UNIVERSITÀ DEGLI STUDI DELL'INSUBRIA

Istituto Nazionale di Fisica Nucleare

New Analytical Approach to Analog Hawking Radiation

for Subcritical and Transcrical Regimes

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Schedule

Analog Gravity

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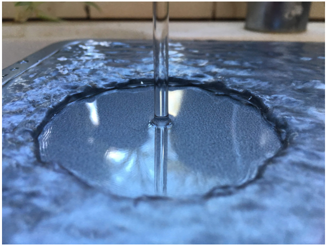
Analog Gravity

William Unruh, 1981:
Also some earth-based systems can present event horizons, similar to those of Black Holes.



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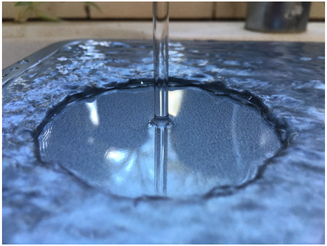

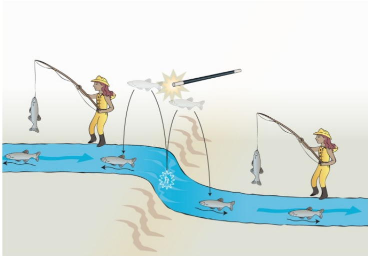

If quantum mechanics plays a significant role in the system, we should be able to observe something analogous to Hawking Radiation[1].

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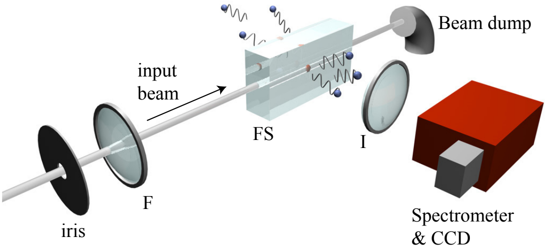
Optical Black Holes - I

[2] Phys. Rev. Lett. 105, 203901 (2010), Como.

Optical Black Holes - I



Optical Black Holes are created by sending a high-energy laser pulse into a dielectric medium.[2]

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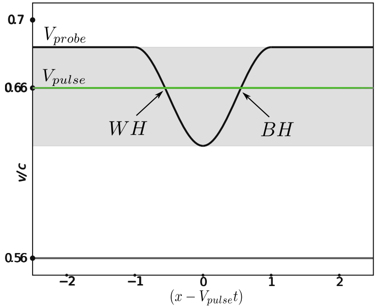

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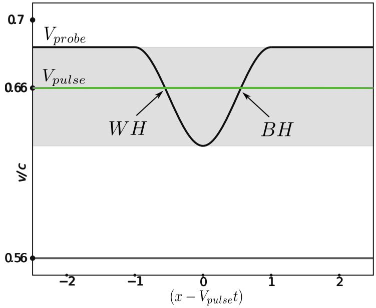

$T \sim 80\mathrm{K}$

$\rightarrow$ The radiation can in principle be observed directly!

The effects of dispersion: different regimes

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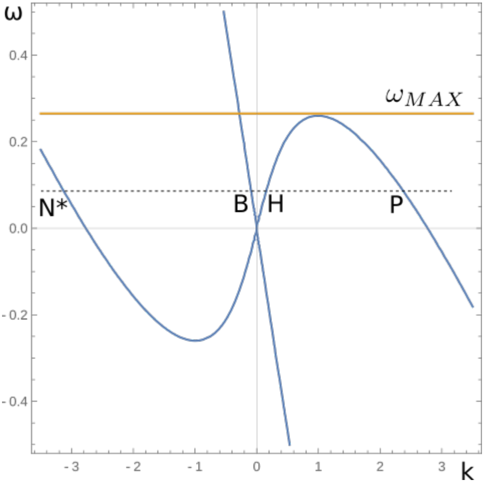
Asymptotic dispersion relation ($x = \pm\infty$)



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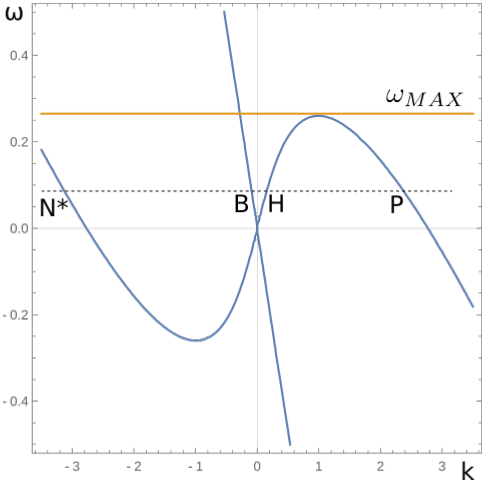

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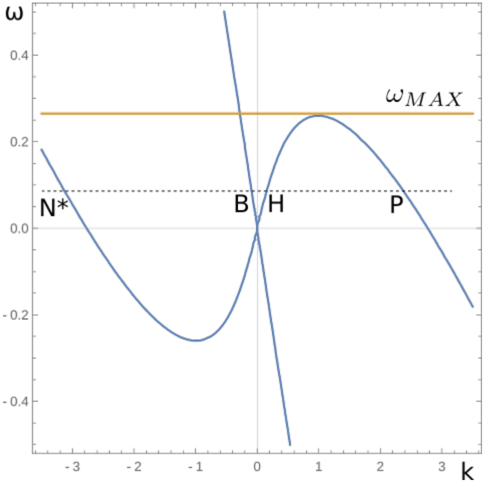

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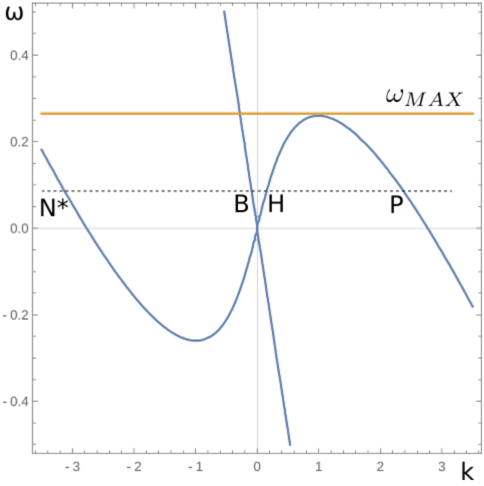

- For $0 < \omega < \omega_{max}$ there are **four real solutions**;
- The H-mode is the Hawking mode, which goes out of the horizon;
- The N^*-mode has **negative norm**.

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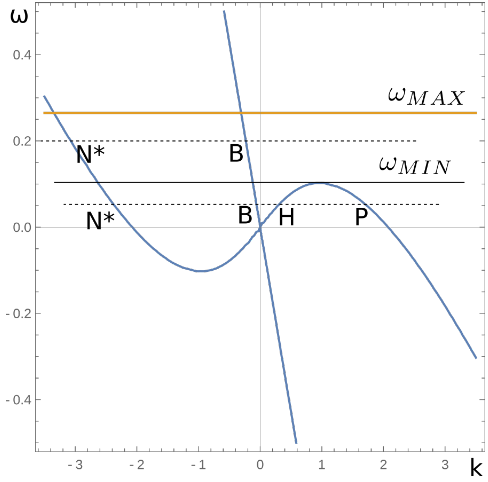

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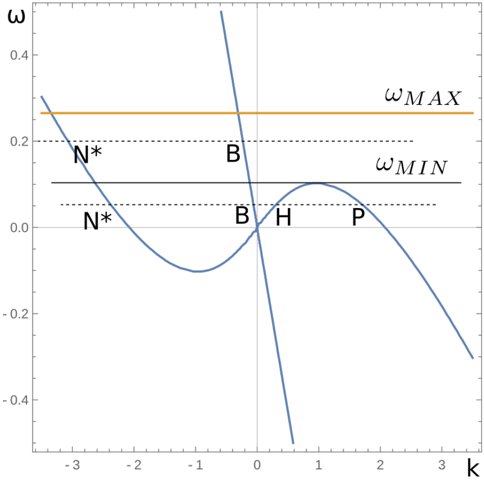

- The frequencies $\omega_{min} < \omega < \omega_{max}$ encounter a (group) event horizon;

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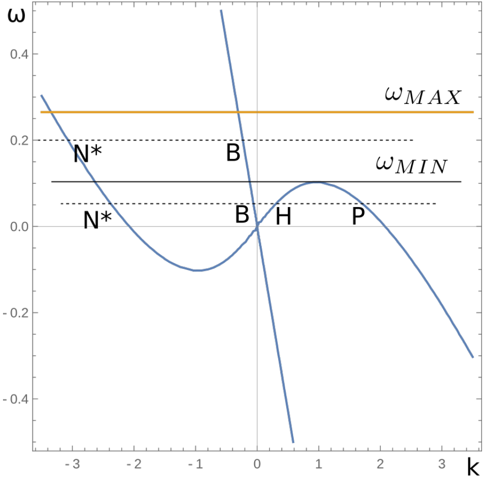

- The frequencies $\omega_{min} < \omega < \omega_{max}$ encounter a (group) event horizon;
- The frquencies $0 < \omega < \omega_{min}$ encounter **no horizon** (subcritical regime).

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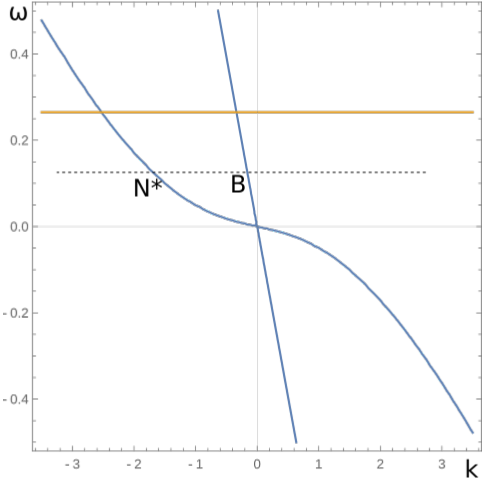

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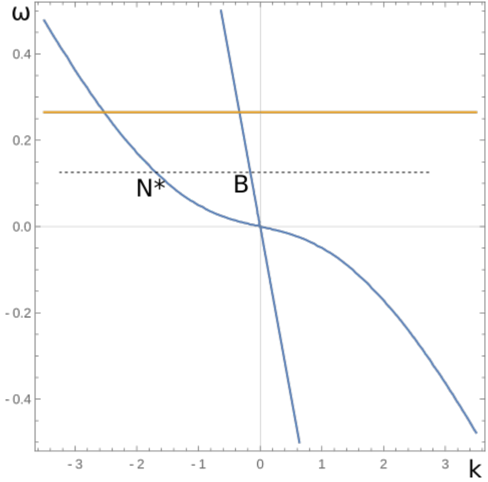

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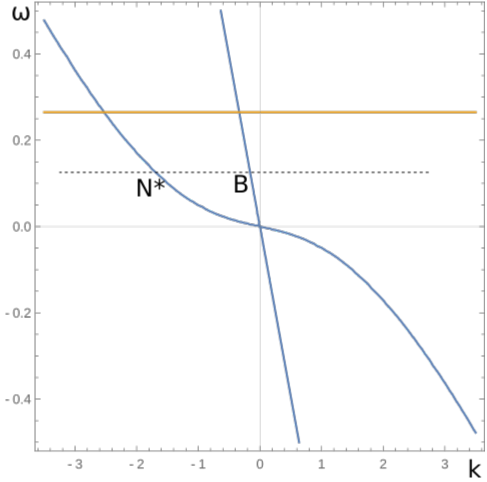

- All frequencies $0 < \omega < \omega_{max}$ encounter an event horizon.
- **Thermal spectrum** predicted, at least at low frequencies.

The model

[3] Belgiorno, Cacciatori, Faccio, *Hawking Radiation: From Astrophysical Black Holes to Analogous Systems in Lab* (World Scientific Publishing, Singapore, 2018).

The model

We consider the following Lagrangian

$$\mathcal{L} = \frac{1}{2}(\partial_0\phi)^2 + \frac{1}{2}\left((\partial_0\psi)^2 + \mu^2\psi^2\right) + g\phi\partial_x\psi - \frac{\lambda}{4!}\psi^4 \qquad (1)$$

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- It is designed so that the free particles of the model ($\lambda = 0$) satisfy the Cauchy dispersion relation:

$$n^2(\omega) := \frac{k^2}{\omega^2} = \frac{\mu^2}{g^2} + \frac{\omega^2}{g^2} =: A + B\omega^2 \qquad (2)$$

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- $\delta n \propto \bar{\psi}(x,t)^2 \propto I(x,t)$: the model reproduces the Kerr effect.

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- This allows to consistently quantize the theory defining a Fock space of free particles which satisfy the Cauchy D.R.

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The EOM have an **exact** solitonic solution

$$\psi_s(x - Vt) = \frac{\alpha}{\cosh(\beta(x - Vt))},$$

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Solitons of this type are also known to show up as approximate solutions of the Maxwell equations inside a Kerr medium (cfr. "nonlinear Schrödinger equation").

The mode equation in soliton background

[4]Belgiorno *et al.*, Phys. Rev. **D** 102, 105003 (2020)

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We consider the linearized EOM around a background field (δn)

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- The solution is in the form $\psi(x, t) = e^{-i\omega t'} f(x')$:
- $f(x')$ satisfies a fourth-order equation of generalized Orr-Sommerfeld type[4].

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$$\begin{aligned}
0 = & \left(V^4\gamma^4\right) z^4 f^{(4)} + \left(6V^4\gamma^4 - 4iV^3\Omega\gamma^4\right) z^3 f^{(3)} \\
& + \left(-G^2\gamma^2 - \mu^2 V^2\gamma^2 + 7V^4\gamma^4 - 12iV^3\Omega\gamma^4 - 6V^2\Omega^2\gamma^4 + \varepsilon\frac{V^2\gamma^2 z^3}{(1-z)^2}\right) z^2 f^{(2)} \\
& + \left(-G^2\gamma^2 - \mu^2 V^2\gamma^2 + 2iG^2 V\Omega\gamma^2 + 2i\mu^2 V\Omega\gamma^2 + V^4\gamma^4 - 4iV^3\Omega\gamma^4 - 6V^2\omega^2\gamma^4 + 4iV\omega^3\gamma^4 \right. \\
& \left. + \varepsilon\left(\frac{3V^2\gamma^2 z^2 - 2iV\Omega\gamma^2 z^2)}{(1-z)^2} + \frac{4V^2\gamma^2 z^3}{(1-z)^3}\right)\right) z\, f^{(1)} \\
& + \left(\mu^2\Omega^2\gamma^2 + G^2 V^2\Omega^2\gamma^2 + \Omega^4\gamma^4 + \varepsilon\left(\frac{V^2\gamma^2 z - 2iV\Omega\gamma^2 z - \Omega^2\gamma^2 z}{(1-z)^2} + \frac{6V^2\gamma^2 z^2 - 4iV\Omega\gamma^2 z^2}{(1-z)^3} + \frac{6V^2\gamma^2 z^3}{(1-z)^4}\right)\right) f
\end{aligned}$$

New perturbative approach

[5] Belgiorno *et al.*, Phys. Rev. **D** 102, 105003 (2020).
[6] Coutant, Weinfurtner, Phys. Rev. **D** 94, 064026 (2016).

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Previous perturbative approaches to similar equations in analog gravity were made:

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- This is suppose to reproduce the subcritical case without any approximation in the dispersion;
- There is no theoretical obstacle in extending the approach to the critical case (maybe just some higher orders in the perturbation).

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The perturbative expansion[7]

[6]A.V. Chueshev, V.V. Chueshev, Journal of Siberian Federal University. Mathematics & Physics 2022, **15**(3), 308318.

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- We express the solution as a power series:

$$u(z) = u_0(z) + \varepsilon u_1(z) + \varepsilon^2 u_2(z) + \dots$$

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$$\ldots\,,$$

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$$\ldots\,,$$
$$u_n^{(4)} + v_1(z)u_n^{(2)} + v_2(z)u_n^{(1)} + v_3(z)u_n = w_1(z)u_{(n-1)}^{(2)} + w_2(z)u_{(n-1)}^{(1)} + w_3(z)u_{(n-1)}$$

- We were able to **solve exactly** the first-order equation using the method of variation of constants.

[6] A.V. Chueshev, V.V. Chueshev, Journal of Siberian Federal University. Mathematics & Physics 2022, **15**(3), 308318.

Boundary conditions for the solution

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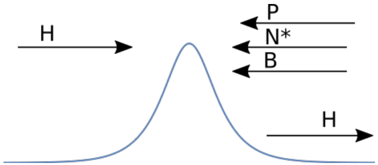
We impose asymptotics conditions that reproduce the inverse scattering of a H-mode:



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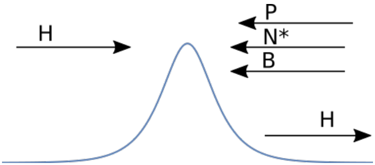

From the conserved scalar product of the theory we get

$$1 = |H| + |P| - |N| + |B|$$

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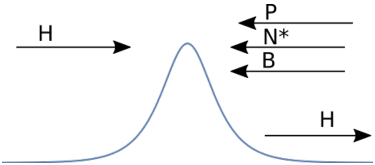

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In the Bogolubov analysis, $|N|$ can be identified as the the rate of **spontaneous particle emission**.

The first-order solution - Exact espression

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The exact expression of $u_1(z)$ with the imposed boundary condition is

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\begin{aligned}
u_1(z) = &-\frac{i}{4V^2\gamma^2 z} \\
&\Bigg(\frac{4i(\frac{\omega}{V}+(k_H+i))^2\,{}_2F_1\big(2,i(k_H-k_P)+1;i(k_H-k_P)+2;\frac{1}{z}\big)}{(k_H-k_P)(k_H-k_P-i)(k_P-k_N)(k_P-k_B)} + \frac{16\big(k_H+\frac{\omega}{V}+\frac{3i}{2}\big)\,{}_2F_1\big(3,i(k_H-k_P)+1;i(k_H-k_P)+2;\frac{1}{z}\big)}{(k_H-k_P)(k_H-k_P-i)(k_P-k_N)(k_P-k_B)} - \frac{24i\,{}_2F_1\big(4,i(k_H-k_P)+1;i(k_H-k_P)+2;\frac{1}{z}\big)}{(k_H-k_P)(k_H-k_P-i)(k_P-k_N)(k_P-k_B)} \\
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&+\frac{4z^2\left(V^2\left(k_H^2(z-1)^2-2ik_H(z-1)-z-1\right)+2V\omega(z-1)(k_H(z-1)-i)+\omega^2(z-1)^2\right)}{V^2(z-1)^3(k_H-k_P)(k_H-k_N)(k_H-k_B)} \Bigg)
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&+ \frac{4z^2\Big(V^2\Big(k_H^2(z-1)^2 - 2ik_H(z-1) - z - 1\Big) + 2V\omega(z-1)(k_H(z-1)-i) + \omega^2(z-1)^2\Big)}{V^2(z-1)^3(k_H-k_P)(k_H-k_N)(k_H-k_B)} \Bigg)
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- From this form one can easily derive the asymptotic expression at $z = \infty$ ($x = +\infty$);

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- From this form one can easily derive the asymptotic expression at $z=\infty$ ($x=+\infty$);
- Thanks to the connection formulas of the Hypergeometric function we also derived the asymptotic expression at $z=0$ ($x=-\infty$).

The first-order solution - Asymptotics

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The asymptotic expression of the solution is

$$f(x') \sim \begin{cases} e^{-ik_H x'} & , x' \to -\infty \\ H\, e^{-ik_H x'} + P\, e^{-ik_P x'} + N\, e^{-ik_N x'} + B\, e^{-ik_B x'} & , x' \to +\infty \end{cases},$$

where

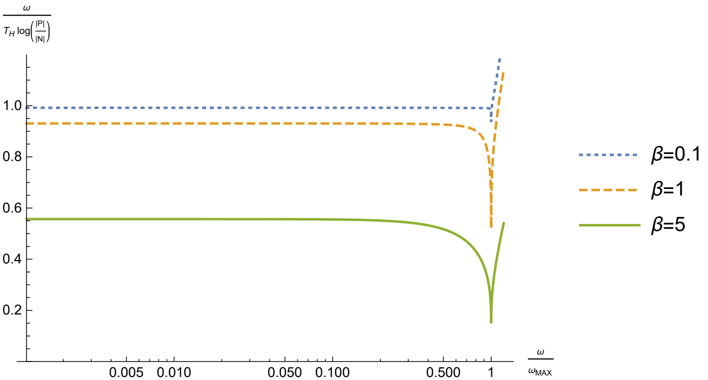
$$H = 1 - i\varepsilon \frac{(k_H V + \omega)^2}{V^4 \gamma^2 (k_H - k_P)(k_H - k_N)(k_H - k_B)} + O(\varepsilon^2)$$

$$P = -i\varepsilon \frac{\pi (k_P V + \omega)^2 (-1)^{-i(k_H - k_P)} \mathrm{csch}[(k_H - k_P)\pi/(2\beta)]}{2\beta V^4 \gamma^2 (k_P - k_N)(k_P - k_B)} + O(\varepsilon^2)$$

$$N = -i\varepsilon \frac{\pi (k_N V + \omega)^2 (-1)^{-i(k_H - k_N)} \mathrm{csch}[(k_H - k_N)\pi/(2\beta)]}{2\beta V^4 \gamma^2 (k_N - k_P)(k_N - k_B)} + O(\varepsilon^2)$$

$$B = -i\varepsilon \frac{\pi (k_B V + \omega)^2 (-1)^{-i(k_H - k_B)} \mathrm{csch}[(k_H - k_B)\pi/(2\beta)]}{2\beta V^4 \gamma^2 (k_B - k_P)(k_B - k_N)} + O(\varepsilon^2)\,.$$

Estimation of the temperature



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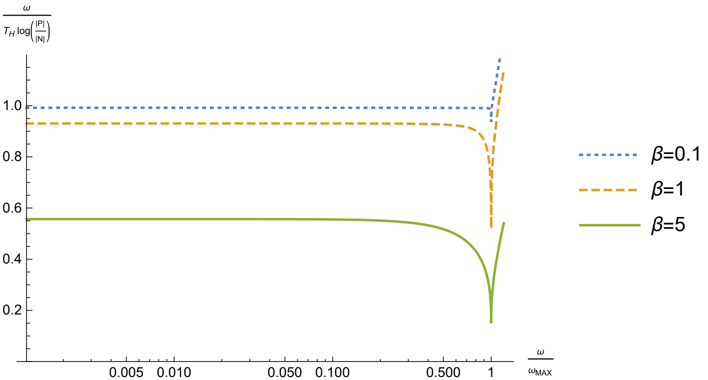
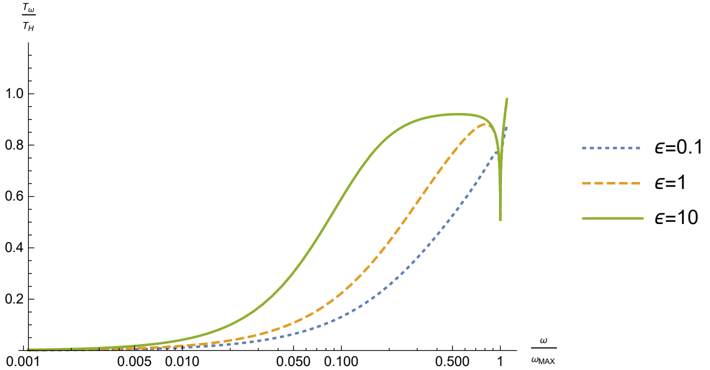


- The ratio $\frac{|P|}{|N|}$ grows exponentially with ω.
- The estimated temperature $T_H = \frac{\beta\gamma^2 V(g^2-\mu^2 V^2)}{2\pi g(2g+\mu V)}$ is less accurate as β increases.

Plot of T_ω



[8] Michel, Parentani, Phys. Rev. **D** 90, 044033 (2014).

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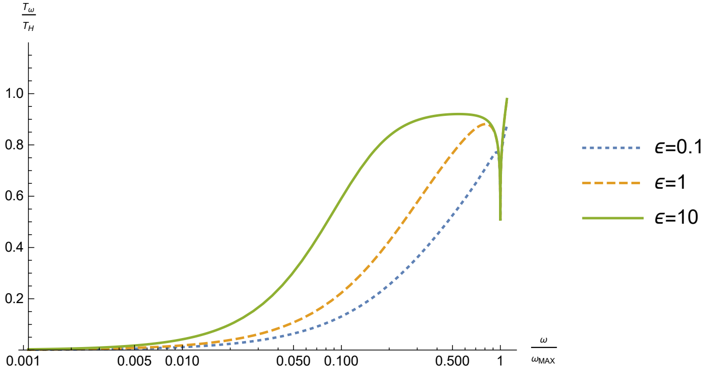

- Plateau appearing at $T_\omega \approx T_H$ for high frequencies.

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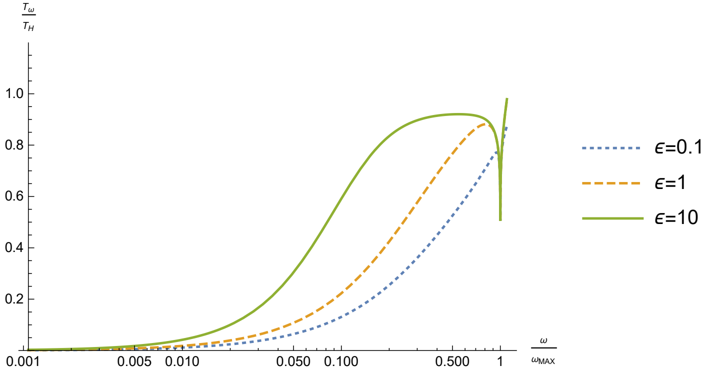
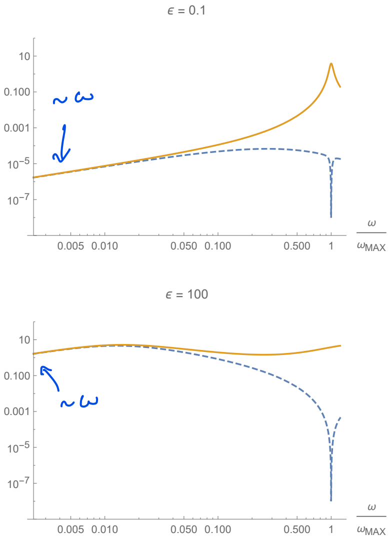


- Plateau appearing at $T_\omega \approx T_H$ for high frequencies.
- The behaviour is similar to what was found by numerical studies of the transcritical flow in shallow water[8].

[8] Michel, Parentani, Phys. Rev. **D** 90, 044033 (2014).

Plot of $|P|$ and $|N|$



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- Our plots show **signs of criticality** (i.e. thermality of the spectrum) for high values of ε: to this values the results shouldn't be trusted, but this fact is qualitatively interesting nonetheless.

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- Investigate the possibility of finding **exact** (non-perturbative) integrals of such equations.

Thank you for the attention!