RECENT RESULTS ON HAMILTONIAN EQUIVALENCE BETWEEN JORDAN AND EINSTEIN FRAME

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partially in collaboration with Matteo Galaverni and based on arXiv:2003.04304 Phys. Rev. D 103 024022 (2021) , arXiv:2110.12222 Phys. Rev. D 105 084008 (2022), arXiv:2112.02098 Universe 8 (2021), and more to come…

Outline

- Jordan and Einstein frame (JF and EF)

- Hamiltonian analysis of Brans-Dicke theory with GHY-boundary term (case $\omega \neq -\frac{3}{2}$ and $\omega = -\frac{3}{2}$).

- Hamiltonian transformations from Jordan to the Einstein Frame. "Vexata Questio": are these transformations canonical? "Anti-Newtonian"transformations as Hamiltonian canonical transformations.

- Confronting and contrasting e.o.m. of Brans- Dicke FLRW flat, k=0, case in JF and EF. Study of the Hamiltonian canonicity on the extended phase space and gauge fixed (N=0) phase space.

- Hamiltonian analysis of JF and EF in spherical symmetric case. Two inequivalent solutions, Janis and BBMB, mapped one into the other upon gauge fixing lapse and radial shift. Physical considerations

- Conclusions.

Jordan-Einstein Frames

- Old paper: Dicke (Phys. Rev. (1962) **125,** 6 2163-2167)

Suppose the proton mass is m_p in mass units m_u and, in "natural units", we scale the unit of measurement by a factor λ^{-1} (length)$^{-1}$ $\tilde{m}_u = \lambda^{-1} m_u$. In the new unit the proton mass $\tilde{m}_p = \lambda^{-1} m_p$.

- Confronting the measurement of the proton mass in the two mass units (Faraoni and Nadeau 2006)

$$\frac{\tilde{m}_p}{\tilde{m}_u} = \frac{\lambda^{-1} m_p}{\lambda^{-1} m_u} = \frac{m_p}{m_u}$$

Jordan-Einstein Frames

- Since $d\tilde{s} = \lambda ds$ and $ds = \left(g_{ij}dx^i dx^j\right)^{\frac{1}{2}}$, then the covariant metric functions scales as

$$\tilde{g}_{\mu\nu} = \lambda^2 g_{\mu\nu}$$

- Invariance under rescaling of unit of measurement implies Weyl (conformal invariance) of the metric tensor

- The starting frame is called "Jordan" frame and the conformal transformed the "Einstein Frame. One observable can be computed in both frames. Its measure, obviously different in the two frames, is related by conformal rescaling according to the observable's dimensions.(e.g. $\tilde{m}_p = \lambda^{-1} m_p$).

Scalar-Tensor Theory

- In general, one starts from a scalar-tensor theory, with GHY-like boundary term, in the Jordan Frame

$$S = \int_M d^n x \sqrt{-g} \left(f(\phi) R - \frac{1}{2} \lambda(\phi) g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi - U(\phi) \right) + 2 \int_{\partial M} d^{n-1} \sqrt{h} f(\phi) K$$

- and passes to the Einstein Frame with the transformation

$$\tilde{g}_{\mu\nu} = \left(16\pi G f(\phi) \right)^{\frac{2}{n-2}} g_{\mu\nu} \ ,$$

- therefore the action becomes

$$S = \int_M d^n x \sqrt{-\tilde{g}} \left(\frac{1}{16\pi G} \tilde{R} - A(\phi) \tilde{g}^{\mu\nu} \partial_\mu \phi \partial_\nu \phi - V(\phi) \right) + \frac{1}{8\pi G} \int_{\partial M} d^{n-1} \sqrt{\tilde{h}} \tilde{K}$$

Scalar-Tensor Theory

where

$$A(\phi) = \frac{1}{16\pi G}\left(\frac{\lambda(\phi)}{2f(\phi)} + \frac{n-1}{n-2}\frac{(f'(\phi))^2}{f^2(\phi)}\right), V(\phi) = \frac{U(\phi)}{[16\pi G f(\phi)]^{\frac{n}{n-2}}}$$

- One is looking for solutions of the equation of motions such that if

$$(g_{\mu\nu}(x), \phi(x))$$

is solution in the Jordan Frame,

$$(\tilde{g}_{\mu\nu}(x,\phi), \phi(x))$$

is solution in the Einstein frame

Brans-Dicke Theory

- Brans-Dicke, with GHY boundary term, is a particular case of Scalar Tensor theory

$$S = \int_M d^4x\sqrt{-g}\left(\phi\,{}^4R - \frac{\omega}{\phi}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - U(\phi)\right) + 2\int_{\partial M} d^3x\sqrt{h}\phi K \quad .$$

- We studied the 3+1 ADM (Hamiltonian) decomposition

$$g = -(N^2 - N_iN^i)dt\otimes dt + N_i(dx^i\otimes dt + dt\otimes dx^i) + h_{ij}dx^i\otimes dx^j \quad ,$$

- The ADM Lagrangian $\mathcal{L}_{ADM}$ is

$$\begin{aligned}\mathcal{L}_{ADM} &= \sqrt{h}\Big[N\phi\left({}^{(3)}R + K_{ij}K^{ij} - K^2\right) - \frac{\omega}{N\phi}\left(N^2h^{ij}D_i\phi D_j\phi - (\dot{\phi} - N^iD_i\phi)^2\right) \\ &+ 2K(\dot{\phi} - N^iD_i\phi) - NU(\phi) + 2h^{ij}D_iND_j\phi\Big]\end{aligned}$$

Brans-Dicke Theory

- Therefore $\mathcal{H}_{ADM}$ is the sum of the Hamiltonian constraint $\mathcal{H}$ and the momentum constraint $\mathcal{H}_i$

$$\mathcal{H}_{ADM} = N\mathcal{H} + N^i\mathcal{H}_i$$

$$\begin{aligned}\mathcal{H} &= \sqrt{h}\Bigg\{ \left[-\phi\, {}^3R + \frac{1}{\phi h}\left(\pi^{ij}\pi_{ij} - \frac{{\pi_h}^2}{2}\right)\right] \\ &+ \frac{\omega}{\phi}D_i\phi D^i\phi + 2D^iD_i\phi + \frac{1}{2h\phi}\left(\frac{1}{3+2\omega}\right)(\pi_h - \phi\pi_\phi)^2 + V(\phi)\Bigg\}\end{aligned}$$

$$\mathcal{H}_i = -2D_j\pi^j_i + D_i\phi\pi_\phi$$

- The constraint algebra is like Einstein's Geometrodynamics

$$\{\mathcal{H}_i(x), \mathcal{H}_j(x')\} = \mathcal{H}_i(x')\partial_j\delta(x,x') - \mathcal{H}_i(x){\partial_j}'\delta(x,x') \qquad \{\mathcal{H}(x), \mathcal{H}_i(x')\} = -\mathcal{H}(x')\partial'_i\delta(x,x')$$

$$\{\mathcal{H}(x), \mathcal{H}(x')\} = \mathcal{H}^i(x)\partial_i\delta(x,x') - \mathcal{H}^i(x')\partial'_i\delta(x,x')$$

Brans-Dicke Theory-Einstein Frame

- Implementing the Weyl (conformal) transformation, we get the following ADM metric $\tilde{g}$ in the Einstein Frame

$$\tilde{g} = -(\tilde{N}^2 - \tilde{N}_i\tilde{N}^i)dt \otimes dt + \tilde{N}_i(dx^i \otimes dt + dt \otimes dx^i) + \tilde{h}_{ij}dx^i \otimes dx^j$$

$$\tilde{N} = (16\pi G f(\phi))^{\frac{1}{n-2}}\, N; \tilde{N}_i = (16\pi G f(\phi))^{\frac{2}{n-2}}\, N_i;$$

$$\tilde{h}_{ij} = (16\pi G f(\phi))^{\frac{2}{n-2}}\, h_{ij}.$$

- Now we recall that in the Brans-Dicke case $f(\phi) = \phi$

$$S = \frac{1}{16\pi G}\int_M dx^4\sqrt{-\tilde{g}}\left[{}^4\tilde{R} - \frac{(\omega + \frac{3}{2})}{\phi^2}\tilde{g}^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - \tilde{V}(\phi)\right]$$

Brans-Dicke Theory-Einstein Frame

- Canonical momenta ($\tilde{\mathcal{L}}_{\rm ADM}$ is ADM-Lagrangian density in the E-F)

$$\tilde{\pi}^{ij} = \frac{\partial \tilde{\mathcal{L}}_{ADM}}{\partial \dot{\tilde{h}}_{ij}} = -\frac{\sqrt{\tilde{h}}}{16\pi G}\left(\tilde{K}^{ij} - \tilde{K}\tilde{h}^{ij}\right) = \frac{\pi^{ij}}{16\pi G\phi}$$

$$\tilde{\pi}_\phi = \frac{\partial \tilde{\mathcal{L}}_{ADM}}{\partial \dot{\phi}} = \frac{\sqrt{\tilde{h}}(\omega + \frac{3}{2})}{8\pi G\tilde{N}\phi^2}\left(\dot{\phi} - \tilde{N}^i\partial_i\phi\right)$$

$$= \frac{1}{\phi}(\phi\pi_\phi - \pi_h).$$

- The ADM Hamiltonian density $\mathcal{H}_{ADM}$ in the E-F is

$$\mathcal{H}_{ADM} = \frac{\sqrt{\tilde{h}}\tilde{N}}{16\pi G}\left[-{}^3\tilde{R} + \frac{(16\pi G)^2}{\tilde{h}}\left(\tilde{\pi}^{ij}\tilde{\pi}_{ij} - \frac{\tilde{\pi}_h^2}{2}\right) + \frac{(\omega + \frac{3}{2})}{\phi^2}\partial_i\phi\partial^i\phi \; + \frac{64(\pi G)^2\phi^2}{h(\omega + \frac{3}{2})}\tilde{\pi}_\phi^2 + \tilde{V}(\phi)\right]$$
$$-2\tilde{N}^i\tilde{D}_j\tilde{\pi}_i^j + \tilde{N}^i\partial_i\phi\tilde{\pi}_\phi \; .$$

Canonical Transformations

- Here, for simplicity, we repeat the transformations from the Jordan to the Einstein Frame in Hamiltonian formalism

$$\tilde{N} = N(16\pi G\phi)^{\frac{1}{2}}\,;\tilde{N}_i = N_i(16\pi G\phi)\,;\tilde{h}_{ij} = (16\pi G\phi)\,h_{ij}\,;\tilde{\pi} = \frac{\pi}{(16\pi G\phi)^{\frac{1}{2}}}\,;$$

$$\tilde{\pi}^i = \frac{\pi^i}{(16\pi G\phi)},;\tilde{\pi}^{ij} = \frac{\pi^{ij}}{16\pi G\phi}\,;\phi = \phi\,;\tilde{\pi}_\phi = \frac{1}{\phi}(\phi\pi_\phi - \pi_h)$$

- One can check they are not Hamiltonian Canonical Transformations

$$\{\tilde{N}, \tilde{\pi}_\phi\} = \frac{8\pi G N}{\sqrt{16\pi G\phi}} \neq 0, \text{and } \{\tilde{N}_i, \tilde{\pi}_\phi\} = 16\pi G N_i \neq 0$$

- Therefore it is meaningless to perform the Dirac's constraint analysis in the Einstein Frame and show that the constraint algebra, for Brans-Dicke, closes in this frame

Hamiltonian Analysis of BD for $\omega \neq -\frac{3}{2}$	
in Jordan Frame	**in Einstein Frame**
constraints	*constraints*
$\pi \approx 0; \pi^i \approx 0; \mathcal{H} \approx 0; \mathcal{H}_i \approx 0;$	$\widetilde{\pi} \approx 0; \widetilde{\pi}_i \approx 0; \widetilde{\mathcal{H}} \approx 0; \widetilde{\mathcal{H}}_i \approx 0;$
constraint algebra	*constraint algebra*
$\{\pi, \pi_i\} = 0; \{\pi, \mathcal{H}\} = 0; \{\pi, \mathcal{H}_i\} = 0; \{\pi_i, \mathcal{H}\} = 0;$	$\{\widetilde{\pi}, \widetilde{\pi}_i\} = 0; \{\widetilde{\pi}, \widetilde{\mathcal{H}}\} = 0; \{\widetilde{\pi}, \widetilde{\mathcal{H}}_i\} = 0; \{\widetilde{\pi}_i, \widetilde{\mathcal{H}}\} = 0;$
$\{\pi_i, \mathcal{H}_j\} = 0;\ \{\mathcal{H}(x), \mathcal{H}_i(x')\} = -\mathcal{H}(x')\partial_i'\delta(x, x');$	$\{\widetilde{\pi}_i, \widetilde{\mathcal{H}}_j\} = 0;\ \left\{\widetilde{\mathcal{H}}(x), \widetilde{\mathcal{H}}_i(x')\right\} = -\widetilde{\mathcal{H}}(x')\partial_i'\delta(x, x');$
$\{\mathcal{H}_i(x), \mathcal{H}_j(x')\} = \mathcal{H}_i(x')\partial_j\delta(x, x') - \mathcal{H}_j(x)\partial_i{}'\delta(x, x');$	$\{\widetilde{\mathcal{H}}_i(x), \widetilde{\mathcal{H}}_j(x')\} = \widetilde{\mathcal{H}}_i(x')\partial_j\delta(x, x') - \widetilde{\mathcal{H}}_i(x)\partial_i{}'\delta(x, x');$
$\{\mathcal{H}(x), \mathcal{H}(x')\} = \mathcal{H}^i(x)\partial_i\delta(x, x') - \mathcal{H}^i(x')\partial_i'\delta(x, x');$	$\{\widetilde{\mathcal{H}}(x), \widetilde{\mathcal{H}}(x')\} = \widetilde{\mathcal{H}}^i(x)\partial_i\delta(x, x') - \widetilde{\mathcal{H}}^i(x')\partial_i'\delta(x, x');$

Main criticism to this non-canonicity argument

- In litterature, people object N, N^i are mere Lagrangian multipliers and canonicity should be checked on the true physical degrees of freedom.

- This could be misleading. Lapse and Shifts cannot be eliminated "ad hoc", they are still canonical variables

- The only way we can "safely" treat them is by making a gauge fixing (ex. $N \approx c_1$, $N^i \approx c^i$ so that $\pi \approx 0$, $\pi_i \approx 0$ becomes second class constraints).

- N, N^i, π , π_i are then eliminated defining Dirac's brackets.

Canonical Transformations

- There exist Hamiltonian Canonical Transformations (Anti-Gravity transformations)

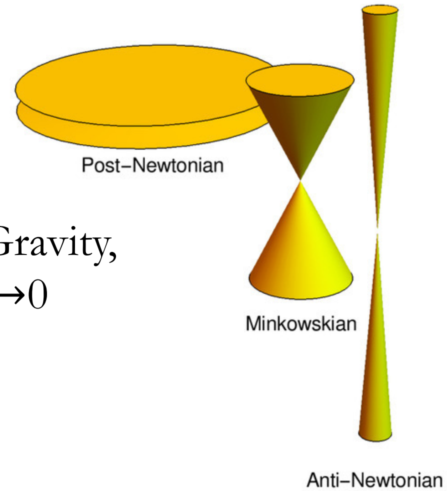
$$N \mapsto N,\;\; N^i \mapsto N^i,\;\; h_{ij} \mapsto \lambda^2 h_{ij}$$ (in two dim. $ds^2 = -dt^2 + \lambda^2 dx^2; \lambda > 1$)

$$\tilde{N}^* = N \,;\, \tilde{N}^*{}_i = N_i \,;\, \tilde{\pi}^* = \pi \,;\, \tilde{\pi}^{*i} = \pi^i$$

- In this case the ADM Hamiltonian $\mathcal{H}_{ADM}$

$$\mathcal{H}_{ADM} = \frac{\sqrt{\tilde{h}}\tilde{N}^*(\phi)^{\frac{1}{2}}}{(16\pi G)^{\frac{1}{2}}}\Bigg[-\,{}^3\tilde{R} + \frac{(16\pi G)^2}{\tilde{h}}\left(\tilde{\pi}^{ij}\tilde{\pi}_{ij} - \frac{\tilde{\pi}_h^2}{2}\right)$$
$$+\frac{(\omega+\frac{3}{2})}{\phi^2}\partial_i\phi\partial^i\phi \;+\; \frac{64(\pi G)^2\phi^2}{h(\omega+\frac{3}{2})}\tilde{\pi}_\phi^2 + \tilde{V}(\phi)\Bigg] - 2\tilde{N^*}^i\tilde{D}_j\tilde{\pi}_i^j + \tilde{N^*}^i\partial_i\phi\tilde{\pi}_\phi\,.$$

Carrolian Gravity,
G→ ∞ , c →0



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- Since this theory is canonically equivalent to B-D, the constraint algebra of secondary first class constraints ($\mathcal{H}$, $\mathcal{H}_i$) is like B-D's one.

BRANS-DICKE PARTICULAR CASE $\omega = -\frac{3}{2}$

- The B-D action for $\omega = -\frac{3}{2}$ is (for consistency reasons here U(ϕ)=$\alpha\phi^2$ α is a constant)

$$S^{(-3/2)} = \int_M d^4x\sqrt{-g}\left(\phi R + \frac{3}{2}\frac{g^{\mu\nu}}{\phi}\partial_\mu\phi\partial_\nu\phi - \alpha\phi^2\right) + 2\int_{\partial M} d^3x\sqrt{h}\phi K\ .$$

- It is invariant under this conformal transformations

$$\widetilde{g}_{\mu\nu} = \Omega^2 g_{\mu\nu} \quad \widetilde{\phi} = \frac{\phi}{\Omega^2}$$

- The ADM Hamiltonian in this particular case is

$$\mathcal{H}^{(-3/2)}_{ADM} = \sqrt{h}\Bigg\{N\left[-\phi\,^3R + \frac{1}{\phi h}\left(\pi^{ij}\pi_{ij} - \frac{{\pi_h}^2}{2}\right)\right]$$
$$- \frac{3N}{2\phi}D_i\phi D^i\phi + N2D^iD_i\phi + NU(\phi)\Bigg\} - 2N^iD_j\pi^j_i + N^iD_i\phi\pi_\phi$$

BRANS-DICKE PARTICULAR CASE $\omega = -\frac{3}{2}$

- Clearly the Hamiltonian and momenta constraints are

$$\mathcal{H}^{(-3/2)} = \sqrt{h}\left\{ \left[-\phi\ {}^3R + \frac{1}{\phi h}\left(\pi^{ij}\pi_{ij} - \frac{{\pi_h}^2}{2}\right)\right] - \frac{3}{2\phi} D_i\phi D^i\phi + 2D^i D_i\phi + U(\phi)\right\}$$

$$\mathcal{H}_i^{(-3/2)} = -2D_j\pi_i^j + D_i\phi\pi_\phi$$

- We also have a further primary constraint due to conformal invariance

$$C_\phi \equiv \pi_h - \phi\pi_\phi \approx 0$$

- All the constraints (shown through lengthy and technically complicated calculations) are first class .

BD PARTICULAR CASE EINSTEIN FRAME

- The B-D action for $\omega = -\frac{3}{2}$ in the Einstein Frame is nothing else but (the potential is now $\widetilde{V}(\phi)$ a constant function)

$$S^{(-3/2)} = \frac{1}{16\pi G}\int_M dx^4 \sqrt{-\widetilde{g}}\left({}^4\widetilde{R} - \widetilde{V}(\phi)\right) + \frac{1}{8\pi G}\int_{\partial M} d^3x \sqrt{\widetilde{h}}\widetilde{K}$$

- Notice that the theory is just Einstein GR and is not (Weyl)-Conformal invariant

- The relative ADM Hamiltonian

$$\begin{aligned}\widetilde{\mathcal{H}}^{(-3/2)}_{ADM} &= \frac{\sqrt{\widetilde{h}}\widetilde{N}}{16\pi G}\left[-\,{}^3\widetilde{R} + \frac{(16\pi G)^2}{\widetilde{h}}\left(\widetilde{\pi}^{ij}\widetilde{\pi}_{ij} - \frac{\widetilde{\pi}_h^2}{2}\right) + \widetilde{V}(\phi)\right] - 2\widetilde{N}^i\widetilde{D}_j\widetilde{\pi}^j_i \\ &= \widetilde{N}\widetilde{\mathcal{H}}^{(-3/2)} + \widetilde{N}^i\widetilde{\mathcal{H}}^{(-3/2)}_i\,, \end{aligned} \qquad (1)$$

- The Dirac primary constraint $\tilde{C}_\phi \approx 0$ becomes $-\tilde{\phi}\tilde{\pi}_\phi \approx 0$. The other Dirac's constraints are the same as Einstein's GR.

Hamiltonian Analysis of BD for $\omega = -\frac{3}{2}$	
in Jordan Frame	**in Einstein Frame**
constraints $\pi \approx 0; \pi^i \approx 0; C_\phi \approx 0; \mathcal{H}^{(-3/2)} \approx 0; \mathcal{H}_i^{(-3/2)} \approx 0;$	*constraints* $\widetilde{\pi} \approx 0; \widetilde{\pi}_i \approx 0; \widetilde{C}_\phi = \widetilde{\pi}_\phi \approx 0; \widetilde{\mathcal{H}}^{(-3/2)} \approx 0; \widetilde{\mathcal{H}}_i^{(-3/2)} \approx 0;$
constraint algebra $\{\pi, \pi_i\} = \{\pi, \mathcal{H}^{(-3/2)}\} = \{\pi, \mathcal{H}_i^{(-3/2)}\} = 0;$ $\{\pi_i, \mathcal{H}^{(-3/2)}\} = \{\pi_i, \mathcal{H}_j^{(-3/2)}\} = 0;$ $\left\{C_\phi(x), \mathcal{H}_i^{(-3/2)}(x')\right\} = -\partial_i\delta(x,x')C_\phi(x');$ $\left\{C_\phi(x), \mathcal{H}^{(-3/2)}(x')\right\} = \frac{1}{2}\mathcal{H}^{(-3/2)}(x)\delta(x,x');$ $\left\{\mathcal{H}^{(-3/2)}(x), \mathcal{H}_i^{(-3/2)}(x')\right\} = -\mathcal{H}^{(-3/2)}(x')\partial_i'\delta(x,x');$ $\left\{\mathcal{H}_i^{(-3/2)}(x), \mathcal{H}_j^{(-3/2)}(x')\right\} = \mathcal{H}_i^{(-3/2)}(x')\partial_j\delta(x,x')$ $-\mathcal{H}_j^{(-3/2)}(x)\partial_i{}'\delta(x,x');$ $\{\mathcal{H}^{(-3/2)}(x), \mathcal{H}^{(-3/2)}(x')\} =$ $\mathcal{H}_i^{(-3/2)}(x)\partial^i\delta(x,x') - \mathcal{H}_i^{(-3/2)}(x')\partial'^i\delta(x,x') +$ $\left((D^i \log\phi)C_\phi\right)(x)\partial_i\delta(x,x')$ $-\left((D^i \log\phi)C_\phi\right)(x')\partial_i'\delta(x,x');$	*constraint algebra* $\{\widetilde{\pi}, \widetilde{\pi}_i\} = \{\widetilde{\pi}, \widetilde{\mathcal{H}}^{(-3/2)}\} = 0; \{\widetilde{\pi}, \widetilde{\mathcal{H}}_i^{(-3/2)}\} = 0;$ $\{\widetilde{\pi}_i, \widetilde{\mathcal{H}}^{(-3/2)}\} = \{\widetilde{\pi}_i, \widetilde{\mathcal{H}}_j^{(-3/2)}\} = 0;$ $\left\{\widetilde{C}_\phi(x), \widetilde{\mathcal{H}}_i^{(-3/2)}(x')\right\} = 0;$ $\left\{\widetilde{C}_\phi(x), \widetilde{\mathcal{H}}^{(-3/2)}(x')\right\} = 0;$ $\left\{\widetilde{\mathcal{H}}^{(-3/2)}(x), \widetilde{\mathcal{H}}_i^{(-3/2)}(x')\right\} = -\widetilde{\mathcal{H}}^{(-3/2)}(x')\partial_i'\delta(x,x');$ $\{\widetilde{\mathcal{H}}_i^{(-3/2)}(x), \widetilde{\mathcal{H}}_j^{(-3/2)}(x')\} = \widetilde{\mathcal{H}}_i^{(-3/2)}(x')\partial_j\delta(x,x')$ $-\widetilde{\mathcal{H}}_i^{(-3/2)}(x)\partial_i{}'\delta(x,x');$ $\{\widetilde{\mathcal{H}}^{(-3/2)}(x), \widetilde{\mathcal{H}}^{(-3/2)}(x')\} =$ $\widetilde{\mathcal{H}}_i^{(-3/2)}(x)\partial^i\delta(x,x') - \widetilde{\mathcal{H}}_i^{(-3/2)}(x')\partial_i'\delta(x,x');$

Finite Dimensional Example

- We can apply these considerations on a finite dimensional example: FLRW case with k=0

$$ds^2 = -N^2(t)dt^2 + a^2(t)\left(dr^2 + r^2 d\theta^2 + sen^2\theta d\phi^2\right)$$

- If we put this metric in the B-D action, we obtain the following finite dimensional Lagrangian

$$\mathcal{L} = -\frac{6a\dot{a}^2}{N(t)}\phi(t) - \frac{6a^2\dot{a}}{N(t)}\dot{\phi}(t) + \frac{\omega a^3}{N\phi(t)}(\dot{\phi}(t))^2 - Na^3U(\phi(t))$$

Transformations from the Jordan to Einstein frame

$$\tilde{N} = N(16\pi G\phi)^{\frac{1}{2}} \; ; \; \tilde{\pi} = \frac{\pi}{(16\pi G\phi)^{\frac{1}{2}}} \; ; \; \tilde{a} = (16\pi G\phi)^{\frac{1}{2}} a \; ;$$

$$\tilde{\pi}_a = \frac{\pi_a}{16\pi G\phi} \; ; \; \phi = \phi \; ; \; \tilde{\pi}_\phi = \frac{1}{\phi}\left(\phi\pi_\phi - \frac{1}{2}a\pi_a\right)$$

Canonical momenta from Lagrangian to Hamiltonian formalism

$$\pi = \frac{\partial\mathcal{L}}{\partial\dot{N}} \approx 0 \; , \pi_a = \frac{\partial\mathcal{L}}{\partial\dot{a}} = -\frac{12a\dot{a}}{N(t)}\phi(t) - \frac{6a^2\dot{a}}{N(t)}\dot{\phi}(t) \; ,$$

$$\pi_\phi = \frac{\partial\mathcal{L}}{\partial\dot{\phi}} = -\frac{6a^2\dot{a}}{N(t)} + \frac{2\omega a^3}{N\phi(t)}\dot{\phi}(t)$$

Non-Equivalence and Equivalence of the Equations of Motion

EXTENDED PHASE SPACE

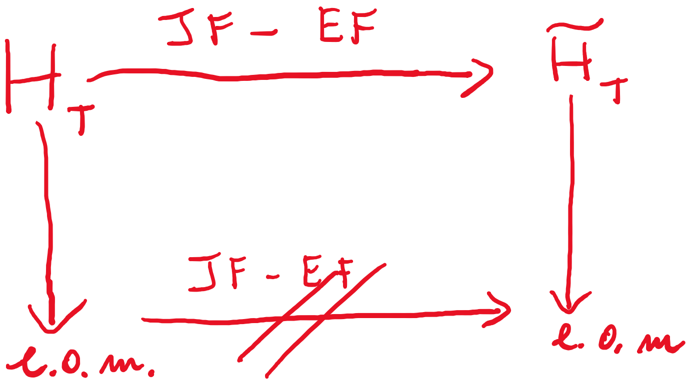
H_T = TOTAL HAMILTONIAN IN THE JF

$\tilde{H}_T$ = TOTAL HAMILTONIAN IN THE EF

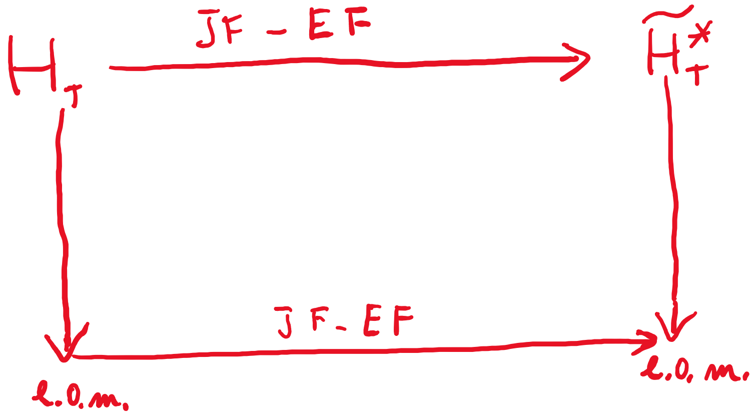
GAUGE FIXING ON N and conf. Transf. on Z

H_T = TOTAL HAMILTONIAN IN THE JF

$\tilde{H}^*_T$ = TOTAL HAMILTONIAN IN THE EF



non-equivalence of the e.o.m.



equivalence of the e.o.m.

CONFRONTING AND CONTRASTING THE E.O.M

$$\dot{N} \approx \lambda_N \,, \tag{1}$$
$$\dot{\pi}_N = -H \approx 0 \,, \tag{2}$$
$$\dot{a} \approx -\frac{N}{2a(2\omega+3)}\left(\frac{\omega\pi_a}{3\phi} + \frac{\pi_\phi}{a}\right) \,, \tag{3}$$
$$\dot{\pi}_a \approx -\frac{N}{2a^2(2\omega+3)}\left(\frac{\omega\pi_a^2}{6\phi} + \frac{2\pi_a\pi_\phi}{a} - \frac{3\phi\pi_\phi^2}{a^2}\right) - 3Na^2U(\phi) \,, \tag{4}$$
$$\dot{\phi} \approx \frac{N}{2a^2(2\omega+3)}\left(-\pi_a + \frac{2\phi\pi_\phi}{a}\right) \,, \tag{5}$$
$$\dot{\pi}_\phi \approx -\frac{N}{2a(2\omega+3)}\left(\frac{\omega\pi_a^2}{6\phi^2} + \frac{\pi_\phi^2}{a^2}\right) - Na^3\frac{dU}{d\phi} \tag{6}$$

$$\dot{N} \approx \frac{\widetilde{\lambda}_N}{(16\pi G\phi)^{\frac{1}{2}}} - \frac{N^2}{2a^2(2\omega+3)}\left(\frac{\pi_\phi}{a} - \frac{\pi_a}{2\phi}\right) \,, \tag{1}$$
$$\dot{\pi}_N \approx -H + \frac{N\pi_N}{2a^2(2\omega+3)}\left(\frac{\pi_\phi}{a} - \frac{\pi_a}{2\phi}\right) \,, \tag{2}$$
$$\dot{a} \approx -\frac{N}{2a(2\omega+3)}\left(\frac{\omega\pi_a}{3\phi} + \frac{\pi_\phi}{a}\right) \,, \tag{3}$$
$$\dot{\pi}_a \approx -\frac{N}{2a^2(2\omega+3)}\left(\frac{\omega\pi_a^2}{6\phi} + \frac{2\pi_a\pi_\phi}{a} - \frac{3\phi\pi_\phi^2}{a^2}\right) - 3Na^2U(\phi) \,, \tag{4}$$
$$\dot{\phi} \approx \frac{N}{2a^2(2\omega+3)}\left(-\pi_a + \frac{2\phi\pi_\phi}{a}\right) \,, \tag{5}$$
$$\dot{\pi}_\phi \approx -\frac{N}{2a(2\omega+3)}\left(\frac{\omega\pi_a^2}{6\phi^2} + \frac{\pi_\phi^2}{a^2}\right) - Na^3\frac{dU}{d\phi} + \frac{H}{2\phi} \,. \tag{6}$$

- Gauge fixing of the lapse N implemented as secondary constraints.

$$\phi_1 = N - 1 \approx 0 \;\; ; \;\; \phi_2 = \pi_N \approx 0 \qquad \text{become second class constraints in the JF}$$
$$\chi_1 = \widetilde{N} - (16\pi G\phi)^{\frac{1}{2}} \approx 0 \;\; ; \;\; \chi_2 = \widetilde{\pi}_N \approx 0 \;\; \text{are also second class constraints in EF}$$

- We can define Dirac's Brackets

E.O.M IN J.F. AND E.F. WITH DIRAC'S BRACKETS

$$\dot{a} = \{a, H_T\}_{DB} \approx -\frac{1}{2a(2\omega+3)}\left(\frac{\omega\pi_a}{3\phi} + \frac{\pi_\phi}{a}\right),$$

$$\dot{\pi}_a = \{\pi_a, H_T\}_{DB} \approx -\frac{1}{2a^2(2\omega+3)}\left(\frac{\omega\pi_a^2}{6\phi} + \frac{2\pi_a\pi_\phi}{a} - \frac{3\phi\pi_\phi^2}{a^2}\right) - 3a^2U(\phi),$$

$$\dot{\phi} = \{\phi, H_T\}_{DB} \approx \frac{1}{2a^2(2\omega+3)}\left(-\pi_a + \frac{2\phi\pi_\phi}{a}\right),$$

$$\dot{\pi}_\phi = \{\pi_\phi, H_T\}_{DB} \approx -\frac{1}{2a(2\omega+3)}\left(\frac{\omega\pi_a^2}{6\phi^2} + \frac{\pi_\phi^2}{a^2}\right) - a^3\frac{dU}{d\phi}$$

$$\dot{\widetilde{a}} \approx -\widetilde{N}\frac{4\pi G\widetilde{\pi}_a}{3\widetilde{a}},$$

$$\dot{\widetilde{\pi}}_a \approx \widetilde{N}\Big[-\frac{(2\pi G)\widetilde{\pi}_a^2}{3\widetilde{a}^2} + \frac{3(8\pi G)\widetilde{\pi}_\varphi^2}{\widetilde{a}^4}$$
$$-3\widetilde{a}^2\frac{1}{(16\pi G)^2}\exp[-\frac{2\varphi}{\sqrt{2\omega+3}}]U(\exp[\frac{\varphi}{\sqrt{2\omega+3}}])\Big],$$

$$\dot{\varphi} \approx \widetilde{N}\frac{(16\pi G)\widetilde{\pi}_\varphi}{\widetilde{a}^3},$$

$$\dot{\widetilde{\pi}}_\varphi \approx \frac{\widetilde{N}\widetilde{a}^3}{(16\pi G)^2}\exp[-\frac{2\varphi}{\sqrt{2\omega+3}}]\left[\frac{2}{\sqrt{2\omega+3}}U(\exp[\frac{\varphi}{\sqrt{2\omega+3}}]) - \frac{dU(\exp[\frac{\varphi}{\sqrt{2\omega+3}}])}{d\varphi}\right]$$

- Here $\varphi = \sqrt{2\omega+3}\,log\phi$ and the Lapses and their conjugate momenta do not evolve

$$\dot{N} \approx \{N, NH\}_{DB} \approx 0$$
$$\dot{\pi}_N \approx \{\pi_N, NH\}_{DB} \approx 0$$

$$\dot{\widetilde{N}} \approx \left\{\widetilde{N}, \widetilde{H}_T\right\}_{DB} \approx 0$$
$$\dot{\widetilde{\pi}}_N \approx \left\{\widetilde{\pi}_N, \widetilde{H}_T\right\}_{DB} \approx 0$$

- We can pass, on the e.o.m, form the EF to the J.F. and we get the same e.o.m as in the J.F. The Hamiltonian canonical transformations from the J.F. to the E.F. are Hamiltonian canonical transformations.

AN EXAMPLE IN SPHERICAL SYMMETRY
JORDAN FRAME

- ADM metric in spherical symmetry

$$ds^2 = -\left(N^2 - \Lambda^2 (N^r)^2\right) dt^2 + 2\Lambda^2 N^r dt dr + \Lambda^2 dr^2 + R^2 d\Omega^2$$

- E-H action in the JF

$$S_{JF} = \frac{1}{16\pi}\int d^4x\sqrt{g}\left[\left(1-\frac{\phi^2}{6}\right){}^{(4)}R - g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi\right] + \frac{1}{8\pi}\int_{\partial M} d^3x\sqrt{h}\left(1-\frac{\phi^2}{6}\right)K$$

- ADM decomposition of E-H action in spherical symmetry and Lagrangian density

$$\begin{aligned} S[h,N,N^r,\phi] &= \frac{1}{16\pi}\int dt\int_\Sigma N\sqrt{h}d^3x\Bigg[\left(1-\frac{\phi^2}{6}\right)\left({}^{(3)}R + K_{ij}K^{ij} - K^2\right) \\ &\quad -g^{00}\partial_0\phi\partial_0\phi - 2g^{0r}\partial_0\phi\partial_r\phi - h^{rr}\partial_r\phi\partial_r\phi\Bigg] \\ &\quad +\frac{1}{16\pi}\int dt\int_\Sigma\left(1-\frac{\phi^2}{6}\right)\left(-(2K\sqrt{h})_{,0} + 2f^i{}_{,i}\right) \\ &\quad +\frac{1}{8\pi}\int_{\partial(\mathbb{R}\times\Sigma)} d^3x\sqrt{h}K\,. \end{aligned}$$

$$\begin{aligned} \mathcal{L}_{JF} &= \left(1-\frac{\phi^2}{6}\right)\left[-\frac{1}{N}\left(\frac{\Lambda}{2}\left(-\dot{R}+R'N^r\right)^2 + R\left(-\dot{\Lambda}+(\Lambda N^r)'\right)\left(-\dot{R}+R'N^r\right)\right)\right. \\ &\quad \left. +N\left(-\frac{RR'}{\Lambda}+\frac{RR'\Lambda'}{\Lambda^2}-\frac{R'^2}{2\Lambda}+\frac{\Lambda}{2}\right)\right] + \frac{1}{4}\left(-\frac{NR^2\phi'^2}{\Lambda}+\frac{\Lambda R^2}{N}(\dot{\phi}-N^r\phi')^2\right) \\ &\quad -\frac{1}{6}\phi\frac{R}{N}\left[R\left(-\dot{\Lambda}+(\Lambda N^r)'\right)+2\Lambda\left(-\dot{R}+R'N^r\right)\right]\left(\dot{\phi}-N^r\phi'\right) \\ &\quad +\frac{1}{6}N\Lambda R^2\left(\frac{\phi'^2}{\Lambda^2}+\frac{\phi\phi''}{\Lambda^2}-\frac{\Lambda'\phi\phi'}{\Lambda^3}+\frac{2\phi R'\phi'}{R\Lambda^2}\right) \end{aligned}$$

AN EXAMPLE IN SPHERICAL SYMMETRY JORDAN FRAME

- Momenta in the Jordan frame

$$
\begin{aligned}
\pi_\Lambda &\equiv \frac{\partial \mathcal{L}_{JF}}{\partial \dot{\Lambda}} = -\frac{1}{N}\left(1-\frac{\phi^2}{6}\right) R\left(\dot{R} - R' N^r\right) + \frac{R^2\phi}{6N}(-N^r\phi' + \dot{\phi})\,, \\
\pi_R &\equiv \frac{\partial \mathcal{L}_{JF}}{\partial \dot{R}} = \frac{1}{N}\left(1-\frac{\phi^2}{6}\right)\left[R\left(-\dot{\Lambda} + (\Lambda N^r)'\right) + \Lambda\left(-\dot{R} + R'N^r\right)\right] + \frac{\phi\Lambda R}{3N}\left(\dot{\phi} - N^r\phi'\right) \\
\pi_\phi &\equiv \frac{\partial \mathcal{L}_{JF}}{\partial \dot{\phi}} = -\frac{\phi R}{6N}\left[R\left(-\dot{\Lambda} + (\Lambda N^r)'\right) + 2\Lambda\left(-\dot{R} + R'N^r\right)\right] + \frac{\Lambda R^2}{2N}\left(\dot{\phi} - N^r\phi'\right)
\end{aligned}
$$

- Hamiltonian density function

$$\mathcal{H}_{JF} = N^r H_r + NH$$

$$H_r = \pi_R R' - \pi'_\Lambda \Lambda + \pi_\phi \phi'$$

$$
\begin{aligned}
H \;=\; & \frac{\phi^2\pi_R^2}{36\Lambda}\left(1-\frac{\phi^2}{6}\right)^{-1} + \frac{\Lambda\pi_\Lambda^2}{2R^2}\left(1+\frac{\phi^2}{18}\right)\left(1-\frac{\phi^2}{6}\right)^{-1} + \frac{\pi_\phi^2}{R^2\Lambda}\left(1-\frac{\phi^2}{6}\right) \\
& -\frac{\pi_R\pi_\Lambda}{R}\left(1-\frac{\phi^2}{18}\right)\left(1-\frac{\phi^2}{6}\right)^{-1} + \frac{\phi\pi_R\pi_\phi}{3R\Lambda} + \frac{\phi\pi_\Lambda\pi_\phi}{3R^2} \\
& + \left(1-\frac{\phi^2}{6}\right)\left(-\frac{\Lambda}{2} + \frac{R'^2}{2\Lambda} - \frac{RR'\Lambda'}{\Lambda^2} + \frac{RR''}{\Lambda}\right) + \frac{R^2\phi'^2}{12\Lambda} + \frac{R^2\phi\Lambda'\phi'}{6\Lambda^2} - \frac{R^2\phi\phi''}{6\Lambda} - \frac{R\phi R'\phi'}{3\Lambda}
\end{aligned}
\qquad (1)
$$

AN EXAMPLE IN SPHERICAL SYMMETRY
EINSTEIN FRAME

- ADM metric in spherical symmetry in EF

$$d\widetilde{s}^2 = -\left(\widetilde{N}^2 - \widetilde{\Lambda}^2(\widetilde{N}^r)^2\right)dt^2 + 2\widetilde{\Lambda}^2\widetilde{N}^r dtdr + \widetilde{\Lambda}^2 dr^2 + \widetilde{R}^2 d\Omega^2$$

- Action in spherical symmetry in the Einstein Frame

$$\begin{aligned} S_{EF} &= \frac{1}{16\pi}\int d^4x\sqrt{-\widetilde{g}}\left({}^{(4)}\widetilde{R} - \widetilde{g}^{\mu\nu}\partial_\mu\widetilde{\phi}\partial_\nu\widetilde{\phi}\right) \\ &\quad + \frac{1}{8\pi}\int_{\partial M} d^3x\sqrt{\widetilde{h}}\widetilde{K} \end{aligned}$$

- Hamiltonian density

$$\widetilde{\pi}_R = -\frac{\widetilde{\Lambda}(\dot{\widetilde{R}} - \widetilde{R}'\widetilde{N}^r) + \widetilde{R}(\dot{\widetilde{\Lambda}} - (\widetilde{\Lambda}\widetilde{N}^r)')}{\widetilde{N}}$$

$$\begin{aligned} \widetilde{H} = \;& \widetilde{N}\Big(-\frac{\widetilde{\pi}_R\widetilde{\pi}_\Lambda}{\widetilde{R}} + \frac{\widetilde{\Lambda}\widetilde{\pi}_\Lambda^2}{2\widetilde{R}^2} + \frac{\widetilde{R}\widetilde{R}''}{\widetilde{\Lambda}} - \frac{\widetilde{R}\widetilde{R}'\widetilde{\Lambda}'}{\widetilde{\Lambda}^2} + \frac{\widetilde{R}'^2}{2\widetilde{\Lambda}} - \frac{\widetilde{\Lambda}}{2} + \frac{\widetilde{\pi}_\phi^2}{\widetilde{\Lambda}\widetilde{R}^2} + \frac{\widetilde{R}^2\widetilde{\phi}'^2}{4\widetilde{\Lambda}}\Big) \\ & + \widetilde{N}^r\left(\widetilde{\pi}_R\widetilde{R}' - \widetilde{\Lambda}\widetilde{\pi}_\Lambda' + \widetilde{\pi}_\phi\widetilde{\phi}'\right) \end{aligned}$$

$$\widetilde{\pi}_\Lambda = -\frac{\widetilde{R}(\dot{\widetilde{R}} - \widetilde{R}'\widetilde{N}^r)}{\widetilde{N}}$$

$$\widetilde{\pi_\phi} = \frac{\widetilde{\Lambda}\widetilde{R}^2}{2\widetilde{N}}(\dot{\widetilde{\phi}} - \widetilde{N}^r\widetilde{\phi}')$$

CANONICAL TRANSFORMATIONS

- Relations among the canonical variables in the JF and EF

$$\widetilde{N} = \left(1 - \frac{\phi^2}{6}\right)^{\frac{1}{2}} N\,;\widetilde{N}^r = N^r\,;\widetilde{\Lambda} = \left(1 - \frac{\phi^2}{6}\right)^{\frac{1}{2}} \Lambda\,;\widetilde{R} = \left(1 - \frac{\phi^2}{6}\right)^{\frac{1}{2}} R\,;\widetilde{\phi} = \sqrt{6}\tanh^{-1}\frac{\phi}{\sqrt{6}}$$

$$\widetilde{\pi}_N = \left(1 - \frac{\phi^2}{6}\right)^{-\frac{1}{2}} \pi_N\,;\widetilde{\pi}_{N^r} = \pi_{N^r}\,;\widetilde{\pi}_\Lambda = \left(1 - \frac{\phi^2}{6}\right)^{-\frac{1}{2}} \pi_\Lambda\,;\widetilde{\pi}_R = \left(1 - \frac{\phi^2}{6}\right)^{-\frac{1}{2}} \pi_R\,;$$

$$\widetilde{\pi}_\phi = \left(1 - \frac{\phi^2}{6}\right)\pi_\phi + \frac{1}{6}R\,\phi\,\pi_R + \frac{1}{6}\Lambda\,\phi\,\pi_\Lambda$$

- Poisson Brackets

$$\left\{\widetilde{N},\widetilde{\pi}_N\right\} = 1\,,\left\{\widetilde{\Lambda},\widetilde{\pi}_\Lambda\right\} = 1\,,\left\{\widetilde{R},\widetilde{\pi}_R\right\} = 1\ \left\{\widetilde{\phi},\widetilde{\pi}_\phi\right\} = 1$$

$$\{\widetilde{\pi}_\Lambda,\widetilde{\pi}_\phi\} = 0\,,\{\widetilde{\pi}_R,\widetilde{\pi}_\phi\} = 0$$

- The transformations from JF to EF result to be not Hamiltonian canonical

$$\left\{\widetilde{N},\widetilde{\pi}_\phi\right\} = -\frac{\phi}{6}\left(1 - \frac{\phi^2}{6}\right)^{\frac{1}{2}} N$$

PHYSICAL IN-EQUIVALENT SOLUTIONS IN JF ANF EF

- BBMB is solution of e.o.m in JF upon gauge fixing the lapse and shift $N = \left(1 - \frac{m}{\rho}\right)\ ; N^r = 0$

$$
\begin{aligned}
ds^2 &= -\left(1 - \frac{m}{\rho}\right)^2 dt^2 + \left(1 - \frac{m}{\rho}\right)^{-2} d\rho^2 + \rho^2 d\Omega^2 \\
\phi &= \frac{\sqrt{6}m}{\rho - m} \\
m &= \frac{b}{4}
\end{aligned}
$$

- The corresponding solution of the e.o.m. in the EF is the Janis solution

$$1 - \frac{b}{r} = \left(1 - \frac{b}{2\rho}\right)^2$$

$$ds^2 = \frac{1}{1 - \phi^2/6} d\tilde{s}^2$$

$$\phi = \sqrt{6} \tanh \frac{\tilde{\phi}}{\sqrt{6}}$$

$$
\begin{aligned}
d\tilde{s}^2 &= -\left(1 - \frac{b}{r}\right)^{\frac{1}{2}} dt^2 + \left(1 - \frac{b}{r}\right)^{-\frac{1}{2}} dr^2 + r^2 \left(1 - \frac{b}{r}\right)^{-\frac{1}{2}} d\Omega^2 \\
\tilde{\phi} &= \sqrt{\frac{3}{8}} \log\left(1 - \frac{b}{r}\right)
\end{aligned}
$$

- As before, gauge fixing the lapse and radial shift make the Hamiltonian transformations from JF to EF canonical transformations.

Conclusions

- Hamiltonian analysis of Brans-Dicke theory for $\omega \neq -\frac{3}{2}$, and $\omega = -\frac{3}{2}$ shows that the Weyl(conformal) transformation from Jordan to Einstein frame are not Hamiltonian canonical transformations on the "extended phase space".

- We have confronted and contrasted the e.o.m. of F.L.R.W. flat metric on Brans-Dicke theory in the Jordan and Einstein frame. Gauge fixing the lapse function N and introducing Dirac's Brackets, the transformation from the Jordan to the Einstein frame are Hamiltonian canonical transformations. (Does it imply JF and EF are physically equivalent?)

- We have studied an example of ADM Hamiltonian analysis in spherical symmetry. The Hamiltonian transformation from JF to EF are still Hamiltonian non-canonical on the extended phase space. Gauge fixing the lapse and the radial shift, we show that we can map the BBMB solution in JF to the Janis solution in EF. The transformations are Hamiltonian canonical upon gauge fixing, but the two solutions are physically inequivalent.