Investigating some aspects of non-minimal coupling in the context of dynamical stability approach

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Theme of today's talk

Works based on Non-minimal coupling: Algebraic Type

arXiv > gr-qc > arXiv:2105.00361

General Relativity and Quantum Cosmology

[Submitted on 2 May 2021 (v1), last revised 14 Oct 2021 (this version, v4)]

Dynamical stability of K-essence field interacting non-minimally with a perfect fluid

Anirban Chatterjee, Saddam Hussain, Kaushik Bhattacharya

We study models of non-minimally coupled relativistic fluid and k-essence scalar field in the background of a flat Friedmann-Lemaitre-Robertson-Walker universe. The non-minimal coupling term is introduced in the Lagrangian level. We employ the variational approach with respect to independent variables that produce modified $k-$essence field equations and the Friedmann equations. We have analyzed the coupled field-fluid framework explicitly using the dynamical system technique considering two different models based on inverse power-law potential. After examining these models it is seen that both models are capable of producing accelerating attractor solutions satisfying adiabatic sound speed conditions.

Comments: Some figures are modified and some text changed. Has 27 pages, 10 figures. This version has been accepted for publication in Physical Review D
Subjects: **General Relativity and Quantum Cosmology (gr-qc)**; High Energy Physics - Theory (hep-th)
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PHYS. REV. D 104, 103505 (2021)

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General Relativity and Quantum Cosmology

[Submitted on 20 Mar 2022 (v1), last revised 22 Jan 2023 (this version, v3)]

Ghost condensates and pure kinetic k-essence condensates in presence of field-fluid non-minimal coupling in the dark sector

Saddam Hussain, Anirban Chatterjee, Kaushik Bhattacharya

In this article we try to find out the conditions when a ghost field in conjunction with a barotropic fluid produces a stable accelerating expansion phase of the universe. It is seen that in many cases the ghost field produces a condensate and drives the fluid energy density to zero in the final accelerating phase, but there can be other possibilities. We have shown that a pure kinetic k-essence field (which is not a ghost field) interacting with fluid can also form an interaction induced condensate and produce a stable accelerating phase of the universe. In the latter case the fluid energy density does not vanish in the stable phase.

Comments: Corrected and modified version. 25 pages, Accepted for publication in Universe
Subjects: **General Relativity and Quantum Cosmology (gr-qc)**; Cosmology and Nongalactic Astrophysics (astro-ph.CO); High Energy Physics - Phenomenology (hep-ph)
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Universe 2023, 9(2), 65

Work based on Non-minimal coupling: Derivative Type

arXiv > gr-qc > arXiv:2206.12398

General Relativity and Quantum Cosmology

[Submitted on 24 Jun 2022]

Dynamical Stability in presence of non-minimal derivative dependent coupling of k-essence field with a relativistic fluid

Kaushik Bhattacharya, Anirban Chatterjee, Saddam Hussain

In this paper we investigate a non-minimal, space-time derivative dependent, coupling between the k-essence field and a relativistic fluid using a variational approach. The derivative coupling term couples the space-time derivative of the k-essence field with the fluid 4-velocity via an inner product. The inner product has a coefficient whose form specifies the various models of interaction. By introducing a coupling term at the Lagrangian level and using the variational technique we obtain the k-essence field equation and the Friedmann equations in the background of a spatially flat Friedmann-Lemaitre-Robertson-Walker (FLRW) metric. Explicitly using the dynamical analysis approach we analyze the dynamics of this coupled scenario in the context of two kinds of interaction models. The models are distinguished by the form of the coefficient multiplying the derivative coupling term. In the simplest approach we work with an inverse square law potential of the k-essence field. Both of the models are not only capable of producing a stable accelerating solution, they can also explain different phases of the evolutionary universe.

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Subjects: **General Relativity and Quantum Cosmology (gr-qc)**; Cosmology and Nongalactic Astrophysics (astro-ph.CO); High Energy Physics - Theory (hep-th)
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arXiv:2206.12398
(Accepted for Publication in EPJC)

Plan for today's talk:

Theme of the presentation

- **Motivation** behind this non-minimally coupled sector.
- **Definition & Constituents** of Non-minimally coupled field-fluid sectors.
- **Theoretical Framework** of this coupled model.
- Essence of **Non-canonical** type scalar field (k-essence).
- Evolution of coupled system in Isotropic-Homogeneous universe (**FLRW background**).
- Techniques of **Dynamical Stability Analysis**.
- Comparative studies on **Three types of scenario** $\Rightarrow$ Algebraic coupling model (Inverse square law potential & constant type potential) & Derivative coupling model (Inverse square law potential).
- **Results & Discussion**.
- **Conclusion**.

Non-minimally coupled field-fluid scenario:

Based on: Phys. Rev. D **104** 103505 (2021) & Universe 2023, 9(2), 65 & arXiv:2206.12398

Motivation & Constituents:

- Can able to solve **cosmological coincidence problem** & Alleviate **Hubble Tension**.
- Explore interacting field-fluid model which can be derived from a **variational approach**.
- Field → **Non-canonical type (k-essence)**; Fluid → **Relativistic fluid**.

Non-minimal coupling:

- Einstein's equations: $R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = T^{\mathrm{tot.}}_{\mu\nu}$
- $T^{\mathrm{tot.}}_{\mu\nu} \Rightarrow T^{(R)}_{\mu\nu} + T^{(b)}_{\mu\nu} + T^{(\mathrm{M})}_{\mu\nu} + T^{(\phi)}_{\mu\nu}$
- At late-time cosmic evolution $\Rightarrow \cancelto{0}{T^{(R)}_{\mu\nu}}$ & $\cancelto{0}{T^{(b)}_{\mu\nu}}$.
- Conservation of total energy-momentum tensor $\Rightarrow \nabla^{\mu}\left(T^{(\mathrm{M})}_{\mu\nu} + T^{(\phi)}_{\mu\nu}\right) \approx 0$.
- For non-minimal coupling scenario,

$$\boxed{\nabla^{\mu} T^{(\phi)}_{\mu\nu} = -\nabla^{\mu} T^{(\mathrm{M})}_{\mu\nu} \equiv Q_{\nu}}$$

Theoretical Framework:

Total action of the coupled system:

$$\boxed{S = \int_\Omega d^4x \left[\sqrt{-g}\,\frac{R}{2\kappa^2} - \sqrt{-g}\,\rho(n,s) + J^\mu(\varphi_{,\mu} + s\theta_{,\mu} + \beta_A \alpha^A_{,\mu}) - \sqrt{-g}\,\mathcal{L}(\phi, X) \;+ S_{\rm int}\right]}$$

- **1st term** $\rightarrow$ Gravitational part of the action.
- **2nd & 3rd term** $\rightarrow$ Action for a perfect fluid.
- **4th term** $\rightarrow$ Action for the k-essence scalar field.
- $S_{\rm int} \Longrightarrow$ I. **Algebraic Coupling:** $-\sqrt{-g}\; f(n, s, \phi, X)$

  II. **Derivative Coupling:** $f(n, s, \phi, X) J^\mu \partial_\mu \phi$

Details on Fluid Sector:

- **Current Density (J^μ)** $\Rightarrow \sqrt{-g}\, n u^\mu$.
- **Velocity four vector (u^μ)** $\Rightarrow u^\mu u_\mu = -1$.
- **Energy-Momentum Tensor of fluid ($T^{(M)}_{\mu\nu}$)** $\Rightarrow \rho u_\mu u_\nu + \left(n\dfrac{\partial \rho}{\partial n} - \rho\right)(u_\mu u_\nu + g_{\mu\nu})$
- **Pressure & energy density (Fluid)** $\Rightarrow P_M = \left(n\dfrac{\partial \rho}{\partial n} - \rho\right)$

I. Theoretical Framework: Algebraic Coupling Model

Details on Field Sector:

- **Modified field equation** $\Rightarrow \mathcal{L}_{,\phi} + \nabla_\mu(\mathcal{L}_{,X}\nabla^\mu\phi) + f_{,\phi} + \nabla_\mu(f_{,X}\nabla^\mu\phi) = 0$
- **Energy-Momentum Tensor of field** ($T^{(\phi)}_{\mu\nu}$) $\Rightarrow -\mathcal{L}_{,X}\,(\partial_\mu\phi)(\partial_\nu\phi) - g_{\mu\nu}\,\mathcal{L}$
- **Pressure & energy density (field)** $\Rightarrow \rho_\phi = \mathcal{L} - 2X\mathcal{L}_{,X}$ **and** $P_\phi = -\mathcal{L}$

Details on Interacting Sector (Field & Fluid):

- **Energy-Momentum Tensor of Int. sector** ($T^{(\text{int})}_{\mu\nu}$) $\Rightarrow n\dfrac{\partial f}{\partial n}u_\mu u_\nu + \left(n\dfrac{\partial f}{\partial n} - f\right)g_{\mu\nu} - f_{,X}(\partial_\mu\phi)(\partial_\nu\phi)$
- **Pressure & energy density (Int.)** $\Rightarrow \rho_{\text{int}} = f - 2Xf_{,X} \quad P_{\text{int}} = \left(n\dfrac{\partial f}{\partial n} - f\right)$
- **Total energy momentum tensors** $\Rightarrow T^{\text{tot.}}_{\mu\nu} = T^{(\phi)}_{\mu\nu} + T^{(M)}_{\mu\nu} + T^{(\text{int})}_{\mu\nu}$
- **Conservation of Total energy momentum tensors** $\Rightarrow \boxed{\nabla^\mu T^{\text{tot.}}_{\mu\nu} = 0.}$
- **Interaction term** $\Rightarrow f_{,\phi}\partial_\nu\phi - f_{,X}\partial_\mu\phi(\nabla^\mu\nabla_\nu\phi)$

CGB, NT and MW, Phys. Rev. D 91 (2015)

II. Theoretical Framework: Derivative Coupling Model

Details on Field Sector:

- **Modified field equation** $\Rightarrow -\mathcal{L}_{,\phi} - \nabla_\mu \left(\mathcal{L}_{,X}\partial^\mu\phi\right)$
  $+ f_{,\phi}\, nU^\alpha\partial_\alpha\phi + \nabla_\mu \left(f_{,X}(\partial^\mu\phi)\right)(nU^\alpha\partial_\alpha\phi) + f_{,X}(\partial^\mu\phi)\nabla_\mu(nU^\alpha\partial_\alpha\phi) - (\nabla_\mu f)(nU^\mu)$
- **Energy-Momentum Tensor of field** $(T^{(\phi)}_{\mu\nu}) \Rightarrow -\mathcal{L}_{,X}(\partial_\mu\phi)(\partial_\nu\phi) - g_{\mu\nu}\,\mathcal{L}$
- **Pressure & energy density (field)** $\Rightarrow \rho_\phi = \mathcal{L} - 2X\mathcal{L}_{,X}$ **and** $P_\phi = -\mathcal{L}$

Details on Interacting Sector (Field & Fluid):

- **Energy-Momentum Tensor of Int. sector**
  $$(T^{(\mathrm{int})}_{\mu\nu}) \Rightarrow \left[-n^2\frac{\partial f}{\partial n}\left[U^\mu U^\nu + g^{\mu\nu}\right] + n\frac{\partial f}{\partial X}(\partial^\mu\phi\partial^\nu\phi)\right]U^\alpha\partial_\alpha\phi$$
- **Pressure & energy density (Int.)** $\Rightarrow \rho_{\mathrm{int}} = nf_{,X}\dot{\phi}^3 \quad P_{\mathrm{int}} = (nf_{,n} - f)$
- **Total energy momentum tensors** $\Rightarrow T^{\mathrm{tot.}}_{\mu\nu} = T^{(\phi)}_{\mu\nu} + T^{(M)}_{\mu\nu} + T^{(\mathrm{int})}_{\mu\nu}$
- **Conservation of Total energy momentum tensors** $\Rightarrow \boxed{\nabla^\mu T^{\mathrm{tot.}}_{\mu\nu} = 0.}$
- **Interaction term** $\Rightarrow \left[f_{,n}(\nabla_\mu n) + f_{,X}(\nabla_\mu X)\right](nU^\mu)\partial^\nu\phi - f_{,X}(\nabla^\nu X)(nU^\alpha\partial_\alpha\phi)$

Details of k-essence Model:

- A Lagrangian with **non-canonical kinetic terms** expressed as $\boxed{L = V(\phi)F(X)}$ with **Kinetic term** $\boxed{X = \frac{1}{2}g^{\mu\nu}\nabla_\mu\phi\nabla_\nu\phi}$, $g^{\mu\nu}$ is the metric, $V(\phi)$ and $F(X)$ are functions of ϕ and X respectively.
- In the background of **FLRW space-time**, k-essence scalar field $\boxed{\phi(t, \vec{x}) = \phi(t)}$.
  Kinetic term $\rightarrow$ $\boxed{X = \frac{1}{2}\dot{\phi}^2}$.
- Stress-energy tensor is equivalent to that of an ideal fluid with **Energy density** $\boxed{\rho = V(\phi)(2XF_{,X} - F)}$ and **Pressure** $\boxed{p = V(\phi)F(X)}$.
- **EOM for k-essence sector** $\rightarrow$ $\boxed{(F_{,X} + 2XF_{,XX})\ddot{\phi} + 3HF_{,X}\dot{\phi} + (2XF_{,X} - F)\frac{V_\phi}{V} = 0}$.
- For **constant potential** and **homogeneous scalar field** in **FLRW background** ensure the **scaling relation** $\rightarrow$ $\boxed{XF_{,X}^2 = Ca^{-6}}$, where C is a constant & $\boxed{F_{,X} = \frac{dF}{dX}}$.

M. Born and L. Infeld, Proc.Roy.Soc.Lond A144(1934)

C. Armendariz-Picon & V.F. Mukhanov Phys. Rev. Lett. 85, 4438–4441 (2000)

Coupled system in FLRW Background: Algebraic & Derivative Coupling

Modified Friedmann Equations:

$3H^2 = \kappa^2(\rho_{\mathrm{M}} + \rho_\phi + \rho_{\mathrm{int}})$ & $2\dot{H} + 3H^2 = -\kappa^2(P_{\mathrm{M}} + P_\phi + P_{\mathrm{int}})$

Conserving Quantities:

Particle number density $\Rightarrow$ $\nabla_\mu(nu^\mu) = 0 \Rightarrow \dot{n} + 3Hn = 0$ & **Entropy** $\Rightarrow$ $\nabla_\mu(nsu^\mu) = 0 \Rightarrow \dot{s} = 0$

Coupled system in Algebraic Coupling:

Modified field equation in FLRW background for arbitrary potential (PHYS. REV. D 104, 103505 (2021))

$$[\mathcal{L}_{,\phi} + f_{,\phi}] - 3H\dot{\phi}\,[\mathcal{L}_{,X} + f_{,X}] + \frac{\partial}{\partial X}(P_{\mathrm{int}} + f)(3H\dot{\phi})$$
$$-\ddot{\phi}\,[(\mathcal{L}_{,X} + f_{,X}) + 2X(\mathcal{L}_{,XX} + f_{,XX})] - \dot{\phi}^2(\mathcal{L}_{,\phi X} + f_{,\phi X}) = 0$$

Modified field equation in FLRW background for constant potential [Universe 2023, 9(2), 65]

$$-3H\dot{\phi}\,[\mathcal{L}_{,X} + f_{,X}] + \frac{\partial}{\partial X}(P_{\mathrm{int}} + f)(3H\dot{\phi})$$
$$-\ddot{\phi}\,[(\mathcal{L}_{,X} + f_{,X}) + 2X(\mathcal{L}_{,XX} + f_{,XX})] = 0$$

Coupled system in Derivative Coupling for arbitrary potential[arXiv:2206.12398 (Accepted for Publication in EPJC)]:

$$-\mathcal{L}_{,\phi} + \mathcal{L}_{,\phi X}\dot{\phi}^2 + \mathcal{L}_{,XX}\dot{\phi}^2\ddot{\phi} + 3H\mathcal{L}_{,X}\dot{\phi} + \mathcal{L}_{,X}\ddot{\phi}$$
$$+\left[3nHf_{,nX}\dot{\phi} - f_{,\phi X}\dot{\phi}^2 - f_{,XX}\dot{\phi}^2\ddot{\phi} - f_{,X}\ddot{\phi}\right]n\dot{\phi}$$
$$-nf_{,X}\dot{\phi}\ddot{\phi} + 3n^2Hf_{,n} - nf_{,X}\dot{X} = 0$$

Technique of Dynamical Stability Analysis:

Motivation & Techniques:

- Apply to any physical system **evolving with time**.
- To investigate the coupled system behavior from **early to late time phase** of the evolution.
- For continuous and finite system, x_i variables that define the dynamical system, expressed as $\frac{dx_i}{dt} = f_i(x_1, x_2, ..., x_i)$.
- Above equation is autonomous equations and fixed or critical points exist at $x_i = y_0$ for $f_i(y_0) = 0$.
- To check stability of the critical points $\rightarrow$ **Jacobian matrix** $\Rightarrow \mathcal{J}_{ij} = \frac{\partial f_i}{\partial x_j}$.
- Eigenvalue at the critical point of Jacobian matrix $\Rightarrow$ stability of the critical points.
- Sign. of eigenvalues (positive) $\Rightarrow$ **unstable / saddle critical points** & Sign. of eigenvalues (negative) $\Rightarrow$ **stable critical points**.
- For $n \times n$ Jacobian matrix, n eigenvalues exist.

CGB, NT and MW, Phys. Rev. D 91 (2015)

AC, SH and KB, Phys. Rev. D 104 (2021)

Algebraic coupling: 3-D & 2-D Autonomous System

3-D System (PHYS. REV. D 104, 103505 (2021))

- Dimensionless variables:
  $x = \dot{\phi}$, $y = \frac{\kappa\sqrt{V(\phi)}}{\sqrt{3}H}$, $z = \frac{\kappa^2 f}{3H^2}$, $\sigma = \frac{\kappa\sqrt{\rho}}{\sqrt{3}H}$,
  $B = \frac{f_{,\phi}k^2}{H^3}$, $C = \frac{\kappa^2 P_{\rm int}}{3H^2}$, $D = \frac{\kappa^2}{3H^2}f_{,X}$,
  $E = \frac{\kappa^2}{H^3}\frac{\partial^2 f}{\partial\phi\partial X}$, $\lambda = -\frac{V_{,\phi}}{\kappa V^{3/2}}$.
- Constraint Eqn:
  $\sigma^2 = 1 - y^2\left(\frac{3}{4}x^4 - \frac{1}{2}x^2\right) - z + x^2 D$.
- Friedmann's Eqn:
  $\frac{\dot{H}}{H^2} = -\frac{3}{2}[\omega\sigma^2 + y^2 F + C + 1]$.
- Other variables:
  $\Omega_\phi = y^2(x^2 F_{,X} - F)$, $\Omega_{\rm int} = z - x^2 D$
- Critical Points:
  $x' = y' = z' = 0$. Prime denotes the derivative of the dynamical variables x, y, z with respect to Hdt.
- Chosen Forms:
  $F(X) = X^2 - X$ & $V(\phi) = \frac{\delta^2}{\kappa^2\phi^2}$
  ($\phi \to$ k-essence scalar field, $\delta \to$ model parameter).
- Form of Interaction:
  $f = \alpha\rho^\epsilon(\frac{\phi}{\kappa})X$ & $f = \alpha\rho(\frac{\phi}{\kappa})^m X^n$.
  (ϵ, m, $n \to$ Model parameters).
- Study in **matter dominated ($\omega = 0$) background**.

2-D System [Universe 2023, 9(2), 65]

- Dimensionless variables:
  $x = \dot{\phi}$, $\sigma = \frac{\kappa\sqrt{\rho}}{\sqrt{3}H}$, $y = \frac{\kappa^2 f}{3H^2}$, $z = \frac{H_0}{H}$,
  $C = \frac{\kappa^2 P_{\rm int}}{3H^2}$, $D = \frac{\kappa^2 f_{,X}}{3H^2}$, $\alpha = \frac{\kappa^2 V_0}{H_0^2}$.
- Constraint Eqn:
  $\sigma^2 = 1 - \frac{\alpha z^2}{3}(x^2 F_{,X} - F) - y + x^2 D$.
- Friedmann's Eqn:
  $\frac{\dot{H}}{H^2} = -\frac{3}{2}\left(\omega\sigma^2 + \frac{\alpha z^2}{3}F + C + 1\right)$.
- Other variables:
  $\Omega_\phi \equiv \frac{\alpha z^2}{3}(x^2 F_{,X} - F)$, $\Omega_{\rm int} \equiv y - x^2 D$
- Critical Points:
  $x' = z' = 0$. Prime denotes the derivative of the dynamical variables x, z with respect to Hdt.
- Chosen Forms:
  $F(X) = AX^2 + BX$ & $V(\phi) = V_0$ (Const.).
- Form of Interaction:
  $f = g\rho X^\beta$ & $f = gV_0\rho^q X^\beta M^{-4q}$
  (g, β, q, $V_0 \to$ Model parameters).
- Study in the context of **matter dominated ($\omega = 0$) background**.

Comparative study: Autonomous Equations of Algebraic coupling

Autonomous Equations in 3-D system (PHYS. REV. D 104, 103505 (2021))

$$
\begin{aligned}
x' = \dot{x}/H &= \frac{(B/3 + \sqrt{3}\lambda y^3 F) + 3x\left(y^2 F_{,X} + C_{,X}\right) - x^2(E/3 + \sqrt{3}\lambda y^3 F_{,X})}{[(D - y^2 F_{,X}) + x^2(D_{,X} - y^2 F_{,XX})]} \\
y' = \dot{y}/H &= -\frac{\sqrt{3}\lambda y^2 x}{2} + \frac{3}{2} y \left[\omega\sigma^2 + y^2 F + C + 1\right] \\
z' = \dot{z}/H &= \left[-3(C+z) + \frac{B}{3}x + Dx\, x'\right] + 3z\left[\omega\sigma^2 + y^2 F + C + 1\right]
\end{aligned}
$$

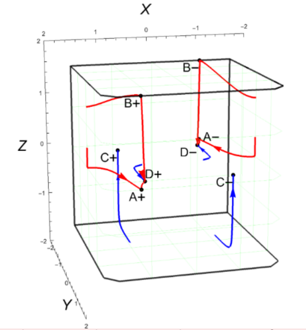
Autonomous Equations in 2-D system [Universe 2023, 9(2), 65]

$$
\begin{aligned}
x' = \dot{x}/H &= \frac{3x\left(\frac{\alpha z^2}{3} F_{,X} + C_{,X}\right)}{\left[(D - \frac{\alpha z^2}{3} F_{,X}) + x^2\left(D_{,X} - \frac{\alpha z^2}{3} F_{,XX}\right)\right]} \\
z' = \dot{z}/H &= \frac{3}{2} z \left[\omega\,\sigma^2 + \frac{\alpha z^2}{3} F + C + 1\right].
\end{aligned}
$$

Results & Discussion (Algebraic Coupling)

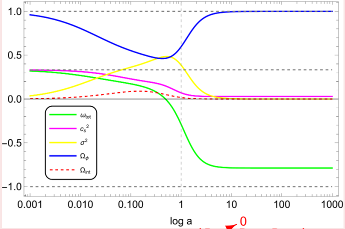
Phase Trajectory and Evolution Plot for Inverse Square Potential ($f = \alpha\rho(\frac{\phi}{\kappa})^m X^n$) [PHYS. REV. D 104, 103505 (2021)]

Phase Trajectory: ($n = 1$, $m = 3$, $\delta = 10$)



- 3-dimensional compact phase space of x, y, z.
- Total 8 critical points.
- Symmetric about x-y plane.
- Red curves $\rightarrow$ Repeller & Blue curves $\rightarrow$ Attractor.
- $A_\pm$ & $B_\pm$ $\Rightarrow$ Saddle points, and, $C_\pm$ & $D_\pm$ $\Rightarrow$ Stable fixed points.
- Critical point $D_\pm$ $\rightarrow$ Global Attractor.

Evolution plot: ($n = 1$, $m = 3$, $\delta = 10$)



- Grand EOS $\Rightarrow \frac{P_{\rm tot}}{\rho_{\rm tot}} = \frac{(\cancel{P_M}^{\,0}+P_\phi+P_{\rm int})}{(\rho_M+\rho_\phi+\rho_{\rm int})}$

  Sound Speed $\Rightarrow c_s^2 = \frac{dP_{\rm tot}/dX}{d\rho_{\rm tot}/dX}$
- Accelerating Universe $\Rightarrow -\frac{1}{3} \leq \omega_{\rm tot.} \leq -1$.

  Fluid density $\Rightarrow 0 \leq \sigma^2 \leq 1$.

  Field density $\Rightarrow 0 \leq \Omega_\phi \leq 1$.

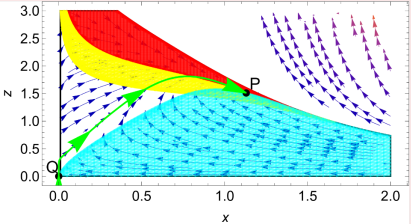
  Sound speed $\Rightarrow 0 \leq c_s^2 \leq 1$.

  Int. energy density $\Rightarrow \Omega_\phi \neq 0$.
- Energy transfer from field to fluid and then fluid to field has also been observed.
- Coupled field-fluid system starting from radiation to matter and end up at accelerating phase with negligible sound speed.

Results & Discussion (Algebraic Coupling)

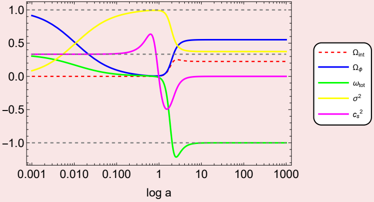
Phase Trajectory and Evolution Plot for Constant Potential ($f = gV_0\rho^q X^\beta M^{-4q}$) [Universe 2023, 9(2), 65]

Phase Trajectory:
($A = \frac{1}{2}, B = \frac{1}{3}, q = -1, g = \frac{1}{2}, \alpha = 1, \beta = \frac{1}{3}, M = 1$)



- 2-dimensional compact phase space of x, z.
- 1 non-trivial stable fixed point (P).
- Constraint on the phase space from $0 \leq \Omega_\phi \leq 1$ & $0 \leq \sigma^2 \leq 1$.
- Red region $\rightarrow$ Phantom Behavior, Yellow region $\rightarrow$ Accelerating Universe & Blue region $\rightarrow$ sound speed is between 0 and 1. Green lines $\rightarrow$ lines of stability go towards stable fixed point (P).

Evolution plot:
($A = \frac{1}{2}, B = \frac{1}{3}, q = -1, g = \frac{1}{2}, \alpha = 1, \beta = \frac{1}{3}, M = 1$)



- Energy density of k-essence sector dominates over the early and late time.
- Fluid density (σ^2) is dominating when $\omega_{\rm tot.}$ crosses zero line.
- Energy transfer from field to fluid and then fluid to field has also been observed.
- Total EOS starts from radiation dominated phase and ended up at accelerating phase with $\omega_{\rm tot} = -1$.
- Interacting energy density ($\Omega_{\rm int.}$) exist at late time.

Dynamical analysis for Derivative coupling system

Analysis in 3-D System [arXiv:2206.12398 (Accepted for Publication in EPJC)]

- Dimensionless variables:
  $x = \dot{\phi}$, $y = \frac{\kappa\sqrt{V(\phi)}}{\sqrt{3}H}$, $z = \frac{\kappa^2 nf}{3H^2}$, $\sigma = \frac{\kappa\sqrt{\rho}}{\sqrt{3}H}$,
  $B = \frac{nf_{,\phi}\kappa^2}{H^3}$, $C = \frac{\kappa^2 P_{\rm int}}{3H^2}$, $D = \frac{n\kappa^2}{3H^2}f_{,X}$,
  $E = \frac{n\kappa^2}{H^3}\frac{\partial^2 f}{\partial\phi\partial X}$, $\lambda = -\frac{V_{,\phi}}{\kappa V^{3/2}}$.
- Constraint Eqn:
  $\sigma^2 = 1 - y^2\left(3x^4/4 - x^2/2\right) - x^3 D$.
- Friedmann's Eqn:
  $\frac{\dot{H}}{H^2} = -\frac{3}{2}[\omega\sigma^2 + y^2F + C + 1]$.
- Other variables:
  $\Omega_\phi = y^2(x^2F_{,X} - F)$, $\Omega_{\rm int} = Dx^3$
- Critical Points:
  $x' = y' = z' = 0$. Prime denotes the derivative of the dynamical variables x, y, z with respect to Hdt.
- Chosen Forms:
  $F(X) = X^2 - X$ & $V(\phi) = \frac{\delta^2}{\kappa^2\phi^2}$
  ($\phi \to$ k-essence scalar field, $\delta \to$ model parameter).
- Form of Interaction:
  $f = \frac{\rho}{n}(\phi/\kappa)^m X^\eta$ & $f = \frac{F}{n\kappa^2\phi^2}$.

Autonomous Equations in 3-D system

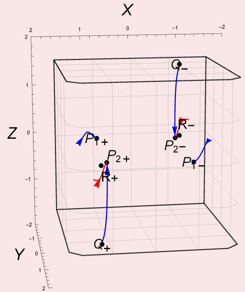
$$x' = \frac{\lambda F - \lambda F_{,X}x^2 - 3xy^2F_{,X} - x^2A_{,X} - \frac{E}{3}x^3 - A}{[y^2F_{,XX} + D_{,X}x]x^2 + [y^2F_{,X} + 3Dx]}$$

$$y' = \frac{\lambda x}{2y} + \frac{3}{2}y\left(\omega\sigma^2 + y^2F + C + 1\right)$$

$$z' = 3z\left(\omega\sigma^2 + y^2F + C\right) + A + \frac{Bx}{3} + Dxx'$$

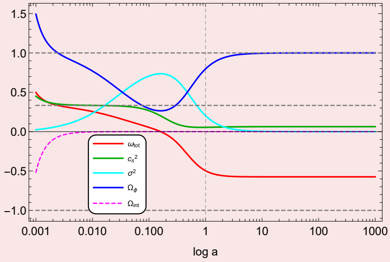
Results & Discussion (Derivative coupling with $f = \frac{\rho}{n}(\phi/\kappa)^m X^\eta$ and $V(\phi) = \frac{\delta^2}{\kappa^2\phi^2}$)

Phase Trajectory: ($\delta = 20$, $m = \frac{1}{2}$, $\eta = 1$)



- 3-dimensional compact phase space of x, y, z.
- Total 8 critical points.
- Symmetric about x-y plane.
- Red curves $\to$ Repeller & Blue curves $\to$ Attractor.
- $P_{1\pm}$ & $P_{2\pm}$ $\Rightarrow$ Stable attractor, and, $Q_\pm$ & $R_\pm$ $\Rightarrow$ Saddle fixed points.

Evolution plot: ($\delta = 20$, $m = \frac{1}{2}$, $\eta = 1$)



- Accelerating Universe $\Rightarrow -\frac{1}{3} \leq \omega_{\text{tot.}} \leq -1$.
- At late time: Fluid density $\Rightarrow 0 \leq \sigma^2 \leq 1$.
  Field density $\Rightarrow 0 \leq \Omega_\phi \leq 1$.
  Sound speed $\Rightarrow 0 \leq c_s^2 \leq 1$.
  Int. energy density $\Rightarrow \Omega_\phi \neq 0$.
- Energy transfer from field to fluid and then fluid to field has also been observed.
- Coupled field-fluid system starting from radiation to matter and end up at accelerating phase with negligible sound speed.

Conclusion: Non-minimal coupling in the context of Dynamical stability analysis

- Investigation of linear order cosmological effects of **non-minimally coupled** scalar field and a pressure-less relativistic fluid using the **variational method**.
- A **non-minimal interaction** term $f(n, s, \phi, X)$ & $f(n, s, \phi, X)J^{\mu}\partial_{\mu}\phi$ depends both on the fluid n, s and field sector's ϕ, X variables.
- Presence of **interaction** term, **Field and Friedmann equations** are modified in the background of **FLRW universe**.
- We develop the phase space using **dimensionless variables** and examine the dynamics of **power law and constant potential** in coupled k-essence sector.
- Evolutionary dynamics reveal **field-to-fluid-to-field** energy transfer.
- A **stable late-time cosmic accelerating scenario** has been observed through this non-minimally coupled field-fluid model.
- From **Early to late time phase** of the universe has been realized through evolutionary dynamics of the non-minimally coupled sectors.


Thank you!