New features of K-essential cosmologies

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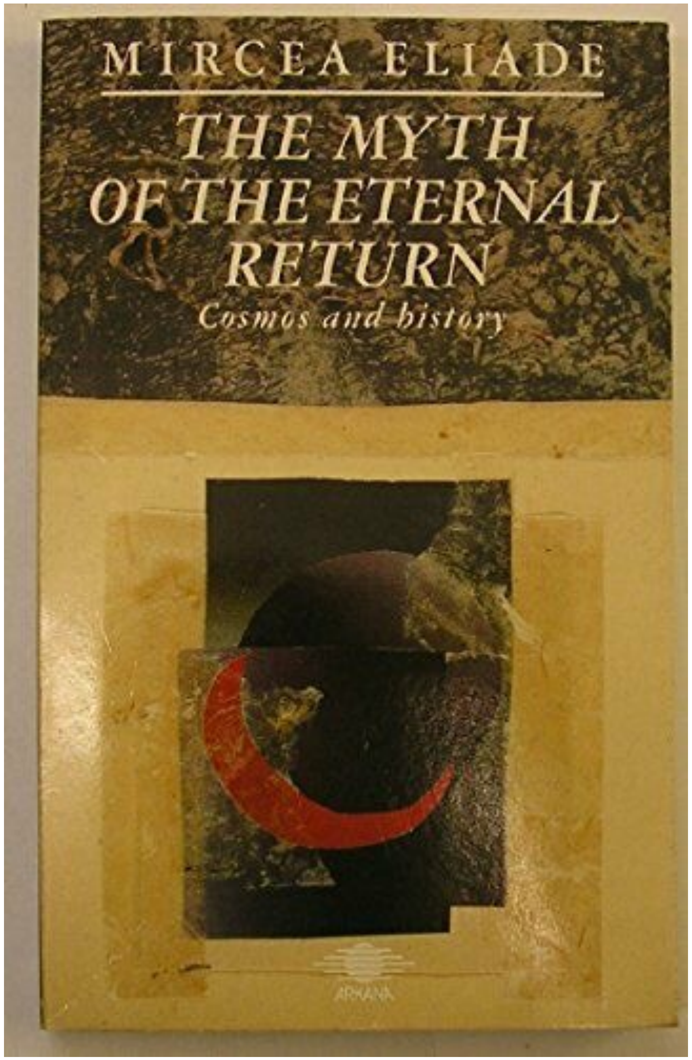
Como

Why did we start this research

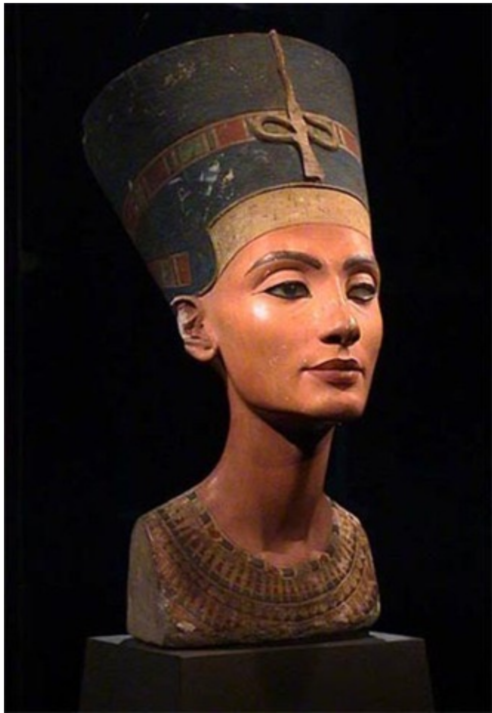
- Can we find a truly cyclic universe in (say) k-essence models of cosmology ? (no, AdS is not a good example…)

Myth of the eternal return



The name: *κόσμος*

- «Pythagoras (570 – 495 BC) was the first man to call himself a philosopher ("lover of wisdom") *Cicero, Tusculans disputations 5.3.8–9.*
- «Pythagoras was the first to call kosmos the encompassing of all things, because of the order that reigns in it» *Aetius, Placita (1st century A.C.).*

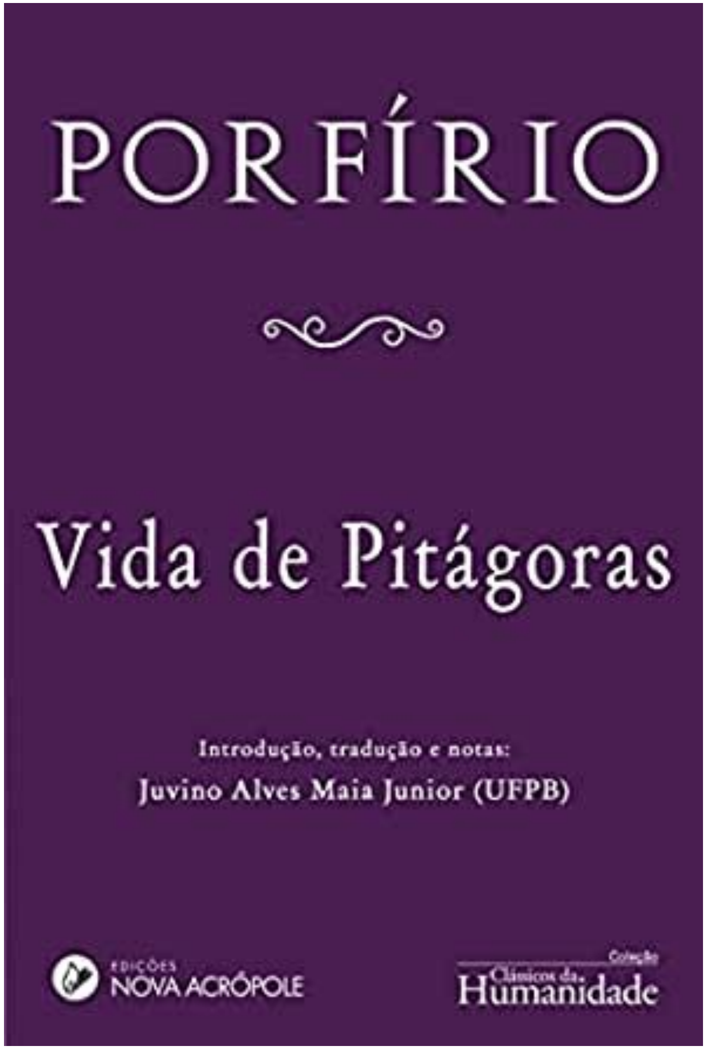


Cosmology

- «The task of the contemplation of the nature **(theoria phusike)** is to examine the substance of the sky and the stars, the power and the quality of generation and corruption and, by Zeus! It is capable of leading demonstrations on the subject of the size of the form and of the order of such things.
- As for the astronomy (*astrologia*) it does not undertake to speak of anything like that but it demonstrates the order (*taxis*) of celestial things, having declared (*apophenasa*) that the sky (*ouranos*) is truly a world (*kosmos*); it speaks of forms, sizes, distances from the Earth to the Sun and the Moon, eclipses, the conjunctions of the stars, on the quantity and the quality that are shown in their revolutions».

Posidonius (135 BC - 51 BC)

The Life of Phytagoras



Dicaearchus of Messina quoted by Porphyry

- **18.** When he reached Italy he stopped at Crotona. His presence was that of a free man, tall, graceful in speech and gesture, and in all things else. Dicaearchus relates that the arrival of this great traveler, endowed with all the advantages of nature, and prosperously guided by fortune, produced on the Crotonians so great an impression, that he won the esteem of the elder magistrates, by his many and excellent discourses. They ordered him to exhort the young men, and then to the boys who flocked out of the school to hear him; and lastly to the women, who came together on purpose.

- **19**. Through this he achieved great reputation, he drew great audiences from the city, not only of men, but also of women, among whom was a specially illustrious person named Theano. He also drew audiences from among the neighboring barbarians, among whom were magnates and kings. What he told his audiences cannot be said with certainty, for he enjoined silence upon his hearers. But the following is a matter of general information. He taught that the soul was immortal and that after death it transmigrated into other animated bodies. After certain specified periods, the same events occur again; that nothing was entirely new; that all animated beings were kin, and should be considered as belonging to one great family. Pythagoras was the first one to introduce these teachings into Greece.

Cosmological Equations

$$ds^2 = dt^2 - a(t)^2 \left(\frac{dr^2}{1-Kr^2} + r^2(d\theta^2 + \sin^2\theta d\phi^2) \right)$$

Raychaudhuri Eq.

$$\frac{\ddot{a}}{a} = -\frac{4}{3}\pi G(\rho + 3p)$$

Friedmann Eq.

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8}{3}\pi G\rho - \frac{K}{a^2}$$

Together imply
(for each component):

$$\dot{\rho}_i = -3\frac{\dot{a}}{a}(\rho_i + p_i)$$

Equation of state

$$p = f(\rho, s)$$

$$p = w\,\rho \qquad w$$

Pure k-essence models

$$L(\phi, \partial\phi) = F((\partial\phi)^2) = F(X)$$

$$T_{\mu\nu} = (\partial_X F)\, \partial_\mu \varphi\, \partial_\nu \varphi - \frac{1}{2}\, F\, g_{\mu\nu}.$$

$$L(\phi, \partial\phi) = V_1(\phi) F(X) + V_2(\phi)$$

Flat k-essence: cosmological equations

$$ds^2 = dt^2 - a^2(t)\left(dx^2 + dy^2 + dz^2\right) \qquad \phi = \phi(t) \quad X = \dot{\phi}^2 \geq 0$$

$$\varrho(X) = \frac{3\dot{a}^2}{a^2} = X\partial_X F - \frac{1}{2}F \qquad\qquad p(X) = \frac{1}{2}F$$

Field equations $(\partial_X F)\sqrt{X} = \pm\left(\frac{a_0}{a}\right)^3, \; a_0 \neq 0$

Dependence on the cosmic time

$$\frac{dX}{dt} = \mp\frac{2\sqrt{3}\,X\,F_X\,\sqrt{XF_X - \frac{1}{2}F}}{2XF_{XX} + F_X} = \mp\frac{(\rho(X) + p(X))\sqrt{3\rho(X)}}{\rho'(X)}$$

Barotropic fluids

$$F(X) = \text{sgn}\left(\gamma - \frac{1}{2}\right)(\partial\phi)^{2\gamma} = 2p \qquad \varrho = \left|\gamma - \frac{1}{2}\right|(\partial\phi)^{2\gamma}$$

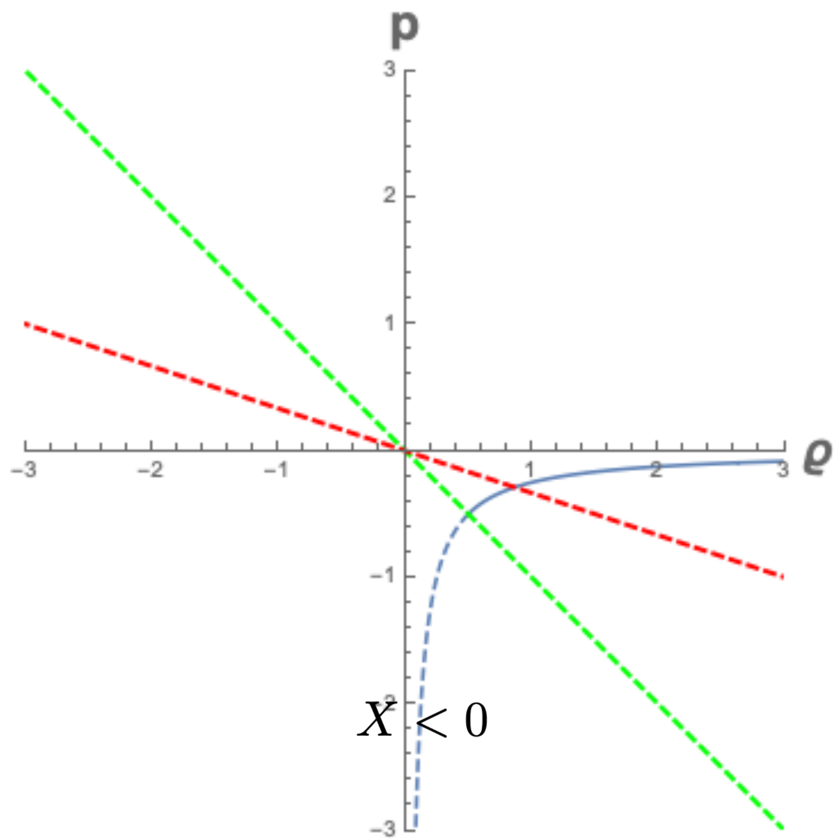
$$p = w\rho = \frac{\rho}{2\gamma - 1}$$

- $\gamma = 0$ corresponds to the cosmological constant
- $\gamma = 1$ corresponds to stiff matter
- $\gamma = 2$ corresponds to radiation
- The limits $\gamma \to \pm\infty$ corresponds to dust $(p = 0)$
- The limit $\gamma \to \frac{1}{2}$ corresponds to empty space

The Chaplygin Gas

$$p = \frac{1}{2}F = -\frac{1}{2}\sqrt{1-X}$$

$$\varrho = \frac{1}{2}\frac{1}{\sqrt{1-X}}$$

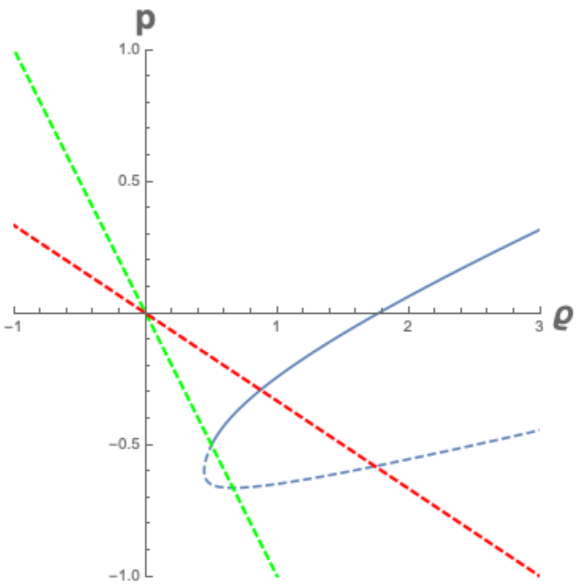


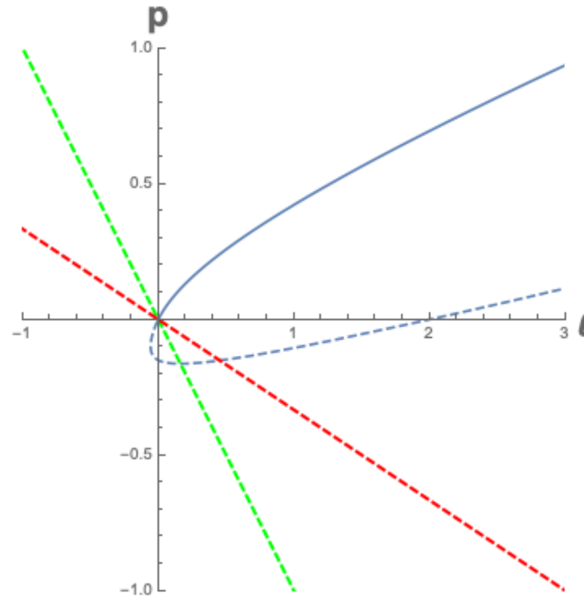
$$p = -\frac{1}{4\varrho}$$

Polynomial interactions: standard case

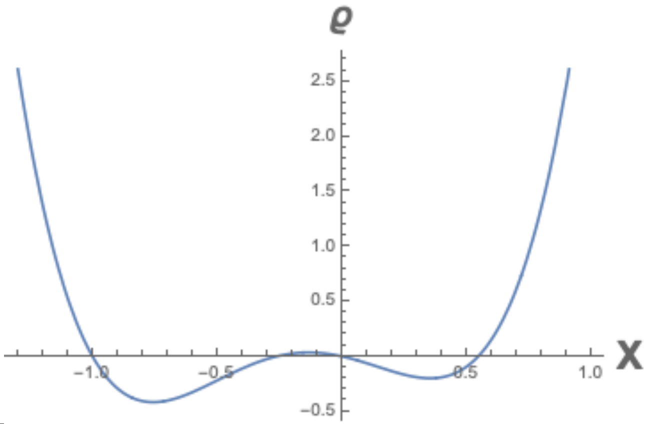
$$F(X) = \sum_{n=0}^{N} c_n X^n. \qquad c_0 \leq 0, \qquad c_n \geq 0 \;\; \text{for} \;\; n \geq 1.$$

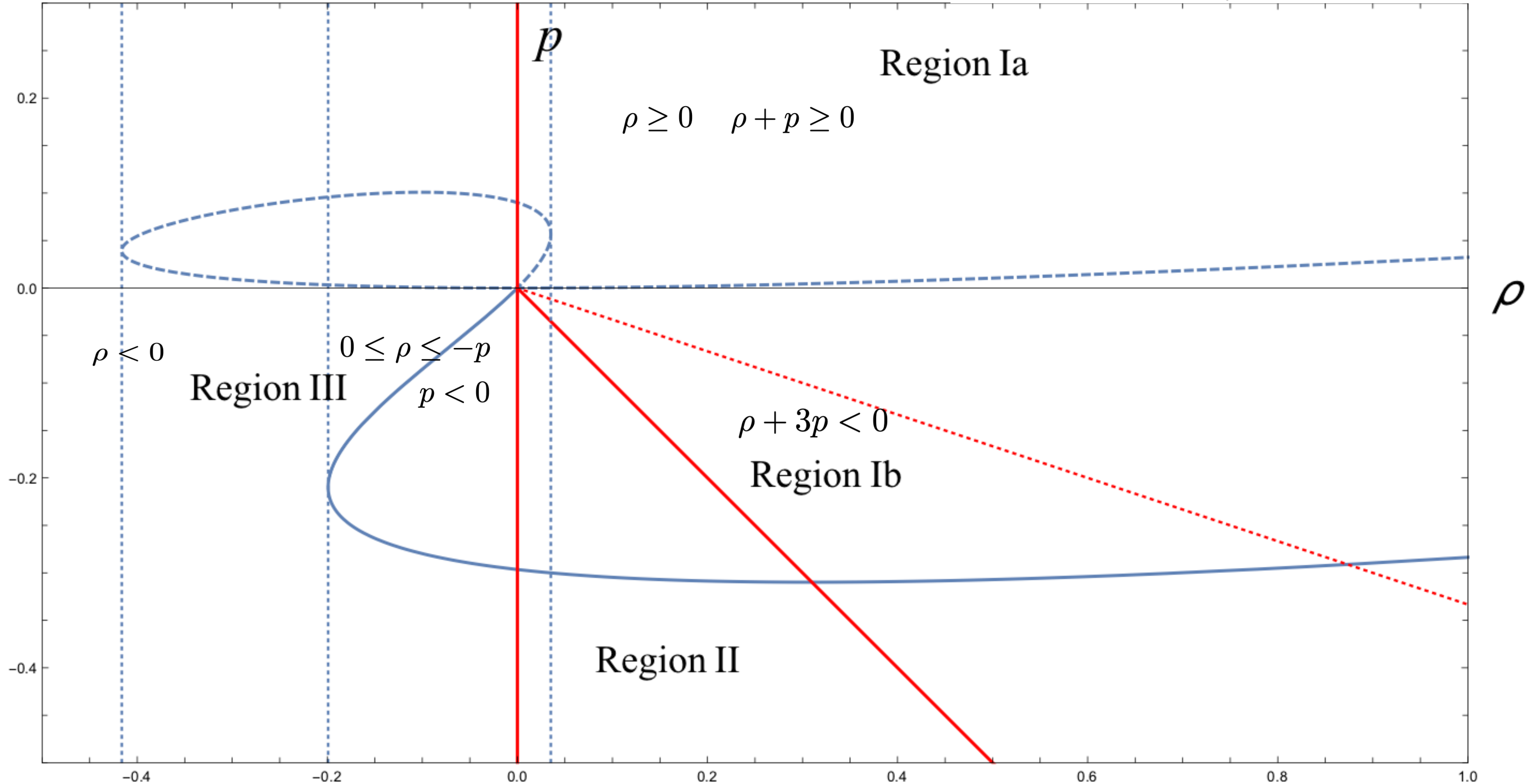
$$\rho(a) \sim \left(N - \frac{1}{2}\right) c_N^{\frac{1}{1-2N}} \left(\frac{a^3 N}{A}\right)^{\frac{2N}{1-2N}}$$





Polynomial interactions: general case





A completely soluble model: lessons form the quartic self-interaction

$$F = -\Lambda + \mu(\partial\varphi)^2 + \lambda(\partial\varphi)^4$$

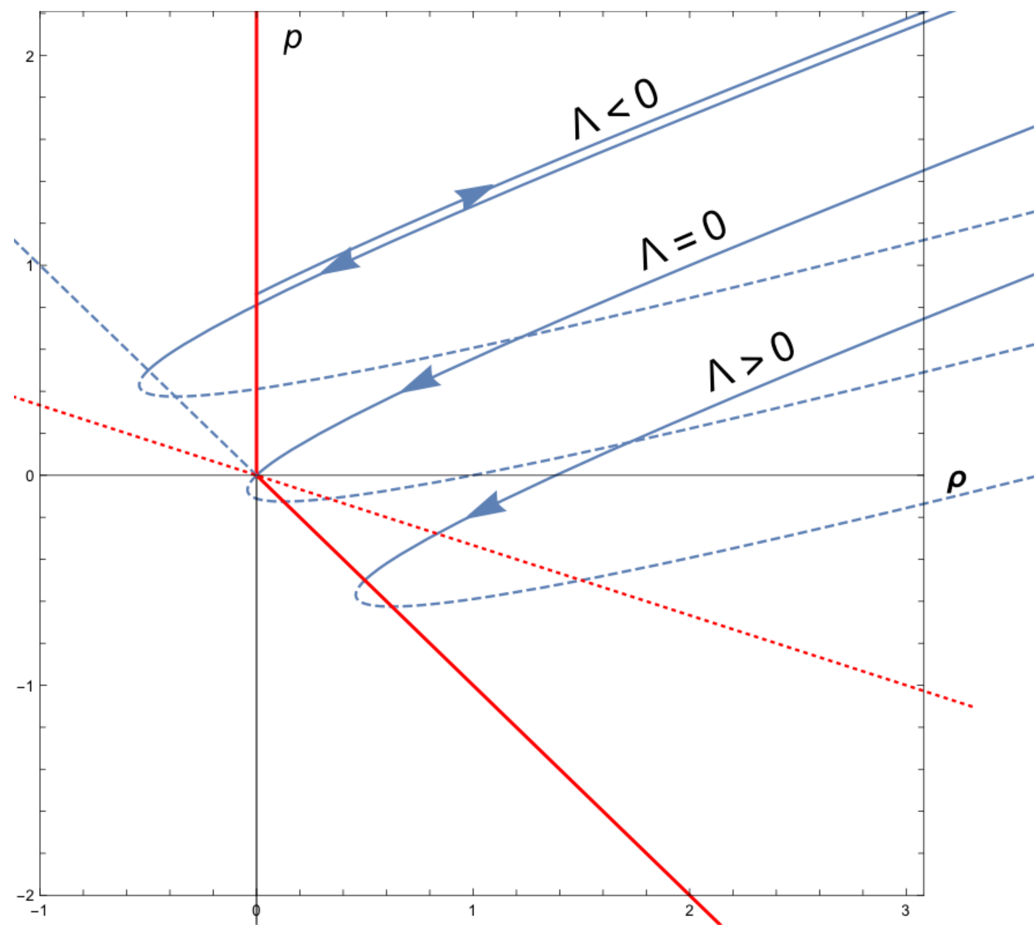
$$F = -\Lambda + \mu X + \lambda X^2$$

K-essence with a cosmological constant

$$F = -\Lambda + \mu X + \lambda X^2 \qquad \mu > 0, \quad \lambda > 0$$

$$\rho(X) = \frac{1}{2}\left(3X^2 + X + \Lambda\right)$$

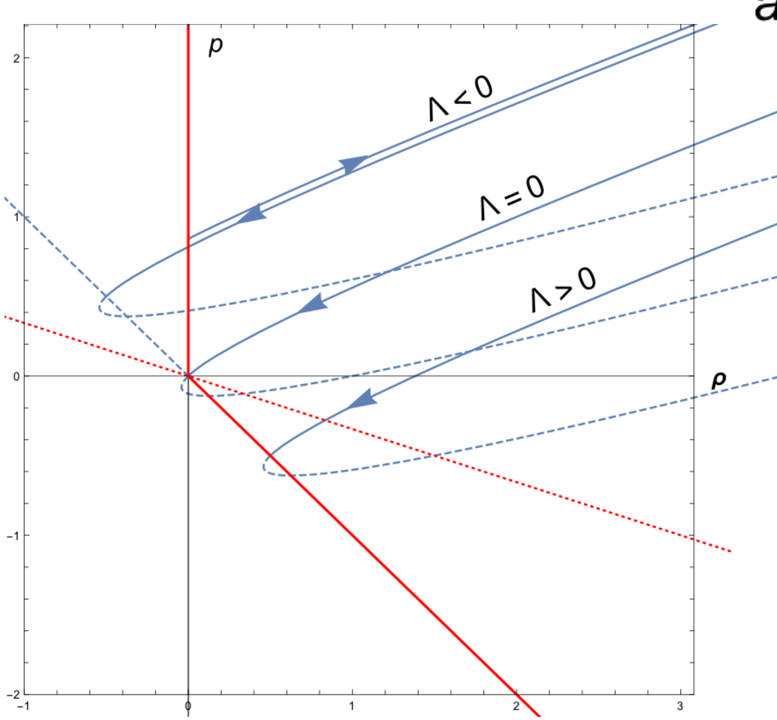
$$p(X) = \frac{1}{2}\left(X^2 + X - \Lambda\right)$$

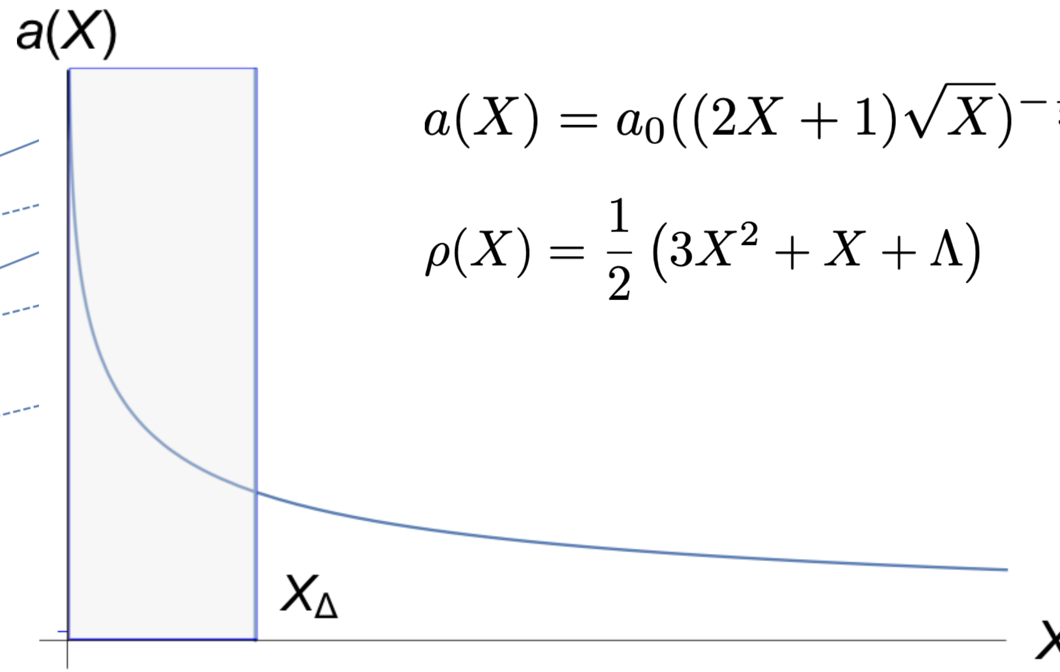


K-essence with a cosmological constant

$$F = -\Lambda + X + X^2$$

Field equations $(\partial_X F)\sqrt{X} = \pm\left(\frac{a_0}{a}\right)^3, \; a_0 \neq 0$





$$a(X) = a_0((2X+1)\sqrt{X})^{-\frac{1}{3}},$$

$$\rho(X) = \frac{1}{2}\left(3X^2 + X + \Lambda\right)$$

$$\frac{dX}{dt} = \mp\frac{2\sqrt{3}\,X\,F_X\sqrt{XF_X - \frac{1}{2}F}}{2XF_{XX} + F_X}$$

$$t(X) = \log\left(\frac{1}{2X} + 1\right)^{\frac{1}{3}\sqrt{2}}$$

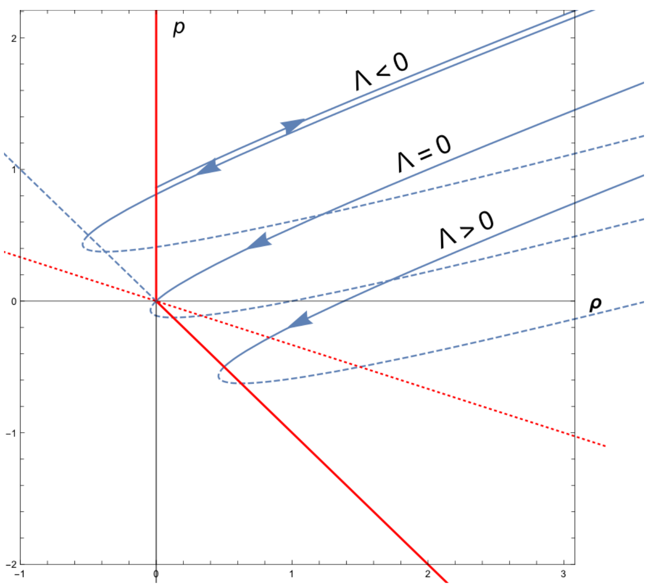
$$\Lambda = 1/12$$

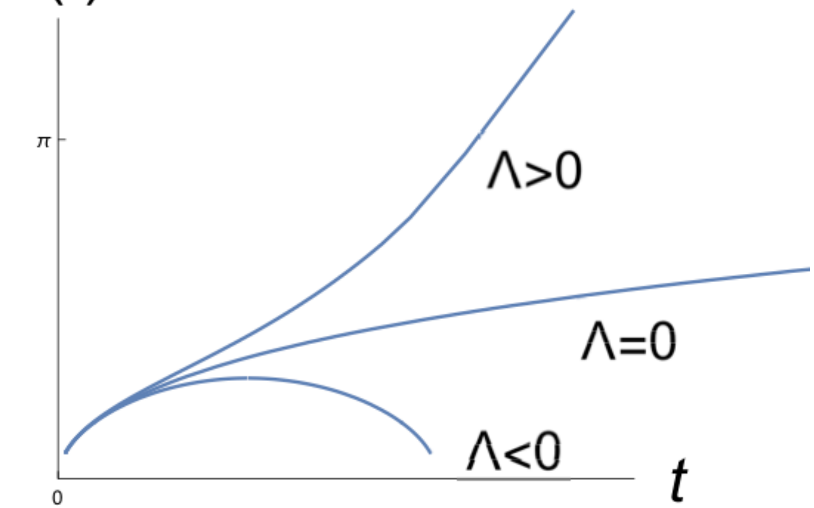
With the cosmological constant

$$F = -\Lambda + X + X^2$$

$$a(X) = a_0\left((2X+1)\sqrt{X}\right)^{-\frac{1}{3}},$$

$$t(X) = \frac{2}{3}\sqrt{\frac{2}{12\Lambda+3}}\left[\tanh^{-1}\left(\frac{2}{\sqrt{12\Lambda+3}}\right) - \tanh^{-1}\left(\frac{2X-2\Lambda+\frac{1}{2}}{\sqrt{4\Lambda+1}\sqrt{\Lambda+3X^2+X}}\right)\right] + \frac{1}{3\sqrt{6\Lambda}}\tanh^{-1}\left(\frac{2\Lambda+X}{2\sqrt{\Lambda}\sqrt{\Lambda+3X^2+X}}\right) - \frac{1}{3\sqrt{6\Lambda}}\coth^{-1}\left(2\sqrt{3\Lambda}\right)$$



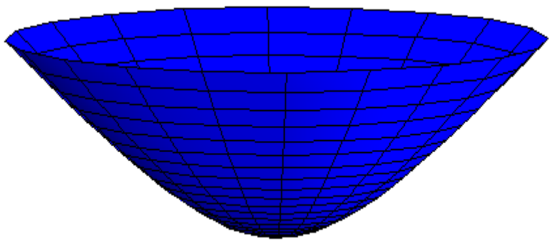


$\Delta > 1/4$ $$t_\Delta = \frac{\pi + 2\tan^{-1}\left(\frac{1}{2\sqrt{3\Delta}}\right)}{6\sqrt{6\Delta}} + \frac{\left(\pi - 2\tan^{-1}\left(\frac{2}{\sqrt{12\Delta-3}}\right)\right)}{3\sqrt{6\Delta-\frac{3}{2}}}$$

$\Delta < 1/4$ $$t_\Delta = \frac{\pi + 2\tan^{-1}\left(\frac{1}{2\sqrt{3\Delta}}\right)}{6\sqrt{6\Delta}} + \frac{2\log\left(\frac{2+\sqrt{3}\sqrt{1-4\Delta}}{2-\sqrt{3}\sqrt{1-4\Delta}}\right)}{3\sqrt{6-24\Delta}}$$

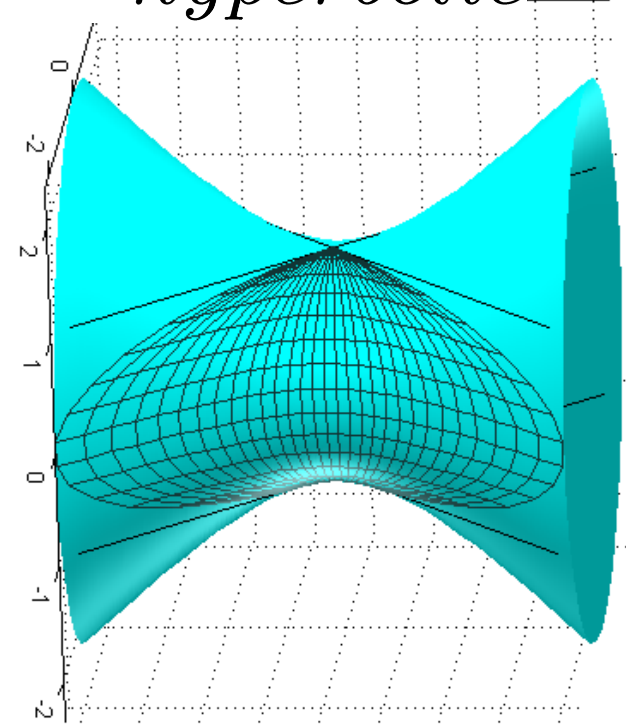
Only Λ<0

$$\ddot{a} = -\frac{1}{3}|\Lambda|\, a \qquad \dot{a}^2 = -\frac{1}{3}|\Lambda|\, a^2 - K$$

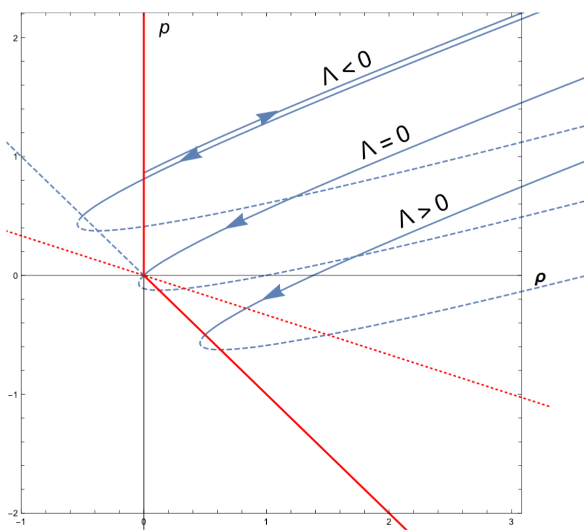


$$K = -1$$

hyperbolic



$$a(t) = \sqrt{\frac{3}{|\Lambda|}} \sin \sqrt{\frac{|\Lambda|}{3}}\, t$$



Without the cosmological constant

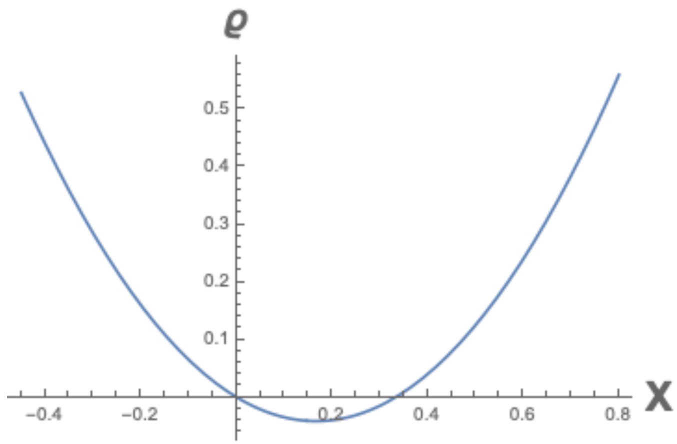
$$\Lambda = 0, \quad \mu < 0, \quad \lambda > 0$$

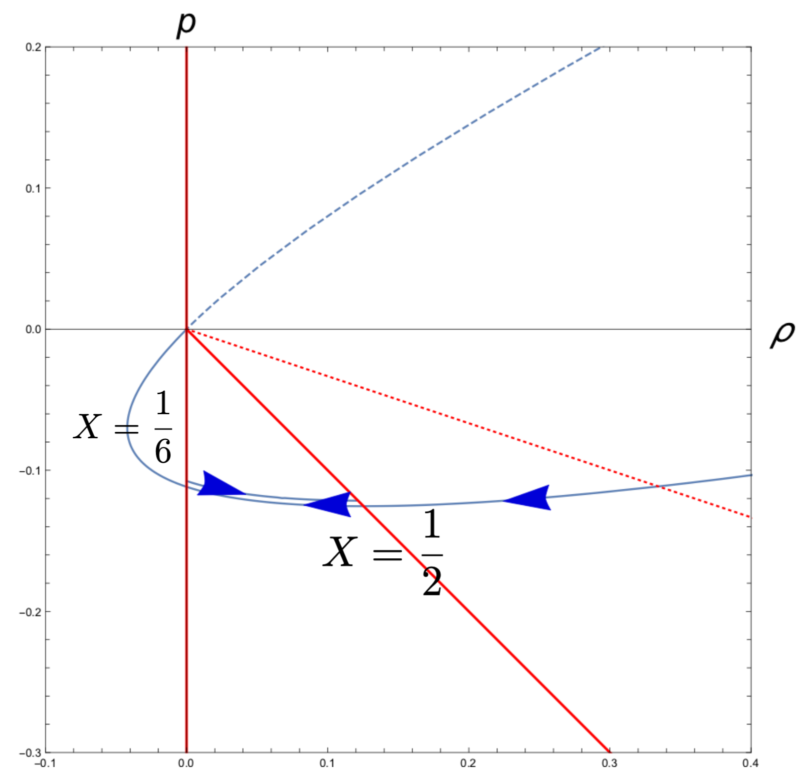
$$F = -X + X^2 = -(\partial\phi)^2 + (\partial\phi)^4$$

$$\rho(X) = \frac{1}{2}\left(3X^2 - X\right)$$

$$p(X) = \frac{1}{2}\left(X^2 - X\right)$$

$$c_s^2 = \frac{1-2X}{1-6X}$$

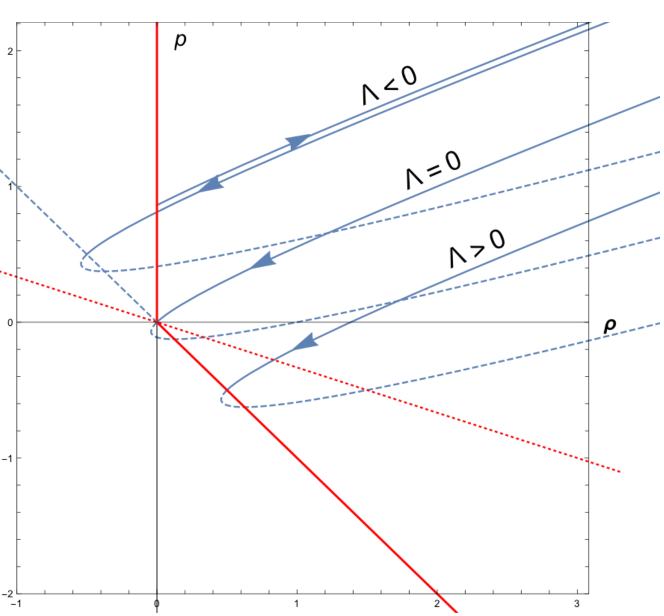


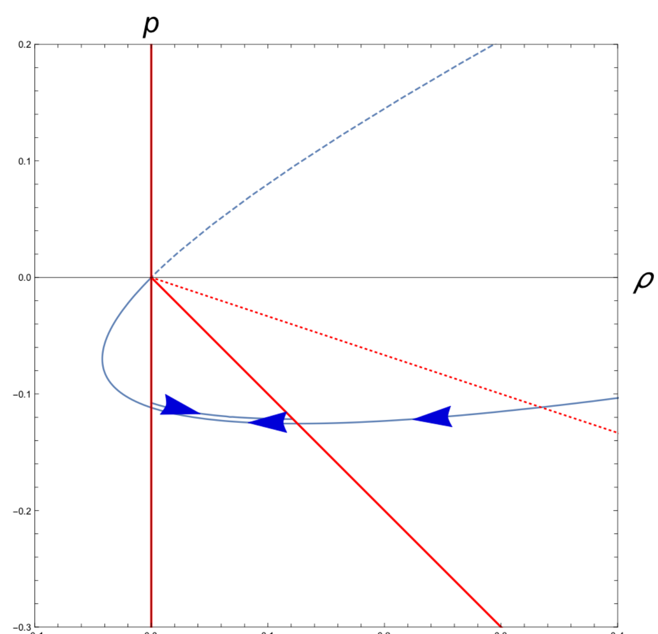


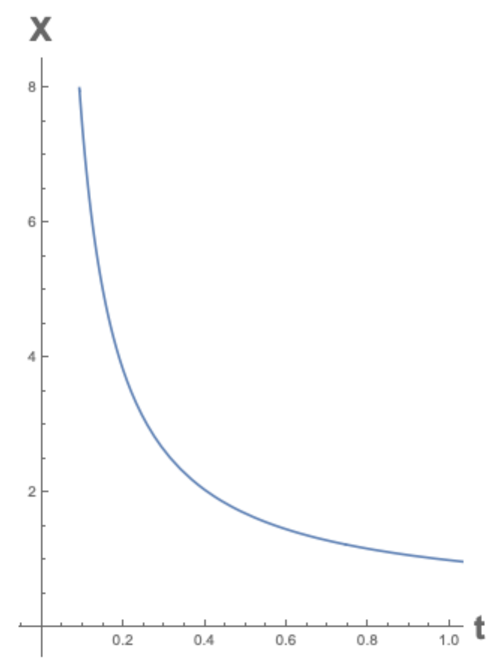
$$F = -X + X^2 = -(\partial\phi)^2 + (\partial\phi)^4$$

$$\rho(X) = \frac{1}{2}\left(3X^2 - X\right) \qquad a(X) = a_0\left((2X-1)\sqrt{X}\right)^{-\frac{1}{3}}$$

$$t(X) = \frac{1}{3}\sqrt{2} - \frac{1}{3}\sqrt{2 - \frac{2}{3X}} + \frac{4}{3}\sqrt{\frac{2}{3}}\left(\tanh^{-1}\left(\frac{\sqrt{X}}{\sqrt{3X-1}}\right) - \tanh^{-1}\left(\frac{1}{\sqrt{3}}\right)\right)$$





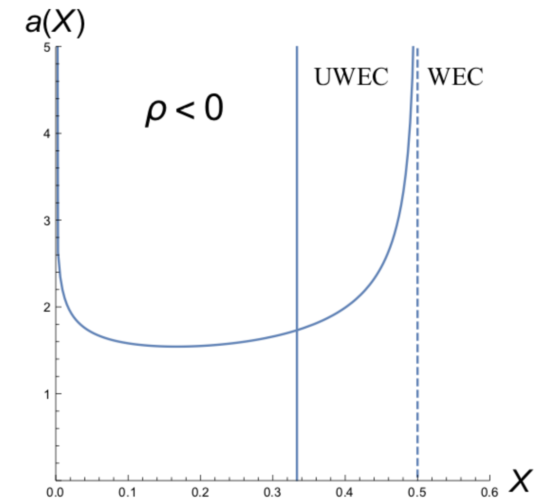


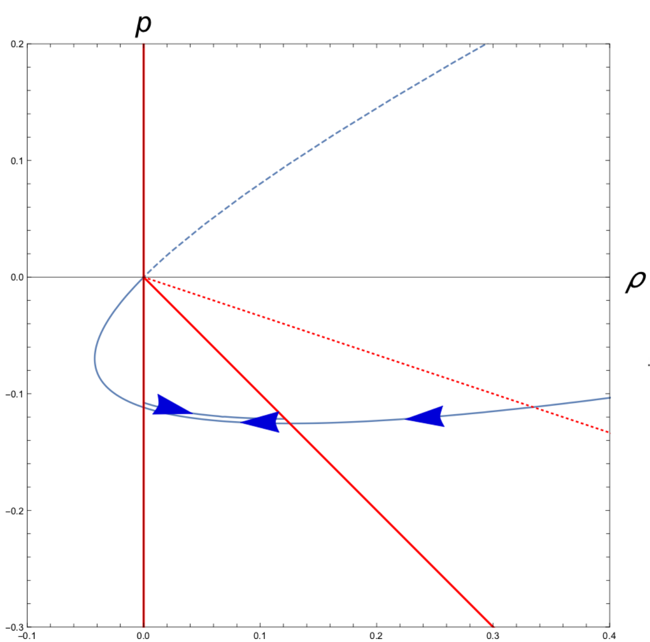
$$F = -X + X^2 = -(\partial\phi)^2 + (\partial\phi)^4$$

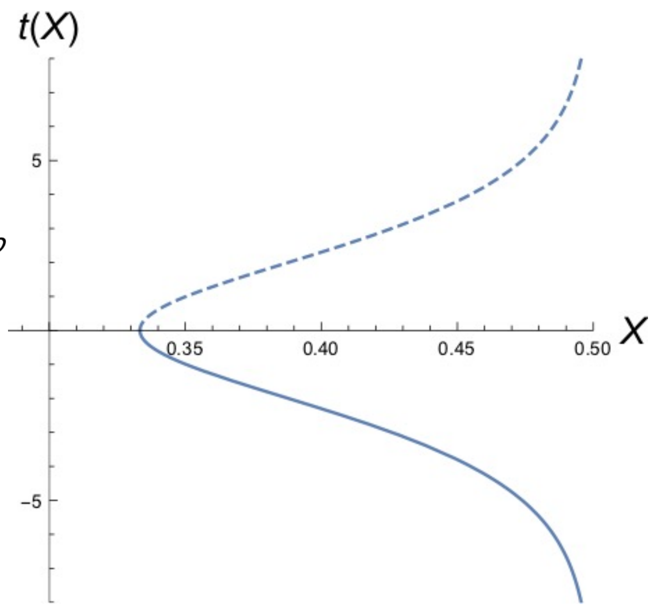
$$a(X) = a_0((1-2X)\sqrt{X})^{-\frac{1}{3}}$$

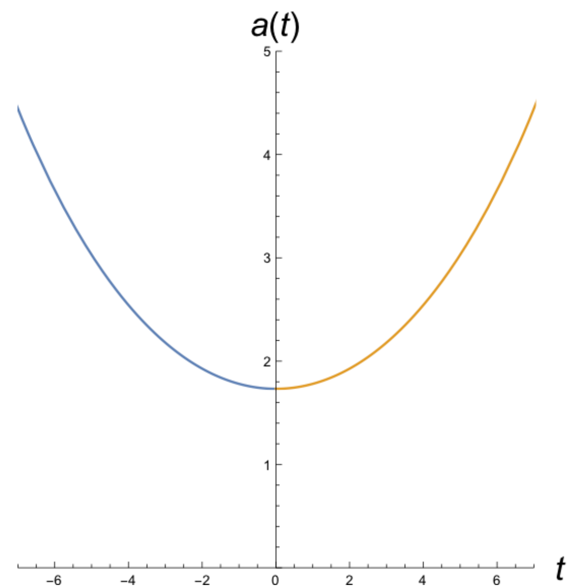
$$\frac{dX}{dt} = \mp\frac{2\sqrt{3}\,X\,F_X\,\sqrt{X F_X - \frac{1}{2}F}}{2X F_{XX} + F_X}$$

$$t(X) = \frac{1}{3}\sqrt{\frac{2(3X-1)}{3X}} - \frac{2}{3}\sqrt{\frac{2}{3}}\log\left(\frac{4X + 2\sqrt{X(3X-1)} - 1}{1-2X}\right).$$

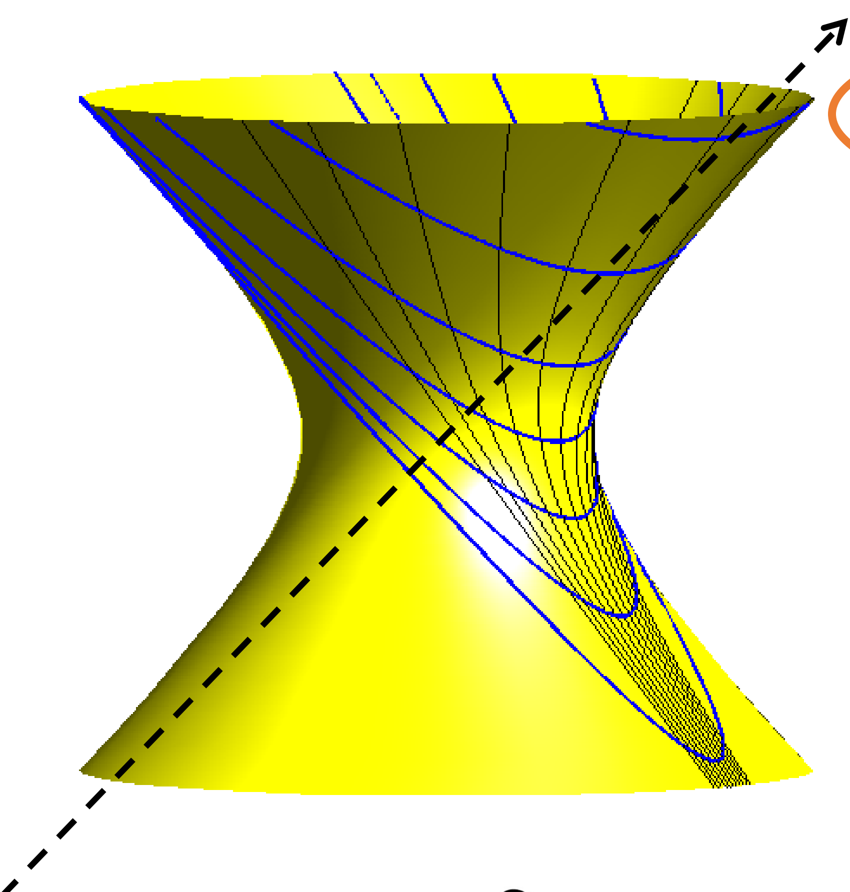








Flat de Sitter model (Lemaître, 1924)



$$X_0 + X_4 = R \exp \frac{t}{R}$$

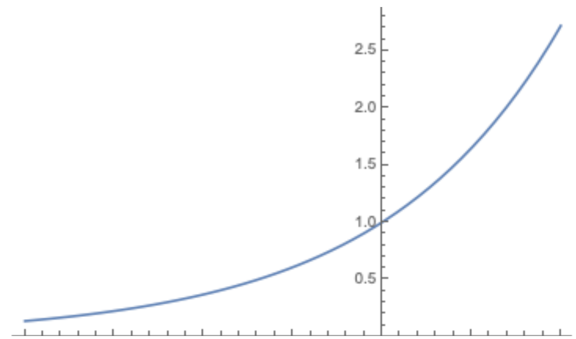
$$\begin{cases} X_0 &= R \sinh \frac{t}{R} + \frac{1}{2R} e^{\frac{t}{R}} |\vec{x}|^2 \\ X_1 &= \exp\left(\frac{t}{R}\right) x_1 \\ X_2 &= \exp\left(\frac{t}{R}\right) x_2 \\ X_3 &= \exp\left(\frac{t}{R}\right) x_3 \\ X_4 &= R \cosh \frac{t}{R} - \frac{1}{2R} e^{\frac{t}{R}} |\vec{x}|^2 \end{cases}$$

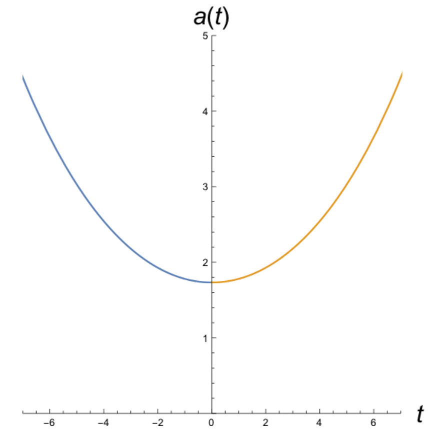
$$ds^2 = dX_0^2 - dX_1^2 - \ldots dX_4^2 \Big|_{dS} =$$

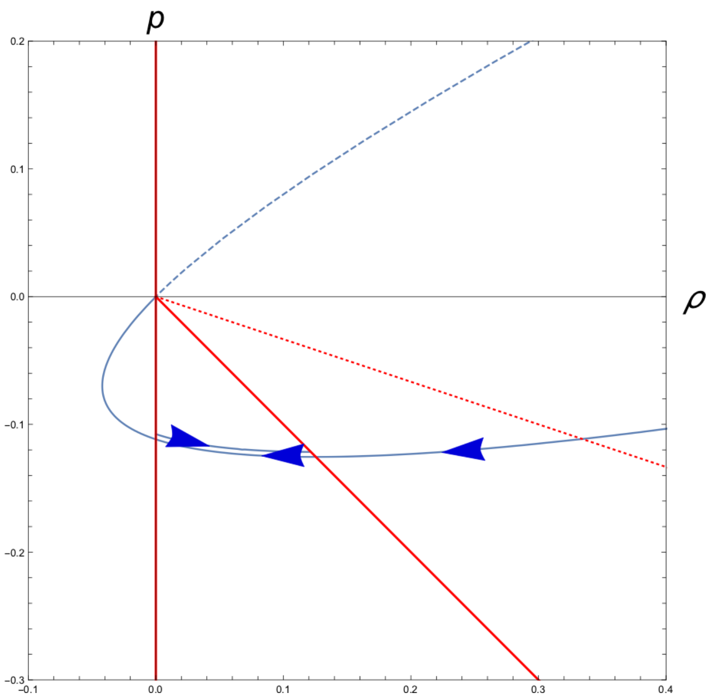
$$= dt^2 - \exp \frac{2t}{R} \left(dx_1^2 + dx_2^2 + dx_3^2\right)$$

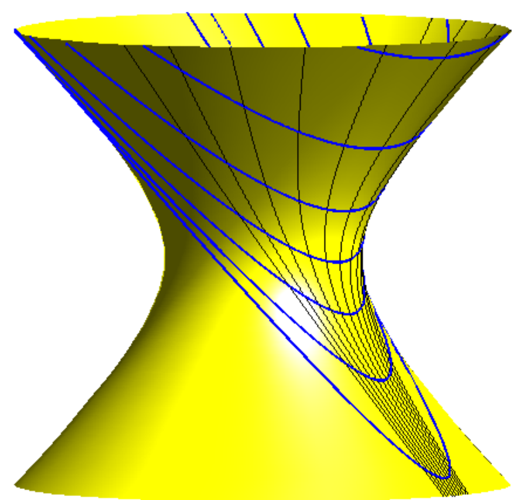
$$a(X) = a_0((1-2X)\sqrt{X})^{-\frac{1}{3}}$$

$$t(X) = \frac{1}{3}\sqrt{\frac{2(3X-1)}{3X}} - \frac{2}{3}\sqrt{\frac{2}{3}}\log\left(\frac{4X + 2\sqrt{X(3X-1)} - 1}{1-2X}\right).$$

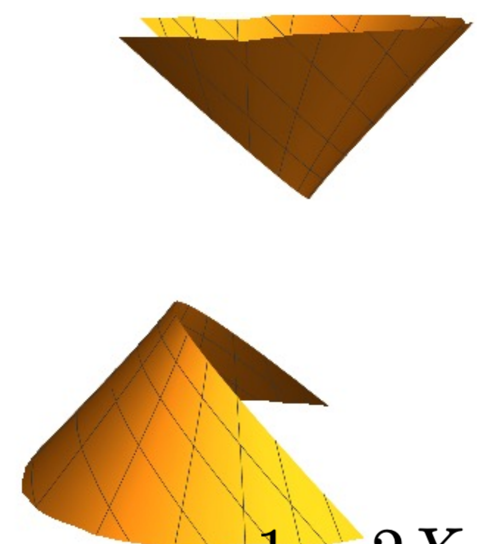








$$c_s^2 = -1$$

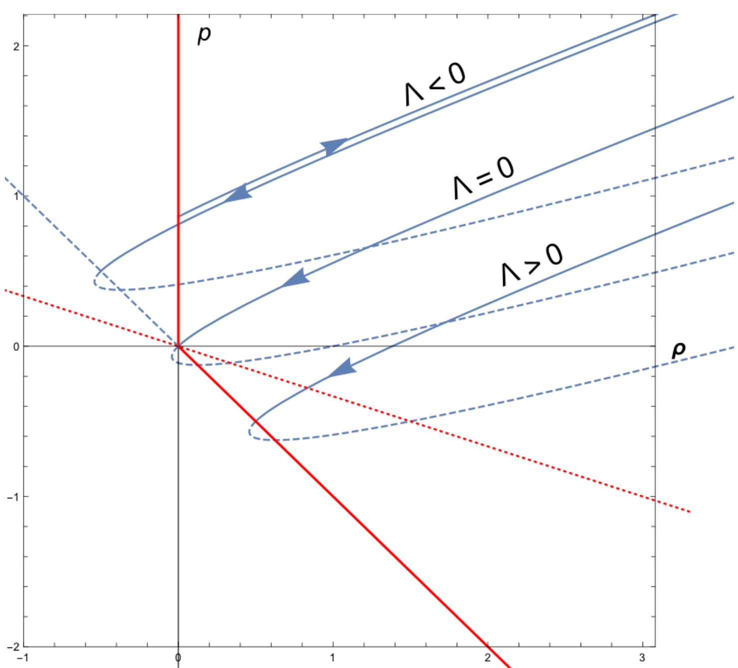


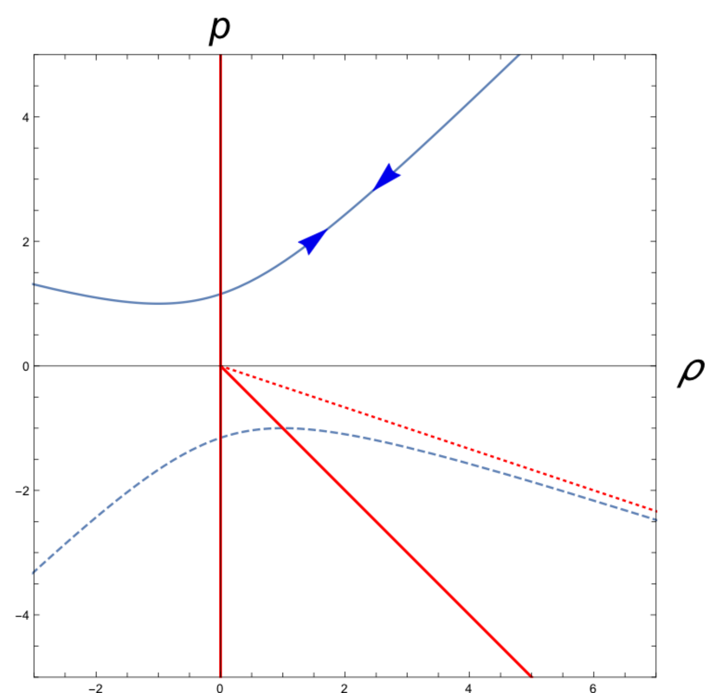
$$c_s^2 = \frac{1-2X}{1-6X}$$

Mimicking a negative cosmological constant

$$F = X + \frac{1}{X} = 2p \qquad \rho(X) = \frac{X^2 - 3}{2X}$$

$$a(X) = \frac{a_0\sqrt{X}}{\sqrt[3]{X^2 - 1}}, \qquad t(X) = \frac{1}{\sqrt{6}}\int_{\sqrt{3}}^{X} \frac{x^2 + 3}{\sqrt{x^3 - 3x}\,(x^2 - 1)}dx.$$





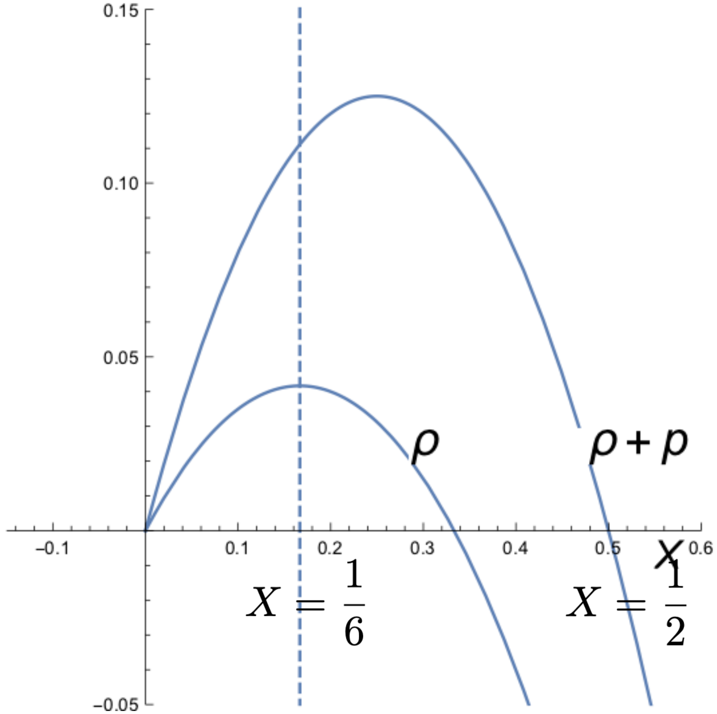
Phenomenology of a branching point

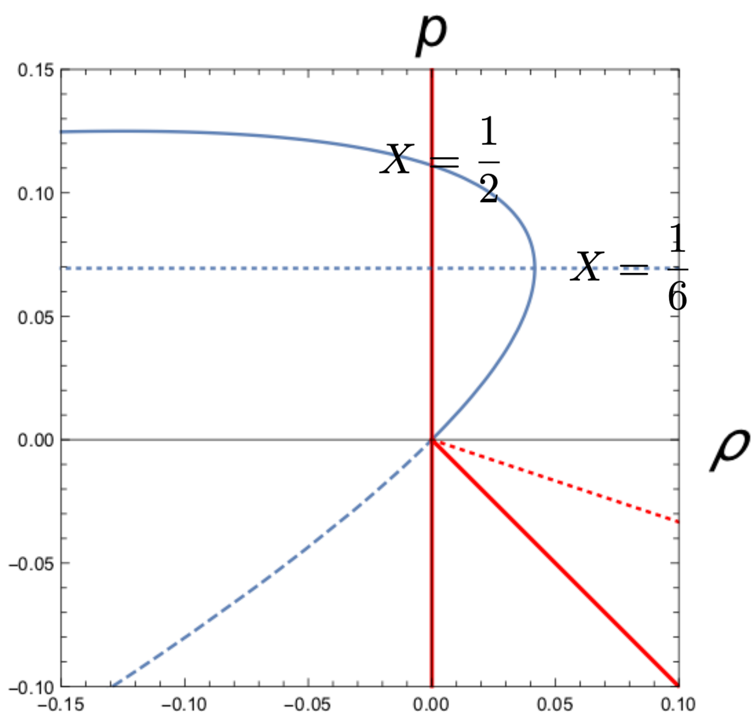
$$\Lambda = 0, \quad \mu < 0, \quad \lambda > 0$$

$$F = X - X^2$$

$$\rho = \frac{1}{2}\left(X - 3X^2\right)$$

$$c_s^2 = \frac{1-2X}{1-6X}$$





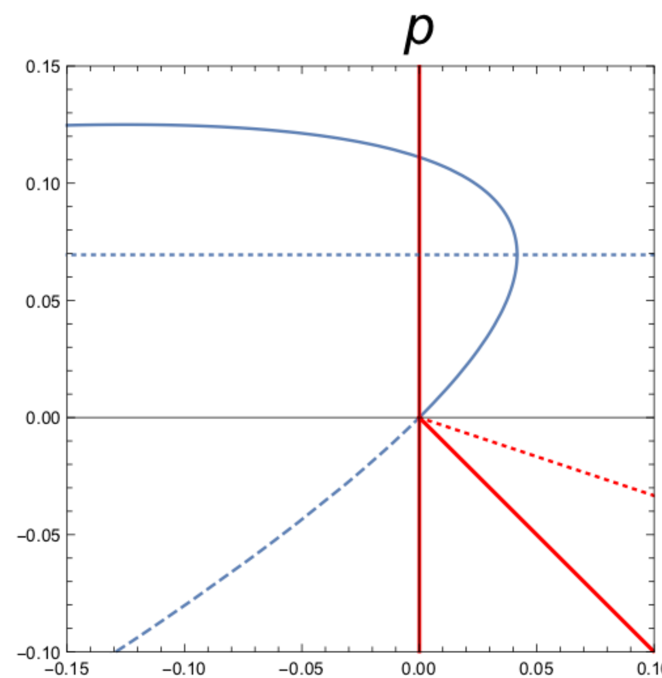
Phenomenology of a branching point

$$\Lambda = 0, \quad \mu < 0, \quad \lambda > 0 \qquad\qquad F = X - X^2$$

The presence of a branching point in the physical region renders the dynamical behaviour subtler
The following equations are the starting point:

$$\begin{aligned} a(X) &= a_0 \left(\sqrt[3]{\sqrt{X}(1-2X)} \right)^{-1}, \\ \dot{X} &= \pm \frac{3\sqrt{6}X(2X-1)\sqrt{X-3X^2}}{6X-1} \end{aligned}$$

the final task is to describe
the scale factor $a(t)$ as a function of the cosmic time.



Phenomenology of a branching point

$$F = X - X^2$$

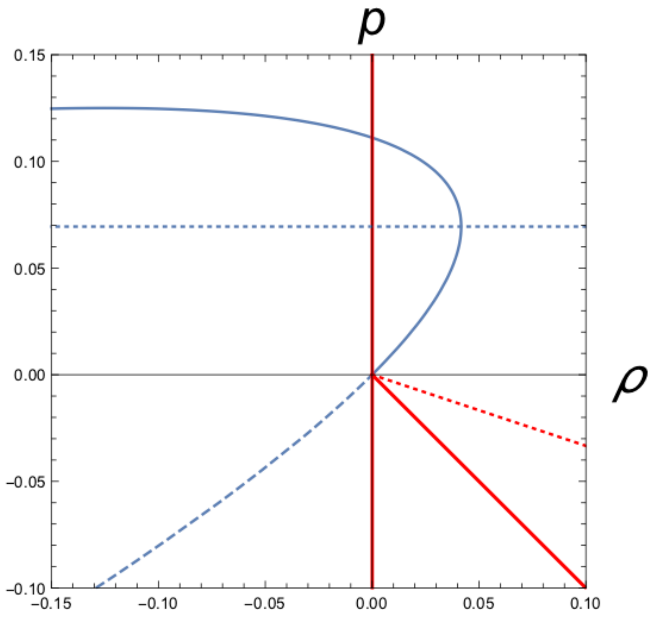
$$\Lambda = 0, \quad \mu < 0, \quad \lambda > 0$$

Initial conditions at $X = 0$ (which means $a \sim \infty$).

$$\dot{a}(X) = \frac{\partial a}{\partial X}\dot{X} = -\frac{\sqrt{3X - 9X^2}}{\sqrt{2}\sqrt[3]{(1-2X)\sqrt{X}}} = -\sqrt{3\rho(X)}a(X)$$

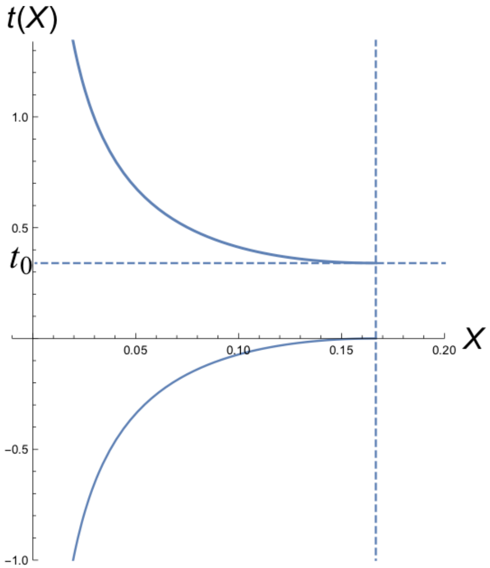
$$\ddot{a}(X) = \frac{\partial \dot{a}}{\partial X}\dot{X} = \frac{3X(3X-2)}{2\sqrt[3]{(1-2X)\sqrt{X}}} = -\frac{3a(X)}{2}(\rho + 3p) < 0.$$

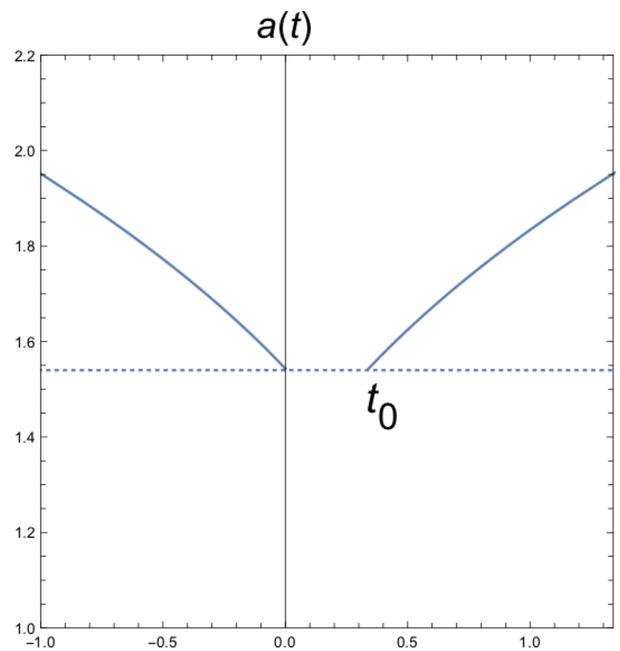
$$t(X) = \frac{1}{27}\sqrt{2}\left(-\frac{3\sqrt{3-9X}}{\sqrt{X}} - 12\sqrt{3}\tan^{-1}\left(\frac{\sqrt{X}}{\sqrt{1-3X}}\right) + 2\sqrt{3}\pi + 9\right),$$

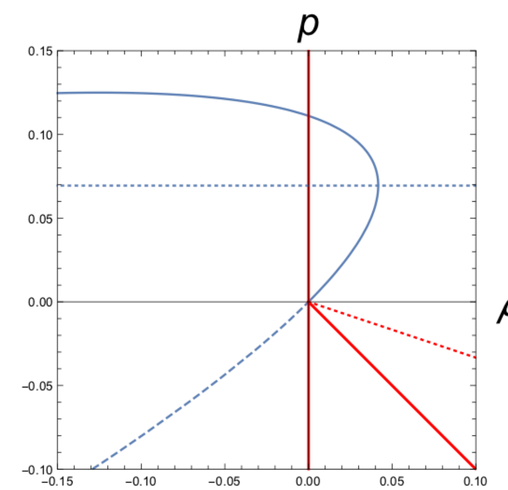


$$a(X_c) = \frac{\sqrt{3}}{\sqrt[6]{2}}, \qquad \dot{a}(X_c) = -\frac{\sqrt{3}}{2^{5/3}}, \qquad \ddot{a}(X_c) = -\frac{3\sqrt{3}}{8\sqrt[6]{2}}.$$

Phenomenology of a branching point







$$a(X_c) = \frac{\sqrt{3}}{\sqrt[6]{2}}, \quad \dot{a}(X_c) = -\frac{\sqrt{3}}{2^{5/3}}, \quad \ddot{a}(X_c) = -\frac{3\sqrt{3}}{8\sqrt[6]{2}}.$$

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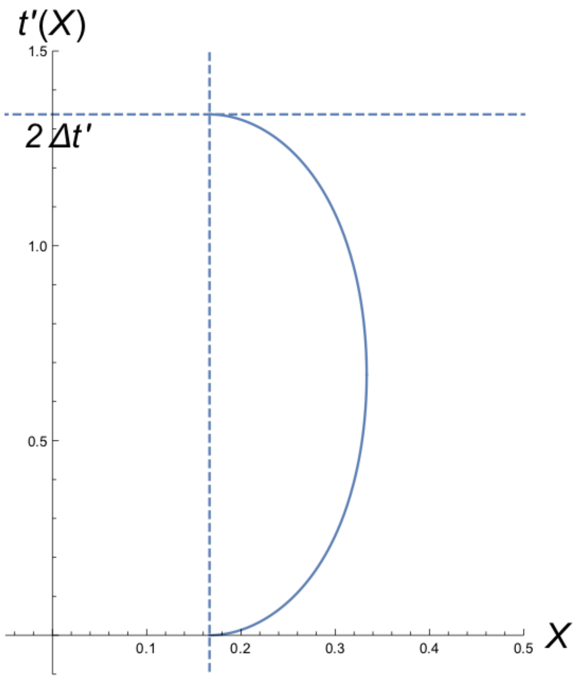
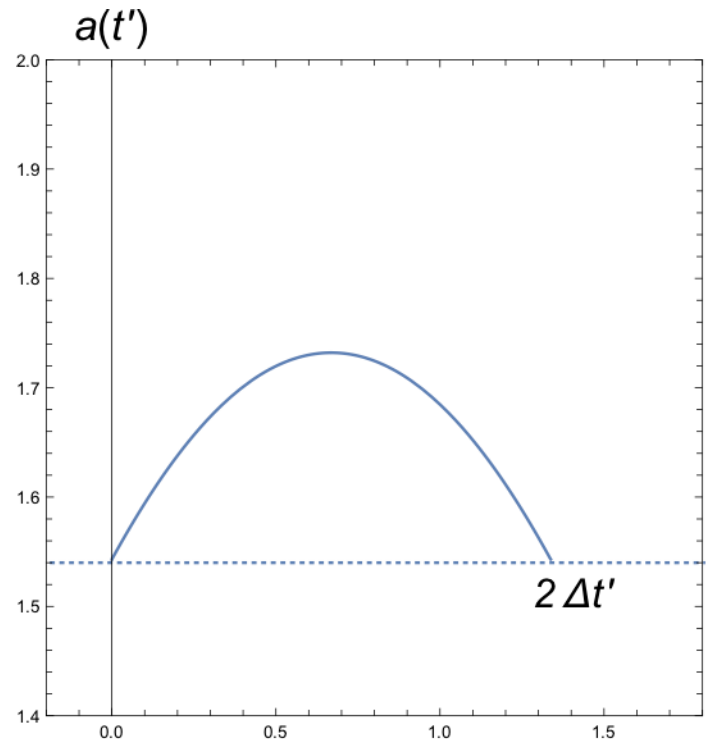
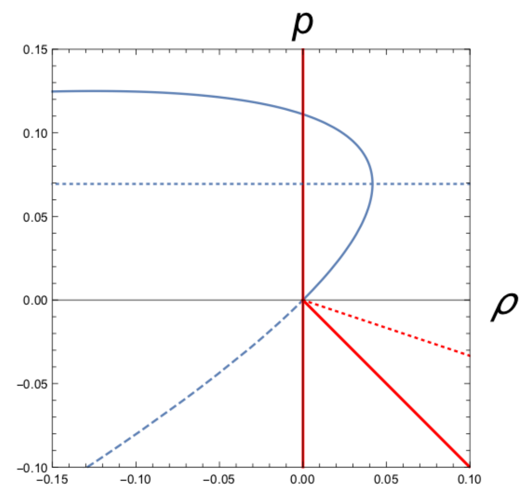
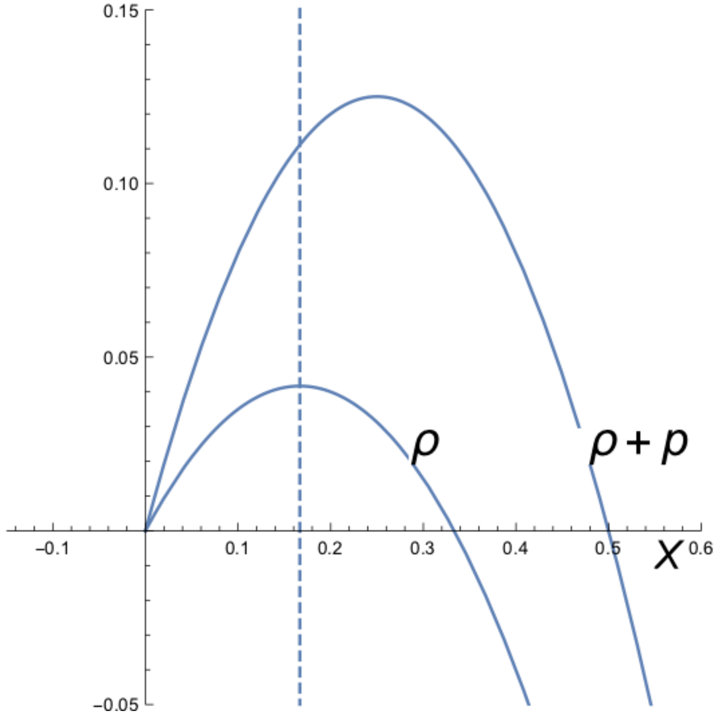


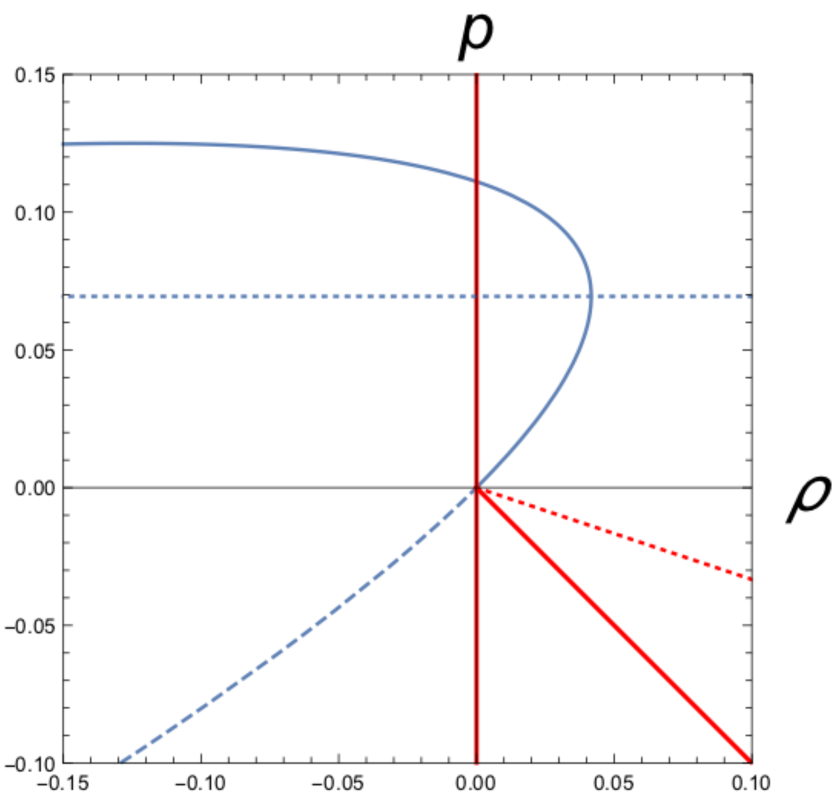




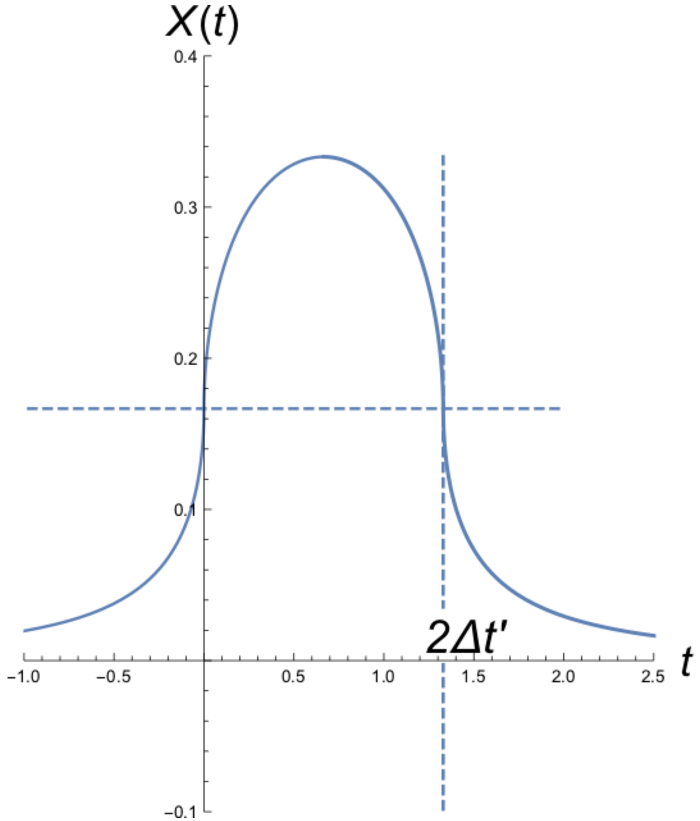
$$t'(X) = \frac{1}{27}\sqrt{2}\left(\frac{3\sqrt{3-9X}}{\sqrt{X}} + 12\sqrt{3}\tan^{-1}\left(\frac{\sqrt{X}}{\sqrt{1-3X}}\right) - 2\sqrt{3}\pi - 9\right)$$

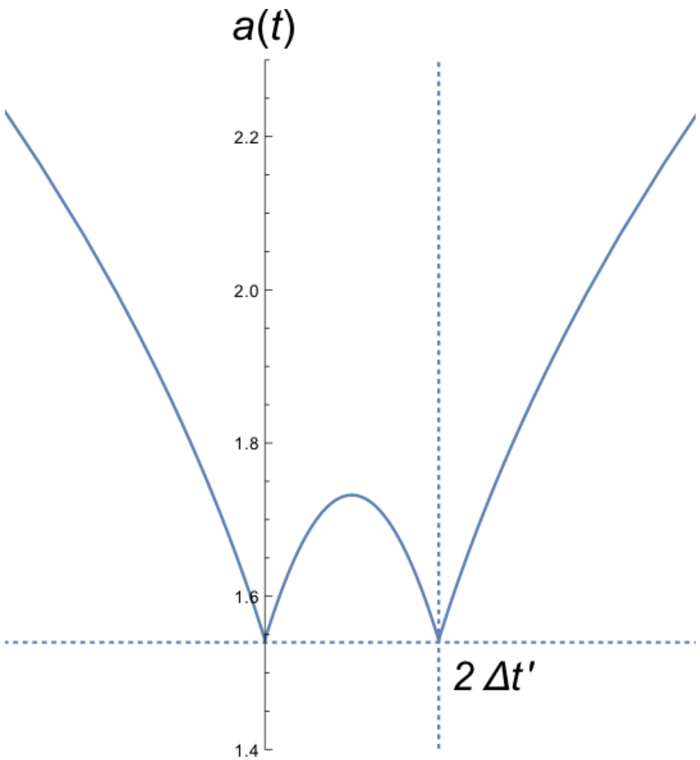
So what happens at the branching point?





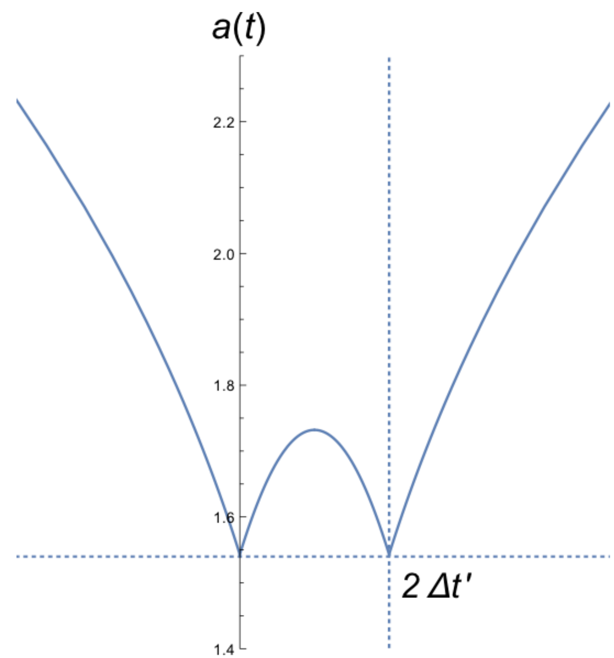
First scenario



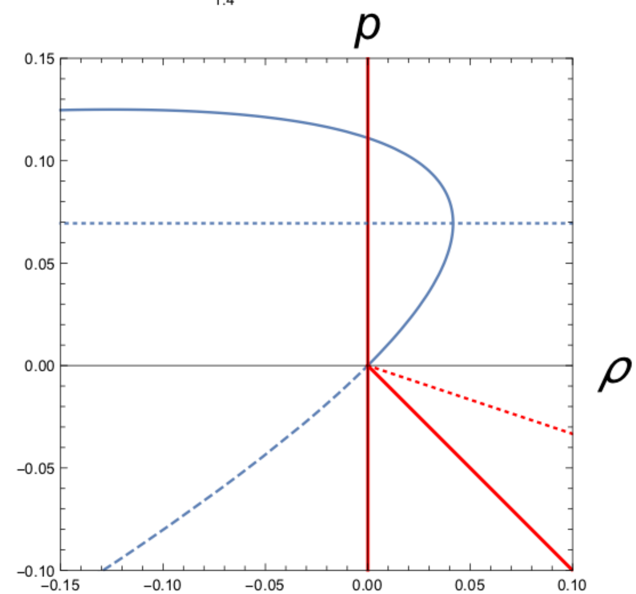


First interpretation

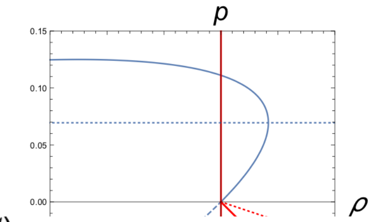
When the field gets at the critical point the density reaches a maximum and the universe cannot shrink anymore. Like in the elastic collision the sign of the velocity is reverted. The expansion following the collision keeps decelerating as the acceleration in this model is always negative.

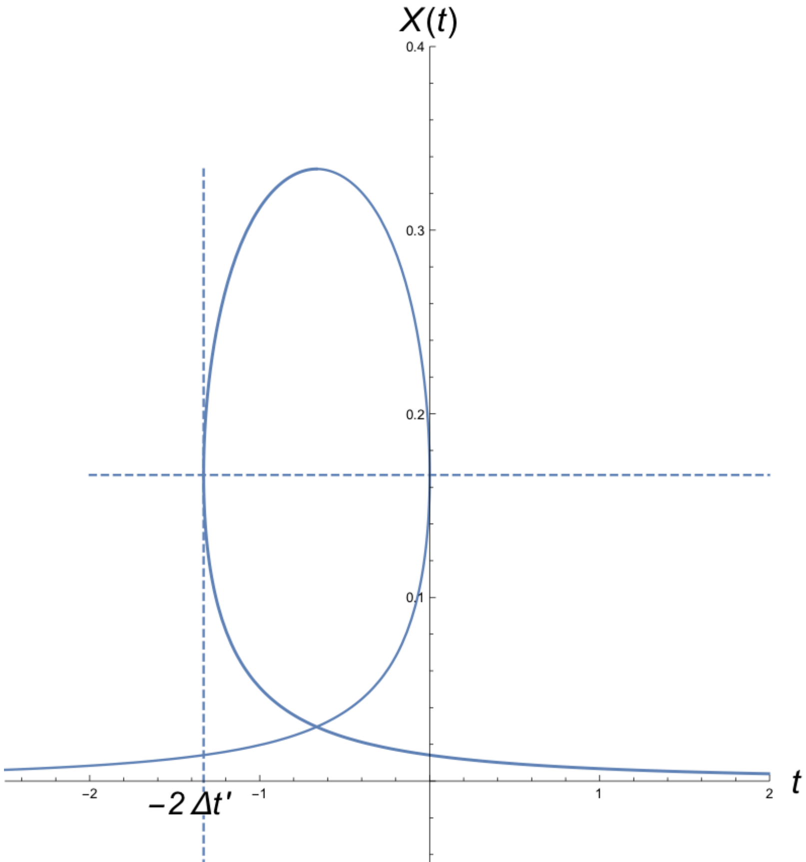


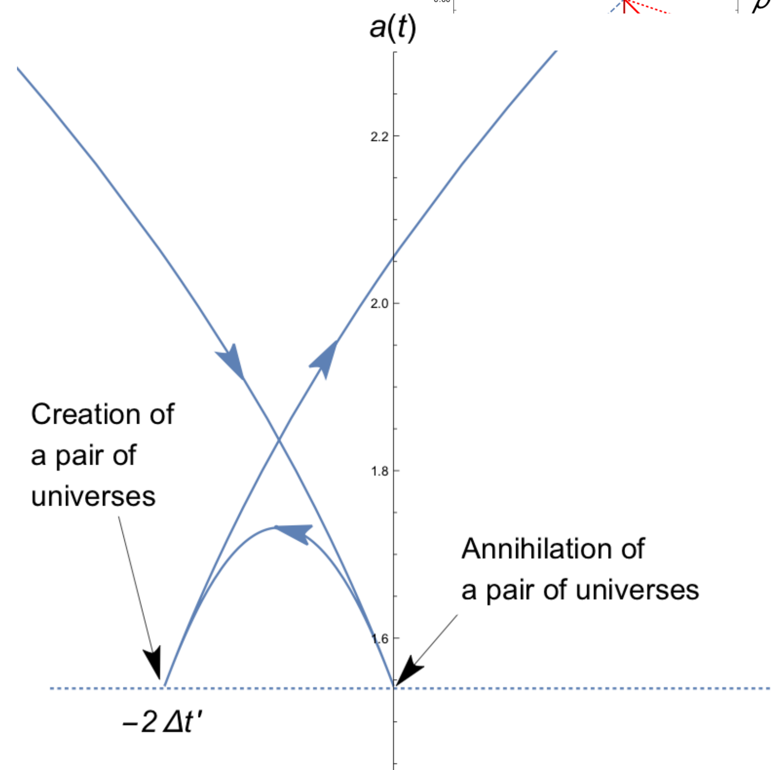
1. The Universe starts at $t = -\infty$ with infinite radius and zero velocity. It is a flat Minkowski spacetime.
2. The evolution of the universe makes the radius shrink with a negative and decreasing velocity $\dot{a}$ up to $t = 0$ when the universe hits the branching point.
3. *The (unstable) bulged bounce.* The branching point acts like a wall and the universe undergoes an elastic collision where the velocity is reverted and becomes positive. The universe enters in a phase of decelerated expansion. At $t = \Delta t'$ the expansion stops and the universe starts again to shrink up to $t = 2\Delta t'$ where it hits the branching point for a second time.
4. At the branching point the velocity is reverted again. The universe enters in a phase of everlasting decelerated expansion that will drive it back to Minkowski space at $t = \infty$.
5. There is no singularity at $t = \infty$.



Second interpretation

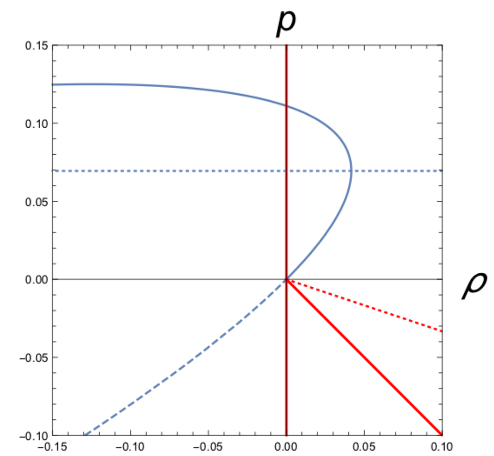






$X(t)$ and $a(t)$ are plotted in *trompe l'œil* . What is the meaning of such diagrams? They represents the clever solution that the universe gives to apparently unsolvable problem it has to face when arriving at the branching point: how could it go from $a(X_c) = \sqrt{3}/\sqrt[6]{2}$ to $a(1/3) = \sqrt{3}$ with a negative velocity and from $\dot{a}(X_c) = -\sqrt{3}/2^{5/3}$ to $\dot{a}(1/3) = 0$ with a negative acceleration? Running backward in time!

Second interpretation



The ramification point in the equation of state is encountered twice during the time evolution. When the universe gets at the ramification point the time starts flowing backward till the universe gets again to the ramification point, when the usual forward orientation of time is recovered. The phenomenology is essentially the same as in the previous conservative description but the inversion of the velocity is caused by an inversion of the sense of the flow of time, inversion that is short-lived.

Borrowing from relativistic quantum mechanics ideas it is tempting to say that a universe pair is annihilated when the ramification point is first encountered; a pair of universes is created when the ramification point is encountered a second time.

There is only one possible difference. Since the time runs backward the instability due to the negative squared velocity of sound needs to be reconsidered.