An introduction to Trace Dynamics

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Incompleteness of QM

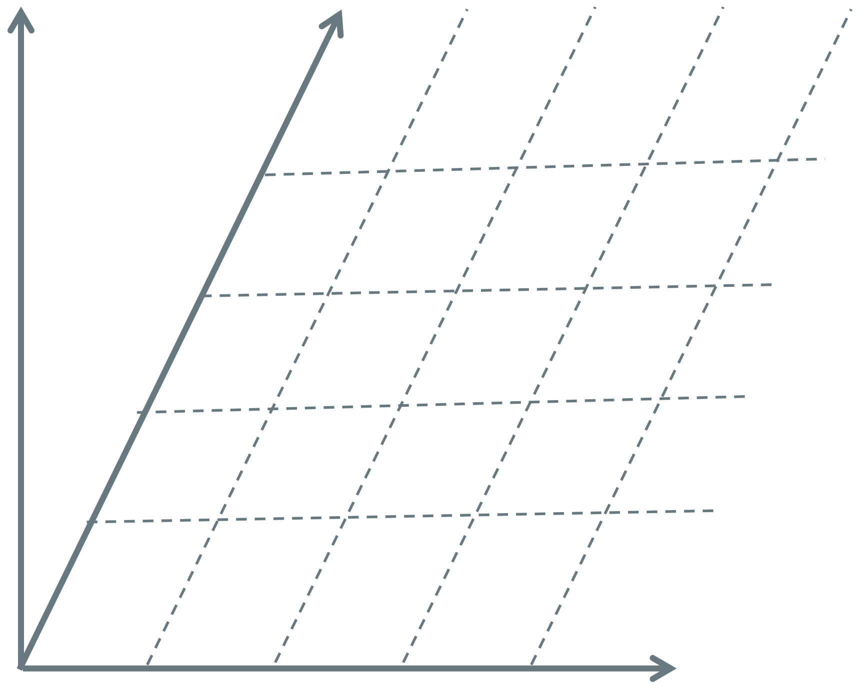
Trace Dynamics

Classical Dynamics

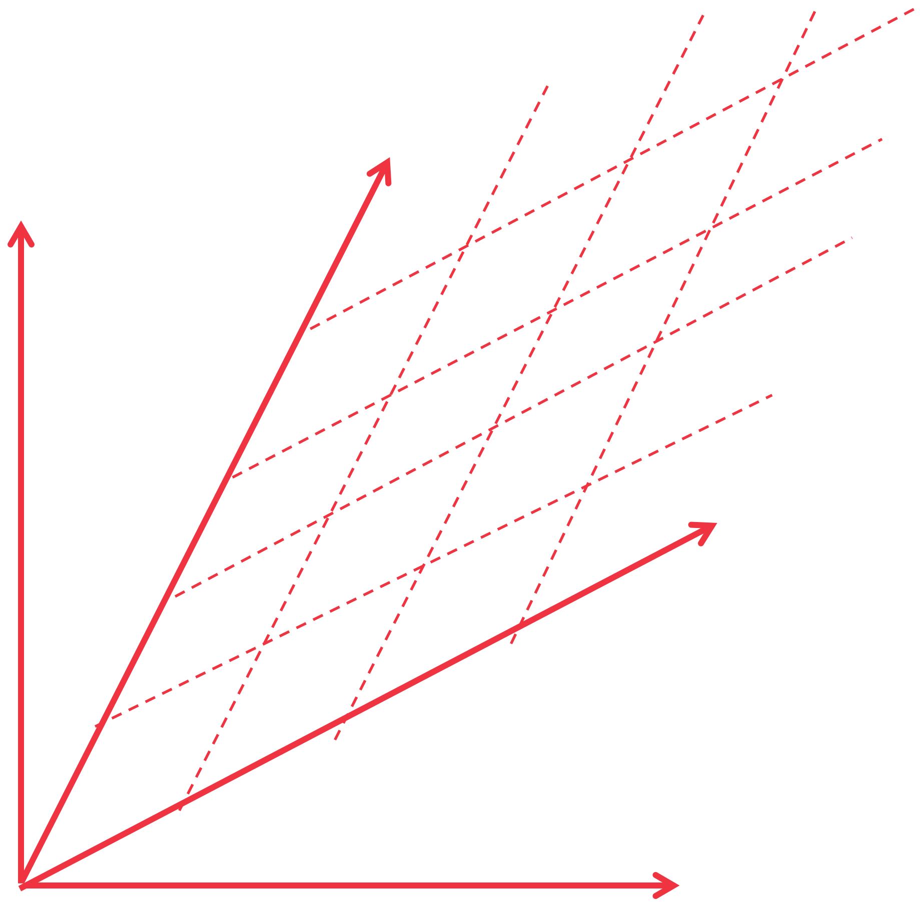
Statistical Thermodynamics

Statistical Fluctuations

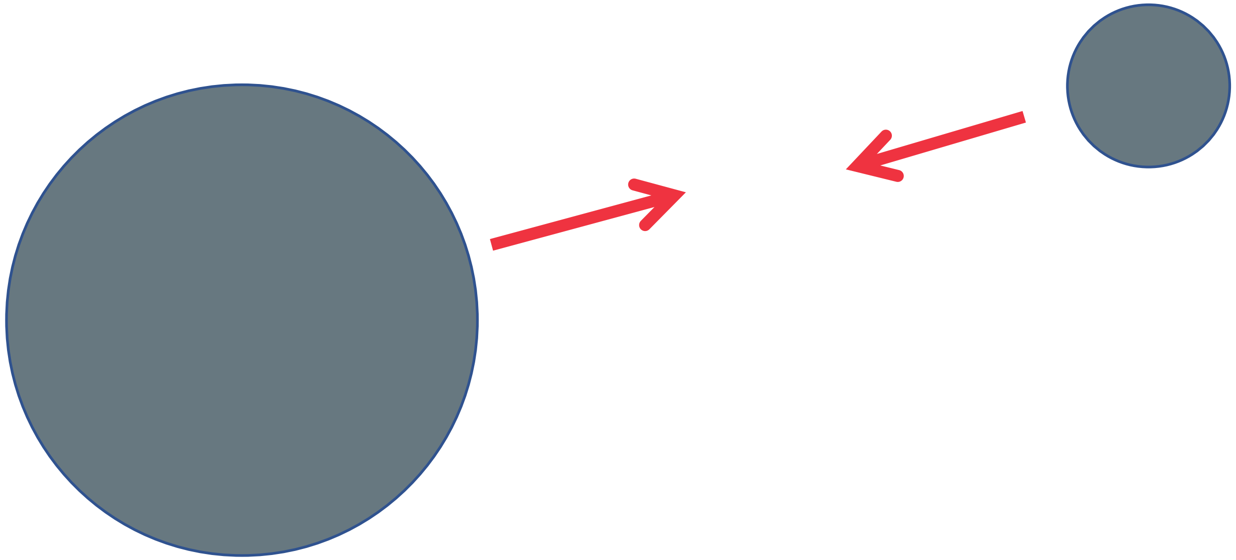
Incompleteness of QM



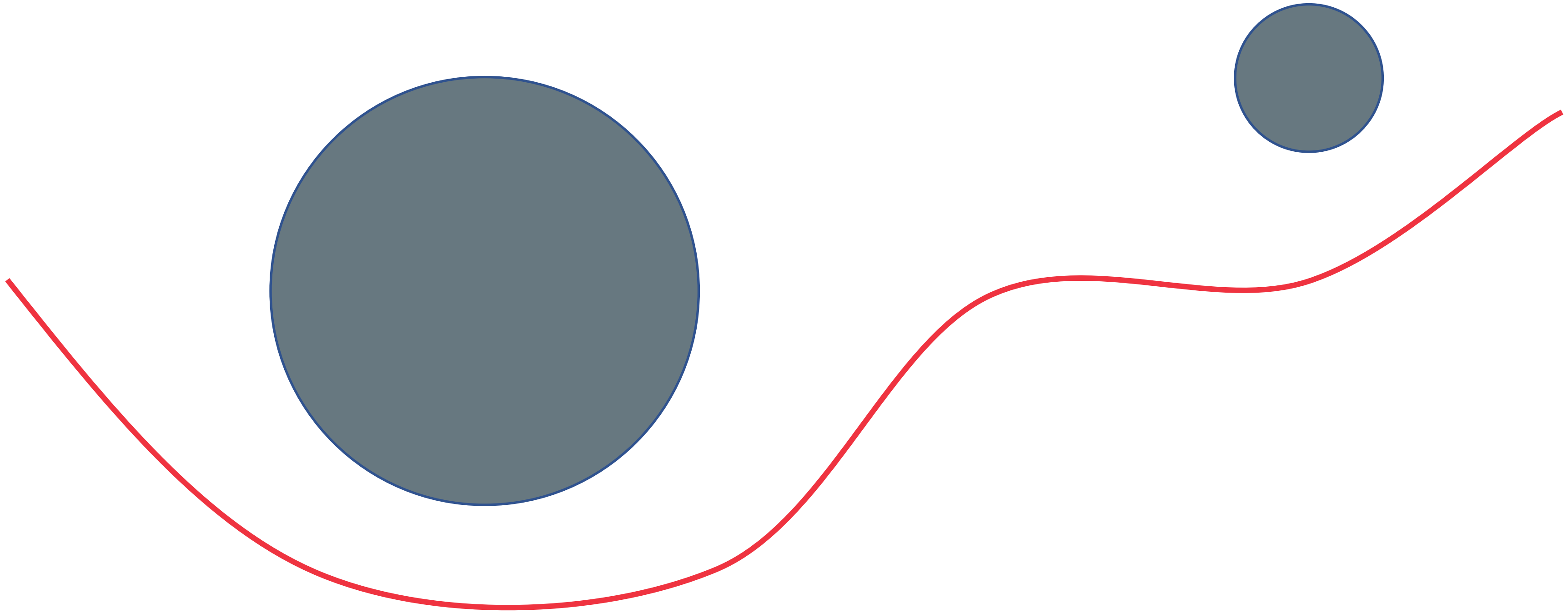
Incompleteness of QM



Incompleteness of QM



Incompleteness of QM



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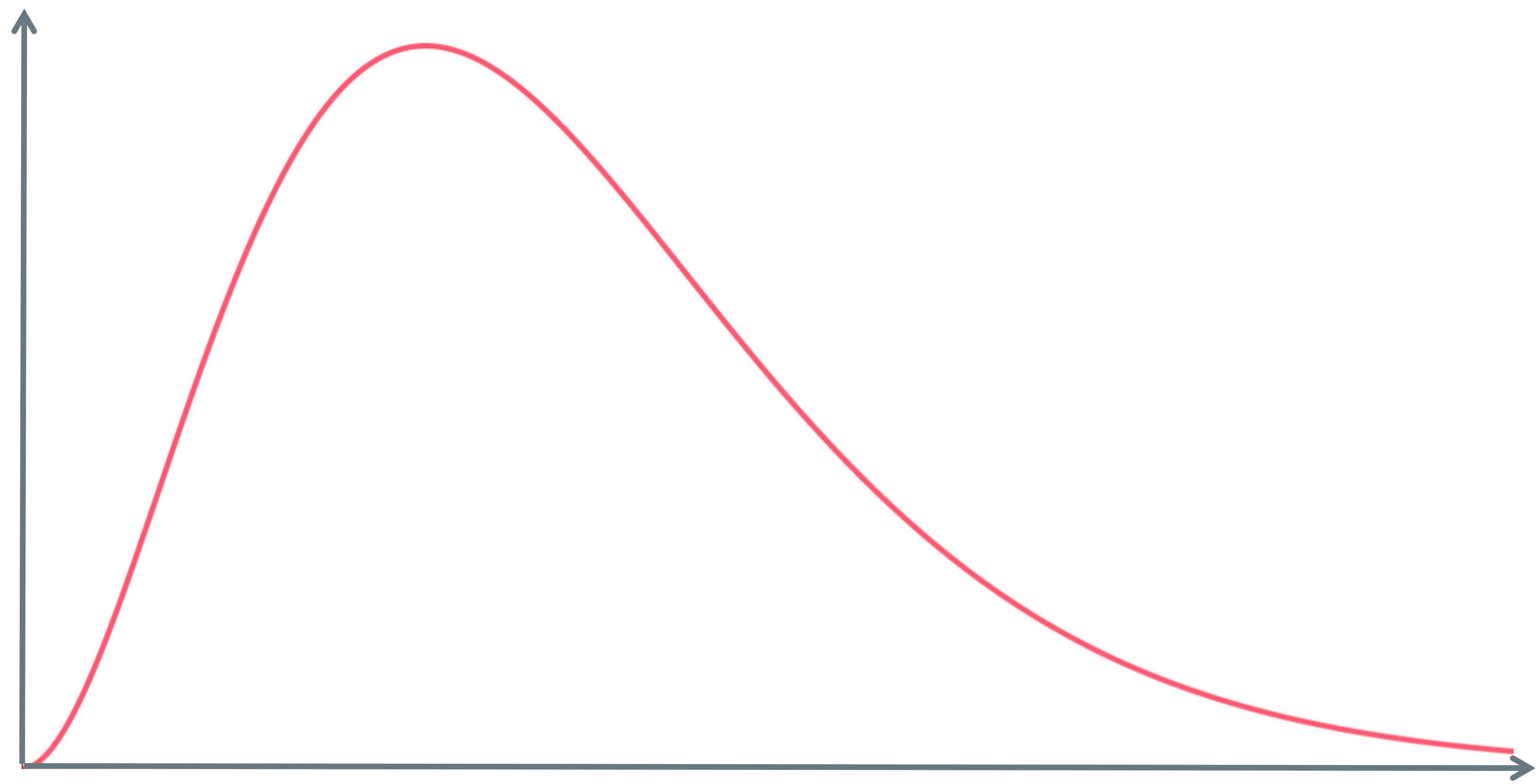


Incompleteness of QM

$$|dead\rangle + |alive\rangle$$

Incompleteness of QM

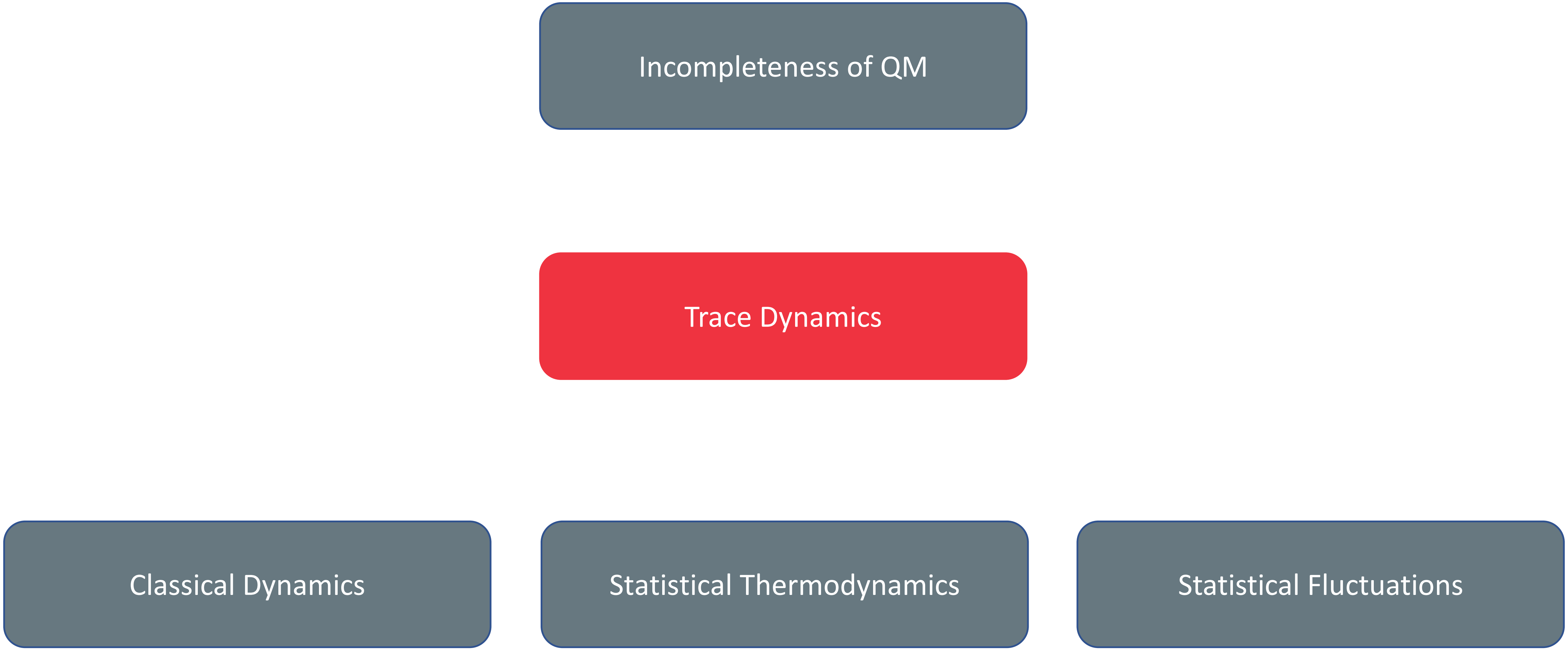
$$\dot{q} = \{q, H\}$$

$$\dot{p} = \{p, H\}$$

$$\dot{q} = \frac{-i}{\hbar}[q, H]$$

$$\dot{p} = \frac{-i}{\hbar}[p, H]$$

Incompleteness of QM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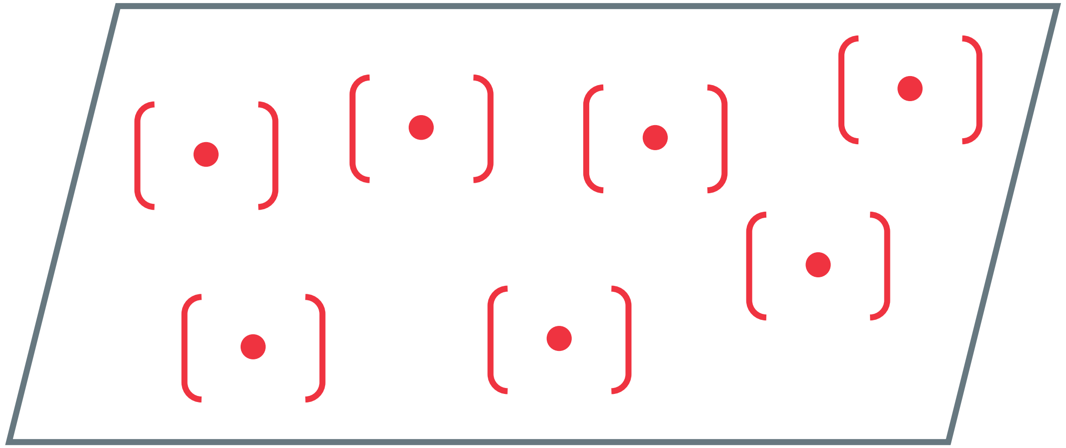
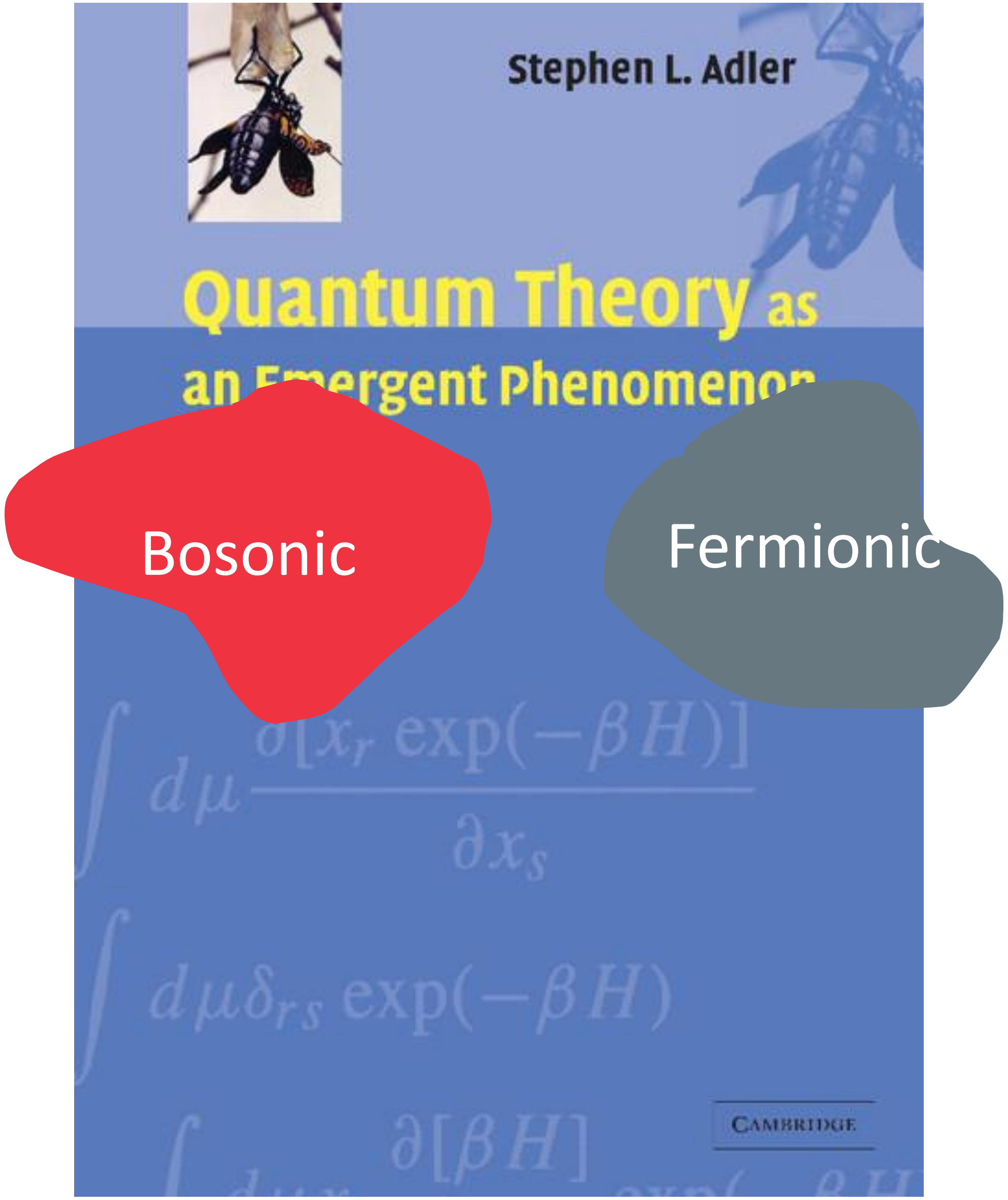
Fundamental degrees of freedom

Grassmann Numbers

$\theta_i\theta_j + \theta_j\theta_i = 0$ $\qquad$ $\theta_i^2 = 0$

$\chi = \theta_R + i\theta_I$ $\qquad$ $\{\chi_r, \chi_s\} = 0$





Trace Derivative

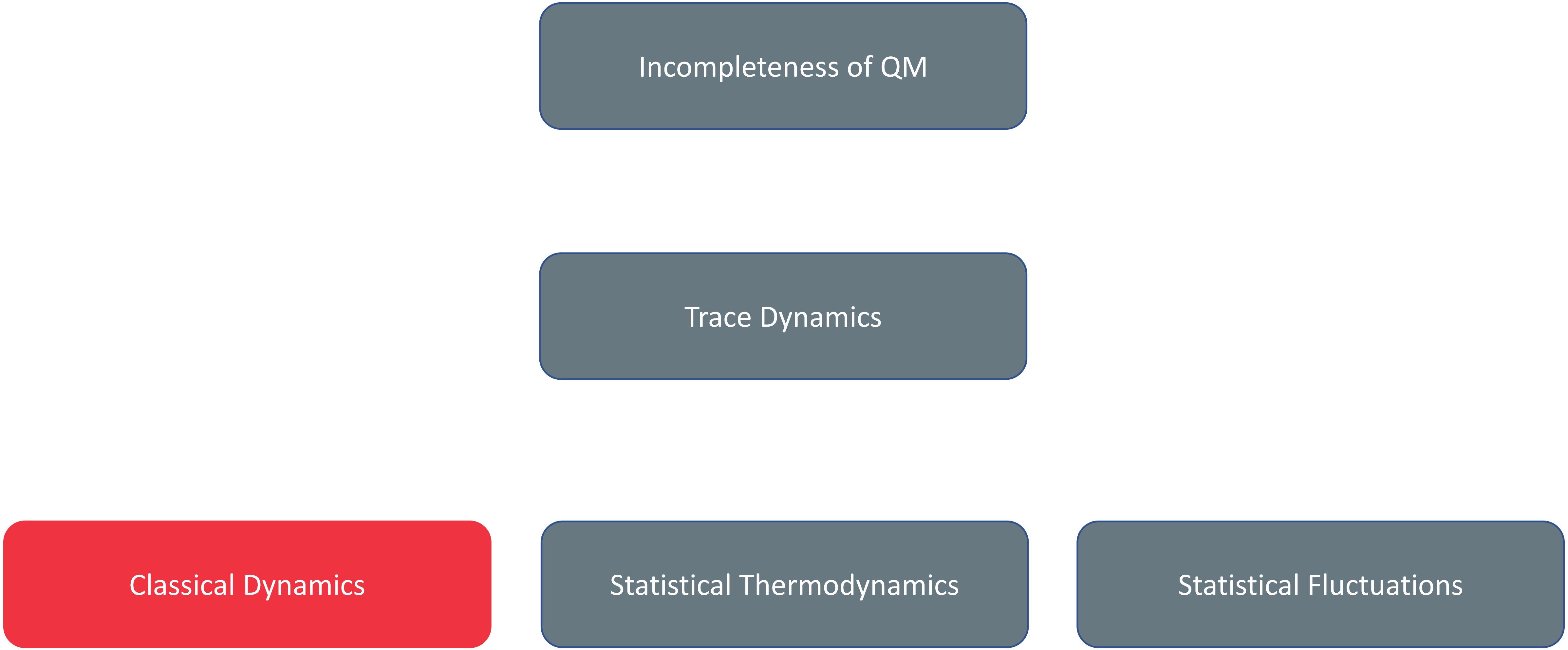
$$P = A\mathcal{O}B\mathcal{O}C$$

$$\delta \mathbf{P} = \mathrm{Tr}\frac{\delta \mathbf{P}}{\delta \mathcal{O}}\delta \mathcal{O} \qquad \frac{\delta \boldsymbol{P}}{\delta \mathcal{O}} \qquad where, \boldsymbol{P} = TrP$$

$$\delta P = A(\delta \mathcal{O})B\mathcal{O}C + A\mathcal{O}B(\delta \mathcal{O})C$$

$$\delta \mathrm{Tr} P = \mathrm{Tr}\left(\epsilon_{A\mathcal{O}} B\mathcal{O}CA(\delta \mathcal{O}) + \epsilon_C CA\mathcal{O}B(\delta \mathcal{O})\right)$$

$$\frac{\delta \mathbf{P}}{\delta \mathcal{O}} = \epsilon_{A\mathcal{O}} B\mathcal{O}CA + \epsilon_C CA\mathcal{O}B,$$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Lagrangian and Hamiltonian dynamics

$$\mathbf{L}[\{q_r\}, \{\dot{q}_r\}] = \mathrm{Tr} L[\{q_r\}, \{\dot{q}_r\}]$$

$$\mathbf{H} = \mathrm{Tr} \sum_r p_r \dot{q}_r - \mathbf{L}$$

$$\mathbf{S} = \int dt \mathbf{L}$$

$$\frac{\delta \mathbf{H}}{\delta q_r} = -\dot{p}_r, \quad \frac{\delta \mathbf{H}}{\delta p_r} = \epsilon_r \dot{q}_r$$

$$\frac{\delta \mathbf{L}}{\delta q_r} - \frac{d}{dt} \frac{\delta \mathbf{L}}{\delta \dot{q}_r} = 0$$

$$\{\mathbf{A}, \mathbf{B}\} = \mathrm{Tr} \sum_r \epsilon_r \left(\frac{\delta \mathbf{A}}{\delta q_r} \frac{\delta \mathbf{B}}{\delta p_r} - \frac{\delta \mathbf{B}}{\delta q_r} \frac{\delta \mathbf{A}}{\delta p_r} \right)$$

Conserved quantities

$$\frac{d}{dt}\mathbf{A} = \frac{\partial \mathbf{A}}{\partial t} + \{\mathbf{A}, \mathbf{H}\}$$

Trace Hamiltonian

$$\boldsymbol{H}$$

Fermion Number

$$\mathbf{N} = \frac{1}{2} i \mathrm{Tr} \sum_{r \in F} [q_r, p_r]$$

Adler-Millard Charge

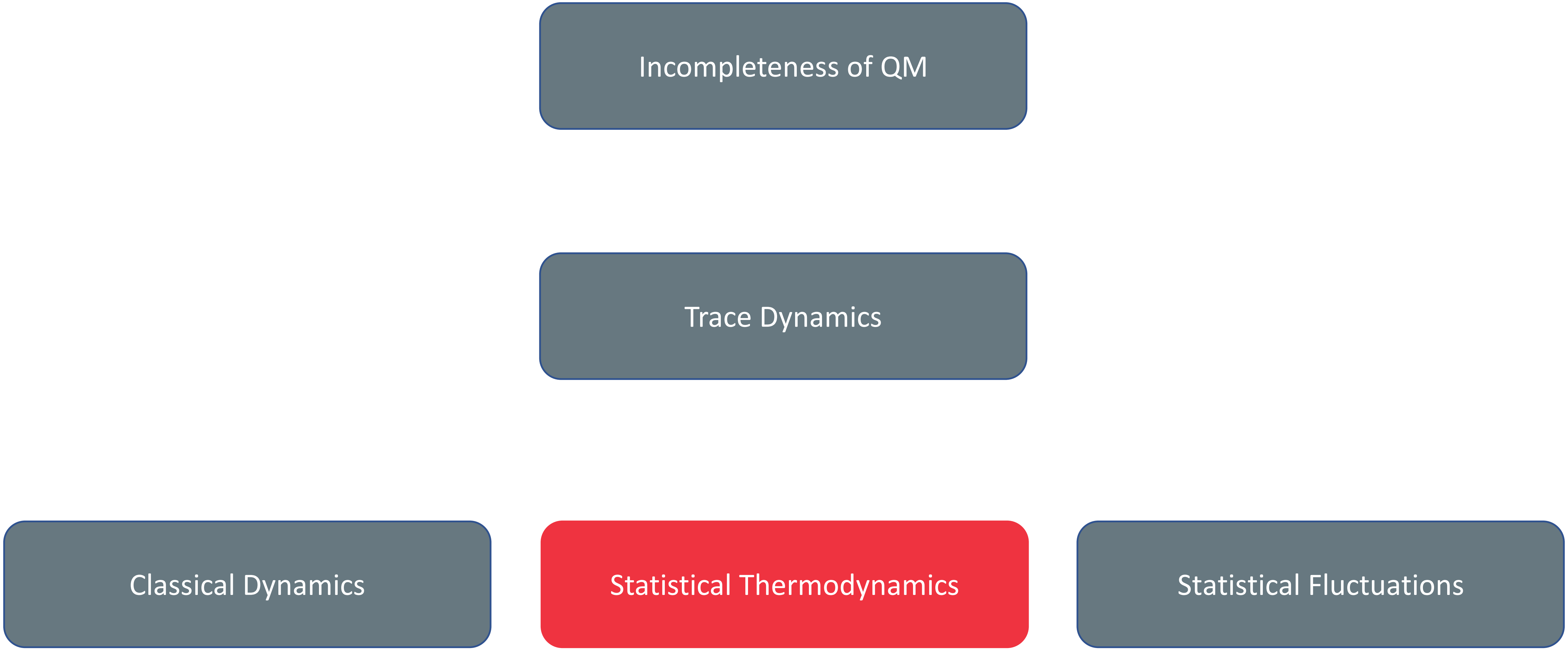
$$\tilde{C} \equiv \sum_{r \in B} [q_r, p_r] - \sum_{r \in F} \{q_r, p_r\}$$

Fermion number is conserved when there are equal numbers of fermionic canonical coordinate and momentum factors in each term in the trace Hamiltonian.

Global Unitary Invariance: When a trace Hamiltonian that is constructed from the matrix dynamical variables using only c-number coefficients Adler-Millard Charge is conserved.

$$\mathbf{L}[\{U^{\dagger}q_r U\}, \{U^{\dagger}\dot{q}_r U\}] = \mathbf{L}[\{q_r\}, \{\dot{q}_r\}] \qquad \mathbf{H}[\{U^{\dagger}q_r U\}, \{U^{\dagger}p_r U\}] = \mathbf{H}[\{q_r\}, \{p_r\}]$$

The Adler-Millard charge plays a vital role in the emergence of quantum dynamics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Operator Phase space

$$d\mu = \prod_{r,m,n} d(x_r)^A_{mn},$$

where $A = 0,\ 1$ for

$$(x_r)_{mn} = (x_r)^0_{mn} + i(x_r)^1_{mn}$$

This measure is invariant under canonical transformations (Louiville’s theorem)

Canonical Ensemble

- Assume large number of degrees of freedom evolving in phase space
- Assume Ergodic hypothesis

$$dP = d\mu[\{x_r\}]\rho[\{x_r\}]$$

- Lorentz invariance is broken; choice of trace Hamiltonian
- $\tilde{C}$ is Poincare invariant and is not affected by choice the trace Hamiltonian
- Canonical averages are determined by $\tilde{C}$ only and so the resulting quantum theory is Poincare invariant

$$\rho = \rho(\tilde{C}, \mathbf{H}, \mathbf{N})$$

$$\rho = \rho(\mathrm{Tr}\tilde{\lambda}\,\tilde{C}; \mathbf{H}, \tau; \mathbf{N}, \eta)$$

Ensemble average of an operator O

$$\langle O \rangle = \int d\mu \rho O$$

$\langle \tilde{C} \rangle$ can be written in terms of real diagonal and non-negative operator D_{eff} and a diagonal phase i_{eff}

$$\langle \tilde{C} \rangle = i_{eff} D_{eff}$$

If the ensemble does not favour any one state over the other then

$$\langle \tilde{C} \rangle = i_{eff} \hbar$$

$$i_{\text{eff}} = i[\text{diag}(1, -1, 1, -1, \ldots, 1, -1)]$$

To obtain equilibrium distribution

$$S = \int d\mu \rho \log \rho,$$

subject to the constraints

$$\int d\mu \rho = 1, \qquad \int d\mu \rho \tilde{C} = \langle \tilde{C} \rangle,$$

$$\int d\mu \rho \mathbf{H} = \langle \mathbf{H} \rangle, \qquad \int d\mu \rho \mathbf{N} = \langle \mathbf{N} \rangle$$

$$\rho_j = Z_j^{-1} \exp\left(-\mathrm{Tr}\tilde{\lambda}\,\tilde{C} - \boldsymbol{\tau}\mathbf{H} - \eta\mathbf{N} - \sum_r \mathrm{Tr} j_r x_r\right)$$

$$Z_j = \int d\mu \exp\left(-\mathrm{Tr}\tilde{\lambda}\,\tilde{C} - \boldsymbol{\tau}\mathbf{H} - \eta\mathbf{N} - \sum_r \mathrm{Tr} j_r x_r\right)$$

Emergence of quantum theory

- Conserved charges in trace dynamics play a role analogous to that of energy in classical statistical physics
- Using this and the equipartition theorem, a general Ward identity is derived for the canonical ensemble
- At low energies we then obtain an effective quantum field dynamics

$$x_r \longrightarrow x_r + \delta x_r,$$

$$\int d\hat{\mu}\, \delta_{x_r}(\rho_j O) = 0$$

$$O = \{\tilde{C}, i_{\text{eff}}\} W$$

$$\langle \Lambda_{u_{\text{eff}}} \rangle_j = \Bigg\langle \Bigg(-\tau \dot{x}_{u\text{eff}} + i\eta \xi_u x_{u\text{eff}} - \sum_s \omega_{us} j_{s\text{eff}} \Bigg) \\ \times \operatorname{Tr} \tilde{C} i_{\text{eff}} W_{\text{eff}} + [i_{\text{eff}} W_{\text{eff}}, x_{u\text{eff}}] \\ + \sum_{s,l} \omega_{us} \epsilon_l \Bigg(W_s^{Rl} \frac{1}{2} \{\tilde{C}, i_{\text{eff}}\} W_s^{Ll} \Bigg)_{\text{eff}} \Bigg\rangle_j = 0$$

Assumptions

- The energy scale τ^{-1} is the Planck scale and we are working at much lower energies
- The chemical potential can be neglected
- The charge $\tilde{C}$ is replaced with its average value $\langle \tilde{C} \rangle = i_{eff}\hbar$
- $W = H$

$$\mathcal{D}x_{u\mathrm{eff}} = i_{\mathrm{eff}}[H_{\mathrm{eff}}, x_{u\mathrm{eff}}] - \hbar\dot{x}_{\mathrm{eff}}$$

$$W = \tilde{\sigma}_v x_v$$

$$\langle\!\langle q_{u\mathrm{eff}}, q_{v\mathrm{eff}} \rangle\!\rangle = \langle\!\langle p_{u\mathrm{eff}}, p_{v\mathrm{eff}} \rangle\!\rangle = 0,$$

$$\langle\!\langle q_{u\mathrm{eff}}, p_{v\mathrm{eff}} \rangle\!\rangle = i_{\mathrm{eff}}\hbar\delta_{uv},$$

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- Consider fluctuations in the Adler-Millard charge

$$\Delta\tilde{C} \simeq \tilde{C} - i_{\text{eff}}\hbar$$
$$= -\hbar(\mathcal{K} + \mathcal{N}),$$

$$|\dot{\Phi}\rangle = (i\hbar^{-1}\{-1 + [\mathcal{K}_0(t) + i\mathcal{K}_1(t)]\}H_{\text{eff}} + \tfrac{1}{2}i[\mathcal{M}_0(t) + i\mathcal{M}_1(t)])|\Phi\rangle,$$

Stochastic,
Linear

$$\mathcal{M}(t) = \sum_{r,l} m_r \mathcal{N}(t)_{r,l}$$

- Considering c-number fluctuations in terms of white noise we get

$$|d\Phi\rangle = \Big[-i\hbar^{-1} H_{\rm eff} dt + i\beta_I dW_t^I H_{\rm eff} + \beta_R dW_t^R H_{\rm eff} + i \int d^3x dW_t^I(\vec{x}) \mathcal{M}_t^I(\vec{x}) + \int d^3x dW_t^R(\vec{x}) \mathcal{M}_t^R(\vec{x}) \Big] |\Phi\rangle.$$

This stochastic differential equation is linear and does not preserve norm

We impose norm preservation by hand and get a nonlinear stochastic modification of the Schrodinger equation

$$
\begin{aligned}
d|\Psi\rangle = \Big[& -i\hbar^{-1} H_{\mathrm{eff}}\, dt + i\beta_I H_{\mathrm{eff}}\, dW_t^I - \frac{1}{2}[\beta_I^2 H_{\mathrm{eff}}^2 \\
& + \beta_R^2 (H_{\mathrm{eff}} - \langle H_{\mathrm{eff}}\rangle)^2] dt + \beta_R (H_{\mathrm{eff}} - \langle H_{\mathrm{eff}}\rangle)^2 dW_t^R \\
& + i \int d^3x\, \mathcal{M}^I(\vec{x}) dW_t^I(\vec{x}) - \frac{\gamma}{2} dt \int d^3x \{\mathcal{M}^I(\vec{x})^2 \\
& + [\mathcal{M}^R(\vec{x}) - \langle \mathcal{M}^R(\vec{x})\rangle]^2\} + \int d^3x [\mathcal{M}^R(\vec{x}) \\
& - \langle \mathcal{M}^R(\vec{x})\rangle]^2 dW_t^R(\vec{x}) \Big] |\Psi\rangle.
\end{aligned}
$$

The CSL model can be obtained by choosing

$$M(\mathbf{x}) = \sum_j m_j N_j(\mathbf{x}),$$

$$N_j(\mathbf{x}) = \int d\mathbf{y} g(\mathbf{y} - \mathbf{x}) \psi_j^\dagger(\mathbf{y}) \psi_j(\mathbf{y}),$$