Conceptual and technical issues in quantum cosmology

Leonardo Chataignier

Dipartimento di Fisica e Astronomia, Università di Bologna
INFN, Sezione di Bologna, I.S. FLAG

Based on:
L.C.,Phys. Rev. D **101**, 086001 (2020); **103**, 026013 (2021);
L.C. and M. Krämer, Phys. Rev. D **103**, 066005 (2021)
$+$ **L.C., PhD Thesis** (University of Cologne 2021) [*Supervisor: Prof. Dr. Claus Kiefer*]
$+$ L.C., Z. Naturforsch. A **77**, 805 (2022);
$+$ L.C., *in preparation*

Today

- Revisit conceptual and technical issues in quantum gravity
- Revisit quantum-gravitational corrections to power spectra of scalar and tensor perturbation in a semiclassical approach to quantum gravity[1]
- Focus on the issue of the definition of the inner product/unitarity
- Argue that
  - Corrections are unitary
  - Inner product is related to a notion of gauge-fixing the time variable
  - Probabilities may be interpreted "relationally" as conditional probabilities
- Relate to previous works in the literature

[1] D. Brizuela, C. Kiefer and M. Krämer, Phys. Rev. D **93** 104035 (2016) & Phys. Rev. D **94** 123527 (2016).

Problems that approaches to quantum gravity face

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- Root of the problems: how to deal with diffeomorphism invariance at the quantum level?
  - Goal: address this issue in mechanical toy models ('worldline') by direct analogy to techniques and concepts used in classical canonical gauge systems

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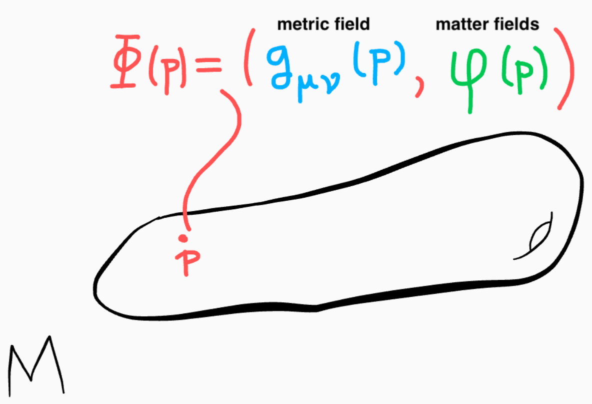
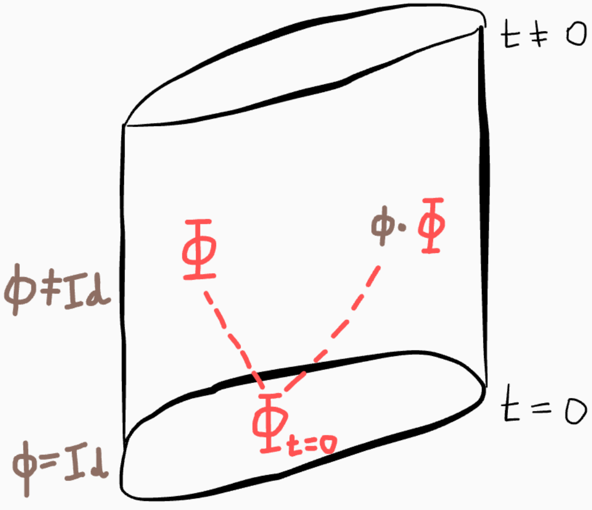
How to deal with diffeomorphism invariance at the classical level?



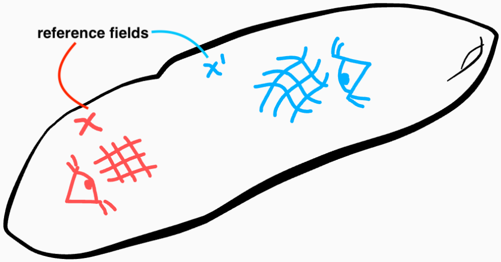
- Background independence
- Fixed elements:
  $\mathcal{B} = \{$topology, dimension, differential structure, signature$\}$

Gauge indeterminism



Determinism is restored by considering diffeomorphism invariants (or equivalence classes) $\Rightarrow$ Observables

Relational observables



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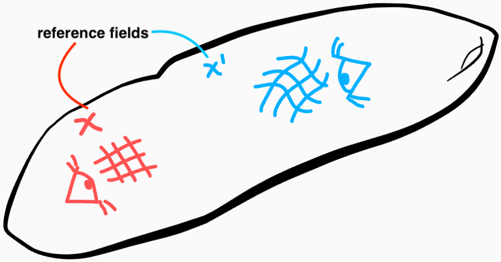

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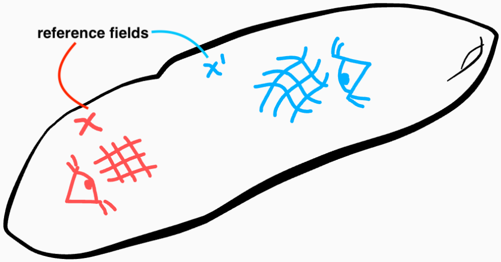

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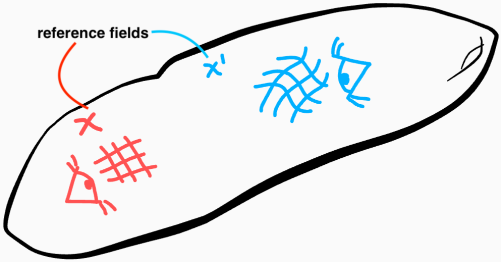

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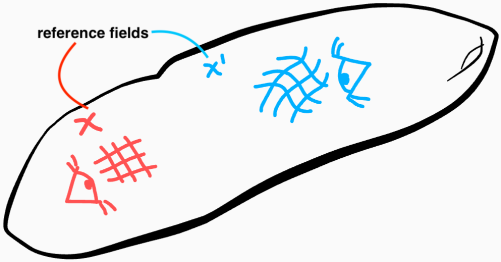

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  - are **conditional** quantities and yield predictions (the value of Φ) based on a certain condition (the observed value of χ).

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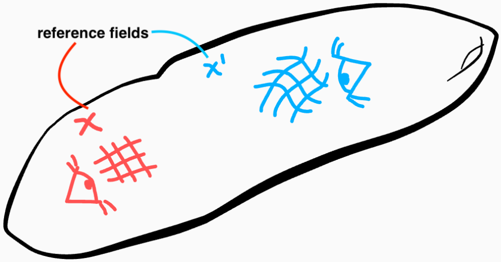

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- The outcomes of their experiments are the values of Φ relative to χ $\Rightarrow$ 'relational observables'
- The relational observables (denoted by $\mathcal{O}[\Phi|\chi]$)
  - completely encode the dynamics in the region of the experiment
  - are **conditional** quantities and yield predictions (the value of Φ) based on a certain condition (the observed value of χ).
  - make no reference to the abstract point $p \in \mathcal{M}$ and can thus be seen as constant spacetime scalars for each fixed value of χ $\Rightarrow$ diffeomorphism invariants!

Phenomenology – What's the problem?

- Quantum gravity (QG) must address the issue of producing testable predictions $\rightarrow$ early universe may be an adequate testing ground
- Inflationary paradigm: quantum fluctuations of the metric and the inflaton give rise to the CMB anisotropies and the conditions for structure formation
- QG effects ($\sim$ Planck scale): some may already be relevant at the high energies present during the inflationary phase
- How to compute these QG effects?
  - Assume that metric/inflaton perturbations couple to a *quantum* FLRW background through a **master Wheeler-DeWitt equation**
  - Assess unitarity of this theory
- Let's see a possible formalism for this.

Classical FLRW background

- Minisuperspace: scale factor + minimally coupled inflaton[3]

$$S = \int_{t_0}^{t_1} \mathrm{d}t \; \left(p_a \dot{a} + p_\phi \dot{\phi} - NC \right) \; ,$$

$$C = -\frac{\kappa}{2a} p_a^2 + \frac{1}{2a^3} p_\phi^2 + a^3 \mathcal{V}(\phi) \; .$$

- Config.-space metric and conformal structure (change of "einbein frame")

$$\mathcal{G} := \mathrm{diag} \left(-\frac{a}{\kappa}, a^3 \right) \; ,$$

$$N(t) = \tilde{N}(t)\Omega(t) \; , \; \mathcal{G}_{ij} = \Omega(t) \tilde{\mathcal{G}}_{ij} \; , \; a^3 \mathcal{V}(\phi) = \frac{a^3 \tilde{\mathcal{V}}(a, \phi)}{\Omega(t)} \; ,$$

for $\Omega(t) > 0$.

[3] $\hbar = c = 1$ and $\kappa = \frac{4\pi G}{3}$.

Classical FLRW background (cont'd)

- Time-reparametrization invariance:

$$\dot{f} = \{f, NC\} \approx N\{f, C\} \ ,$$

eqs. of motion (e.o.m.) have the same form for any choice of einbein $N(t)$.

- Gauge-fixing: choose the level sets of a phase-space function χ to be time. This determines the einbein via the Faddeev-Popov determinant (FPdet) Δ_χ,

$$\frac{1}{N} \approx \Delta_\chi := \{\chi, C\} \ .$$

- The FPdet changes under a change of einbein frame (minisuperspace conformal transf.)

$$\tilde{\Delta}_\chi := \{\chi, \tilde{C}\} \approx \Omega(t)\Delta_\chi \ .$$

The de Sitter case

- Let's specialize to the de Sitter case for simplicity. This is the "no-roll" limit of inflation, in which the inflaton remains constant.

- One finds from the inflaton e.o.m.:

$$\phi = \text{const.} \Rightarrow p_\phi = \frac{\partial \mathcal{V}}{\partial \phi} = 0 \ .$$

The flat potential corresponds to a constant Hubble rate H_0,

$$\mathcal{V}(\phi) = \frac{H_0^2}{2\kappa} \Rightarrow \ C = -\frac{\kappa}{2a} p_a^2 + a^3 \frac{H_0^2}{2\kappa} \ .$$

- Since ϕ doesn't roll, the only dynamical degree of freedom in the background is the scale factor, from which we can define conformal time as

$$\eta(a) := \int \frac{\mathrm{d}a}{H_0 a^2} = -\frac{1}{H_0 a} \in (-\infty, 0) \ .$$

Conformal time as a gauge-fixing of the time variable

- If we describe the dynamics w.r.t. conformal time, the associated FPdet reads

$$\Delta_\eta = \{\eta(a), C\} = -\frac{\kappa}{H_0 a^3} p_a \ .$$

- But, since

$$C = -\frac{\kappa}{2a} p_a^2 + a^3 \frac{H_0^2}{2\kappa} \approx 0 \Rightarrow p_a \approx -\frac{H_0}{\kappa} a^2 \ ,$$

this implies that

$$\Delta_\eta \approx \frac{1}{a} = \frac{1}{N} \ .$$

- If we change the einbein basis to one adapted to conformal time, $N = \tilde{N} a$, we obtain

$$\tilde{\Delta}_\eta = -\kappa H_0^2 \eta^4 p_\eta \approx 1 = \frac{1}{\tilde{N}} \ ,$$

where p_η is the momentum conjugate to $\eta(a)$.

Quantum perturbations on a classical de Sitter background

- Given the scalar A, B, ψ and E and tensor h_{ij} perturbations, as well as the inflaton perturbation φ, we can define the usual Mukhanov-Sasaki variables

$$v(\eta,\mathbf{x}) := a\left\{\varphi + \frac{\dot{\phi}}{\mathcal{H}}\left[A + 2\mathcal{H}(B-\dot{E}) + \frac{\mathrm{d}}{\mathrm{d}\eta}(B-\dot{E})\right]\right\}$$

$$= \int_{\mathbb{R}^3}\frac{\mathrm{d}^3k}{(2\pi)^{\frac{3}{2}}}v_{\mathbf{k}}(\eta)\mathrm{e}^{\mathrm{i}\mathbf{k}\cdot\mathbf{x}} \ ,$$

$$v_{\mathbf{k}}^{(+,\times)} := \frac{a}{\sqrt{12\kappa}}h_{\mathbf{k}}^{(+,\times)} \ .$$

- The usual quantum field theory (QFT) of these perturbations is governed by the Schrödinger equation

$$\mathrm{i}\frac{\partial\tilde{\psi}}{\partial\eta} = \hat{H}\tilde{\psi} \ , \ \hat{H} := \frac{1}{2}\sum_{\mathbf{k}}\left\{-\frac{\partial^2}{\partial v_{\mathbf{k}}^2} + \left(k^2 - \frac{2}{\eta^2}\right)v_{\mathbf{k}}^2\right\} \ .$$

Master Wheeler-DeWitt equation

- One can go beyond usual QFT by considering the direct interaction of the perturbations with a *quantum* background in a time-reparametrization invariant way.
- This is achieved via the master Wheeler-DeWitt (WDW) constraint

$$\left\{\frac{\mathrm{e}^{-2\alpha}}{a_0^2}\left[\frac{\kappa}{2}\frac{\partial^2}{\partial\alpha^2}+a_0^6\mathrm{e}^{6\alpha}\frac{H_0^2}{2\kappa}\right]+\hat{H}\right\}\Psi(\alpha,v)=0$$

- This is a timeless theory: it does not depend on any external/preferred time parameter.
- However, motivated by recent developments in relational approaches[4], we consider that the theory is not strictly timeless, but relational. This may lead to phenomenological differences with a strictly timeless theory.

[4] L. C., Phys. Rev. D **101** 086001 (2020); L.C., arXiv:2006.05526 [gr-qc], P. A. Hoehn *et al.*, arXiv:2007.00580 [gr-qc].

Relational quantum dynamics

- How can one interpret the master WDW equation relationally? What does “relational” even mean in the quantum theory? There are many possible (and provisional) answers.[5]

- Let’s be pragmatic: relational $=$ in relation to $=$ relative. So the quantum theory is about probabilities of certain measurement outcomes *relative* to some other observation(s).

- This suggests that we should define **conditional probabilities**[6] from the wave function of the universe $\Psi(\alpha, v)$.

[5] L.C., arXiv:2006.05526 [gr-qc], P. A. Hoehn *et al.*, arXiv:2007.00580 [gr-qc].

[6] D. N. Page and W. K. Wootters, Phys. Rev. D **27** 2885 (1983).

Relational quantum dynamics (cont'd)

- But, first, we need an inner product (IP). Inspired by the usual Faddeev-Popov gauge-fixing procedure, let's define[7]

$$\left(\Psi_{(1)}\middle|\Psi_{(2)}\right) := \int \mathrm{d}\alpha \mathrm{d}v \ \left(\hat{\mu}^{\frac{1}{2}}\Psi_{(1)}\right)^* |J|\delta(\chi - t)\hat{\mu}^{\frac{1}{2}}\Psi_{(2)} \ ,$$

  where χ is the gauge-fixing condition, $J = \frac{\partial\chi}{\partial\alpha}$ and $\hat{\mu}$ is a measure analogous to the FPdet such that the IP is conserved w.r.t t (unitarity) and positive-definite.

- We then *define*

$$p_\Psi := \frac{1}{\left(\Psi\middle|\Psi\right)} \left(\hat{\mu}^{\frac{1}{2}}\Psi\right)^* \hat{\mu}^{\frac{1}{2}}\Psi\Big|_{\chi(\alpha,v)=t}$$

  to be a conditional probability, i.e., the probability of obtaining a certain v configuration conditioned on the value of χ.

[7] See also A. Barvinsky, Phys. Rept. **230** 237 (1993).

The weak-coupling expansion

- In general, we can only solve the master WDW equation perturbatively. The formal expansion parameter is κ ($\sim$ the inverse Planck mass squared).
- This formal expansion should be valid when all energy scales are below the Planck scale or when v is weakly coupled to the background ($\kappa \ll 1$).
- In order to determine the wave function of the universe $\Psi(\alpha, v)$, we make the "Wentzel-Kramers-Brillouin (WKB)-like" ansatz[8]

$$\Psi(\alpha, v) = \exp\left\{\frac{\mathrm{i}}{\kappa} S(\alpha, v)\right\} = \exp\left\{\frac{\mathrm{i}}{\kappa} \sum_{n=0}^{\infty} \kappa^n S_n\right\} .$$

- Analogously, we assume that the measure $\hat{\mu}$ can be found in perturbation theory using the expansion

$$\hat{\mu} \equiv \sum_{n=0}^{\infty} \kappa^n \hat{\mu}_n \left(\alpha; v, -\mathrm{i}\frac{\partial}{\partial v}\right) .$$

[8] C. Kiefer and T. P. Singh, Phys. Rev. D **44**, 1067 (1991).

The weak-coupling expansion (cont'd)

- The lowest-orders of the κ-expansion imply that $S_0(\alpha, v) \equiv S_0(\alpha)$ is only a function of the scale factor and solves the Hamilton-Jacobi equation for the de Sitter background,

$$-\frac{1}{2}\left(\frac{\partial S_0}{\partial \alpha}\right)^2 + \frac{a_0^6}{2}\mathrm{e}^{6\alpha} H_0^2 = 0 \ .$$

- This also implies that $\frac{\mathrm{e}^{-2\alpha}}{a_0^2}\frac{\partial S_0}{\partial \alpha}\frac{\partial}{\partial \alpha} \stackrel{!}{=} \frac{\partial}{\partial \eta}$, i.e., the directional derivative along the trajectories associated with S_0 coincides with the conformal time derivative.

- In this way, the conformal time gauge is singled-out by the κ-expansion, and we choose $\chi = \eta(a)$ in the definition of conditional probabilities.

The weak-coupling expansion (cont'd II)

- The fact that $S_0(\alpha, v) \equiv S_0(\alpha)$ also implies that we can factorize the background, $\Psi(\alpha, v) =: \mathrm{e}^{\frac{\mathrm{i}}{\kappa} S_0(\alpha)} \psi(\alpha; v)$. The equation for $\psi(\alpha; v)$ is found to be

$$\mathrm{i}\frac{\partial\psi}{\partial\eta} = \hat{H}\psi + \frac{3\mathrm{i}}{2\eta}\psi + \frac{\kappa}{2} H_0^2 \eta^3 \frac{\partial}{\partial\eta}\left(\eta\frac{\partial\psi}{\partial\eta}\right) .$$

- The idea now is: terms with imaginary coefficients do not violate unitarity, but rather define the measure in perturbation theory.[9] Indeed, we find

$$\mathrm{i}\frac{\partial\psi}{\partial\eta} = \hat{H}\psi + \frac{3\mathrm{i}}{2\eta}\psi + \mathcal{O}(\kappa) \Rightarrow \mathrm{i}\frac{\partial}{\partial\eta}\tilde{\psi} = \hat{H}\tilde{\psi} + \mathcal{O}(\kappa) ,$$

where $\tilde{\psi} := \frac{1}{H_0}|\eta|^{-\frac{3}{2}}\psi$. This is just the usual QFT Schrödinger equation for a classical background, and $\tilde{\psi}$ is to be identified with the usual QFT wave function(al).

[9] B. S. DeWitt, Rev. Mod. Phys. **29** 377 (1957); C. Lämmerzahl, Phys. Lett. A **203**, 12 (1995).

Unitarity

- So, if we define the lowest-order measure $\hat{\mu}_0^{\frac{1}{2}} := \frac{1}{H_0}|\eta|^{-\frac{3}{2}}$, we obtain the conditional probabilities

$$p_\Psi(v|\eta) = \frac{\left(\hat{\mu}_0^{\frac{1}{2}}\psi\right)^* \hat{\mu}_0^{\frac{1}{2}}\psi}{\int \mathrm{d}v \, \left(\hat{\mu}_0^{\frac{1}{2}}\psi\right)^* \hat{\mu}_0^{\frac{1}{2}}\psi} + \mathcal{O}(\kappa) = \frac{\tilde{\psi}^*\tilde{\psi}}{\int \mathrm{d}v \, \tilde{\psi}^*\tilde{\psi}} + \mathcal{O}(\kappa) \, ,$$

  and the dynamics is manifestly unitary at this order. What about the next?

- At order κ, we need to deal with the higher η-derivatives. Perturbatively, we find

$$\kappa|\eta|^{\frac{3}{2}}\frac{\partial}{\partial\eta}\left(\eta\frac{\partial\psi}{\partial\eta}\right) = -\mathrm{i}\kappa\frac{\partial}{\partial\eta}\left(\eta|\eta|^{\frac{3}{2}}\hat{H}\psi\right) + \frac{9\kappa|\eta|^{\frac{3}{2}}}{4\eta}\psi + \mathcal{O}(\kappa^2) \, .$$

  Again, let's use the terms with imaginary coefficients to define the measure.

Unitarity (cont'd)

- From

$$\kappa|\eta|^{\frac{3}{2}}\frac{\partial}{\partial\eta}\left(\eta\frac{\partial\psi}{\partial\eta}\right) = -\mathrm{i}\kappa\frac{\partial}{\partial\eta}\left(\eta|\eta|^{\frac{3}{2}}\hat{H}\psi\right) + \frac{9\kappa|\eta|^{\frac{3}{2}}}{4\eta}\psi + \mathcal{O}(\kappa^2)\ ,$$

the equation for $\psi(\alpha; v)$ can be written as

$$\mathrm{i}\frac{\partial}{\partial\eta}\left(\hat{\mu}^{\frac{1}{2}}\psi\right) = \left(\hat{H} + \kappa\frac{9H_0^2\eta^2}{8}\right)\frac{|\eta|^{-\frac{3}{2}}}{H_0}\psi + \mathcal{O}(\kappa^2)\ ,$$

where we defined $\hat{\mu}^{\frac{1}{2}} := \frac{|\eta|^{-\frac{3}{2}}}{H_0}\left(1 + \frac{\kappa H_0^2\eta^4}{2}\hat{H}\right) + \mathcal{O}(\kappa^2)$.

- If we again define $\tilde{\psi} := \hat{\mu}^{\frac{1}{2}}\psi$, we obtain the Schrödinger equation

$$\mathrm{i}\frac{\partial\tilde{\psi}}{\partial\eta} = \hat{H}_{\text{eff}}\tilde{\psi} = \left[\hat{H} - \kappa\frac{H_0^2\eta^4}{2}\hat{H}^2 + \mathcal{O}(\kappa^2)\right]\tilde{\psi}\ .$$

Unitarity (cont'd II)

- Dynamics is then dictated by the corrected Schrödinger equation

$$\mathrm{i}\frac{\partial\tilde{\psi}}{\partial\eta} = \hat{H}_{\mathrm{eff}}\tilde{\psi} = \left[\hat{H} - \kappa\frac{H_0^2\eta^4}{2}\hat{H}^2 + \mathcal{O}(\kappa^2)\right]\tilde{\psi}\ .$$

and the conditional probabilities

$$p_\Psi(v|\eta) = \frac{\left(\hat{\mu}^{\frac{1}{2}}\psi\right)^*\hat{\mu}^{\frac{1}{2}}\psi}{\int \mathrm{d}v\ \left(\hat{\mu}^{\frac{1}{2}}\psi\right)^*\hat{\mu}^{\frac{1}{2}}\psi} + \mathcal{O}(\kappa^2) = \frac{\tilde{\psi}^*\tilde{\psi}}{\int \mathrm{d}v\ \tilde{\psi}^*\tilde{\psi}} + \mathcal{O}(\kappa^2)\ ,$$

which are manifestly conserved at this order (the η-derivative of the denominator vanishes).
- Thus, there is no violation of unitarity, if we define conditional probabilities with the measure $\hat{\mu}$.
- **Important**: Note that the definition of $\hat{\mu}$ follows naturally from the κ-expansion (it's not *ad hoc*) and its deduction is independent from any "ontological commitment" to gauge-fixing and relationalism.

Relation to a notion of gauge-fixing the time variable

- But *can* we relate $\hat{\mu}$ in the conditional probabilities to a FPdet? Yes.
- First, recall that the background minisuperspace is endowed with the metric

$$\mathcal{G} := \mathrm{diag}\left(-\frac{a}{\kappa}, a^3\right) .$$

- Second, we note that the lowest-order measure can be written as

$$\hat{\mu}_0 = \frac{1}{H_0^2|\eta|^3} = \frac{\sqrt{\kappa}}{a}\left|\frac{\partial a}{\partial \eta}\right|\sqrt{|\det \mathcal{G}|} = \left|\frac{\partial a}{\partial \eta}\right|\sqrt{|\det \tilde{\mathcal{G}}|} ,$$

where the factor of $\frac{\sqrt{\kappa}}{a}$ has been absorbed into the config.-space metric via a conformal transformation (classically allowed). The result is then the square root of the determinant of the background minisuperspace metric with respect to the (η, ϕ) coordinates. Note that this is the "natural" measure when one is quantizing a theory with a curved config.-space. So the meaning of $\hat{\mu}_0$ is geometrical. What about the next order?

Relation to a notion of gauge-fixing the time variable (cont'd)

At the next order, we find

$$\begin{aligned}
\tilde{\psi}^*_{(1)}\tilde{\psi}_{(2)} &= \psi^*_{(1)}\hat{\mu}\psi_{(2)} = \psi^*_{(1)}\hat{\mu}_0^{\frac{1}{2}}\left(1+\kappa H_0^2\eta^4\hat{H}\right)\hat{\mu}_0^{\frac{1}{2}}\psi_{(2)} \\
&= \psi^*_{(1)}\hat{\mu}_0^{\frac{1}{2}}\left(1+\mathrm{i}\kappa H_0^2\eta^4\frac{\partial}{\partial\eta}\right)\hat{\mu}_0^{\frac{1}{2}}\psi_{(2)} = \Psi^*_{(1)}\hat{\mu}_0^{\frac{1}{2}}\left(\mathrm{i}\kappa H_0^2\eta^4\frac{\partial}{\partial\eta}\right)\hat{\mu}_0^{\frac{1}{2}}\Psi_{(2)} \\
&= \Psi^*_{(1)}\hat{\mu}_0\left(-\kappa H_0^2\eta^4\hat{p}_\eta\right)\Psi_{(2)}\ ,
\end{aligned}$$

where $\hat{p}_\eta := -\mathrm{i}\hat{\mu}_0^{-\frac{1}{2}}\frac{\partial}{\partial\eta}\hat{\mu}_0^{\frac{1}{2}}$ is the operator for the momentum conjugate to η w.r.t. the geometrical measure $\hat{\mu}_0$[10]. But the operator in parenthesis is exactly a quantization of $\tilde{\Delta}_\eta$, the classical FPdet for conformal time that we found before. Thus, $\int \mathrm{d}v\ \tilde{\psi}^*_{(1)}\tilde{\psi}_{(2)} = \int \mathrm{d}v\ \Psi^*_{(1)}\hat{\mu}_0\hat{\tilde{\Delta}}_\eta\Psi_{(2)}$. So the role of the FP operator is to connect wave functions of the universe to conditional wave functions.

[10] B. S. DeWitt, Rev. Mod. Phys. **29** 377 (1957)

Corrections to primordial power spectra

- One of the main applications of the BO weak-coupling expansion is the calculation of corrections to primordial power spectra of scalar and tensor modes.

- Let's focus on a single Fourier mode ("random phase approximation")

$$\mathrm{i}\frac{\partial\tilde{\psi}_{\mathbf{k}}}{\partial\eta} = \left[\hat{H}_{\mathbf{k}} - \kappa\frac{H_0^2\eta^4}{2}\hat{H}_{\mathbf{k}}^2\right]\tilde{\psi}_{\mathbf{k}} \ .$$

  This is actually a non-trivial approximation, since it requires neglecting interaction terms $\hat{H}_{\mathbf{k}}\hat{H}_{\mathbf{k}'}$ for $\mathbf{k} \neq \mathbf{k}'$.

- As $\hat{H}_{\mathbf{k}}$ is at most quadratic in $v_{\mathbf{k}}$, the correction $\hat{H}_{\mathbf{k}}^2$ contains terms of order v^4. So we make the ansatz

$$\tilde{\psi}_{\mathbf{k}} = \mathcal{N}_{\mathbf{k}}(\alpha)\exp\left\{-\frac{1}{2}\Omega_{\mathbf{k}}(\alpha)v_{\mathbf{k}}^2 - \frac{\kappa}{4}\Gamma_{\mathbf{k}}(\alpha)v_{\mathbf{k}}^4\right\} \ .$$

- The goal is to find out what $\Omega_{\mathbf{k}}(\alpha)$ and $\Gamma_{\mathbf{k}}(\alpha)$ are by inserting this ansatz into the corrected Schrödinger equation.

Corrections to primordial power spectra (cont'd)

- We find

$$\Omega_{\mathbf{k}}(\eta) = \Omega_{\mathbf{k};0}(\eta) + \kappa\Omega_{\mathbf{k};1}(\eta) \ ,$$

$$\Omega_{\mathbf{k};0}(\eta) = \frac{k^3\eta^2}{1+k^2\eta^2} + \frac{\mathrm{i}}{\eta(1+k^2\eta^2)} \ , \ \text{(usual Bunch-Davies result)}$$

$$\Omega_{\mathbf{k};1}(\eta) = \frac{\mathrm{e}^{2\mathrm{i}\arctan(k\eta)}H_0^2\eta^2}{k\eta+\mathrm{i}}\left[\frac{10\mathrm{i}+6k\eta-3\mathrm{i}k^2\eta^2}{2(k\eta-\mathrm{i})(k\eta+\mathrm{i})} - \frac{4\Gamma(0,-4\mathrm{i}k\eta)}{(k\eta+\mathrm{i})}\mathrm{e}^{-4\mathrm{i}k\eta} - \frac{2\Gamma(0,-2\mathrm{i}k\eta)}{(k\eta-\mathrm{i})}\mathrm{e}^{-2\mathrm{i}k\eta}\right] \ ,$$

$$\Gamma_{\mathbf{k}}(\eta) = \frac{H_0^2\eta\left(4\mathrm{i}k^2\eta^2+4k\eta+\mathrm{i}\right)\mathrm{e}^{4\mathrm{i}\arctan(k\eta)}}{6\left(k^2\eta^2+1\right)^2} - \frac{8H_0^2\eta^4k^3\Gamma(0,-4\mathrm{i}k\eta)\mathrm{e}^{-4\mathrm{i}[k\eta-\arctan(k\eta)]}}{3\left(k^2\eta^2+1\right)^2} \ .$$

- Not very illuminating, but the power spectra are proportional to the conditional correlation function

$$\langle v_{\mathbf{k}}^2\rangle = \frac{\int \mathrm{d}v \ \tilde{\psi}^* v_{\mathbf{k}}^2\tilde{\psi}}{\int \mathrm{d}v \ \tilde{\psi}^*\tilde{\psi}} = \frac{1+\kappa\delta_{\mathbf{k}}}{2\mathfrak{Re}\Omega_{\mathbf{k};0}} \ .$$

where $1/\mathfrak{Re}\Omega_{\mathbf{k};0}$ is the usual Bunch-Davies result, whereas the correction term is

$$\delta_{\mathbf{k}} = -\frac{\mathfrak{Re}\Omega_{\mathbf{k};1}}{\mathfrak{Re}\Omega_{\mathbf{k};0}} - \frac{3\mathfrak{Re}\Gamma_{\mathbf{k}}}{2(\mathfrak{Re}\Omega_{\mathbf{k};0})^2} \ .$$

Superhorizon limit

- Following the standard procedure[13], we must evaluate the power spectra in the superhorizon limit $k\eta \to 0^-$, which is also equivalent to the late-time limit $\eta \to 0^-$ (end of inflation in quasi-de Sitter).
- Plugging in the previous functions, in this limit, we find

$$\delta_{\mathbf{k}}(\eta) = H_0^2 \left(\frac{k_\star}{k}\right)^3 \left[4 - 2\gamma_E - 2\log(-2k\eta)\right] \,,$$

where $k_\star$ is the pivot scale (a reference scale used in the CMB data analysis).

- This is different from the result of[13], because the would-be unitarity violating terms, which were neglected there, were taken into account here.
- One can explicitly check that, with these results, the norm of $\tilde{\psi}_{\mathbf{k}}(\eta, v)$ is conserved, so the theory is unitary.

[13] See, for example, D. Brizuela, C. Kiefer and M. Krämer, Phys. Rev. D **93** 104035 (2016).

Late-time large logarithms

- The logarithmic correction term in

$$\delta_{\mathbf{k}}(\eta) = H_0^2 \left(\frac{k_\star}{k}\right)^3 \left[4 - 2\gamma_E - 2\log(-2k\eta)\right] \; ,$$

  diverges as $k\eta \to 0^-$. This might jeopardize the validity of perturbation theory and the interpretation of the corrections. How do we deal with this?
- Such large logarithms are actually quite common in the computation of usual QFT corrections in de Sitter effective field theory.[14] There, one uses the **dynamical renormalization group** techniques to resum the leading large logarithms such that the validity of perturbation theory is restored. It remains to be seen whether this can be done also in the BO expansion (future work).
- (Future work) Other possibilities to cure the log: (1) different ansatz for $\tilde{\psi}_{\mathbf{k}}$; (2) take into account the interaction between different Fourier modes.

[14] See, for example, T. Cohen and D. Green, arXiv:2007.03693 [hep-th] and references therein.

Conclusions and Outlook

- We have argued that
  - Weak-coupling expansion corrections are unitary
  - Inner product may be related to a notion of gauge-fixing the time variable
  - Probabilities may be interpreted “relationally” as conditional probabilities

- Correction terms are different from previous work due to the inclusion of would-be unitarity-violating terms

- Large logarithms appear at late times, which may:
  - enhance the size of the corrections (good for observability)
  - invalidate perturbation theory (so some resummation technique is needed – future work).