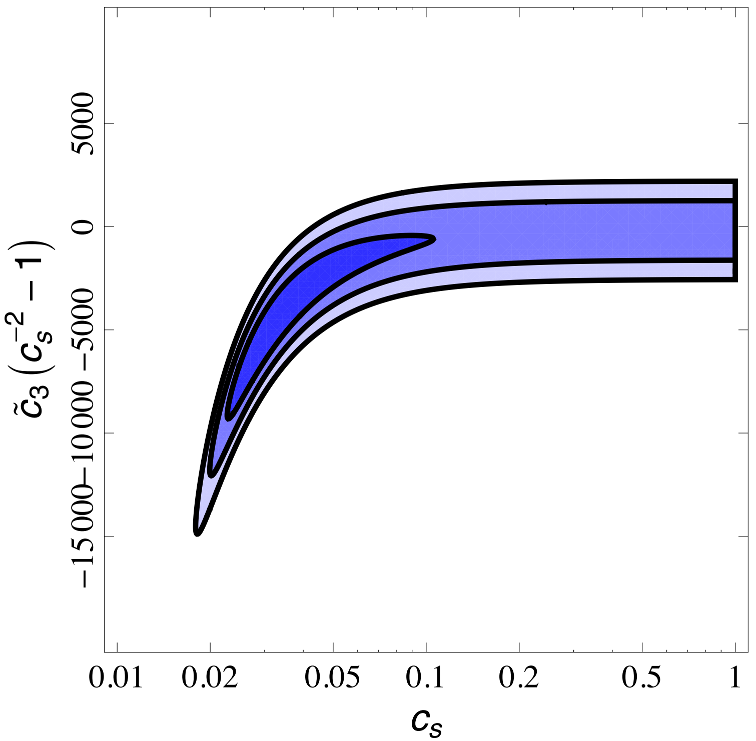
Measuring the of sound speed of the inflation

- General single-field models of inflation: Implications for Effective Field Theory of Inflation

$$S = \int d^4x\,\sqrt{-g}\left[-\frac{M_{\rm Pl}^2\dot{H}}{c_s^2}\left(\dot{\pi}^2 - c_s^2\frac{(\partial_i\pi)^2}{a^2}\right) - M_{\rm Pl}^2\dot{H}(1-c_s^{-2})\dot{\pi}\frac{(\partial_i\pi)^2}{a^2} + \left(M_{\rm Pl}^2\dot{H}(1-c_s^{-2}) - \frac{4}{3}M_3^4\right)\dot{\pi}^3\right]$$

$$f_{\rm NL} \propto \frac{1}{c_s^2}$$

(Cheung et al. 08; Weinberg 08)
for extensions see also N.B., Fasiello, Matarrese, Riotto 10)



68% CL constraints from *Planck*

$$f_{\rm NL}^{\rm equil} = -26 \pm 47$$

$$f_{\rm NL}^{\rm ortho} = -38 \pm 24$$

$$\boxed{c_s \geq 0.021 \qquad (95\,\%,\ T{+}E)}$$

SLIDES OF BACKUP WITH MORE DETAILS ABOUT THE TOPICS OF THE FIRTS LECTURE

INFLATION:

a period of accelerated expansion in the very early universe +....

- Let us quantify what means that the (comoving)) **Hubble radius (a H)$^{-1}$** decreases in time.

$$r_H = \frac{1}{\dot{a}}$$

$$\dot{r_H} = \left(\frac{1}{\dot{a}}\right)^{\cdot} = -\frac{\ddot{a}}{\dot{a}^2} < 0 \Leftrightarrow \boxed{\ddot{a} > 0}$$

- Who is kind enough to provide such an acceleration??

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}(\rho + 3p) > 0 \Rightarrow \boxed{p < -\frac{1}{3}\rho \Rightarrow w < -\frac{1}{3}}$$

$$(p = w\rho)$$

A huge expansion

$$\text{number of e-foldings}\, N = \ln(a_f/a_i) = \int_{t_i}^{t_f} H dt \geq 60$$

$\Rightarrow$

Take a region that at the beginning of inflation has typical size λ_i

Since $\lambda_{phys} \propto a(t)$, it gets inflated by $\dfrac{a_f}{a_i} \geq e^{60} \simeq 10^{26}$!!!!

INFLATION: kinematics

- Be careful: inflation cannot be just pure de-Sitter, for the very same reason that inflation must come to an end. Moreover acceleration can be obtained in ways other than de-Sitter

- Notice that we can write down a simple expression for the acceleration of the scale factor

$$\ddot{a} = \dot{\dot{a}} = (\dot{aH}) = \dot{a}H + a\dot{H} = aH^2 + a\dot{H} = aH^2\left(1 + \frac{\dot{H}}{H^2}\right)$$

- This means that, just from a pure kinematical point if view

$$\ddot{a} > 0 \iff \begin{cases} \dot{H} < 0, \dot{H} < H^2, \text{ sub-exponential inflation} \\ \dot{H} = 0 \quad \text{de-Sitter} \\ \dot{H} > 0, \text{ super-exponential inflation, or pole inflation} \end{cases}$$

- Now take component with eq. of state w=p/ρ → $a(t) = a_* \left[1 + \frac{H_*}{\alpha}(t - t_*)\right]^{\alpha}$

$$\begin{cases} \alpha > 1 & \iff -1 < \omega < -(1/3) \;\; a(t) \propto t^{\alpha}, \text{ power law inflation} \\ \alpha \to \infty & \iff \omega \to -1 \to a(t) = e^{Ht} \text{ de-Sitter} \\ \alpha < 0 & \iff \omega < -1 \to a(t) \propto |t - t_{asy}|^{\alpha}, t_{asy} = t_* - \frac{\alpha}{H_*} \end{cases} \qquad \alpha = \frac{2}{3(1+\omega)}$$

INFLATION: WHY SO IMPORTANT?

- Inflationary paradigm is one of the most relevant development in modern cosmology. Introduced to solve some shortcomings of the standard Hot Big-Bang model (Guth '81)
  e.g.: why the universe is so nearly spatially flat? (flatness problem)
  why the temperature of CMB photons on opposite sides of the sky is so accurately the same even if they were never in causal contact? (horizon problem)

- ***most importantly:*** inflation offers an elegant ***explanation for the origin of the first density perturbations*** which are the seeds for the CMB anisotropies and the Large-Scale-Structures of the Universe we observe today.

INFLATION

$$\ddot{a} > 0 \Rightarrow p < -\frac{1}{3}\rho$$

For sure Inflation takes place before primordial nucleosythesis (T≈1MeV): for radiation and (collisionless) matter dominated epochs p=1/3 ρ and p=0.

An example of accelerated expansion: a de Sitter phase

$$p = -\rho \quad \Longrightarrow \quad \begin{array}{rcl} \rho & = & const. \\ H & = & const. \end{array} \quad \Longrightarrow \quad a(t) \propto \exp(Ht)$$

(equation of state of a cosmological constant or vacuum energy)

An exponential expansion sourced by an energy density that does not dilute away

INFLATION: WHY SO IMPORTANT?

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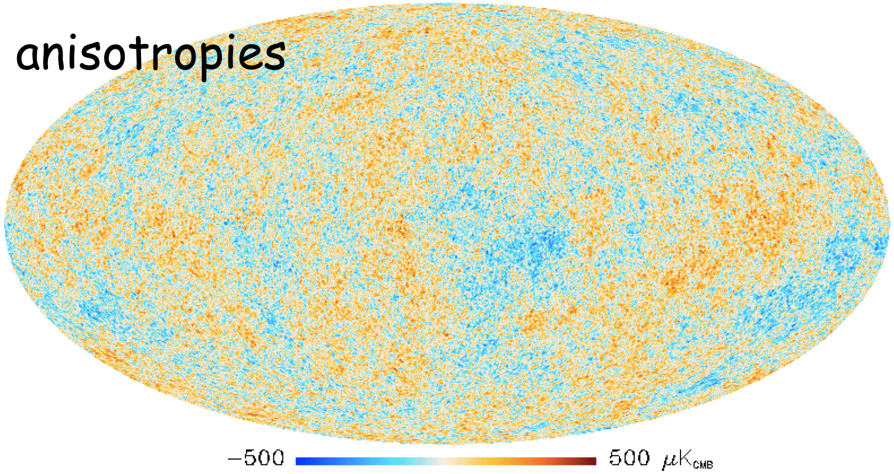
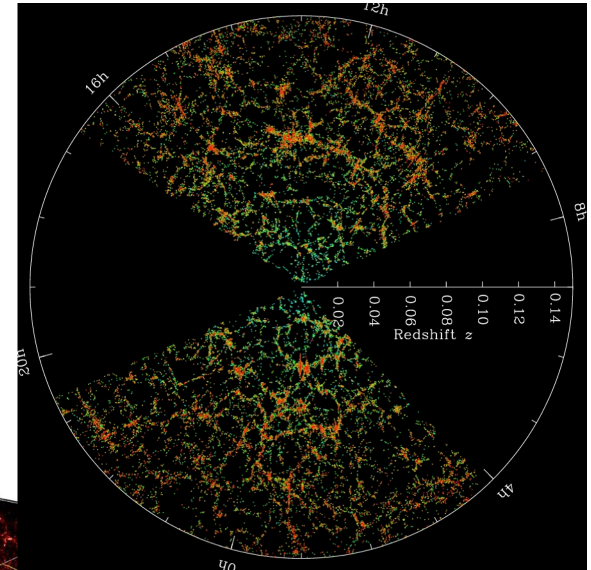
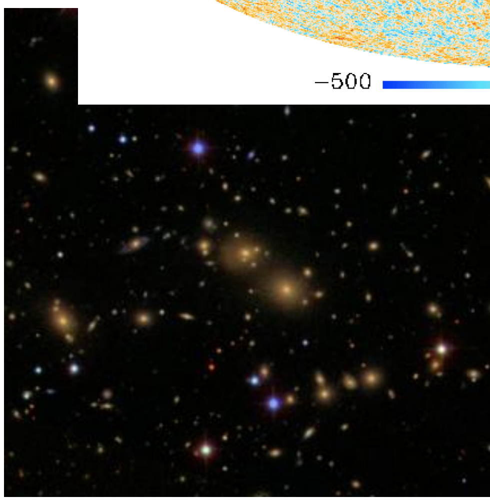
CMB anisotropies

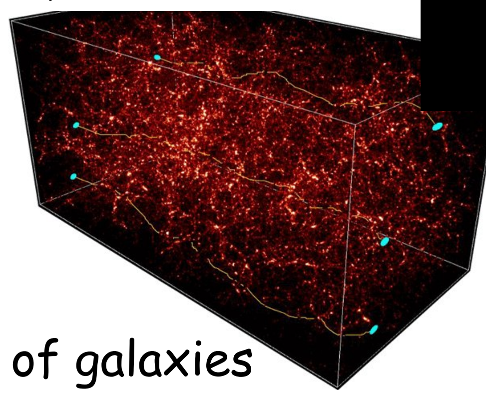




Large-Scale Structures



Clusters of galaxies



Weak gravitational lensing

Different cosmological observables that probe different scales......but all these structures require some initial fluctuations......

$$\frac{\delta\rho}{\rho} \sim \frac{\Delta T}{T} \sim 10^{-5}$$

THE MAIN COMPONENTS OF THE UNIVERSE:
THE ΛCDM MODEL (+ initial conditions from inflation)

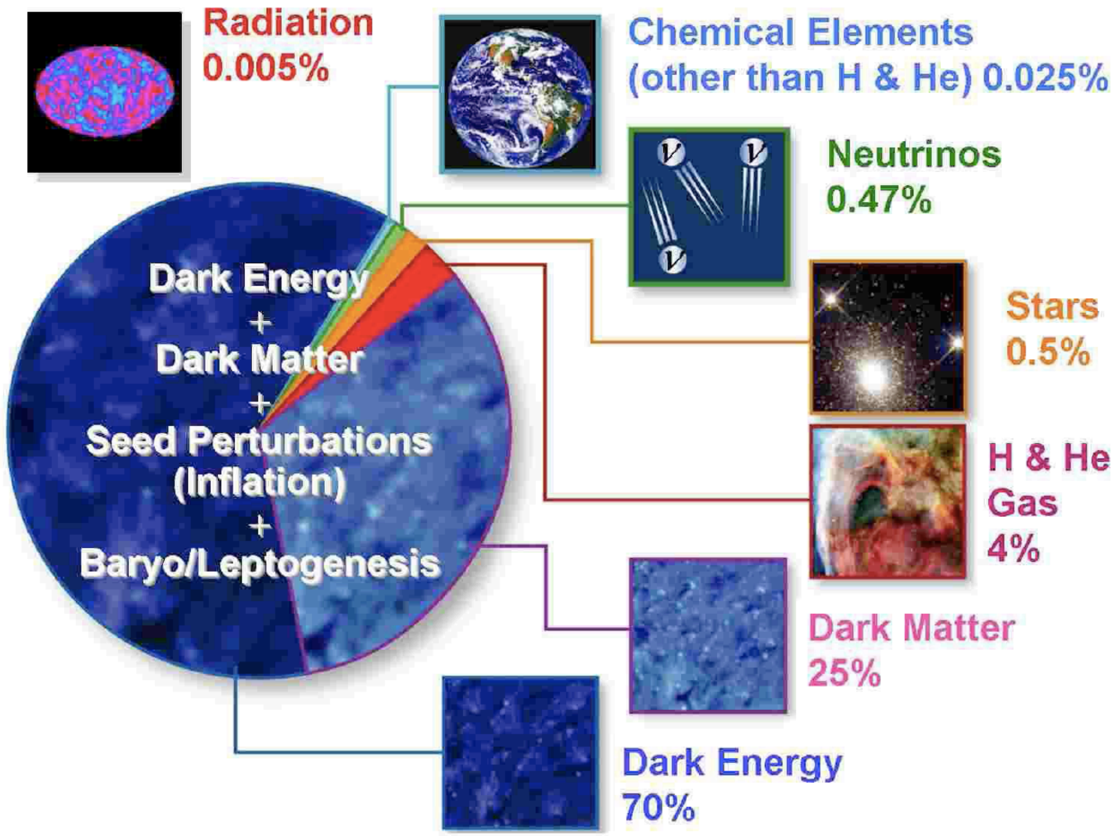
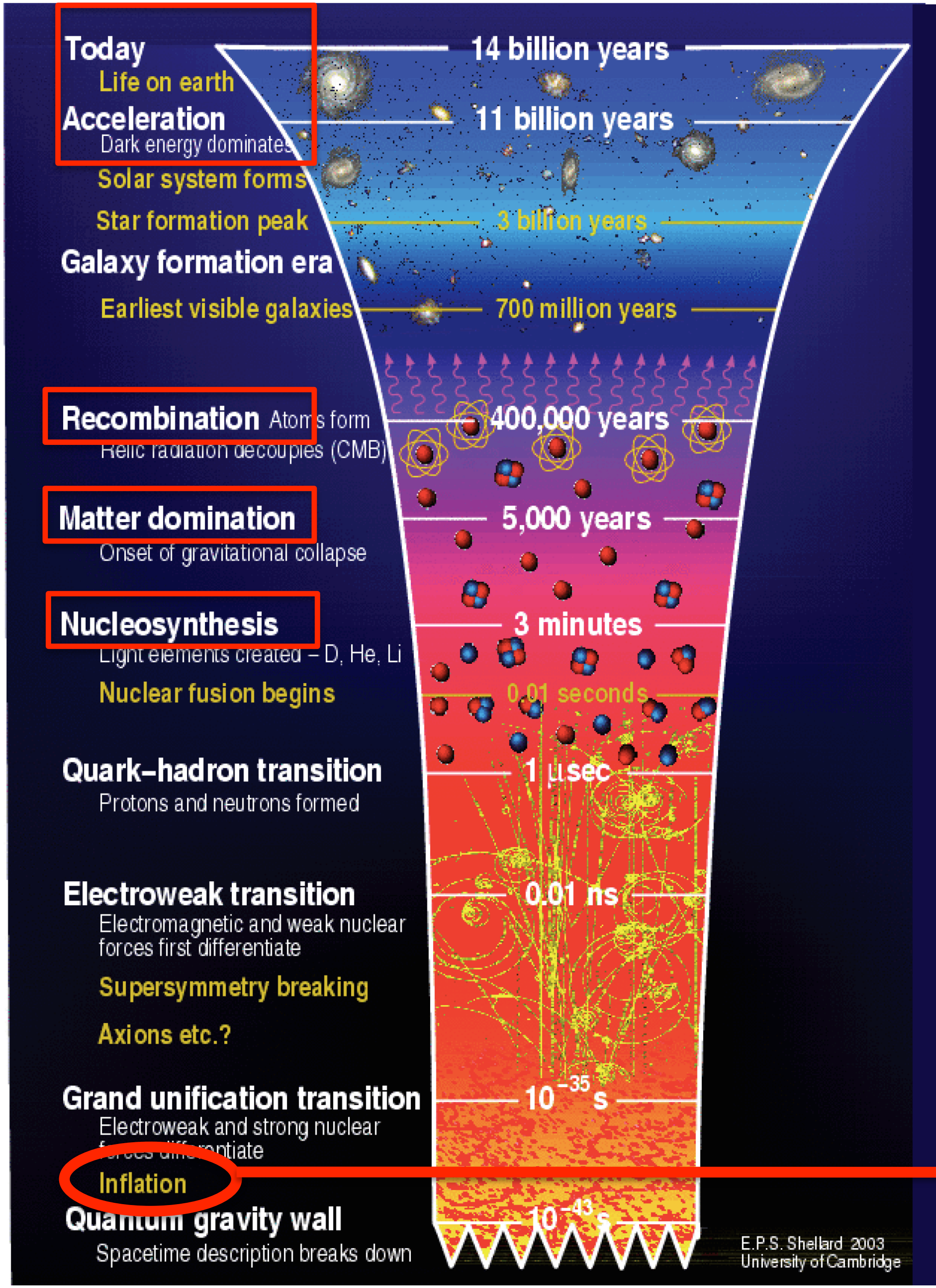


FIG. 1: Elements of the standard ΛCDM cosmological model. Illustrated by the pie is the fraction of the mass-energy of the Universe in various components. Also noted are some other necessary parts of the standard model, namely seed perturbations and baryo/leptogenesis. From E.W. Kolb, arXiv:0709.3102 ``Cosmology and the unexpected''.

N.B.: for the most updated values see the latest *Planck* satellite measurements
https://arxiv.org/abs/1807.06209

We are here

Z_{rec}~0



Z_{rec}~1100

Z_{eq}~3500

T~1 MeV

We seek information about ***very early times and very high energies E up to 10^{16} GeV***

Lecture 1+2

- Dynamics of inflation

- The quantum origin of cosmological perturbations

INFLATION and THE INFLATON

Who is a candidate to provide

$$p < -\frac{1}{3}\rho \quad ??$$

INFLATION and THE INFLATON

Consider a simple real scalar field:

$$S = S_{EH} + S_\phi = \int d^4x\sqrt{-g}\frac{1}{16\pi G}R + S_\phi$$

$$S_\phi = \int d^4x\sqrt{-g}\mathcal{L}_\phi[\phi, g_{\mu\nu}] = \int d^4x\sqrt{-g}\left[-\frac{1}{2}g^{\mu\nu}\phi_{,\mu}\phi_{,\nu} - V(\phi)\right]$$

3 ingredients:

- The scalar field (the so called inflaton field)
- the gravitational field (i.e., the metric)
- the ``rest of the world'': fermions, gauge bosons, other scalars.
  Usually, in the simplest models, these additional components turns out to be subdominant w.r.t. the inflaton field (because e.g., we know that for pressurless matter $\rho_m \sim a^{-3}$ while radiation $\rho_r \sim a^{-4}$, and so they decrease almosr exponentially during inflation).

INFLATION and THE INFLATON

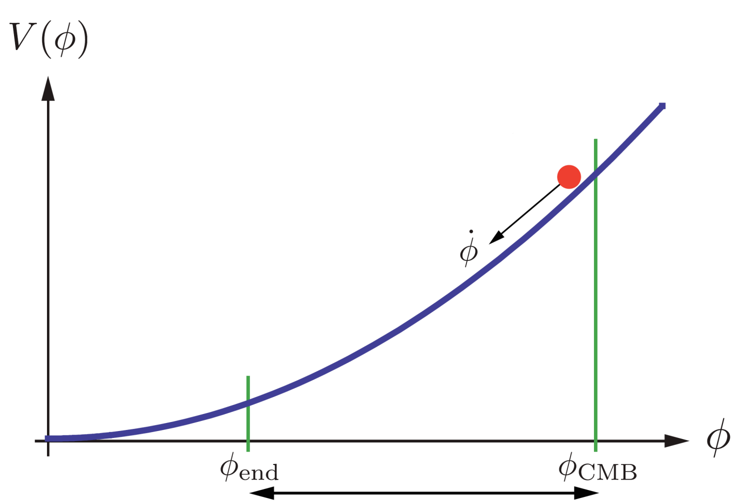
$$\mathcal{L}_\phi[\phi, g_{\mu\nu}] = -\frac{1}{2} g^{\mu\nu} \phi_{,\mu}\phi_{,\nu} - V(\phi)$$

Standard kinetic term

Inflaton potential: describes the self-interactions of the inflaton field and its interactions with the rest of the world

Just think of the inflaton field as a particle that moves under a force induced by the potential

Ex: $V(\phi) = \frac{m^2}{2}\phi^2$



INFLATION and THE INFLATON

Let us see

1. What is the dynamics of a scalar field in an expanding universe?

2. Why a scalar field works well in driving inflation

3. How we characterize the different inflationary models (see also Lectures 7+8)

4. From quantum fluctuations of the inflaton field to primordial density perturbations (which then grow by subsequent gravitational instability to give rise to CMB anisotropies and the Large-Scale structures we observe in the Universe).

1. Dynamics of a scalar field in a curved space-time

$$\frac{\delta S_\phi}{\delta \phi} = 0 \rightarrow \Box\phi = \frac{\partial V}{\partial \phi} \rightarrow \boxed{\ddot{\phi} + 3H\dot{\phi} - \frac{\nabla^2 \phi}{a^2} = -\frac{\partial V}{\partial \phi}}$$

$$\Box\phi \equiv \frac{1}{\sqrt{-g}} \left(g^{\mu\nu} \sqrt{-g}\, \phi_{,\mu} \right)_{,\nu}$$

- This is our master equation: the Klein-Gordon equation for a scalar field in a RW metric

We can associate to the scalar field and energy-momentum tensor

Contributions from derivatives w.r.t to higher order derivatives of the metric tensor

Energy momentum tensor

$$T^{\phi}_{\mu\nu} = -\frac{2}{\sqrt{-g}}\frac{\delta S}{\delta g^{\mu\nu}} = \frac{2}{\sqrt{-g}}\left[-\frac{\partial(\sqrt{-g}\mathcal{L}_\phi)}{\partial g^{\mu\nu}} + \partial_\alpha \frac{\partial(\sqrt{-g}\mathcal{L}_\phi)}{\partial\partial_\alpha g^{\mu\nu}} + \ldots.\right]$$

For a real scalar field, minimally coupled (i.e. without coupling to gravity like $\xi\, R\, \phi^2$)

$$T^{\phi}_{\mu\nu} = -2\frac{\partial \mathcal{L}_\phi}{\partial g^{\mu\nu}} + g_{\mu\nu}\mathcal{L}_\phi = \partial_\mu\phi\,\partial_\nu\phi + g_{\mu\nu}\left(-\frac{1}{2}g^{\alpha\beta}\partial_\alpha\phi\,\partial_\beta\phi - V(\phi)\right)$$

N.B.: try to obtain the expression above by making use of the following expression

$\dfrac{\partial\left(\sqrt{-g}\right)}{\partial g^{\mu\nu}} = -\dfrac{1}{2}\sqrt{-g}g_{\mu\nu}$ which can be obtained from this useful property: $\mathrm{Tr}(\ln M) = \ln(\det M)$

A leitmotiv of these lectures..........

split the scalar field into **a ``classical'' background** expectation value (on the vacuum state) and **quantum fluctuations** around the mean value

$$\phi(\mathbf{x},t) = \phi(t) + \delta\phi(\mathbf{x},t)$$

$$\boxed{\ddot{\phi} + 3H\dot{\phi} = -\frac{\partial V}{\partial \phi}}$$

$$\phi(t) = \langle \phi(\mathbf{x},t) \rangle \equiv \langle 0|\phi(\mathbf{x},t)|0\rangle \neq 0$$

$$\langle \delta\phi(\mathbf{x},t) \rangle = 0$$

$$\langle \delta\phi(\mathbf{x},t)^2 \rangle \ll \phi^2(t)$$

2. Why a scalar field works well in driving inflation?

The scalar field can provide an energy density that remains almost constant in time

✓ Take a homogeneous and isotropic scalar field $\phi(\mathbf{x}, t) = \phi(t)$

$$T^0_{\ 0} = -\rho_\phi = -\left(\frac{1}{2}\dot{\phi}^2 + V(\phi)\right)$$

$$T^i_{\ j} = p_\phi\, \delta^i_{\ j} = \left(\frac{1}{2}\dot{\phi}^2 - V(\phi)\right) \delta^i_{\ j}$$

✓ So:

1. A scalar field behaves like a perfect fluid
2. ***Most importantly: if the potential energy density dominates over the kinetic energy density…….***

$$\boxed{V(\phi) >> \frac{1}{2}\dot{\phi}^2 \Longrightarrow p_\phi \simeq -\rho_\phi}$$

INFLATION and THE INFLATON

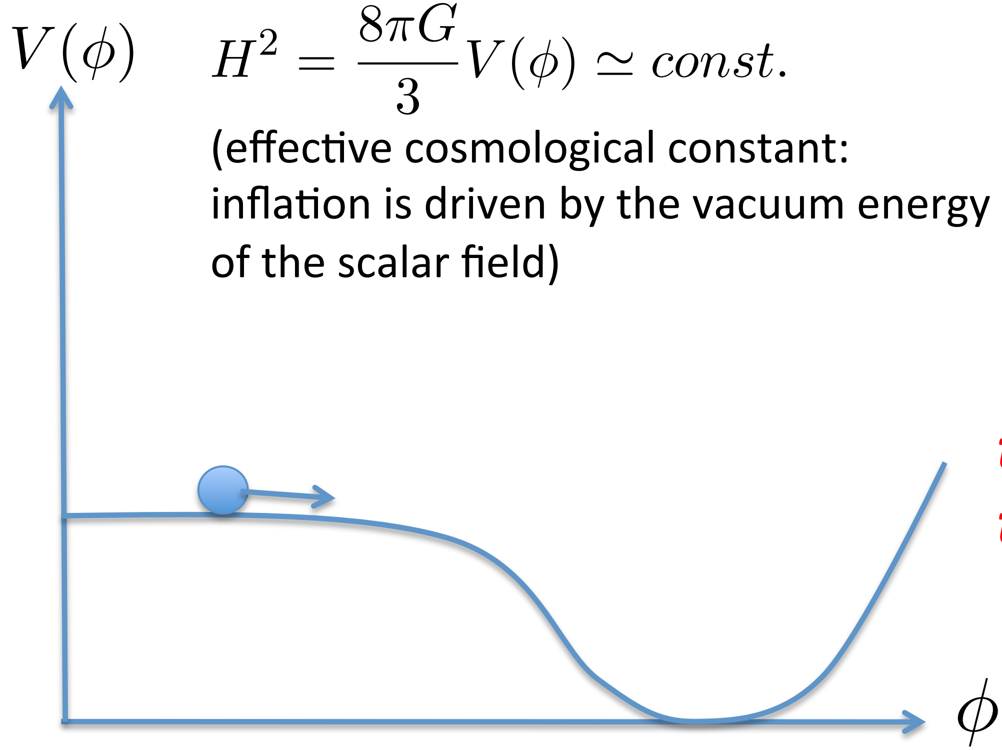
Inflation is attained if the energy density of the universe is dominated by the potential energy of a scalar field (the inflaton)

$$V(\phi) >> \frac{1}{2}\dot{\phi}^2 \Longrightarrow p_\phi \simeq -\rho_\phi$$

If $V(\phi) >> \frac{1}{2}\dot{\phi}^2$ ***the inflaton is slowly rolling its potential***: $\phi(t) \approx const.$

$$H^2 = \frac{8\pi G}{3} V(\phi) \simeq const.$$

(effective cosmological constant: inflation is driven by the vacuum energy of the scalar field)



the potential V(ϕ) must be flat to achieve inflation

INFLATION and THE INFLATON

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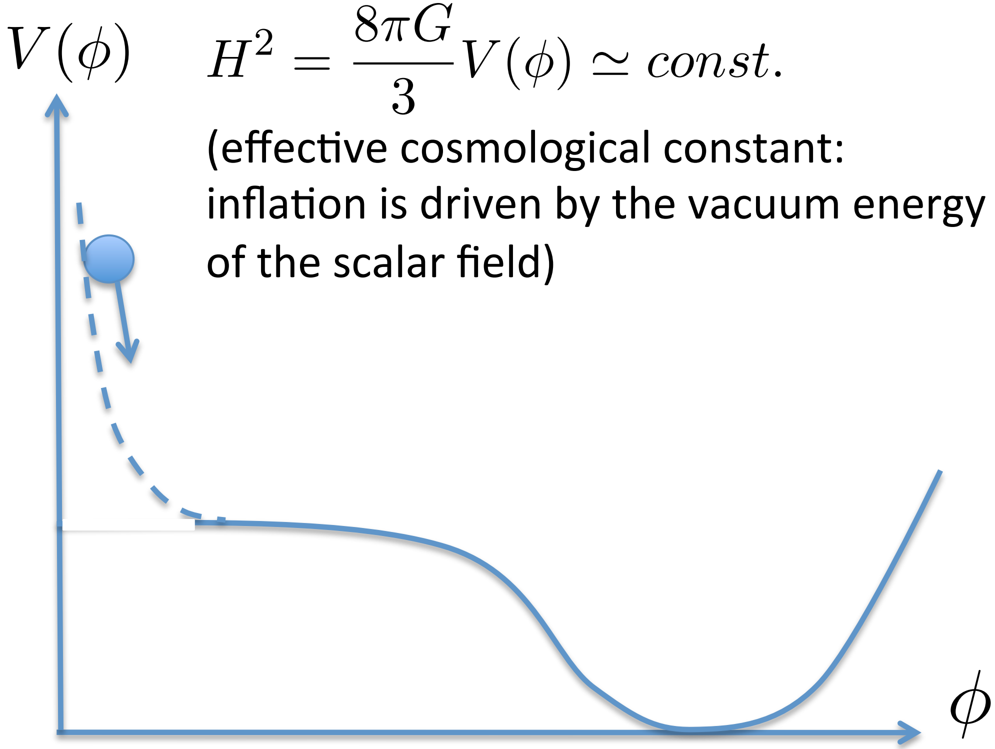


Notice that even if initially the kinetic energy density $(1/2)\dot{\phi}^2$ dominates over the potential term, then

$$\rho_\phi \propto a^{-3(1+w_\phi)} \propto a^{-6}$$

so it decreases fast and if the potential is sufficiently flat inflation starts and takes place. Also this example shows the characteristic of inflation of being an attractor mechanism.

Let us look in more details into the dynamics of a scalar field in a curved space-time

✓ Take the inflaton field which is slow-rolling along its potential $V(\phi) >> \frac{1}{2}\dot{\phi}^2$

$$H^2 = \frac{8\pi G}{3}\left(V(\phi) + \frac{1}{2}\dot{\phi}^2\right) \simeq \frac{8\pi G}{3}V(\phi)$$

✓ This is due to a flat enough potential → we expect that V and its derivatives w.r.t. ϕ vary slowly with ϕ

→ we expect $\ddot{\phi}$ to be negligible as well

It is like to consider the equation of motion of a particle rolling down its potential and subject to a friction term

$$\ddot{x} + 3\,\Gamma\dot{x} = F/m \xrightarrow{t>>1/\Gamma} \dot{x} \simeq \frac{F}{3\Gamma m}$$ ***attractor solution***

$$\ddot{\phi} + 3H\dot{\phi} = -\frac{\partial V}{\partial \phi} \longrightarrow \dot{\phi} \simeq -\frac{V_{,\phi}}{3H}$$

2. Let us look in more details into the dynamics of a scalar field in a curved space-time: ***Slow-roll parameters***

$$\begin{cases} H^2 \simeq \dfrac{8\pi G}{3} V(\phi) \\ \dot{\phi} \simeq -\dfrac{V_{,\phi}}{3H} \end{cases} \longrightarrow$$

$$V(\phi) \gg \frac{1}{2}\dot{\phi}^2 \Rightarrow \frac{(V_{,\phi})^2}{V} \ll H^2$$

AND

$$\ddot{\phi} << 3H\dot{\phi} \Rightarrow V_{,\phi\phi} \ll H^2$$

So the slow-roll conditions, as expected, means that the ***inflaton potential is very flat***

It is then customary to parametrize inflationary models (i.e. the form of the inflaton potential) in a sort of model-independent way by introducing the slow-roll parameters

$$\epsilon = -\frac{\dot{H}}{H^2} = 4\pi G \frac{\dot{\phi}^2}{H^2} \simeq \frac{3}{2}\frac{\dot{\phi}^2}{V} \simeq \frac{1}{16\pi G}\left(\frac{V_{,\phi}}{V}\right)^2 \ll 1$$: the Hubble rate change slowly

$$\eta = -\frac{\ddot{\phi}}{H\dot{\phi}}$$ with $|\eta| \ll 1$: attractor solution

2. Let us look in more details into the dynamics of a scalar field in a curved space-time

N.B.: instead of η one can use a slow-roll parameter η_V

$$\eta_V = \frac{1}{3}\frac{V_{,\phi\phi}}{H^2} = \frac{1}{8\pi G}\left(\frac{V_{,\phi\phi}}{V}\right) \quad \text{with} \quad \eta = -\frac{\ddot{\phi}}{H\dot{\phi}} \simeq \eta_V - \epsilon$$

and so we can also say that $|\eta_V| << 1$

So the slow-roll conditions, as expected, means that the ***inflaton potential is very flat***

Let's summarize:
Inflation takes place if the inflaton potential is sufficiently flat, i.e. if the slow-roll parameters ε, |η| <<1

N.B.:
Remember that we found:

$$\ddot{a} = aH^2\left(1 + \frac{\dot{H}}{H^2}\right) = aH^2(1-\varepsilon)$$

So you see that inflation takes place if ε <1.
So why we say that also |η|<1?
Because this guarantees that the velocity of the scalar field $\dot{\phi}$ changes very slowly with time, and therefore that the condition ε <1 (i.e. inflation) is mantained for long enough (in order to solve the horizon and the flatness problems).

Let us pause for a moment. A couple of comments are in order here.

- When studying the quantum fluctuations of the scalar field, one usually employs a Taylor expansion in the slow-roll parameters, since they are small. Moreover one can treat them as constant (at lowest-order in the slow-roll parameters) since one can show that their time derivatives are higher-order in the slow-roll parameters, with, e.g.,

$$\frac{\dot{\epsilon}}{H} \quad \text{or} \quad \frac{\dot{\eta}}{H} \sim \mathcal{O}(\epsilon^2, \eta^2)$$

- We have considered the ε and η slow-roll parameters. Indeed there exist a full hierarchy of slow-roll parameters, built, e.g., from higher-order derivatives of the potential V w.r.t. to the inflaton field. E.g.,

$$\xi^2 = \left(\frac{1}{4\pi G}\right)^2 \left(\frac{V'V'''}{V^2}\right)$$

which is second-order in the slow-roll parameters.

Two simple but very important examples

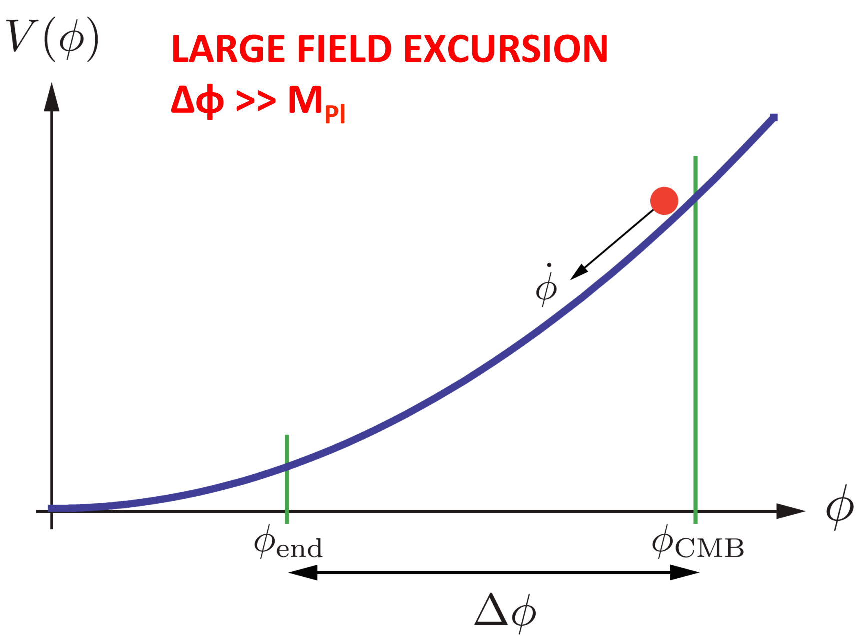
``Large field'' like potential

$$V(\phi) \propto \phi^{\alpha}$$

$$\epsilon \sim \frac{1}{\pi G}\left(\frac{V_{,\phi}}{V}\right)^2 \sim \alpha^2 \frac{1}{\pi G}\frac{1}{\phi^2} \sim \alpha^2 \frac{M_{\rm Pl}^2}{\phi^2}$$

$$\epsilon \ll 1 \Rightarrow \phi \gg M_{\rm Pl}$$

$M_{Pl} = (\hbar c/G)^{1/2} \equiv G^{-1/2} \simeq 10^{19} GeV$



``Small field'' like potential

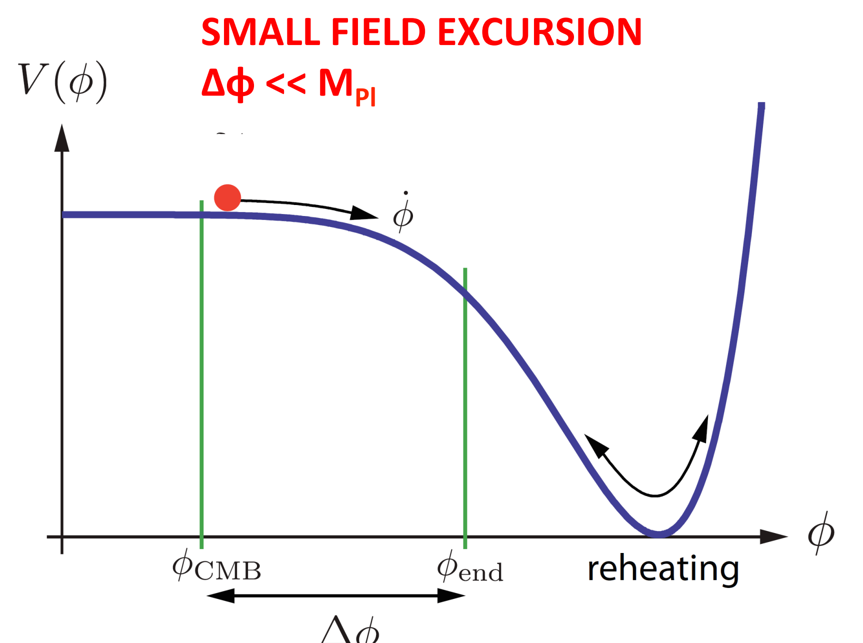
$$V(\phi) = V_0\left[1 - \left(\frac{\phi}{\mu}\right)^p\right]$$

$\phi < \mu < M_{\rm Pl}$

p>2

$$\epsilon \sim p^2 \frac{\phi^{2p}}{\mu^{2p}}\frac{M_{\rm Pl}^2}{\phi^2}\left[1 - \left(\frac{\phi}{\mu}\right)^p\right]^{-1}$$

$\epsilon \to 0$ for $\phi \to 0$

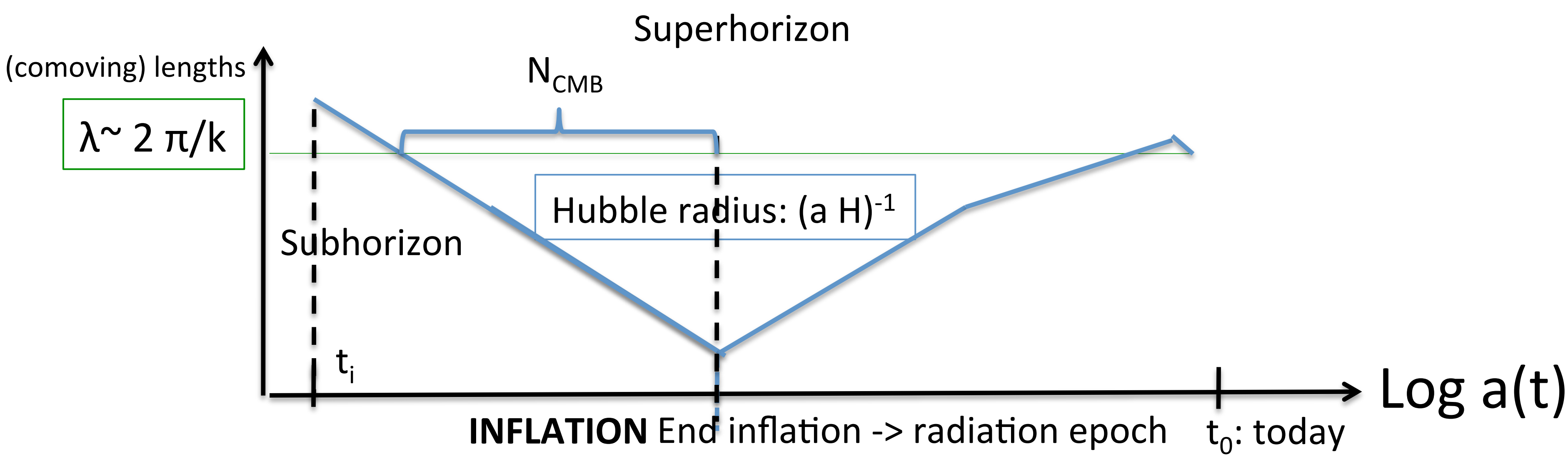


Excursion of the inflaton field (in the observable window):

$$\Delta\phi = \int_{\phi_{CMB}}^{\phi_{end}} d\phi = \int_{t_{CMB}}^{t_{end}} \dot{\phi}dt \simeq \frac{\dot{\phi}}{H}\int_{Ht_{CMB}}^{Ht_{end}} d(Ht) = \frac{\dot{\phi}}{H} N_{CMB} \sim \epsilon^{1/2} N_{CMB} M_{Pl}$$

Therefore in the case $\varepsilon \sim 1/N_{CMB}$ (as it usually happens in large-field models) and not too small the excursion of the field is $\Delta\phi > M_{PL}$ → large-field models (here N_{CMB} defines the "observable window" during inflation, the one that we can observationally probe, that is to say, N_{CMB} corresponds to the 60-70 e-folds (counted from the end of inflation) in correspondence to which the largest observable scale, i.e. the cosmological horizon today, leaves the horizon during Inflation (always think to the plot of the cosmological horizon as a function of time).
The largest observable scales can be probed through CMB measurements, that's why we used the suffix "CMB".

Instead in the case in which cui $\varepsilon \ll 1$ then one gets $\Delta\phi < M_{PL}$ → small-field models.



Why we expect large-scale fluctuations to be generated during inflation ?

$$\phi(\mathbf{x},t) = \phi(t) + \delta\phi(\mathbf{x},t)$$

Negligible on super-horizon scales (k<< a H)

$$\delta\ddot{\phi} + 3H\delta\dot{\phi} - \frac{\nabla^2\delta\phi}{a^2} = -\frac{\partial^2 V}{\partial\phi^2}\,\delta\phi$$

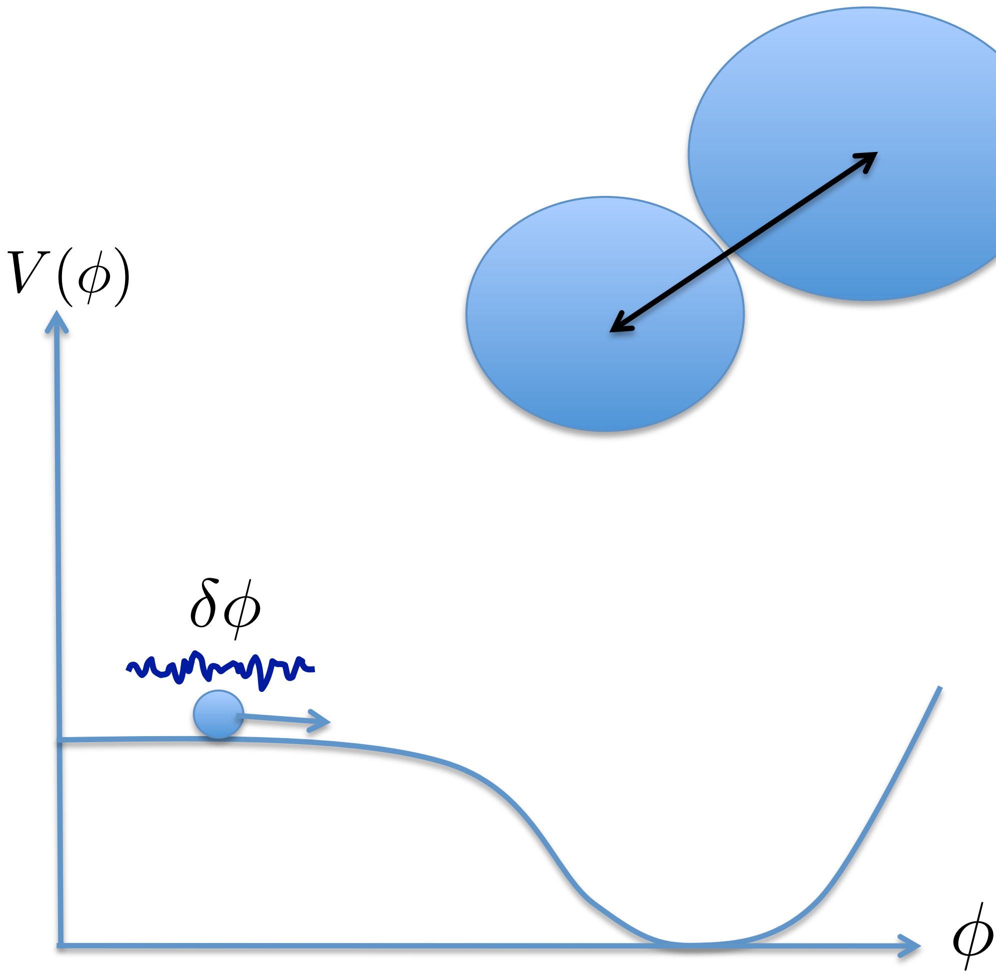
$$\ddot{\phi} + 3H\dot{\phi} = -\frac{\partial V}{\partial\phi} \xrightarrow[\text{Take the time derivative}]{} (\dot{\phi})^{\cdot\cdot} + 3H(\dot{\phi})^{\cdot} = -\frac{\partial^2 V}{\partial\phi^2}\,\dot{\phi}$$

So $\delta\phi$ and $\dot{\phi}$ obey the same equation: compute the Wronskian, and you get $W(\delta\phi,\dot{\phi}) = \delta\dot{\phi}\dot{\phi} - \delta\dot{\phi}\ddot{\phi} \propto e^{-3Ht} \to 0$. Thus the two solutions are related by a constant of proportionality that depends upon the space point (we are neglecting the gradient)

$$\delta\phi(\mathbf{x},t) = -\delta t(\mathbf{x})\,\dot{\phi}(t) \to \boxed{\phi(\mathbf{x},t) = \phi(t - \delta t(\mathbf{x}))}$$

Regions by regions (large - superhorizon – distance apart) the scalar field passes through the same history but at slight different times because of quantum fluctuations

Regions by regions (large - superhorizon – distance apart) the scalar field passes through the same history but at slight different times because of quantum fluctuations



So let us solve this equation with some useful tricks........

✓ $u = a(t)\delta\phi$ $\longrightarrow$ $$u_k'' + \left(k^2 - \frac{a''}{a} + \cancel{V_{,\phi\phi}} a^2\right) u_k = 0$$

N.B.: I am using conformal time $d\tau = dt/a(t)$

$a(t) \propto e^{Ht} \to \tau = -1/(aH)$ and $a''/a = 2/\tau^2 = 2a^2H^2$

✓

SUBHORIZON SCALES: -k τ=k /(aH)>>1;

$$u_k'' + k^2 u_k = 0 \to u_k(\tau) \to \frac{1}{\sqrt{2k}} e^{-ik\tau}$$

SUPERHORIZON SCALES: -k τ=k /(aH)<<1

$$u_k'' - \frac{a''}{a} u_k = 0 \to u_k(\tau) = B(k) a(\tau)$$

✓ **Match the two solutions at horizon-crossing k=a H (-k τ=1)**

$$|B(k)|a = \frac{1}{\sqrt{2k}} \to |B(k)| = \frac{1}{a\sqrt{2k}} = \frac{H}{\sqrt{2k^3}}$$

And hence: $|\delta\phi_{\mathbf{k}}| \simeq \dfrac{H}{\sqrt{2k^3}}$ ON SUPERHORIZON SCALES

Quantum fluctuations of a massless scalar field in de-Sitter

$$u_k'' + \left(k^2 - \frac{a''}{a} + \cancel{V_{,\phi\phi}\, a^2}\right) u_k = 0 \leftrightarrow \boxed{u_k'' + \left(k^2 - \frac{\nu^2 - \frac{1}{4}}{\tau^2}\right) = 0}$$

Bessel equation (with, for the specific case under examination of a massless scalar field and in de Sitter $\nu = 3/2$)

✓ The solutions for a generic ν constant are well known, Hankel functions

$$u_k(\tau) = \sqrt{-\tau}\left[c_1(k)\, H_\nu^{(1)}(-k\tau) + c_2(k)\, H_\nu^{(2)}(-k\tau)\right]$$

✓ Well inside the horizon (-k τ>>1)

$$H_\nu^{(1)}(x \gg 1) \sim \sqrt{\frac{2}{\pi x}}\, e^{i\left(x - \frac{\pi}{2}\nu - \frac{\pi}{4}\right)} \quad , H_\nu^{(2)}(x \gg 1) \sim \sqrt{\frac{2}{\pi x}}\, e^{-i\left(x - \frac{\pi}{2}\nu - \frac{\pi}{4}\right)}$$

So we set: $c_2(k) = 0$ and $c_1(k) = \frac{\sqrt{\pi}}{2}\, e^{i\left(\nu + \frac{1}{2}\right)\frac{\pi}{2}}$

to recover the plane-wave behaviour well inside the horizon

Quantum fluctuations of a massless scalar field in de-Sitter

✓ So the final solution is $u_k(\tau) = \frac{\sqrt{\pi}}{2} e^{i\left(\nu+\frac{1}{2}\right)\frac{\pi}{2}} \sqrt{-\tau}\, H_\nu^{(1)}(-k\tau)$

which for $\nu = 3/2$ gives (remember $\delta\phi_k = u_k(\tau)/a$ and τ= -1/(H a))

$$u_k(\tau) = \frac{e^{-ik\tau}}{\sqrt{2k}}\left(1 - \frac{i}{k\tau}\right) \rightarrow \delta\phi_k = \frac{H}{\sqrt{2k^3}} i e^{-ik\tau}\left(1 + ik\tau\right)$$

On subhorizon scales: -k τ=k /(aH)>>1

$$\delta\phi_k = \frac{1}{a}\frac{e^{-ik\tau}}{\sqrt{2k}}$$

ON SUPERHORIZON SCALES: -k τ=k /(aH)<<1

$$|\delta\phi_{\mathbf{k}}| = \frac{H}{\sqrt{2k^3}}$$

The power spectrum of cosmological perturbations: a quick definition

$$\langle f_{\mathbf{k_1}} f^*_{\mathbf{k_2}} \rangle = \frac{2\pi^2}{k^3} \mathcal{P}_f(k_1) \delta^{(3)}(\mathbf{k_1} - \mathbf{k_2})$$

- Of course to statistically characterize the level of perturbations one cannot take simply, e.g., $\langle \delta\phi(\mathbf{x}, t) \rangle$ given that $\langle \delta\phi(\mathbf{x}, t) \rangle = 0$

- The power spectrum depends only on the modulus of **k** because of isotropy, and the delta Dirac is there because of homogeneity

- You can show that $P(k) = \frac{2\pi^2}{k^3} \mathcal{P}(k)$ is (proportional to) the Fourier transform of the two-point correlation function in real space $\xi(r) = \langle f(\mathbf{x} + \mathbf{r}, t) f(\mathbf{x}, t) \rangle$

- You can easily show that the variance of the fluctuation is given by

$$\sigma^2 = \langle f(\mathbf{x}, t) f(\mathbf{x}, t) \rangle = \frac{1}{2\pi^2} \int_0^\infty dk k^2 P(k) = \int_0^\infty \frac{dk}{k} \mathcal{P}(k)$$

The power spectrum of cosmological perturbations: a quick definition

- For example:

$$
\begin{aligned}
\xi(r) &= \langle f(\mathbf{x}+\mathbf{r},t) f(\mathbf{x},t)\rangle = \frac{1}{(2\pi)^3}\int d^3\mathbf{k}\int d^3\mathbf{k}' e^{i\mathbf{k}\cdot(\mathbf{x}+\mathbf{r})} e^{i\mathbf{k}'\cdot\mathbf{x}} \langle f_{\mathbf{k}} f_{\mathbf{k}'}\rangle \\
&= \frac{1}{(2\pi)^3}\int d^3\mathbf{k}\int d^3\mathbf{k}' e^{i\mathbf{k}\cdot(\mathbf{x}+\mathbf{r})} e^{i\mathbf{k}'\cdot\mathbf{x}} P(k)\delta^{(3)}(\mathbf{k}+\mathbf{k}') \\
&= \frac{1}{(2\pi)^{3/2}}\int d^3\mathbf{k} e^{i\mathbf{k}\cdot\mathbf{r}} \frac{P(k)}{(2\pi)^{3/2}}
\end{aligned}
$$

So that the variance turns out to be

$$
\sigma^2 = \xi(o) = \frac{1}{(2\pi)^{3/2}}\int d^3\mathbf{k}\frac{P(k)}{(2\pi)^{3/2}} = \frac{1}{2\pi^2}\int_0^\infty dk k^2 P(k) = \int_0^\infty \frac{dk}{k}\mathcal{P}(k)
$$

EXERCISE: alternatively you can compute the scalar spectral tilt also in the following way:

$$\mathcal{P}_\zeta = \frac{H^2}{\dot\phi^2}\mathcal{P}_{\delta\phi} = \left(\frac{H^2}{2\pi\dot\phi}\right)^2_{t^{(1)}(k)}$$

(remember $\mathcal{P}_{\delta\phi} \sim H^2$)

To compute the spectral tilt

$$n_s - 1 = \frac{d\ln P_\zeta}{d\ln k}$$

Use the following trick: consider the number of e-foldings between the time a given scale k leaves the horizon during inflation and the end of inflation

$$N_k = \int_{t^{(1)}(k)}^{t_{end}} H dt = \ln(a_{end}/a_k) \simeq \ln\left(\frac{H a_{end}}{k}\right)$$

(N.B: $H_k a_k = k$ by definition)

then $d\ln k \simeq H dt$ $\longrightarrow$

$$\boxed{n_s - 1 = \frac{1}{\mathcal{P}_\zeta}\frac{d\mathcal{P}_\zeta}{H dt} = -2\frac{\ddot\phi}{H\dot\phi} + 4\frac{\dot H}{H^2} = 2\eta_V - 6\epsilon}$$

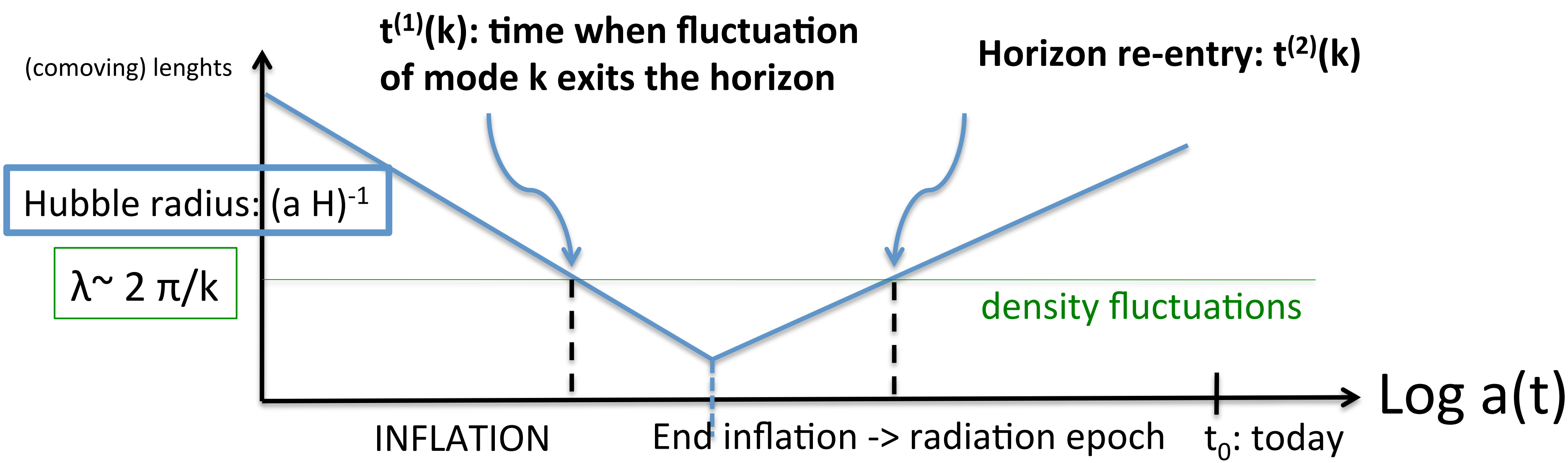
(N.B.: $-\frac{\ddot\phi}{H\dot\phi} \simeq \eta_V - \epsilon$)

Generating the primordial density perturbations

E.g. : if a given fluctuation mode re-enters the horizon during the radiation epoch

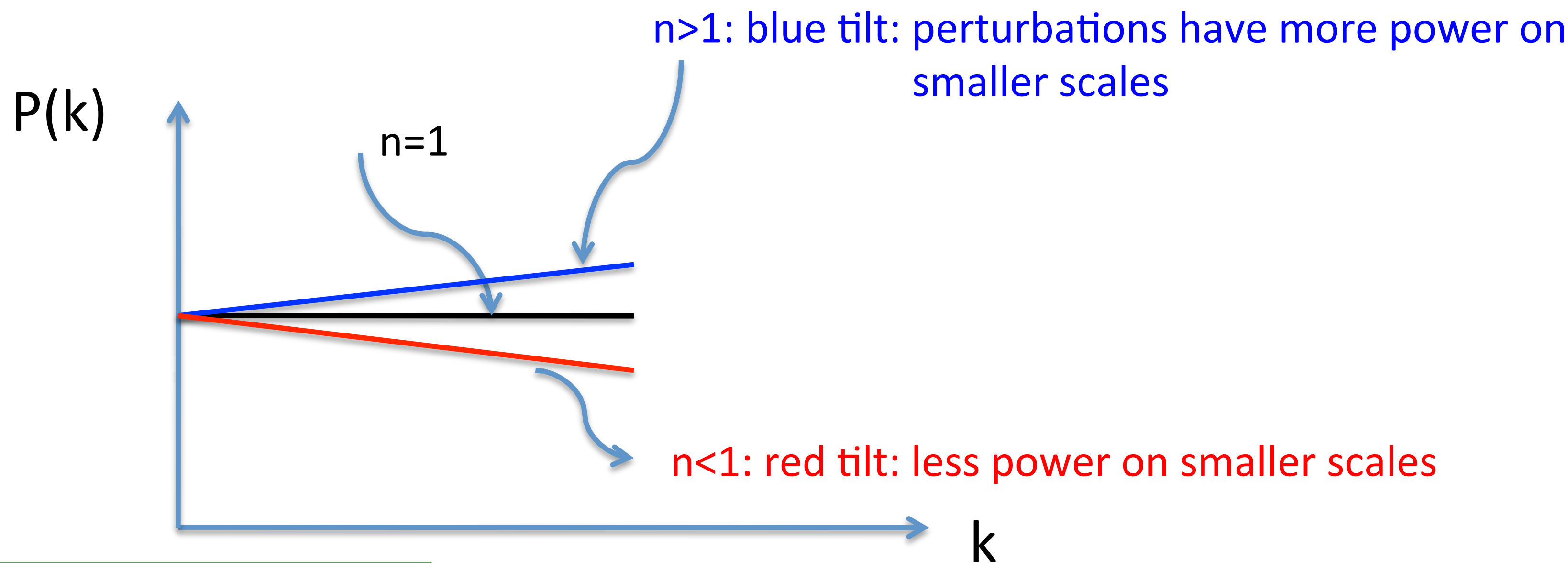
$$\frac{1}{4}\frac{\delta\rho_\gamma}{\rho_\gamma}\Big|_{t^{(2)}(k)} \simeq H\frac{\delta\phi}{\dot{\phi}}\Big|_{t^{(1)}(k)}$$

$\frac{\Delta T}{T}$ CMB anisotropies since CMB photons are black body radiation $\rho_\gamma \propto T^4$



Varying the Spectral index

If n=1: Harrison-Zel’ dovich spectrum (exact scale-invariance)



$$\boxed{n_s - 1 = 2\eta_V - 6\epsilon}$$

parametrizes deviation from scale-invariance:

n=1 would signal some underlying symmetry;

measuring n ≠ 1 would signal a dynamical process for generating the initial density fluctuations (inflation??)

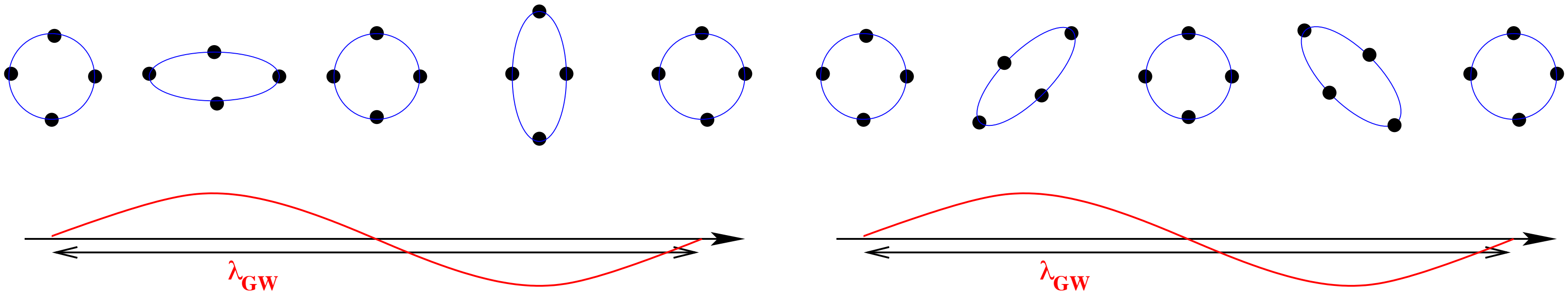


Fig. 1. We show how point particles along a ring move as a result of the interaction with a GW propagating in the direction perpendicular to the plane of the ring. The left panel refers to a wave with $+$ polarization, the right panel with $\times$ polarization.

Quantum fluctuations of a massless scalar field in (quasi) de-Sitter

$$u_k'' + \left(k^2 - \frac{a''}{a} + \cancel{V_{,\phi\phi}} a^2\right) u_k = 0 \;\longleftrightarrow\; \boxed{u_k'' + \left(k^2 - \frac{\nu^2 - \frac{1}{4}}{\tau^2}\right) = 0}$$

Bessel equation

$$a(\tau) = -\frac{1}{H\tau(1-\epsilon)} \longrightarrow \; a''/a \simeq \frac{2}{\tau^2}\left(1 + \frac{3}{2}\epsilon\right)$$

$$\nu \simeq \frac{3}{2} + \epsilon$$

Primordial gravitational waves

And hence, summing over the 2 polarization states:

$$\mathcal{P}_T = \frac{8}{M_{\rm Pl}^2}\left(\frac{H}{2\pi}\right)^2\left(\frac{k}{aH}\right)^{-2\epsilon}$$ with spectral index $$n_T = \frac{d\ln\mathcal{P}_T}{d\ln k} = -2\epsilon$$

- Notice that, since $(H/M_{Pl})^2 \sim V(\phi)$, then the amplitude of the gravitational waves is proportional to the energy scale of inflation: $E_{infl}=V^{1/4}$.

Observational predictions (I)

➢ **Primordial density (scalar) perturbations**

$$\mathcal{P}_\zeta(k) = \frac{1}{2M_{\rm Pl}^2\epsilon}\left(\frac{H}{2\pi}\right)^2\left(\frac{k}{aH}\right)^{n_s-1}$$

amplitude

spectral index: $n_s - 1 = 2\eta_V - 6\epsilon$
(or ``tilt'')

N.B: both depend on the dynamics of the scalar field during inflation and hence on the potential

Recall: $H^2 \simeq (M_{\rm Pl}^2/3)V(\phi);\quad 3H\dot{\phi} \simeq -V'(\phi);\quad \epsilon \simeq \frac{1}{16\pi G}\left(\frac{V_{,\phi}}{V}\right)^2 \ll 1 \quad \eta_V = \frac{1}{8\pi G}\left(\frac{V_{,\phi\phi}}{V}\right) \ll 1$

➢ **Primordial (tensor) gravitational waves**

$$\mathcal{P}_{\rm T}(k) = \frac{8}{M_{\rm Pl}^2}\left(\frac{H}{2\pi}\right)^2\left(\frac{k}{aH}\right)^{n_{\rm T}}$$

Tensor spectral index: $n_{\rm T} = -2\epsilon$

Observational predictions (II)

- The overall amplitude will be fixed by the normalization with observations, so just consider the relative amplitude

$$\boxed{r = \frac{\mathcal{P}_{\mathrm{T}}(k_0)}{\mathcal{P}_{\zeta}(k_0)} = 16\epsilon}$$

tensor-to-scalar perturbation ratio

- **Consistency relation** (valid for all single field of slow-roll inflation). At lowest order in the slow-roll parameters

$$\boxed{r = -8n_T}$$

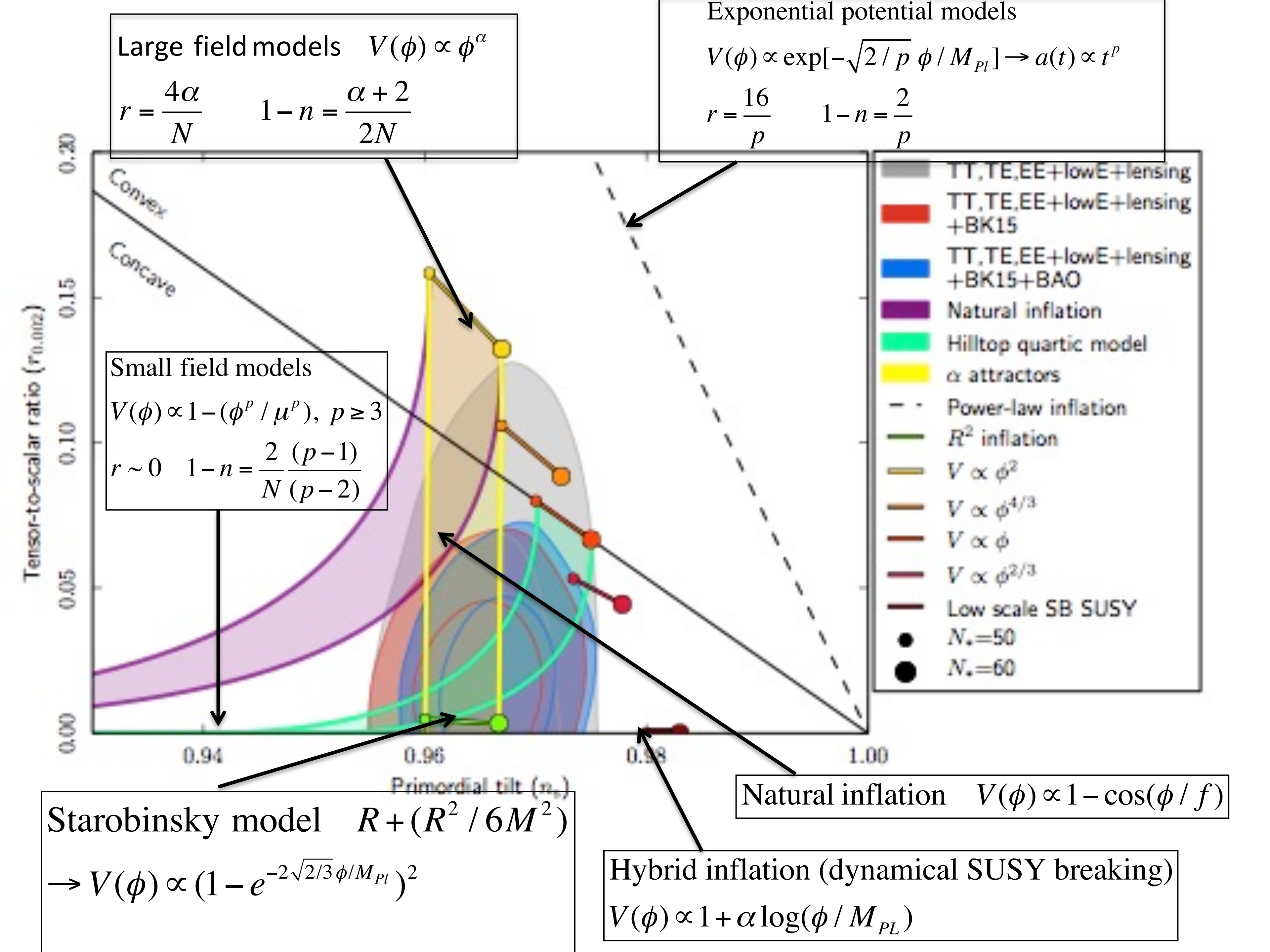
Observational predictions

One can also consider a running of the spectral index and a running of the running

$$\mathcal{P}_\zeta = A_\zeta \left(\frac{k}{k_*}\right)^{n_s - 1 + \frac{1}{2} dn_s/d\ln k \ln(k/k_*) + \frac{1}{8} d^2 n_s/d\ln k^2 (\ln(k/k_*))^2 + \ldots.}$$

$$n_s - 1 = 2\eta_V - 6\varepsilon$$

$$dn_s / d\ln k = -(1/2)\xi^2 + 16\varepsilon\eta_V - 24\varepsilon^2 \qquad \xi^2 = \left(\frac{1}{4\pi G}\right)^2 \left(\frac{V'V'''}{V^2}\right)$$



Large field models $V(\phi) \propto \phi^{\alpha}$
$r = \frac{4\alpha}{N} \quad 1-n = \frac{\alpha+2}{2N}$
Exponential potential models
$V(\phi) \propto \exp[-\sqrt{2/p}\,\phi/M_{Pl}] \to a(t) \propto t^{p}$
$r = \frac{16}{p} \quad 1-n = \frac{2}{p}$
Small field models
$V(\phi) \propto 1-(\phi^{p}/\mu^{p}),\ p \geq 3$
$r \sim 0 \quad 1-n = \frac{2}{N}\frac{(p-1)}{(p-2)}$
Starobinsky model $R + (R^{2}/6M^{2})$
$\to V(\phi) \propto (1-e^{-2\sqrt{2/3}\phi/M_{Pl}})^{2}$
Natural inflation $V(\phi) \propto 1-\cos(\phi/f)$
Hybrid inflation (dynamical SUSY breaking)
$V(\phi) \propto 1+\alpha\log(\phi/M_{PL})$
Convex
Concave
Tensor-to-scalar ratio ($r_{0.002}$)
Primordial tilt (n_s)
0.00
0.05
0.10
0.15
0.20
0.94
0.96
0.98
1.00
TT,TE,EE+lowE+lensing
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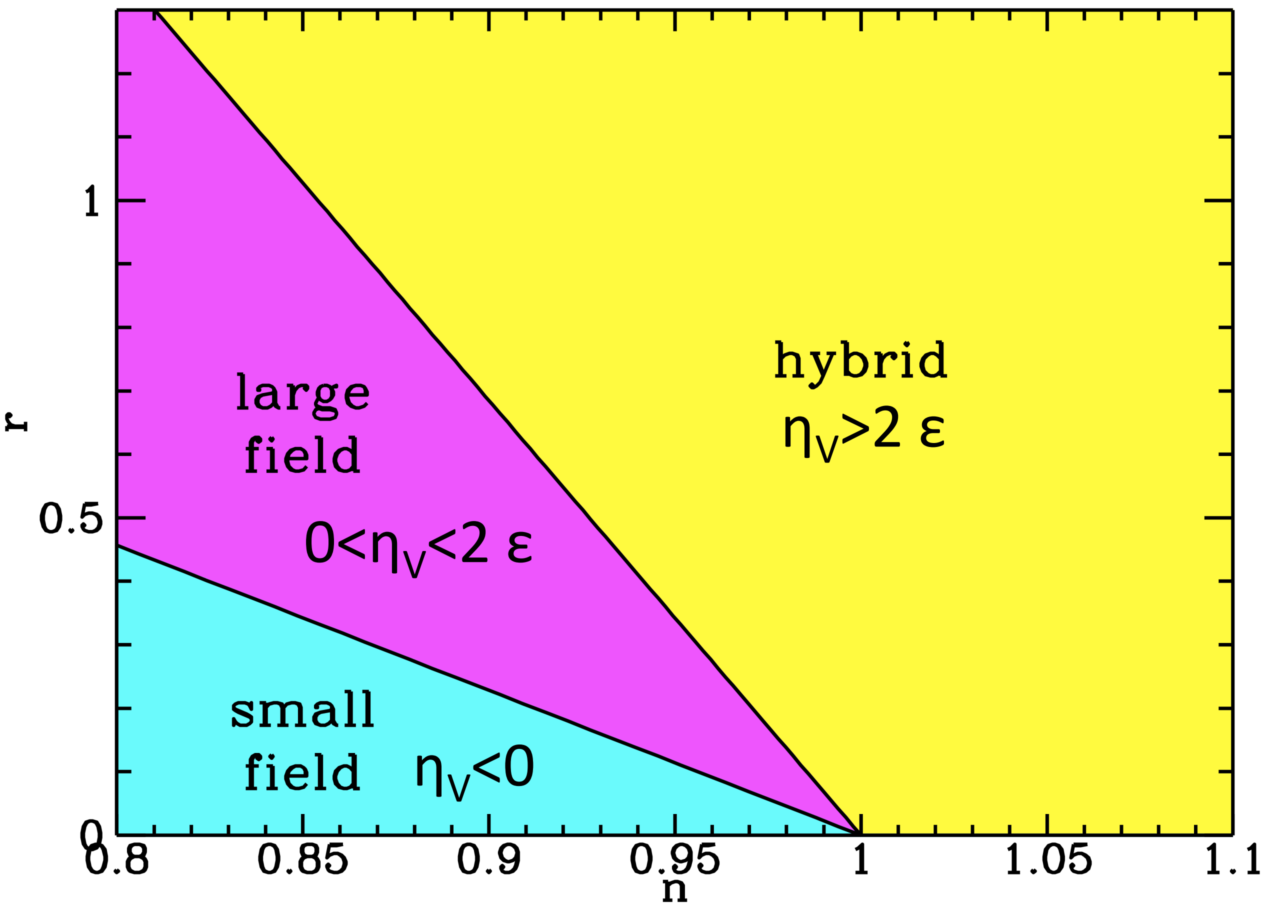
$V \propto \phi^2$
$V \propto \phi^{4/3}$
$V \propto \phi$
$V \propto \phi^{2/3}$
Low scale SB SUSY
$N_*=50$
$N_*=60$

Zoology of inflationary models



$$r = \frac{8}{3}(1-n_s) + \frac{2m_{Pl}^2}{3\pi}\frac{V_{,\phi\phi}}{V}$$

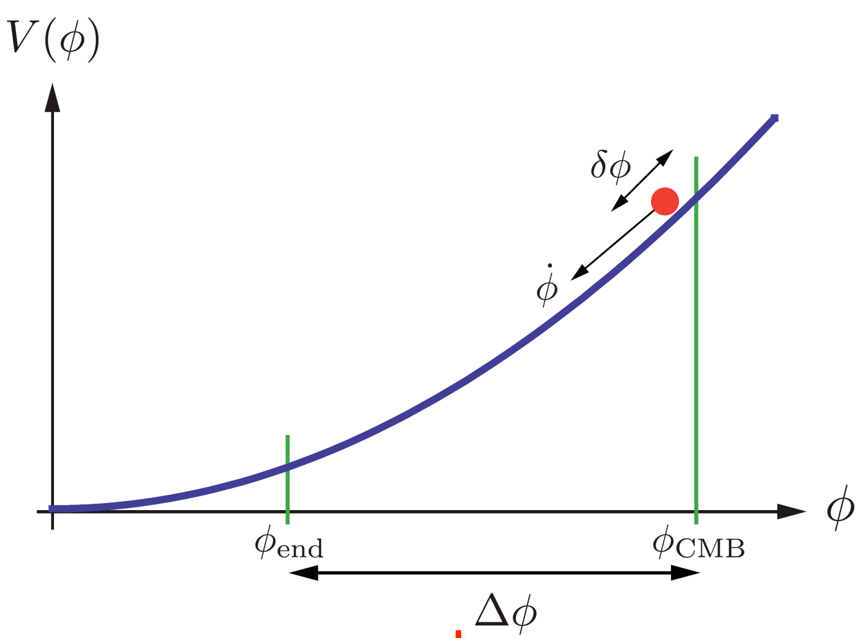
$$n_s - 1 = 2\eta_V - 6\epsilon$$

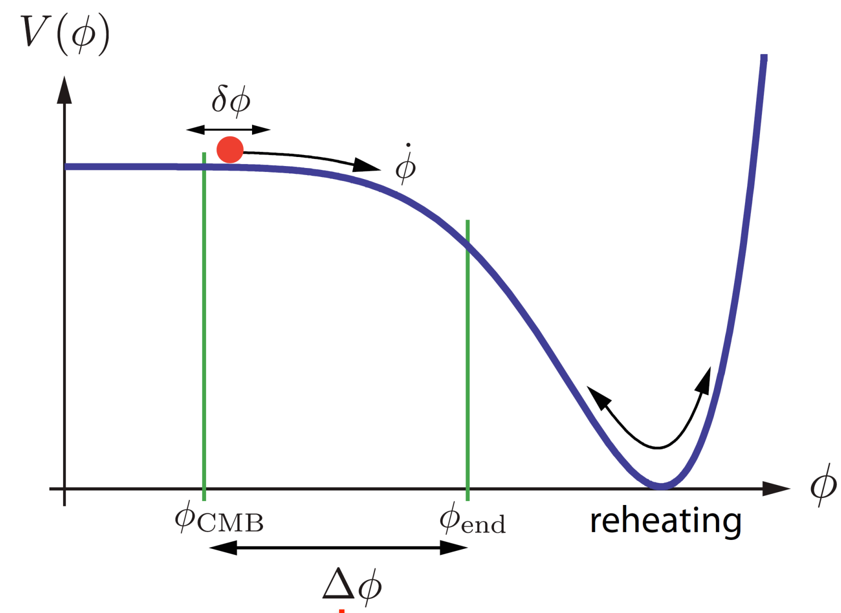
See,e.g., Kinney et al. astro-ph/0007375

Classifying inflationary models

Roughly speaking: ``Large field'' models can produce a high level of gravity waves;
``small field'' models produce a low level of gravity waves

$$r = \frac{\mathcal{P}_{\mathrm{T}}}{\mathcal{P}_{\zeta}} = \epsilon$$





``Large field'' like potential

``Small field'' like potential
predict ε very very small

"Large field" models can produce a high level of gravity waves (r>0.01)

"Small field" models produce a low level of gravity waves (r<0.01)

Take the previously derived formula for the excursion of the scalar field

$$\Delta\phi = \int_{\phi_{CMB}}^{\phi_{end}} d\phi = \int_{t_{CMB}}^{t_{end}} \dot\phi dt \simeq \frac{\dot\phi}{H}\int_{Ht_{CMB}}^{Ht_{end}} d(Ht) = \frac{\dot\phi}{H} N_{CMB} \sim \epsilon^{1/2} N_{CMB} M_{\rm Pl}$$

But remember the tensor-to-scalar ratio $r \sim \epsilon$

The precise relation one obtains is

$$\boxed{\frac{\Delta\phi}{M_{\rm Pl}} \simeq \left(\frac{r}{0.01}\right)^{1/2}}$$

So the bigger is the field excursion during inflation the bigger is the amplitude of the gravity waves

Classifying inflationary models

- ***``Large-field'' models $0<\eta_V<2\varepsilon$:***

$$V(\phi) \propto \phi^p$$

typical of ``caothic inflation scenario'' (Linde `83)

$$V(\phi) \propto \exp[\phi/\mu]$$

``power law inflation'' (Lucchin, Matarrese '85)

- ***``small-field models'': $\eta_V<0$***

$$V(\phi) \propto \left[1 - \left(\frac{\phi}{\mu}\right)^p\right]$$

$$p > 2; \quad \phi < \mu < M_{\rm Pl}$$

from spontaneous symmetry breaking or Goldstone, axion modes (Linde; Albrecht, Steinhardt `82; Freese et al '90)

- ***hybrid models $\eta_V>2\,\varepsilon$:***

$$V(\phi) \propto \left[1 + \left(\frac{\phi}{\mu}\right)^p\right]$$

supersimmetry; typically involve a second field to end inflation (Linde '91; '94)

Classifying inflationary models

Two more interesting models (as an example):

- Natural inflation

$$V(\phi) = V_0 \left[1 - \cos\left(\frac{\phi}{\mu}\right)\right]$$

Related to a shift symmetry of the inflaton: φ → φ+c, where c is a constant. If exact this symmetry would imply that φ is massless (the potential would be exactly flat). Usually the symmetry gets broken → a small mass is generated → pseudo Nambu-Goldstone field (axion).

For $\mu > M_{Pl}$ it is a large field models (Freese et al. 1990)

For $\mu < M_{Pl}$ it is a small field models

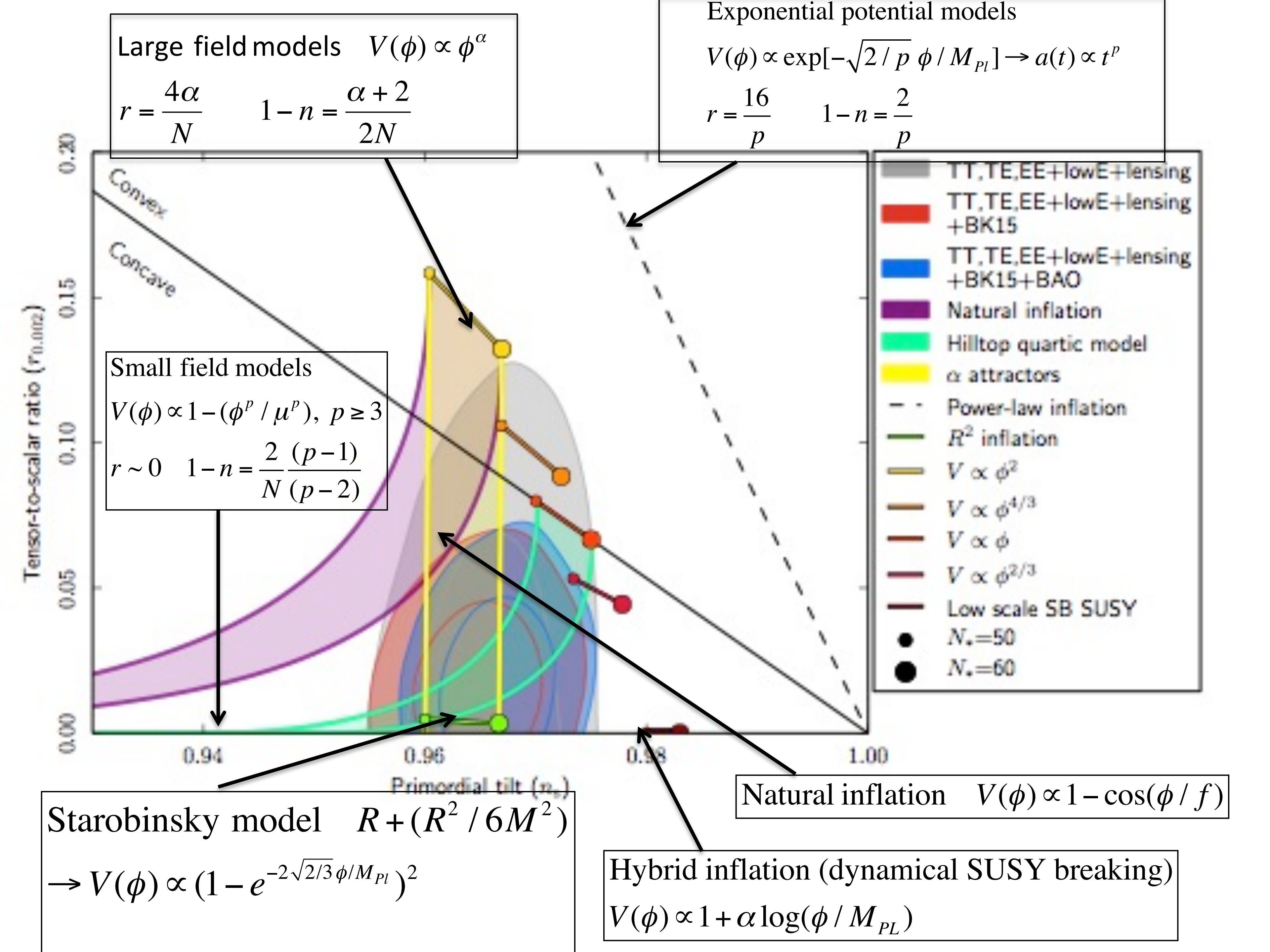
- R^2 inflation

$$S = \int d^4x \sqrt{-g}\, \frac{M_{pl}^2}{2} \left(R + \frac{R^2}{6M^2}\right)$$

Predicts a tiny amount of gravity waves (Starobinsky 1980)

Motivation: a modified gravity theory arising from quantum corrections. The R^2 term corresponds to an additional scalar degree of freedom that plays the role of the inflaton. In fact via a conformal transformation $g_{\mu\nu} \to \mathrm{e}^{-2\omega} g_{\mu\nu}$ with $\sqrt{6} M_{\mathrm{Pl}} \omega = \varphi_1$ one can rewrite this action in the so called Einstein frame, where, besides the Ricci scalar R of the usual Hilbert-Einstein action, there is the action of a minimally coupled scalar field with standard kinetic term and a potential

$$M^2 M_{\mathrm{Pl}}^4 \left(1 - \mathrm{e}^{-2\varphi_1/\sqrt{6} M_{\mathrm{Pl}}}\right)^2$$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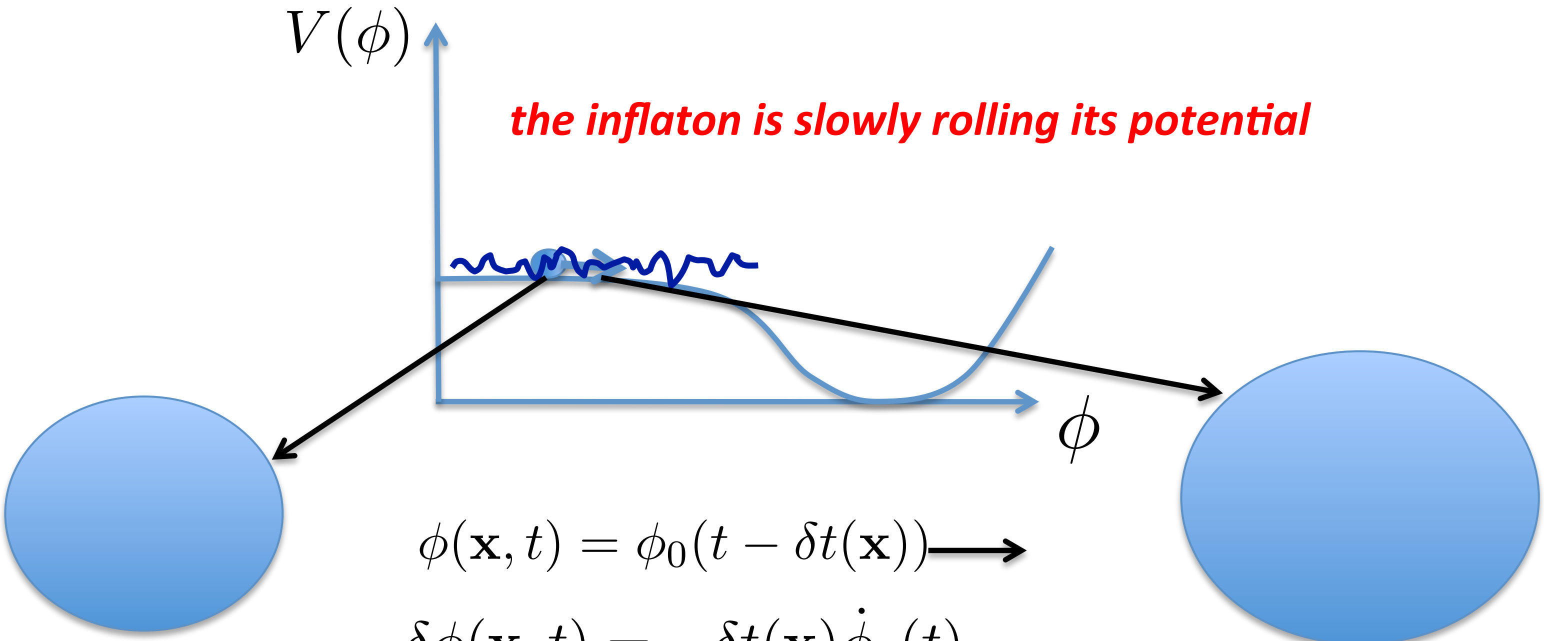
Inflation



$$\phi(\mathbf{x},t) = \phi_0(t - \delta t(\mathbf{x})) \longrightarrow$$

$$\delta\phi(\mathbf{x},t) = -\delta t(\mathbf{x})\dot{\phi}_0(t)$$

- On large (super-horizon scales) so that each region in the universe goes through the same expansion history but at slightly different times

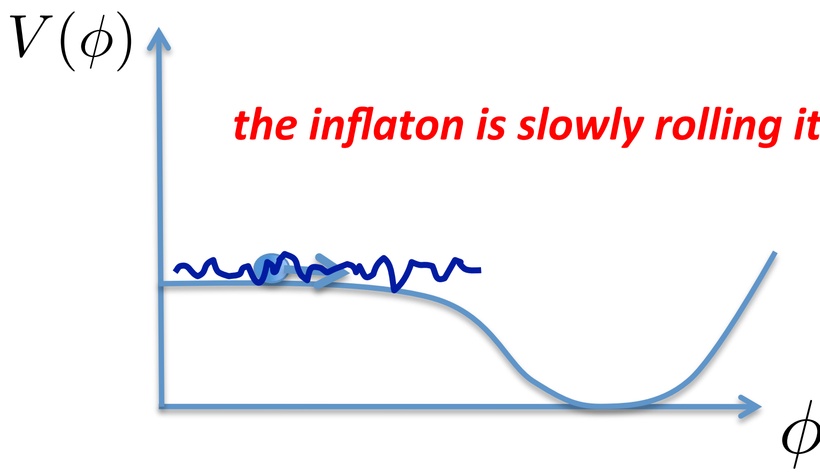
- Fluctuations in the inflaton produce fluctuations in the universe expansion from place to place

$$H^2 \simeq \frac{8}{3}\pi G V(\phi)$$

$$\longrightarrow \text{number of e-foldings } N = \ln(a_f/a_i) = \int_{t_i}^{t_f} H\,dt \longrightarrow$$

$$\boxed{\zeta = H\delta t \quad = -H\frac{\delta\phi}{\dot{\phi}} \simeq -H\frac{\delta\rho}{\dot{\rho}}}$$

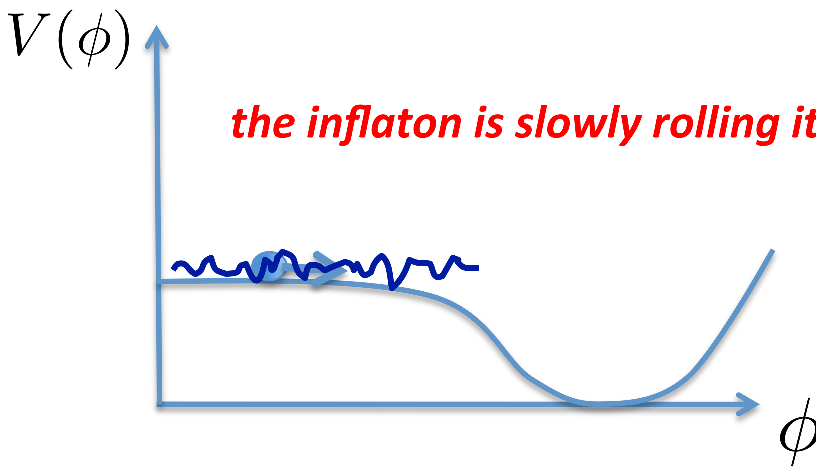
Inflation



Fluctuations in the inflaton produce fluctuations in the universe expansion from place to place, so that each region in the universe goes through the same expansion history but at slightly different times:

$$\zeta \sim \frac{\Delta T}{T} \sim \frac{\delta\rho}{\rho} \qquad \zeta \simeq \frac{H\delta\phi}{\dot{\phi}}$$

Inflation



- On large (super-horizon scales) so that each region in the universe goes through the same expansion history but at slightly different times:

$$\phi(\mathbf{x},t) = \phi_0(t - \delta t(\mathbf{x})) \longrightarrow \delta\phi(\mathbf{x},t) = -\delta t(\mathbf{x})\dot{\phi}_0(t)$$

- Fluctuations in the inflaton produce fluctuations in the universe expansion from place to place $H^2 \simeq \frac{8}{3}\pi G V(\phi) \longrightarrow$

$$\text{number of e-foldings } N = \ln(a_f/a_i) = \int_{t_i}^{t_f} H dt \longrightarrow \boxed{\zeta = H\delta t \quad = -H\frac{\delta\phi}{\dot{\phi}} \simeq -H\frac{\delta\rho}{\dot{\rho}}}$$

Additional expansion