# Gravitational Wave Observations: Exercises

Problem 1. Verify that the GW strain from a pair of $30\,M_\odot$ black holes, separated by a few Schwarzschild radii, orbiting around each other at a frequency of $100\,\mathrm{Hz}$, and located $400\,\mathrm{Mpc}$ away, is of the order of 10^{-21}. What is the GW luminosity (order of magnitude)?

Problem 2. *Newtonian chirp.* In this problem, we will compute the leading-order gravitational waveform for an orbiting binary. This problem has been adapted from the textbook by Creighton and Anderson.

Consider a binary of masses m_1 and m_2 in a circular orbit on the x-y plane so that the angular momentum points along the z direction. Assume uniform circular motion, where the orbital phase $\varphi \equiv \omega t$ increases uniformly with time t; $\omega \equiv 2\pi f_{\mathrm{orb}}$ is the orbital angular frequency.
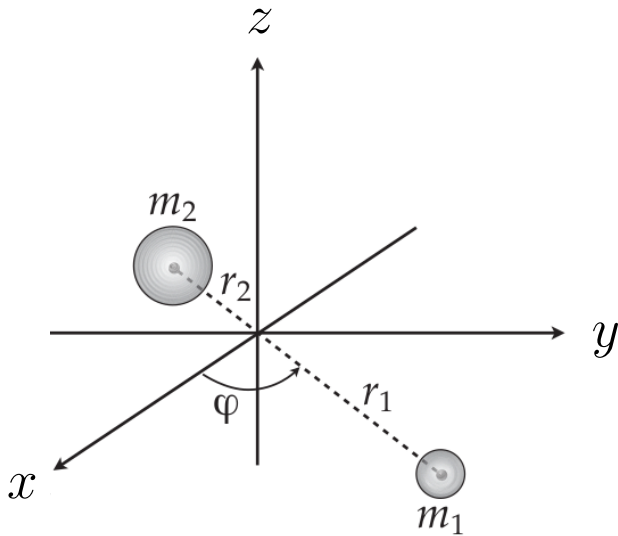
(i) Show that the quadrupole tensor for this system, defined as $I_{ij}(t) \equiv \int x_i x_j \rho(t)\, d^3x$, where ρ is the mass density, when computed in the centre-of-mass frame, evaluates to

$$I_{ij} = \frac{1}{2}\mu\, r_{12}^2 \begin{bmatrix} 1 + \cos 2\varphi & \sin 2\varphi & \\ \sin 2\varphi & 1 - \cos 2\varphi & \\ & & 0 \end{bmatrix}, \tag{1}$$

where $\mu \equiv m_1 m_2/(m_1 + m_2)$ is the usual *reduced mass* of the system, and r_{12} is the separation between the two masses.

(ii) The gravitational wave strain in the transverse traceless gauge is given by $h_{ij}^{\mathrm{TT}} \equiv (2G/c^4 d_L)\, \ddot{\bar{I}}_{ij}$ at a distance d_L from the source (in the radiation zone $d_L \gg r_{12}$); here the overbar denotes the trace-free part of a tensor. Note that when viewed from the z direction, the quadrupole tensor I_{ij} for this system is already transverse. Compute its second time derivative $\ddot{I}_{ij}$ and note that it is traceless. Evaluate the gravitational wave strain along the z axis (in the radiation zone). Can you read off the two polarizations?

$$h_+ = -\frac{4G\,\mu\, r_{12}^2\,\omega^2}{c^4\, d_L} \cos 2\varphi\,,$$
$$h_\times = -\frac{4G\,\mu\, r_{12}^2\,\omega^2}{c^4\, d_L} \sin 2\varphi\,. \tag{2}$$

Note that the gravitational-wave frequency is twice the orbital frequency, $f_{\mathrm{GW}} = 2\, f_{\mathrm{orb}}$. Why would you expect this behaviour?

(iii) In gravitational-wave literature, it is customary to rewrite the above expressions in terms of the velocity parameter $v \equiv r_{12}\,\omega$. For an observer not on the orbital axis, but at an angle of inclination ι from the axis, the gravitational waveform is

$$h_+ = -\frac{2\,G\,\mu}{c^2\,d_L}\,(1 + \cos^2 \iota)\,\left(\frac{v}{c}\right)^2 \cos 2\varphi\,,$$

$$h_\times = -\frac{4\,G\,\mu}{c^2\,d_L}\,\cos \iota\,\left(\frac{v}{c}\right)^2 \sin 2\varphi\,. \tag{3}$$

Use Kepler's third law to show that v can be related to the orbital separation as $v = \sqrt{GM/r_{12}}$, where $M \equiv m_1 + m_2$ is the *total mass* of the system. Also show that v can be related to the gravitational-wave frequency as $v = (\pi G M f_{\mathrm{GW}})^{1/3}$. Show that the total Newtonian energy of the system is

$$E = -\frac{1}{2}\mu v^2\,. \tag{4}$$

(iv) The gravitational-wave luminosity is given by,

$$L_{\mathrm{GW}} = \frac{1}{5}\frac{G}{c^5}\langle \dddot{\bar{I}}_{ij}\,\dddot{\bar{I}}^{ij}\rangle\,, \tag{5}$$

where the angle brackets $\langle \cdot \rangle$ denote a time average, and repeated indices are asummed to be summed over (via the Einstein summation convention). Show that the gravitational-wave luminosity from an orbiting binary is

$$L_{\mathrm{GW}} = \frac{32}{5}\frac{c^5}{G}\,\eta^2\left(\frac{v}{c}\right)^{10}\,, \tag{6}$$

where $\eta \equiv \mu/M$ is the *symmetric mass ratio*.

(v) The energy emitted in gravitational waves is lost from the Newtonian orbital energy of the system. Using $L_{\mathrm{GW}} = -dE/dt$, show that system collapses in a finite time. Find an expression for the frequency evolution of the system. You should obtain

$$\begin{aligned}
\frac{df_{\mathrm{GW}}}{dt} &= \frac{96}{5}\pi^{8/3}\eta\left(\frac{GM}{c^3}\right)^{5/3} f_{\mathrm{GW}}^{11/3}\\
&\equiv \frac{96}{5}\pi^{8/3}\left(\frac{G\mathcal{M}_c}{c^3}\right)^{5/3} f_{\mathrm{GW}}^{11/3}\,.
\end{aligned} \tag{7}$$

This is the expression for the Newtonian chirp. Here it is customary to define yet another mass combination, the *chirp mass*, $\mathcal{M}_c \equiv \eta^{3/5}M$, which is the unique mass combination that characterizes the inspiral of the system. Written out in terms of the component masses, $\mathcal{M}_c \equiv (m_1 m_2)^{3/5}(m_1 + m_2)^{-1/5}$. With a starting gravitational-wave frequency $f_{\mathrm{GW}} = f_0$, show that the system collapses in a time

$$t_c = \frac{5}{256}\frac{G\mathcal{M}_c}{c^3}\left(\frac{\pi G \mathcal{M}_c f_0}{c^3}\right)^{-8/3}\,. \tag{8}$$

Note: In order to obtain this last expression, it is okay to assume an upper frequency $f_{\mathrm{GW}} \to \infty$ at coalescence, rather than a cutoff merger frequency.

Problem 3. *GW waveform templates.* In this problem we will code up a symmetric leading-order "Newtonian chirp" waveform for use in simple data analysis. *Units:* All expressions in this problem are written in geometrized units, in which $G = c = 1$. Mass and distance have units of seconds. Physical units can be obtained by replacing a mass $\mathcal{M}_c$ by $G\mathcal{M}_c/c^3$, and a distance d_L by $c\,d_L$. In geometrized units, $1M_\odot = 4.92549095 \times 10^{-6}$ s and $1\text{pc} = 1.0292712503 \times 10^8$ s.

In the leading order post-Newtonian approximation, the observed GW signal, which is a linear combination of the two polarizations $h_+(t)$ and $h_\times(t)$, can be written out as:

$$h(t) = A(t) \cos \varphi(t). \tag{9}$$

The amplitude $A(t)$ depends on a particular combination of the masses, called the *chirp mass* $\mathcal{M}_c$, the instantaneous frequency $F(t)$ of GWs, the luminosity distance d_L to the source, and a geometric factor $\mathcal{C}$ that depends on the location of the source in the sky and its orientation with respect to the detector.

$$A(t) = \mathcal{C}\,\frac{4\mathcal{M}_c^{5/3}\pi^{2/3}F(t)^{2/3}}{d_L}. \tag{10}$$

For simplicity, we shall assume $\mathcal{C} = 1$ which implies that the binary is conveniently oriented giving circular polarization and the source is located along the direction where the detector shows maximum directional sensitivity. The chirp mass can be expressed in terms of the total mass $M \equiv m_1 + m_2$ and reduced mass $\mu \equiv m_1 m_2/M$ as $\mathcal{M}_c = \mu^{3/5}M^{2/5}$. The frequency evolution $F(t)$ is given by

$$F(t) = \frac{(\mathcal{M}_c F_0^9)^{1/8}}{[(\mathcal{M}_c F_0)^{1/3} - 256\,F_0^3\mathcal{M}_c^2\pi^{8/3}\,t/5]^{3/8}} \tag{11}$$

where F_0 is the starting frequency of the signal: $F_0 \equiv F(t = 0)$. It can be seen that the frequency sweeps from lower to higher frequencies, until the approximation breaks down at $t = t_c$. The *coalescence time* t_c can be computed as

$$t_c = \frac{5}{256\,(\pi F_0)^{8/3}\mathcal{M}_c^{5/3}}. \tag{12}$$

Finally, the phase $\varphi(t)$ of the GW signal can be expressed as

$$\varphi(t) = \varphi_0 - 2\left(\frac{1}{256\,(\pi\mathcal{M}_c F_0)^{8/3}} - \frac{t}{5\mathcal{M}_c}\right)^{5/8}, \tag{13}$$

where φ_0 is the phase at $t = 0$. A Newtonian chirp waveform for chirp mass $\mathcal{M}_c = 10M_\odot$, distance $d_L = 100$ Mpc, initial phase $\varphi_0 = 0$ and start frequency $F_0 = 40$ Hz. is shown below.
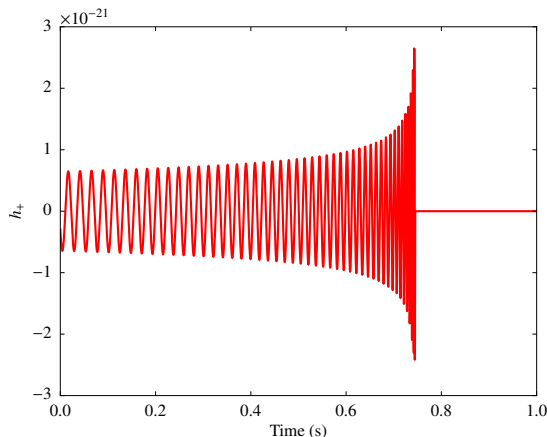
Problem 4. *Matched filtering.* In the case a known signal $h(t)$ buried in stationary Gaussian, white noise, the optimal technique for signal extraction is the *matched filtering*, which involves cross-correlating the data with a *template* of the signal. The correlation function between two time series $x(t)$ and $\hat{h}(t)$ for a time shift τ is defined as:

$$R(\tau) = \int_{-\infty}^{\infty} x(t+\tau)\, \hat{h}^{\star}(t)\, dt. \tag{14}$$

Above, $\star$ denotes complex conjugation, and $\hat{h(t)} \equiv h(t)/\|h\|$, where the norm $\|h\|$ of the template is defined by

$$\|h\|^2 = \int_{0}^{t_c} |h(t)|^2/\sigma^2\, dt,$$

where σ^2 is the variance of the noise. The optimal signal-to-noise ratio (SNR) is obtained when the template exactly matches with the signal.

$$\mathrm{SNR} = \|h\| \tag{15}$$

If the SNR is greater than a predetermined threshold (which corresponds to an acceptably small false alarm probability), a detection can be claimed. Note that the actual detector data is neither white and is only approximately Gaussian, which makes actual GW detection a siginificantly more complex exercise than mentioned above!

A data set containing a Newtonian GW signal with $d_L = 100$ Mpc, $\varphi_0 = 0$, $F_0 = 40$ Hz, but unknown $\mathcal{M}_c$ can be downloaded from https://www.dropbox.com/s/cjs6oh3cx8f6m4p/gw_data.dat.gz. The data $d(t)$ is comprised of the signal $h(t)$ and Gaussian white noise $n(t)$ of standard deviation $\sigma = 10^{-21}$. That is, $d(t) = h(t) + n(t)$. Write a code to detect the signal using the simple matched filtering method mentioned in the previous section. Since you don't know the chirp mass of the signal, choose a grid of chirp masses in the interval $\mathcal{M}_c \in (8, 12)M_\odot$ with some appropriate grid spacing. This is your "template bank"!

Problem 5. *Simple exercises with real data!*

Download the data for the first GW detection GW150914 from the website of the Gravitational Wave Open Science Center (GWOSC): https://www.gw-openscience.org/

Plot the spectrogram (time-frequency map) of the event, confirming that it has the characteristics we would expect, i.e. an increasing frequency and amplitude. From the frequency give a back-of-the-envelope estimate on the maximal possible characteristic size and mass of the GW source, and check whether it is consistent with the values reported by the LIGO-Virgo Collaboration.

GWOSC has an excellent tutorial page which can help in how to read and interpret GW data. You can find the tutorials at: https://www.gw-openscience.org/tutorials/