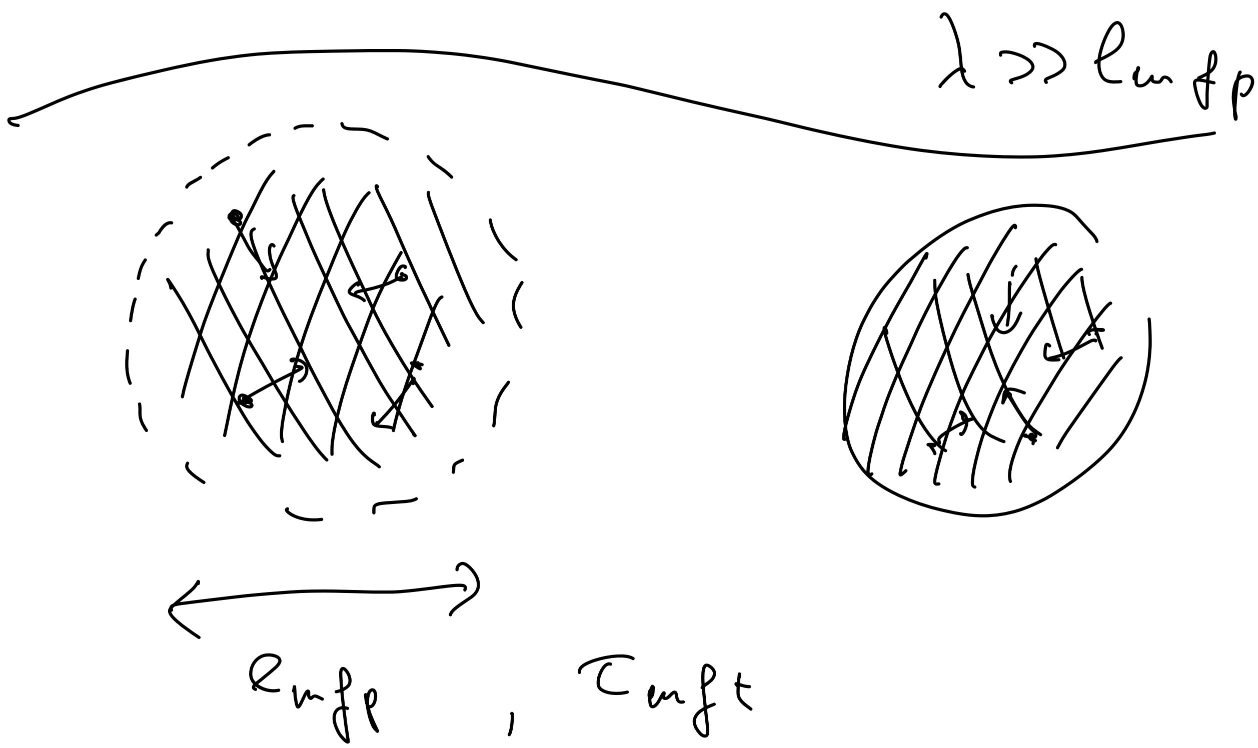
Ordinary fluids

$(T \neq 0)$

Perfect fluid approx.:



thermal reached instantaneously (locally),
dissipation is negligible (suppressed
by $\left(\ell_{mfp}/\lambda\right)^{\#}$)

EOM :
$$\begin{cases} \dot{\rho} + \vec{\nabla}\cdot(\rho\vec{v}) = 0 \\ \dot{\vec{v}} + (\vec{v}\cdot\vec{\nabla})\vec{v} = -\frac{1}{\rho}\vec{\nabla}p \end{cases}$$

$\rho = \rho(\vec{x},t)$ mass density

$\vec{v} = \vec{v}(\vec{x},t)$ velocity field

$p = p(\vec{x},t) = p_{eq}(\rho(\vec{x},t))$ eq. of state

$\uparrow$ same function as at equilibrium

<u>Relativistic fluid</u> $(-,+,+,+)$

$\rho(x)$ $u^\mu(x) = \gamma(v)(1,\vec{v})$

$\uparrow$ total energy ($\supset$ mass) density in local rest frame

$(c^2=1)$

$u^2 = -1$

$$T^{\mu\nu}(x) = \big(\rho(x) + p(x)\big)\, u^{\mu}(x)\, u^{\nu}(x) + p(x)\, \eta^{\mu\nu}$$

$$T^{\mu\nu} u_\mu u_\nu = (\rho + p) - p = \rho \quad ✓$$

$$\text{EOM} \iff \partial_\mu T^{\mu\nu} = 0$$

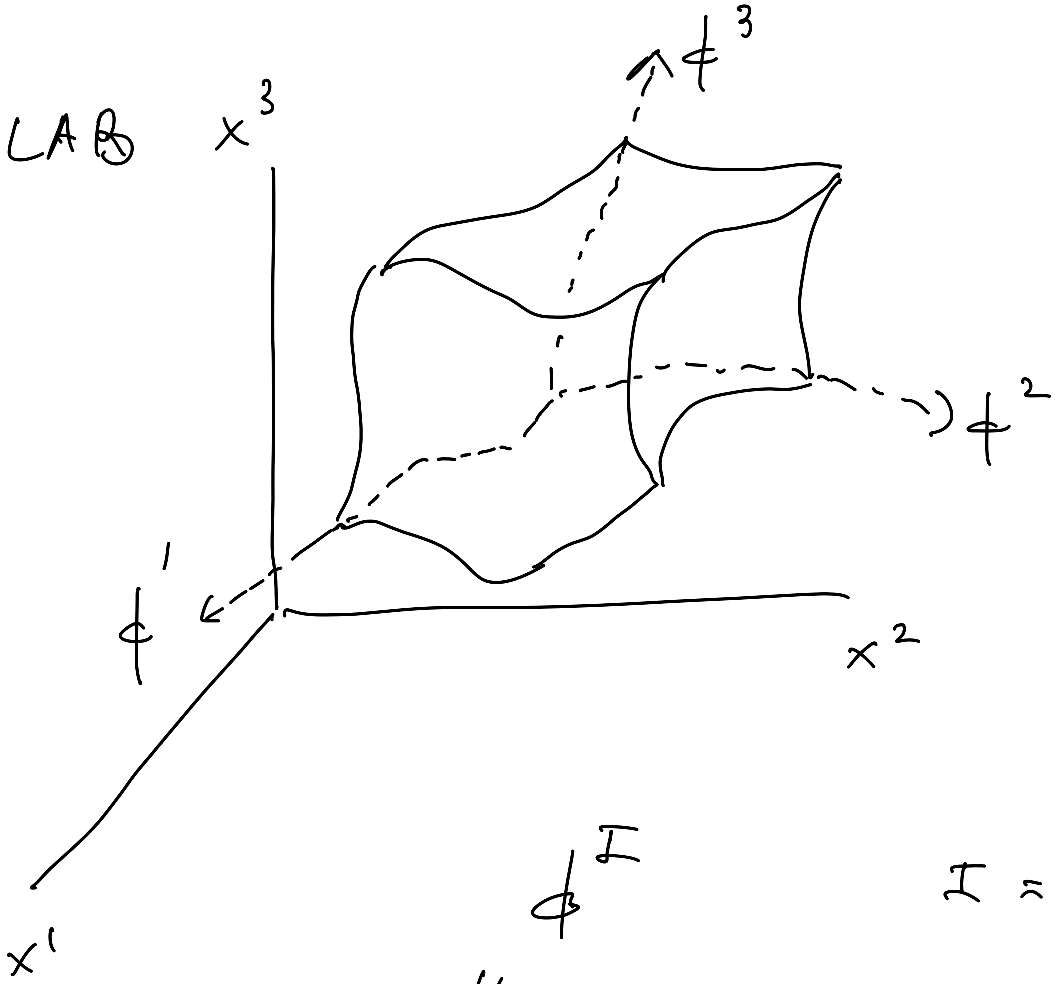
$$\left(\text{In GR: } \nabla_\mu T^{\mu\nu} = 0\right)$$

EFT ?

EFT = relevant d.o.f. + syms

$\cancel{\rho, \vec{v}}$ (or $\cancel{\rho,}\, u^\mu$)

Positional d.o.f. of a continuous medium (solid, fluid, ...)



ϕ^I $\quad I = 1, 2, 3$

"Comoving coordinates"

Configuration space $= \{ \phi^I(\vec{x}, t) \}$ (scalars: ϕ^I; x^μ: $\vec{x}, t$)

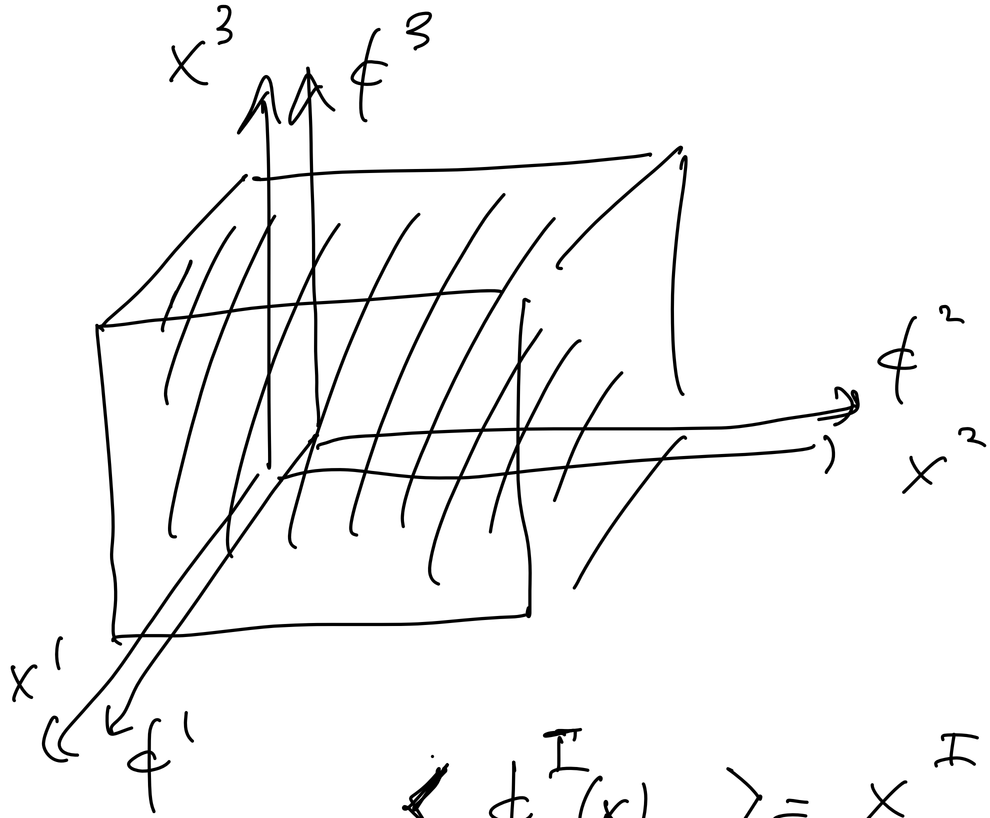
$= \{ \vec{x}(\phi^I, t) \}$

3 scalar fields

$S = S[\phi^I] = ?$

At eq (for given external pressure / bdy conditions),

for static, homogeneous, isotropic state



$$\langle \phi^I(x)_{eq} \rangle = x^I \qquad (*)$$

breaks & rotations

$\Rightarrow$ Need internal translational and rotational syms:

$$
\begin{array}{l}
(\mathrm{IT}) \\
\Longrightarrow \\
(\mathrm{IR})
\end{array}
\begin{cases}
\phi^I(x) \longrightarrow \phi^I(x) + a^I \\
\phi^I(x) \longrightarrow R^I{}_J \cdot \phi^J(x)
\end{cases}
$$

$$R \in SO(3)$$

$(*)$ is invariant under the combined action of spatial and internal translations (rotations).

Syms:

Poincaré, IT, IR

$$B^{IJ} \equiv \partial^\mu \phi^J \partial_\mu \phi^I$$

$$\mathrm{tr}\, B,\ \mathrm{tr}\, B^2,\ \mathrm{tr}\, B^3,\ \det B,\ \ldots$$

symmetric $d \times d$ matrix $\Rightarrow$ d independent invariants.

For us: (eigenvalues)

$d = 3$

$\det B, \ \mathrm{tr}\, B, \ \mathrm{tr}\, B^2$

OR

$\mathrm{tr}\, B, \ \mathrm{tr}\, B^2, \ \mathrm{tr}\, B^3$

OR

...

So far, no difference w/ isotropic solids.

Fluid: can be deformed adiabatically slowly as long as we don't compress/dilate volume elements

have same energy
$$\begin{cases} \phi^I = x^I \\ \phi^I = \xi^I(\vec{x}) \end{cases}$$

volume-preserving diff $\det \frac{\partial \xi^I}{\partial x^i} = 1$

Proposal: $\phi^I \to \xi^I(\phi)$ w/ $\det \frac{\partial \xi^I}{\partial \phi^J} = 1$

$$B^{IJ} = \partial_\mu \phi^I \partial_\nu \phi^J$$

$$\longrightarrow \frac{\partial \xi^I}{\partial \phi^K} \frac{\partial \xi^J}{\partial \phi^L} B^{KL}$$

$$\det B \longrightarrow \det B \cdot \left(\det\left(\frac{\partial \xi}{\partial \phi}\right)\right)^2 = \det B$$

$$\mathrm{Tr}\, B \longrightarrow \mathrm{Tr}\left(\frac{\partial \xi}{\partial \phi} \cdot B \cdot \frac{\partial \xi}{\partial \phi}^T\right) \neq \mathrm{Tr}\, B$$

unless $\left(\frac{\partial \xi}{\partial \phi}\right) \in SO(3)$

$$\Rightarrow \quad S_{\text{fluid}} = \int d^4x \left(F(\det B) + \text{higher } \partial\text{'s} \right)$$

F : generic function

Standard ρ, u^μ language:

$$T^{\mu\nu} = ? \qquad d^4x \to \sqrt{g}\, d^4x$$

$$B^{IJ} \to g^{\mu\nu} \underbrace{\partial_\mu \phi^I}_{= \nabla_\mu \phi^I} \partial_\nu \phi^J$$

$$\frac{\delta}{\delta g^{\mu\nu}} \xrightarrow{HW} T_{\mu\nu} = -2 \underbrace{F'}_{= \frac{\partial F}{\partial \det B}} \cdot B \cdot B^{-1}_{IJ} \partial_\mu \phi^I \partial_\nu \phi^J + F \eta_{\mu\nu}$$

$$\det B^{IJ} \equiv B$$

$$\overset{?}{=} (\rho + p) u_\mu u_\nu + p\, \eta_{\mu\nu}$$

$u^\mu(x)$ = vector field along which ϕ^I's do <u>not</u> change:

$$u^\mu(x)\,\partial_\mu \phi^I(x) = 0 \qquad \forall I = 1,2,3$$

$$u^\mu \propto J^\mu = \epsilon^{\mu\nu\rho\sigma}\,\partial_\nu\phi^1\,\partial_\rho\phi^2\,\partial_\sigma\phi^3$$

$$= \frac{1}{3!}\,\epsilon^{\mu\nu\rho\sigma}\,\epsilon_{IJK}\,\partial_\nu\phi^I\,\partial_\rho\phi^J\,\partial_\sigma\phi^K$$

$$J^2 = J_\mu J^\mu = \dots \underset{(HW)}{=} -\det(B^{IJ})$$

$$u^2 = -1 \implies u^\mu = \frac{J^\mu}{\sqrt{\det(B^{IJ})}}$$

$$\left(\langle \phi^I \rangle_{eq} = x^I \implies u^\mu = (1, \vec{0}) \right)$$

$$T_{\mu\nu} = -2F' \cdot B \cdot B^{-1}_{IJ}\,\partial_\mu\phi^I\,\partial_\nu\phi^J + F\,\eta_{\mu\nu}$$

$$\left(B^{-1}_{IJ}\,\partial_\mu\phi^I\,\partial_\nu\phi^J \right) \cdot u^\mu = 0$$

$$\Rightarrow \quad \# \propto (\eta_{\mu\nu} + u_\mu u_\nu)$$

trace: $\left(B^{-1}_{IJ}\, \partial_\mu \phi^I \partial_\nu \phi^J\right)\eta^{\mu\nu} =$

$$= B^{-1}_{IJ}\, \underbrace{\partial_\mu \phi^I \partial^\mu \phi^J}_{B^{IJ}} = \mathrm{tr}\left(\hat{B}^{-1}\cdot\hat{B}\right)$$

$$= \mathrm{tr}\, \mathbb{1}_{3\times 3} = 3$$

$$(\eta_{\mu\nu} + u_\mu u_\nu)\eta^{\mu\nu} = 4 - 1 = 3$$

$$\Rightarrow T_{\mu\nu} = -2F'\cdot B\,(\eta_{\mu\nu} + u_\mu u_\nu) + F\eta_{\mu\nu}$$

$$= \underbrace{-2F'B}_{(\rho + p)}\cdot u_\mu u_\nu + (\underbrace{F - 2F'B}_{p})\,\eta_{\mu\nu}$$

$$\Rightarrow \begin{cases} \rho = -F \\ p = F - 2F'B \end{cases} \qquad \left(\text{cf: } \begin{array}{l} \rho = 2P'(x)x - P(x) \\ p = P(x) \end{array} \right)$$

Eq. of state: $\quad p = p_{eq}(\rho)$

$$F(B) - 2F'(B)B = p_{eq}(-F)$$

Non-linear, 1st order ODE for F

$\Rightarrow$ F determined by eq. of state

$$\left(\langle \phi^I \rangle_{eq} = \alpha x^I \right)$$

$$p_{eq}(\rho) \longleftrightarrow F(B)$$

Small perturbations

$$\phi^I(x) = x^I + \pi^I(x)$$

$$B^{IJ} = \left(\delta^I_\mu + \partial_\mu \pi^I\right)\left(\delta^J_\nu + \partial_\nu \pi^J\right)\eta^{\mu\nu}$$

$$= \delta^{IJ} + \underbrace{\partial^I \pi^J + \partial^J \pi^I + \partial_\mu \pi^I \partial^\mu \pi^J}$$

δB^{IJ} ,

$$B = \det B^{IJ} = \det\left(\delta^{IJ} + \delta B^{IJ}\right)$$

$$= \exp\left(\mathrm{tr}\,\log\left(\delta^{IJ} + \delta B^{IJ}\right)\right)$$

HW $\vdots$

$$S_{\text{fluid}} \rightarrow \int d^4x \left[F(1) + F'(1)\,\delta B + \frac{1}{2} F''(1)\,\delta B^2 + \ldots \right]$$

HW $\vdots$

$$= \int d^4x \, \underbrace{\left(-2F'(1)\right)}_{(\rho + p)_{eq}} \frac{1}{2} \left[\dot{\vec{\pi}}^2 - c_s^2 \left(\vec{\nabla} \cdot \vec{\pi}\right)^2 \right]$$

$$+ \left(\text{total } \partial' s + O(\pi^3)\right)$$

w/ $\quad c_s^2 = \left.\frac{2F''\cdot B + F'}{F'}\right|_{B=1} \stackrel{?}{=} \left.\frac{dp}{d\rho}\right|_{eq}$ ✓

$$p = F - 2F' \cdot B \qquad \frac{dp}{dB} = (\cancel{F'} - 2F''B - \cancel{F'})$$

$$\rho = -F \qquad \frac{d\rho}{dB} = -F'$$

EOM:

$$-\ddot{\vec{\pi}} + c_s^2 \vec{\nabla}(\vec{\nabla}\cdot\vec{\pi}) = 0$$

$\mathcal{F}_{\vec{x}\to\vec{k}}$: $\quad -\ddot{\vec{\pi}} - c_s^2 \vec{k}(\vec{k}\cdot\vec{\pi}) = 0$

$(\vec{\pi} \propto e^{i\vec{k}\cdot\vec{x}})$

$$\underset{\parallel \vec{k}}{\underset{\uparrow}{\vec{\pi}}} = \vec{\pi}_L + \vec{\pi}_T$$

$$\vec{\pi} = \underset{\parallel \vec{k}}{\underset{\uparrow}{\vec{\pi}_L}} + \underset{\perp \vec{k}}{\underset{\uparrow}{\vec{\pi}_T}}$$

$$\begin{cases} \ddot{\vec{\pi}}_L + c_s^2 k^2 \vec{\pi}_L = 0 \\ \ddot{\vec{\pi}}_T = 0 \end{cases}$$

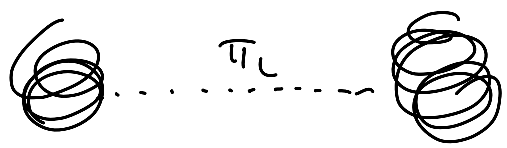
Sound wave at speed c_s

$$\vec{\pi}_T = \vec{c} + \vec{d} \cdot t$$

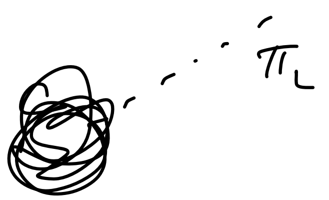
w/ $\vec{c}, \vec{d} \perp \vec{k}$

linear progenitors of vortices

"the incompressible fluid revisited"



$\longrightarrow$ $V_{\text{vortex}-\text{vortex}}(r)$



$\longrightarrow$ Sound emission by turbulence

...

Conservation laws

1) $\partial_\mu T^{\mu\nu} = 0$ (translational invariance)

2) $\phi^I \longrightarrow \xi^I(\phi) \simeq \phi^I + \varepsilon^I(\phi)$

$\det \frac{\partial \xi}{\partial \phi} = 1$ $\qquad \frac{\partial \varepsilon^I}{\partial \phi^I} = 0$

$\left(\det(\hat{\mathbb{1}} + \hat{M}) = 1 + \operatorname{tr} M + \mathcal{O}(M^2) \right)$

Noether:

$$\Rightarrow J^\mu_{(\varepsilon)} = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi^I)} \cdot \varepsilon^I(\phi)$$

$$= 2F' B\, B^{-1}_{IJ}\, \partial^\mu \phi^J\, \varepsilon^I(\phi)$$

$\partial_\mu J^\mu_{(\varepsilon)} = 0$ on EOM

EOM? $\quad S = \int F(B = \det \partial_\mu \phi^I \partial^\mu \phi^J)$

$$\partial_\mu \left(\frac{\partial F}{\partial(\partial_\mu \phi^I)} \right)$$

$$= \partial_\mu \left(2F' B \, B^{-1}_{IJ} \, \partial^\mu \phi^J \right) = 0$$

$$\partial_\mu J^\mu_{(\varepsilon)} = 0 \cdot \varepsilon^I(\phi)$$

$$+ \, 2F' B \underbrace{\underbrace{B^{-1}_{IJ} \underbrace{\partial^\mu \phi^J \, \partial_\mu \phi^K}_{B^{JK}}}_{\delta^{IK}} \frac{\partial \varepsilon^I}{\partial \phi^K}}_{\frac{\partial \varepsilon^I}{\partial \phi^I} = 0}$$

$$= 0$$

conservation of $J^\mu_{(\varepsilon)}$ $\forall \varepsilon^I(\phi)$ equivalent to

Kelvin's theorem

In non-relativistic limit:

$$\oint_{\gamma(t)} \vec{v} \cdot d\vec{\ell} = \text{const} \qquad \forall \gamma$$

$\Rightarrow \infty$ conservation laws.

$$\left(\text{cf.} \quad \frac{d}{dt} Q = 0 \qquad Q = \int J^0 \, d^3x \right)$$

<u>Useful limits</u>

⊛ Conformal theory (~ UV fixed point
T >> masses)

$$T^{\mu}{}_{\mu} = 0$$

$$-2F' B \underbrace{B^{-1}_{IJ} \underbrace{\partial_\mu \phi^I \partial^\mu \phi^J}_{B^{IJ}}}_{3} + 4F = 0$$

$$\overset{2}{\cancel{4}}F - \overset{3}{\cancel{6}}F' B = 0$$

$$\Rightarrow F(B) \propto B^{2/3}$$

$$S_{fluid} = \int d^4x \, \underset{\substack{\uparrow \\ \text{const}}}{(F_0)} B^{2/3} + \text{higher } \partial\text{'s}$$

const
(depends on CFT, central charge)

check :

$$c_s^2 = \frac{2F''B + F'}{F'} = \frac{2(\tfrac{2}{3})(-\tfrac{1}{3})\cancel{B^{-1/3}} + \tfrac{2}{3}\cancel{B^{-1/3}}}{\tfrac{2}{3}\cancel{B^{-1/3}}}$$

$$= \frac{1-\tfrac{2}{3}}{1} = \tfrac{1}{3}$$

CFT : all non-linear interactions are completely fixed.

⊛ Cosmologist's case

$$p = \underset{\substack{\uparrow \\ \text{const.}}}{w} \cdot \rho$$

$$F - 2F'B = -w \cdot F$$

$$2F'B = (1+w)F$$

$$\frac{d\log F}{d\log B} = \frac{1+w}{2}$$

$$\Rightarrow \boxed{F(B) = F_0\, B^{\frac{1+w}{2}}}$$

Finite T superfluid

= superfluid + gas of phonons

= superfluid + normal fluid component

$\phi(x) \to \phi(x) + a$

$\phi^I(x) \to \xi^I(\phi)$

$\det \partial\xi/\partial\phi = 1$

Invariants:

$$X = (\partial\phi)^2$$

$$B = \det \partial_\mu \phi^I \partial^\mu \phi^J$$

$$y = \partial_\mu \phi \, J^\mu$$

$$J^\mu = \frac{\epsilon^{\mu\nu\rho\sigma}}{3!} \epsilon_{IJK} \partial_\nu \phi^I \partial_\rho \phi^J \partial_\sigma \phi^K$$

$$S = \int d^4x \; F(X, B, y)$$

$$= \int d^4x \; \underbrace{P(X)}_{\text{input}} + \underbrace{\delta F(X, B, y)}_{\text{computable in pert. theory.}}$$