Modeling of the nonlinear dynamics of Josephson Traveling-Wave Parametric Amplifiers

DR. C. GUARCELLO

PHYSICS DEPARTMENT "E.R. CAIANIELLO", UNIVERSITY OF SALERNO, FISCIANO (SA), ITALY

Istituto Nazionale di Fisica Nucleare
Laboratori Nazionali di Frascati

cQED@Tn - "Circuit QED: From Quantum Devices to Analogues on Superconducting Circuits"
Oct 3 – 5, 2022
FBK (Trento, Italy)

Istituto Nazionale di Fisica Nucleare

DARTWARS

Detector Array Readout with Traveling Wave AmplifieRS

C. Guarcello, G. Avallone, C. Barone, M. Borghesi, S. Capelli, G. Carapella, A.P. Caricato, I. Carusotto, A. Cian, D. Di Gioacchino, E. Enrico, P. Falferi, L. Fasolo, M. Faverzani, E. Ferri, G. Filatrella, C. Gatti, A. Giachero, D. Giubertoni, V. Granata, A. Greco, D. Labranca, A. Leo, C. Ligi, G. Maccarrone, F. Mantegazzini, B. Margesin, G. Maruccio, C. Mauro, R. Mezzena, A.G. Monteduro, A. Nucciotti, L. Oberto, L. Origo, S. Pagano, V. Pierro, L. Piersanti, M. Rajteri, A. Rettaroli, S. Rizzato, A. Vinante, and M. Zannoni

The DARTWARS collaboration

INFN Istituto Nazionale di Fisica Nucleare · CSN5 Technological research · TIFPA Trento Institute for Fundamental Physics and Applications · COLD · INRiM Istituto Nazionale di Ricerca Metrologica · FBK Fondazione Bruno Kessler · Università degli Studi di Milano Bicocca · Università degli Studi di Trento · DARTWARS Detector Array Readout with Traveling Wave AmplifieRS · Università del Salento · CAPP IBS · Università degli Studi di Salerno · Università degli Studi del Sannio · Politecnico di Torino · CNR IFN Istituto di Fotonica e Nanotecnologie

https://dartwars.unimib.it/

Josephson Traveling-Wave Parametric Amplifiers (JTWPA)

A weak signal traveling in a metamaterial can interact with a strong pump tone at a different frequency, activating the so-called **parametric amplification**.

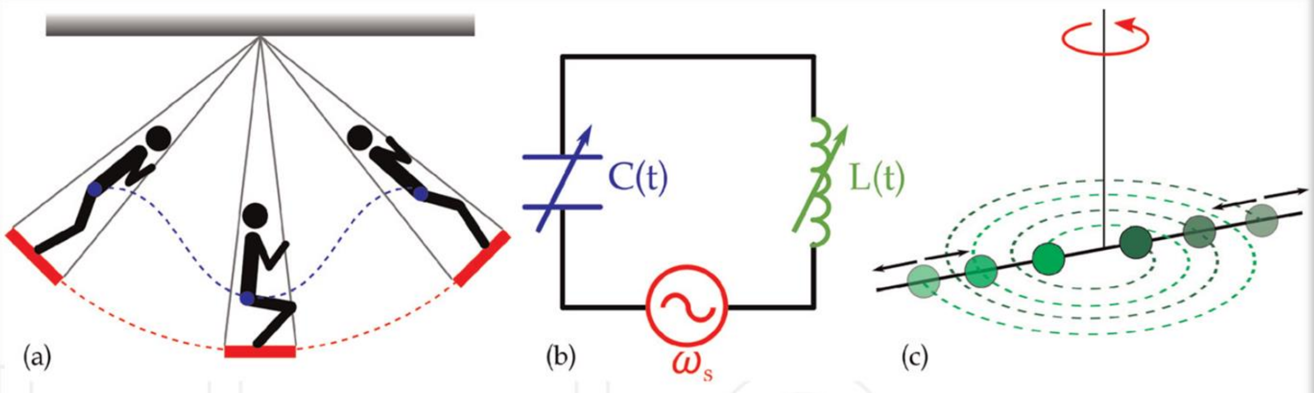


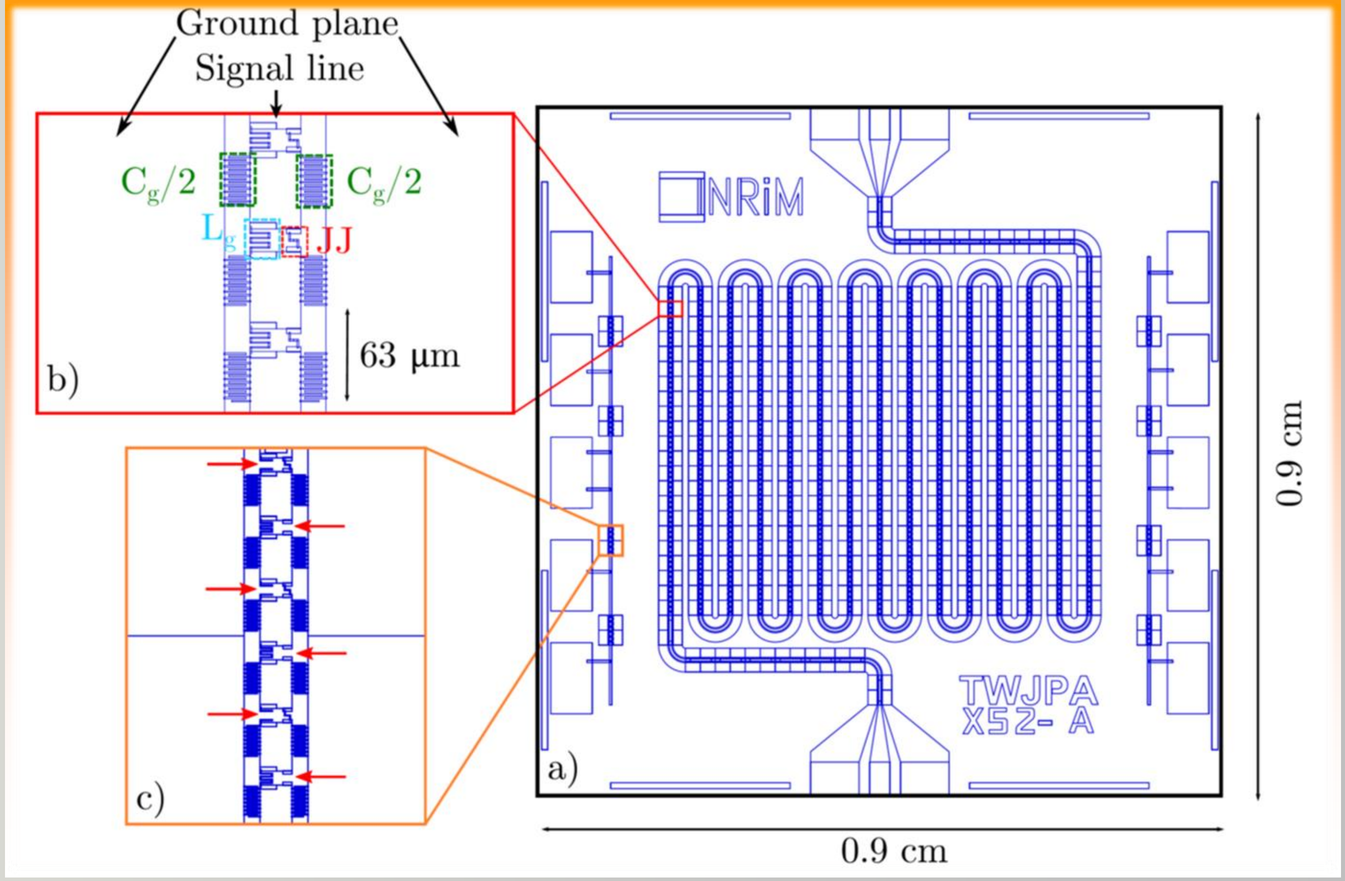
Figure 1.
(a) Sketch of a swing process. An oscillating system at a frequency ω_s is excited by parametric amplification via periodical changes of the center of mass position at a frequency $\omega_p = 2\omega_s$. (b) LC circuit with variable (nonlinear) C and L components. The case in which the capacitance C is periodically changed in time is the circuit analogous to the mechanical system represented in (a), while the case having an oscillating inductance L mimics the condition sketched in (c), consisting in a torque pendulum with variable inertia momentum [18].

The class of devices where these phenomena are promoted is commonly known as **Josephson traveling-wave parametric amplifiers (JTWPA)**

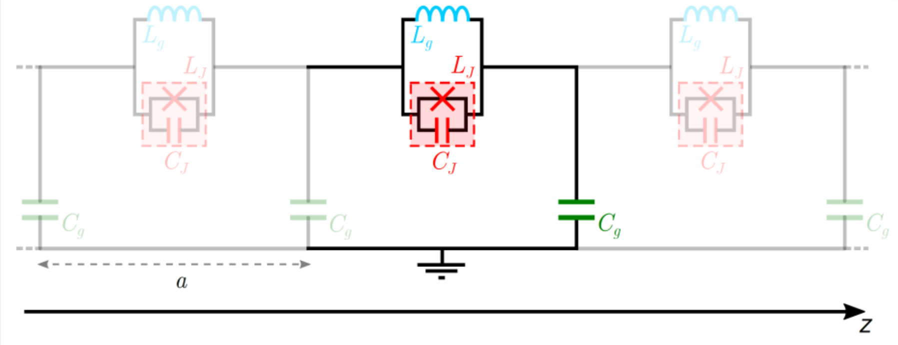
L. Fasolo, *et al.* "Superconducting josephson-based metamaterials for quantum-limited parametric amplification: A review." Adv. Condensed-Matter Materials Physics–Rudimentary Res. Topical Technol.. IntechOpen, 2019.

Layout of a TWJPA Chip TWJPA_X52

Layout of a the **Traveling Wave Josephson Parametric Amplifiers (TWJPA)** Chip TWJPA_X52 (size 10x10 mm2) based on a sequence of 990 elementary cells each formed by an RF-SQUID in series and an interdigital capacitor to ground.



Equivalent circuit of 3 elementary cells

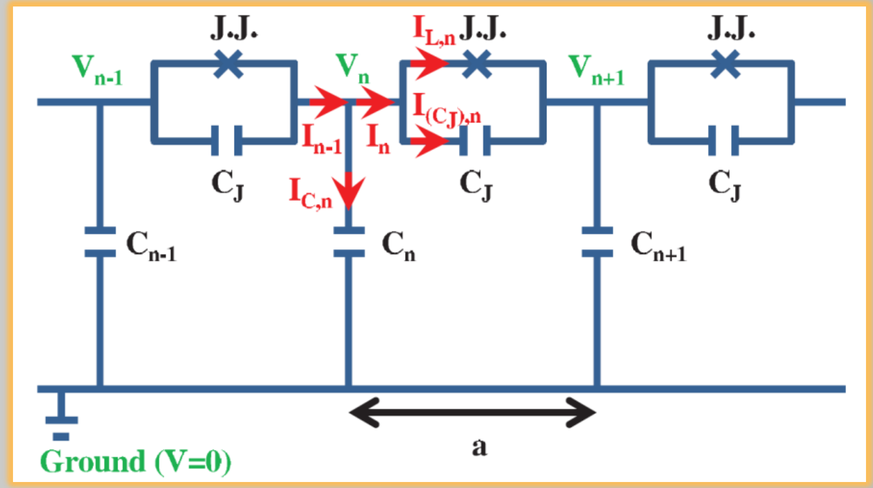


S. Pagano *et al.*, "Development of Quantum Limited Superconducting Amplifiers for Advanced Detection," in IEEE Transactions on Applied Superconductivity, **32**, 4, 1-5 (2022)

DART WARS

State of the art: the theoretical model

The TWJPA is a series of current-biased JJs with an input voltage control, used to amplify a weak signal.



$$V_{n+1} - V_n = -\frac{d\Phi_n}{d\tilde{t}} \qquad I_{L,n} = I_J \sin\left(\frac{\Phi_n}{\varphi_0}\right) \qquad \frac{dI_{L,n}}{dt} = \frac{I_J}{\varphi_0}\cos\left(\frac{\Phi_n}{\varphi_0}\right)\frac{d\Phi_n}{d\tilde{t}}$$

$$\frac{d\Phi_n}{d\tilde{t}} = \frac{\varphi_0}{I_J}\left[1 - \sin^2\left(\frac{\Phi_n}{\varphi_0}\right)\right]^{-\frac{1}{2}} \frac{dI_{L,n}}{dt} \qquad \frac{d\Phi_n}{d\tilde{t}} = \frac{\varphi_0}{I_J}\left[1 - \left(\frac{I_{L,n}}{I_J}\right)\right]^{-\frac{1}{2}} \frac{dI_{L,n}}{dt}$$

For a **weak nonlinearity**, $I_{L,n} \ll I_J$:

$$\frac{d\Phi_n}{d\tilde{t}} = \frac{\varphi_0}{I_J}\left[1 + \frac{1}{2}\left(\frac{I_{L,n}}{I_J}\right)^2\right] \frac{dI_{L,n}}{dt}$$

from which, **expanding further the nonlinear term**, one obtain

$$I_{L,n} = -\frac{1}{L}(\tilde{\varphi}_{n+1} - \tilde{\varphi}_n) + \frac{\varphi_0}{6I_J^3 L^4}(\tilde{\varphi}_{n+1} - \tilde{\varphi}_n)^3$$

The usual strategy is to seek a solution for the obtained "weakly nonlinear wave equation" as a **superposition of three waves (pump, signal, and idler)**

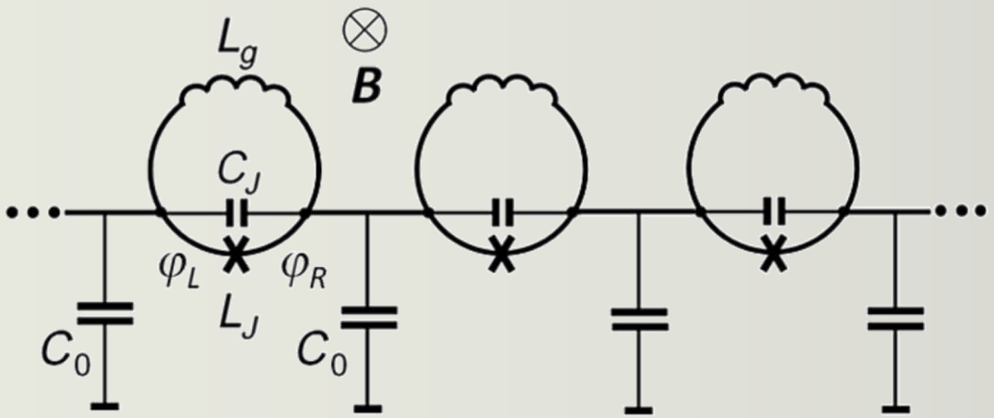
$$\tilde{\varphi}(x,t) = \left[\tilde{A}_p(x)e^{i\psi_p} + \tilde{A}_s(x)e^{i\psi_s} + \tilde{A}_i(x)e^{i\psi_i} + \text{c. c.}\right]/2$$

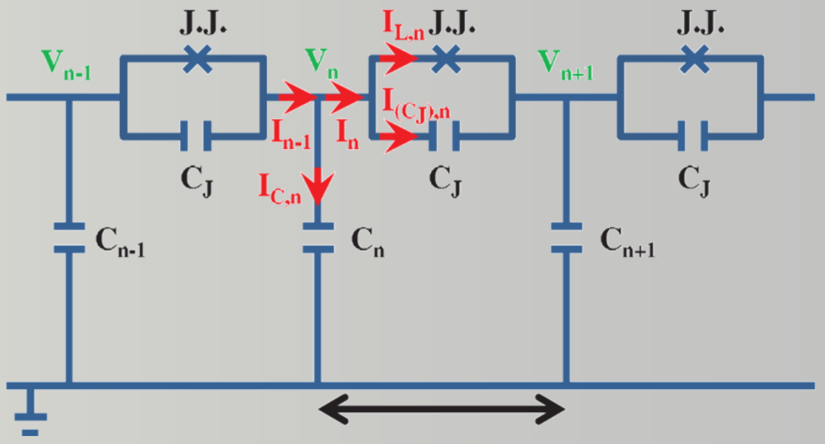
Here, the first 3 terms describe weakly dispersive linear waves with spatially dependent phase velocity, while the 4th and 5th terms represent the nonlinearity and dissipation.
It is the **combination of the weak dispersion and cubic nonlinearity** which allows efficient parametric amplification via the **four-wave mixing (4WM)** process.

Yaakoby, *et al.*, PRB **87**, 144301 (2013)

State of the art: the amplification mechanisms

Two different operative modes:





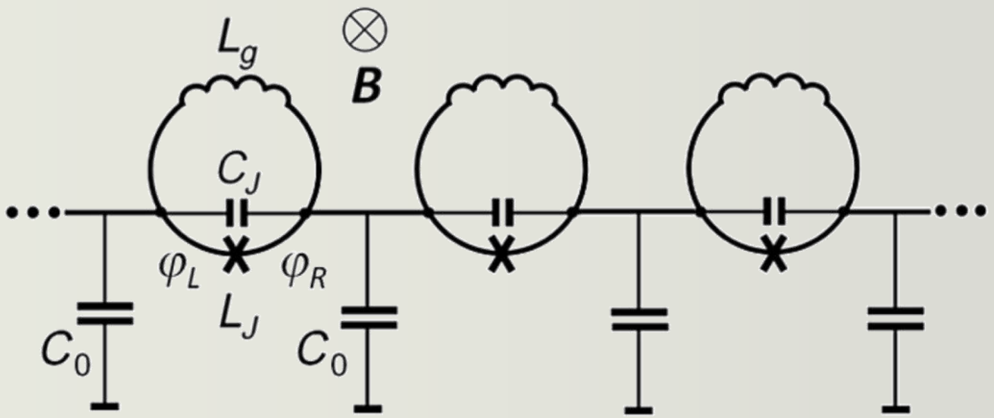
Zorin showed that by embedding a chain of rf-SQUIDs into a coplanar waveguide, it is possible to tune both the 2nd and 3rd order nonlinearities of their CPR. This is a novel approach to the TWJPA, for the possibility to use a **quadratic term** as a source of nonlinearity allows to work in the so called **3-Wave Mixing (3WM)** regime.

Zorin, PRAppl. **6**, 034006 (2016)
Zorin, PRAppl. **12**, 044051 (2019)

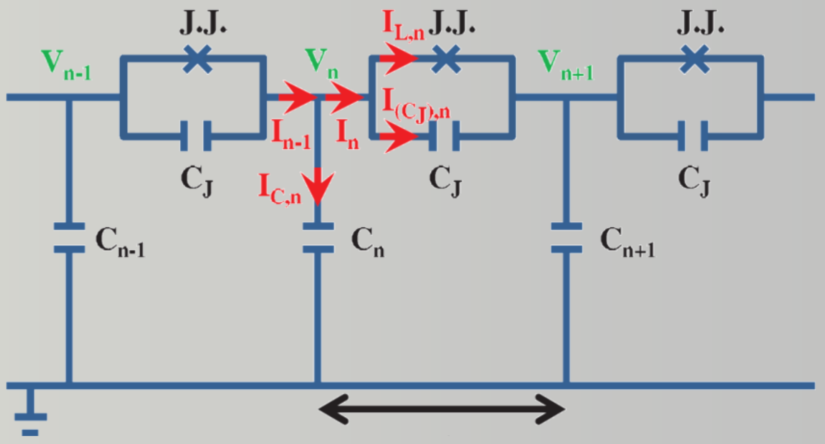
Yaakoby, *et al.*, PRB **87**, 144301 (2013)

State of the art: the amplification mechanisms

Two different operative modes:



3-Wave Mixing (3WM)
$\omega_p = \omega_s + \omega_i$
biased transmission line



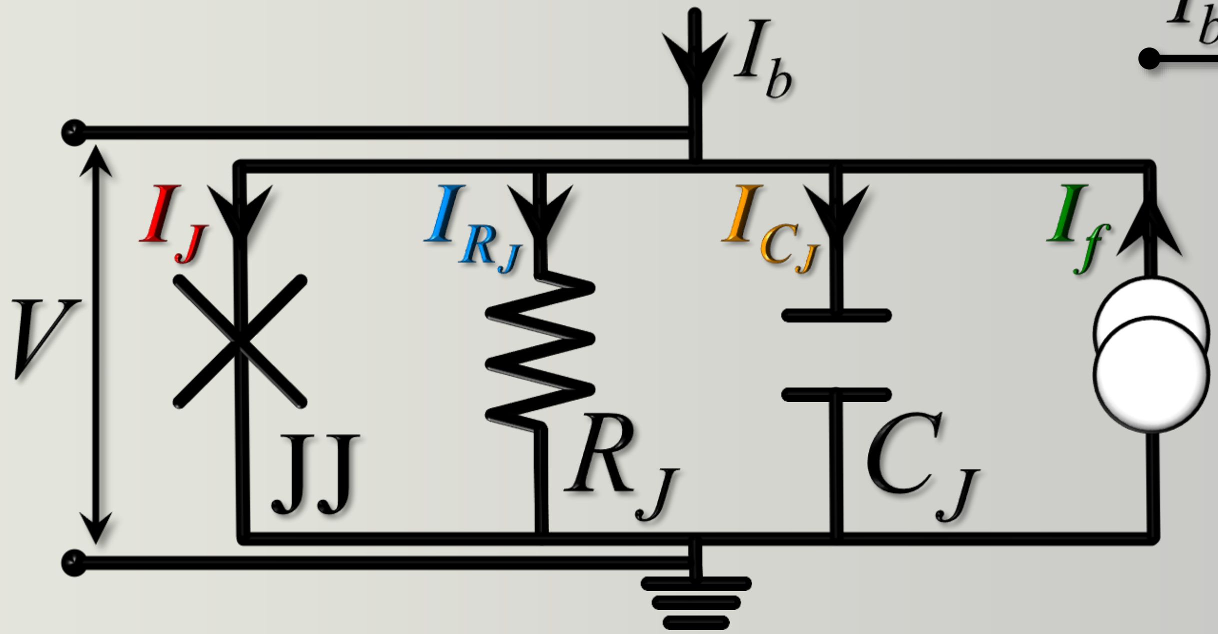
4-Wave Mixing (4WM)
$2\omega_p = \omega_s + \omega_i$
unbiased transmission line

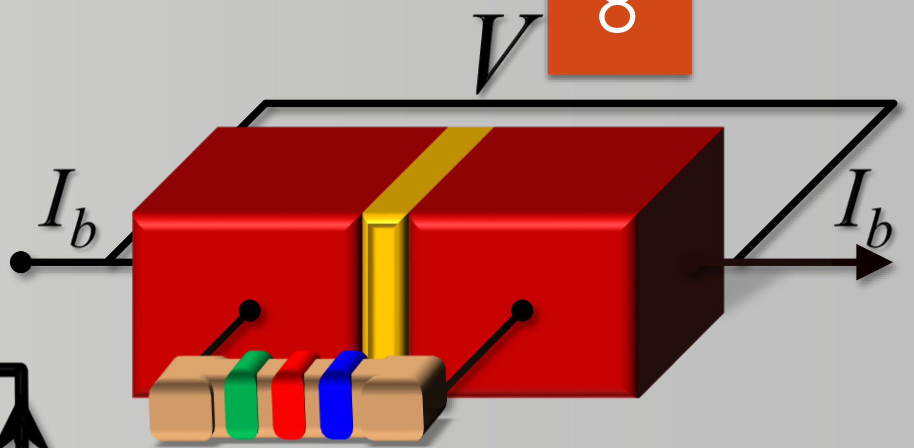
Zorin, PRAppl. **6**, 034006 (2016)
Zorin, PRAppl. **12**, 044051 (2019)

Yaakoby, *et al.*, PRB **87**, 144301 (2013)

The Resistively Capacitance Shunted Junction (RCSJ) model

Short Josephson junction





$$I_J = I_c \sin\varphi$$

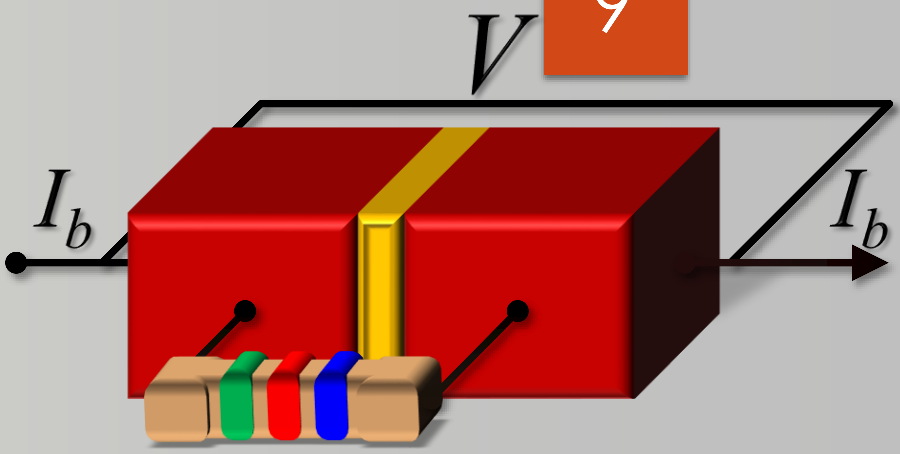
$$I_f = \text{White noise}$$

$$I_{R_J} = \frac{V}{R_J} = \frac{1}{R_J}\frac{\Phi_0}{2\pi}\frac{d\varphi}{dt}$$

$$I_{C_J} = C_J\frac{dV}{dt} = \frac{\Phi_0}{2\pi}C_J\frac{d^2\varphi}{dt^2}$$

$$I_b + I_f = I_J + I_{R_J} + I_{C_J}$$

The Resistively Capacitance Shunted Junction (RCSJ) model



$|I_b| > I_c$

finite voltage state

$\varphi(t)$ evolves in time – dynamic state of the JJ

The total current is composed by

I_J ➢ ***Josephson «dissipationless» channel***

I_{R_J} ➢ **Resistive channel**

At $T > 0$, there is a finite probability for Cooper pairs to be broken up by thermal excitation thereby generating unpaired "normal" electrons (i.e., quasiparticles). If $V \neq 0$, these normal electrons contribute to the current. In contrast to the Josephson current, this normal current channel is resistive.

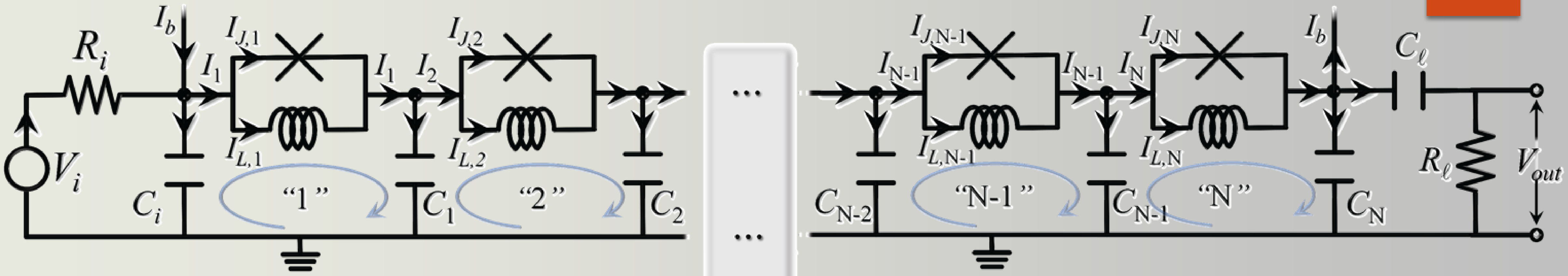
I_{C_J} ➢ ***Capacitive channel***

An SIS tunnel JJ just represents a parallel plate capacitor. In the presence of $V(t) \neq 0$ we have a finite displacement current across this capacitor.

I_f ➢ ***Noise contribution***

A temperature-dependent fluctuating current contribution

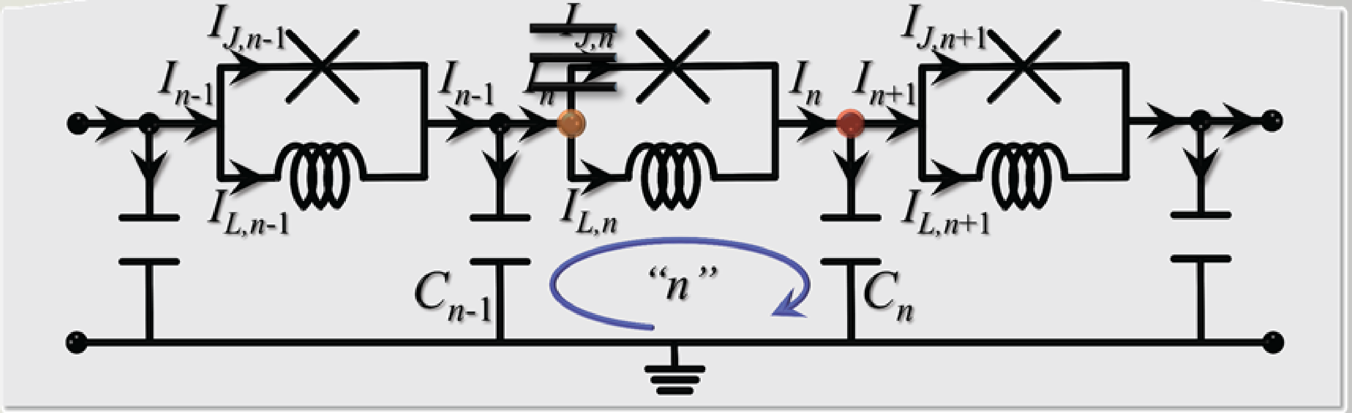
The TWJPA: the design and the modelling



Chip TWJPA_X52 specifics

$R_i = 50\ \Omega$
$C_i = 24\ \text{fF}$
$R_\ell = 50\ \Omega$
$C_\ell = 1\ \text{nF}$

$C_n = 24\ \text{fF}$
$L_n = 120\ \text{pH}$

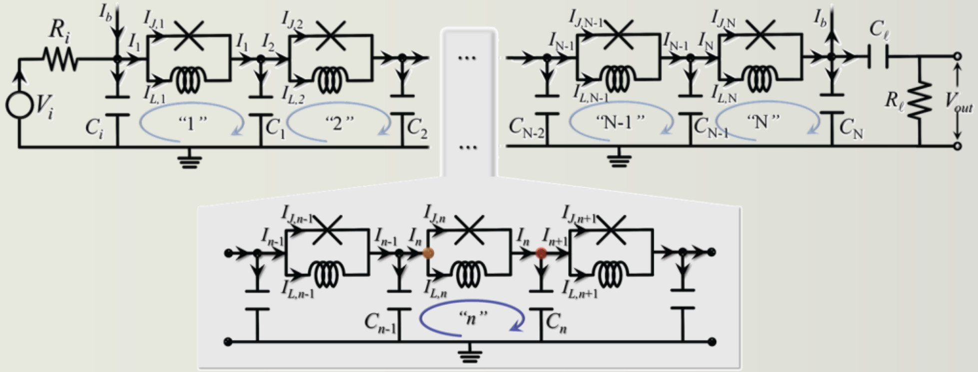


$N = 990$ JJs

$R_J = 20\ \text{k}\Omega$
$C_J = 200\ \text{fF}$
$I_c = 2\ \mu\text{A}$

The TWJPA: the design and the modelling

$$\left\| \begin{matrix} a_{1,2} & a_{1,3} & 0 & \dots & 0 \\ a_{2,1} & a_{2,2} & a_{2,3} & \dots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \dots & a_{N-1,1} & a_{N-1,2} & a_{N-1,3} \\ 0 & \dots & 0 & a_{N,1} & a_{N,2} \end{matrix} \right\| \left\| \begin{matrix} \varphi_1^{m+1} \\ \varphi_2^{m+1} \\ \vdots \\ \varphi_{N-1}^{m+1} \\ \varphi_N^{m+1} \end{matrix} \right\| = \left\| \begin{matrix} A_1 \\ A_2 \\ \vdots \\ A_{N-1} \\ A_N \end{matrix} \right\|$$



$\equiv$

$$A_n = b_{n,1} f_{n-1}^m + b_{n,2} \widetilde{f}_n^m + b_{n,3} f_{n+1}^m + c_{n,1} \varphi_{n-1}^{m-1} + c_{n,2} \varphi_n^{m-1} + c_{n,3} \varphi_{n+1}^{m-1}$$
$$for \quad n = 1, \dots N, \quad m = 1, 2, \dots M,$$

$$A_1 = b_{1,1} I_i^m + b_{1,2} \widetilde{f}_1^m + b_{1,3} f_2^m + c_{1,2} \varphi_1^{m-1} + c_{1,3} \varphi_2^{m-1} + C_1^- I_b$$
$$for \quad n = 0, \quad m = 1, 2, \dots M,$$

$$A_N = b_{N,1} f_{N-1}^m + b_{N,2} \widetilde{f}_N^m + b_{n,3} I_\ell^m + c_{N,1} \varphi_{N-1}^{m-1} + c_{N,2} \varphi_N^{m-1} + I_b.$$
$$for \quad n = N, \quad m = 1, 2, \dots M.$$

$$I_i^{m+1} = I_i^{m-1} + \left[\varphi_1^{m+1} \alpha_1^+ - I_i^m + f_1^m + \varphi_1^{m-1} \alpha_1^- + \left(C_i \dot{V}_i^m - I_b \right) \right] (2k\omega_i)$$
$$I_\ell^{m+1} = I_\ell^{m-1} + \left[f_N^m - \left(1 + \frac{C_N}{C_\ell} \right) I_\ell^m - \varphi_N^{m+1} \alpha_N^+ + \varphi_N^{m-1} \alpha_N^- - I_b \right] \frac{2k}{C_N R_\ell}$$

1) Transmission Line

First, we assume an input voltage equal to

$$V_i = V_{pump}\sin(2\pi\omega_{pump}t) + \cancel{V_{sign}\sin(2\pi\omega_{sign}t)}$$

in the specific case of $V_{pump} \neq 0$ and $V_{sign} \rightarrow 0$

2) TWPA

Then, we assume an input voltage equal to

$$V_i = V_{pump}\sin(2\pi\omega_{pump}t) + V_{sign}\sin(2\pi\omega_{sign}t)$$

in the specific case of $V_{pump} \neq 0$ and $V_{sign} \neq 0$

Transmission Line with Josephson elements

Here, we assume an input voltage equal to

$$V_i = V_{pump} \sin(2\pi\omega_{pump} t) + V_{sign} \sin(2\pi\omega_{sign} t)$$

in the specific case of $V_{pump} \neq 0$ and $V_{sign} \to 0$

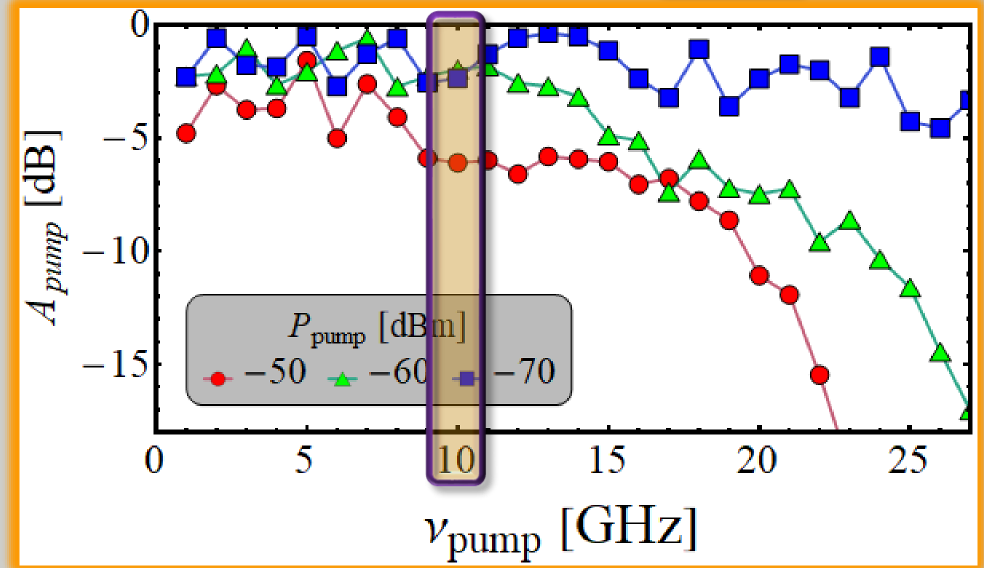
where $V = \sqrt{2R_i 0.001} \times 10^{\frac{P}{20}}$ and $R_i = 50\Omega$.

Next steps…

	ω_{pump} [GHz]	P_{pump} [dBm]	I_{bias} [μA]
1)	$[1 \div 27]$	$[-50, -60, -70]$	0
2)	7	$[-50, -60, -70]$	$[0 \div 25]$

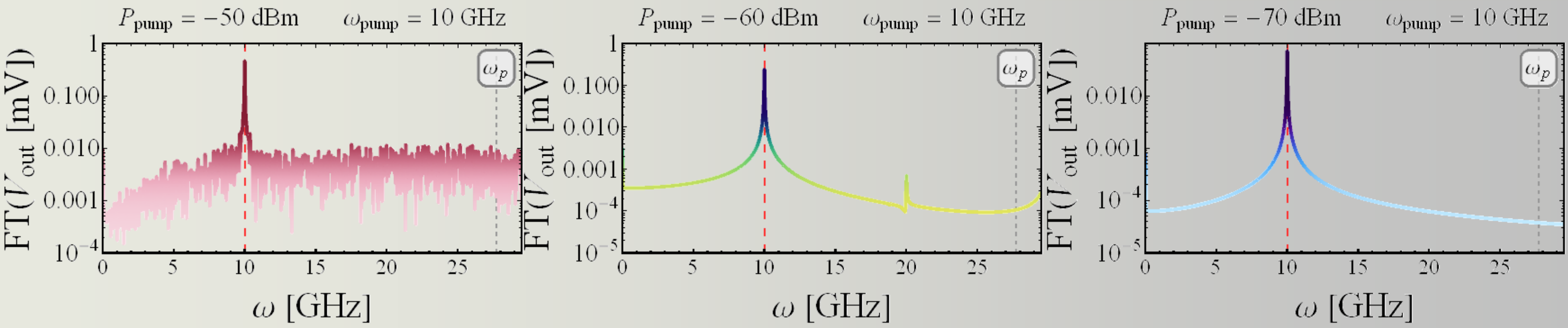
The limit response vs ν_{pump}

	ω_{pump} [GHz]	P_{pump} [dBm]	I_{bias} [μA]
1)	$[1 \div 27]$	$[-50, -60, -70]$	0



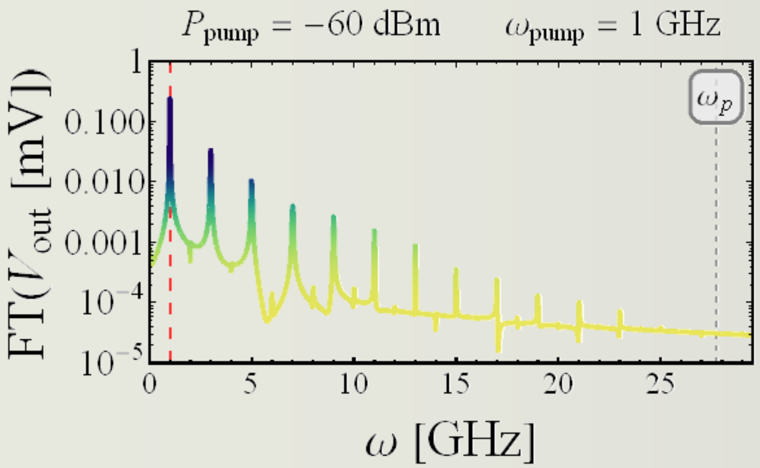
The **attenuation** is calculated as

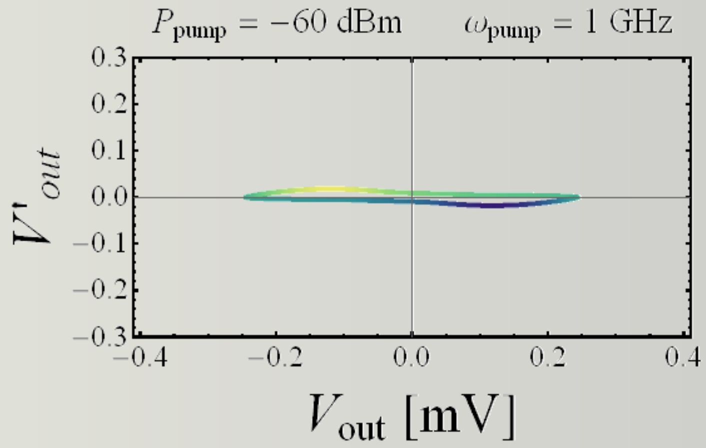
$$A_{pump} = 20\log_{10}\left(\frac{V_{out}}{V_{pump}}\right)\ \text{dB}.$$

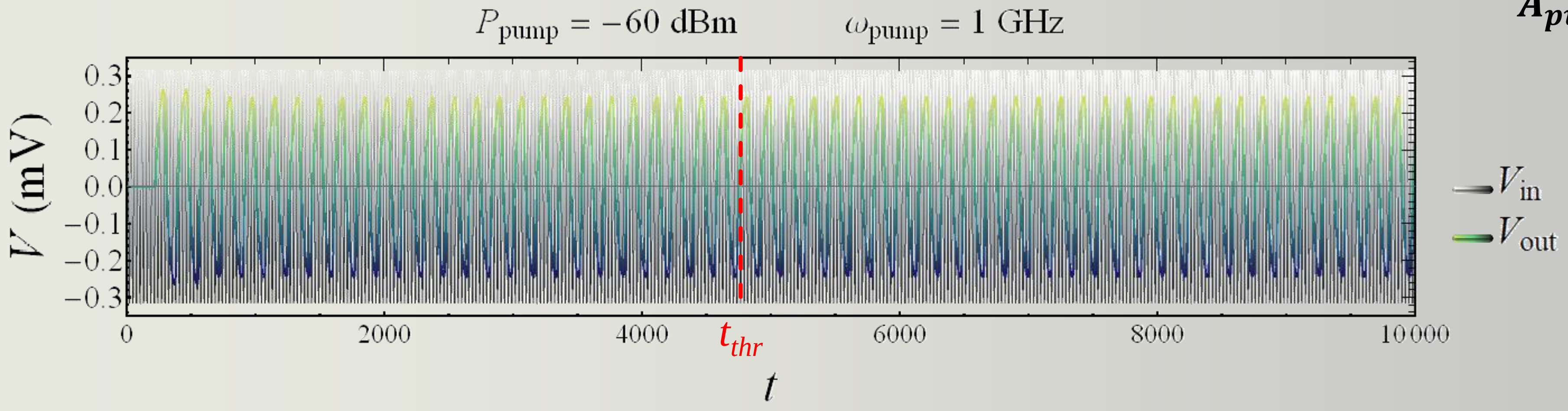


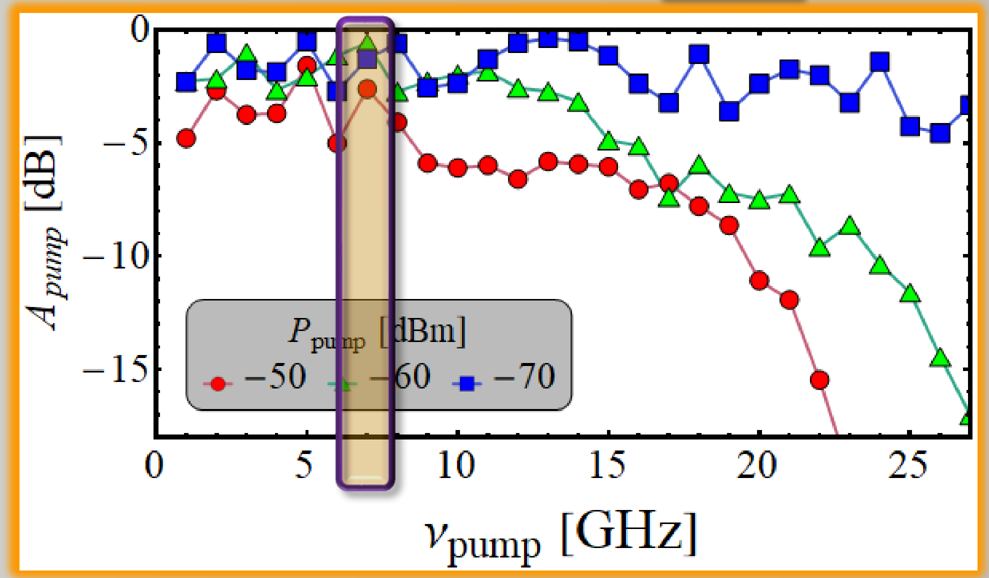
The limit response vs ν_{pump}

	ω_{pump} [GHz]	P_{pump} [dBm]	I_{bias} [μA]
1)	[1 ÷ 27]	−60	0





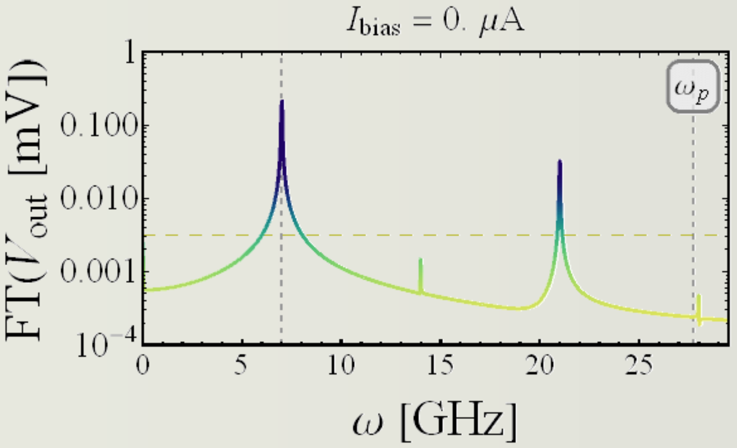


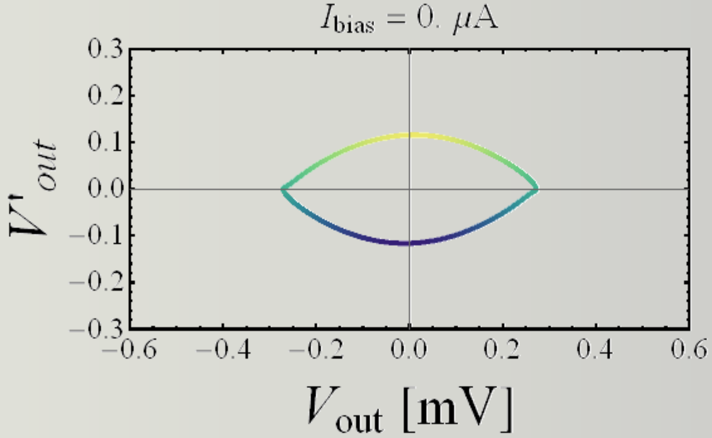


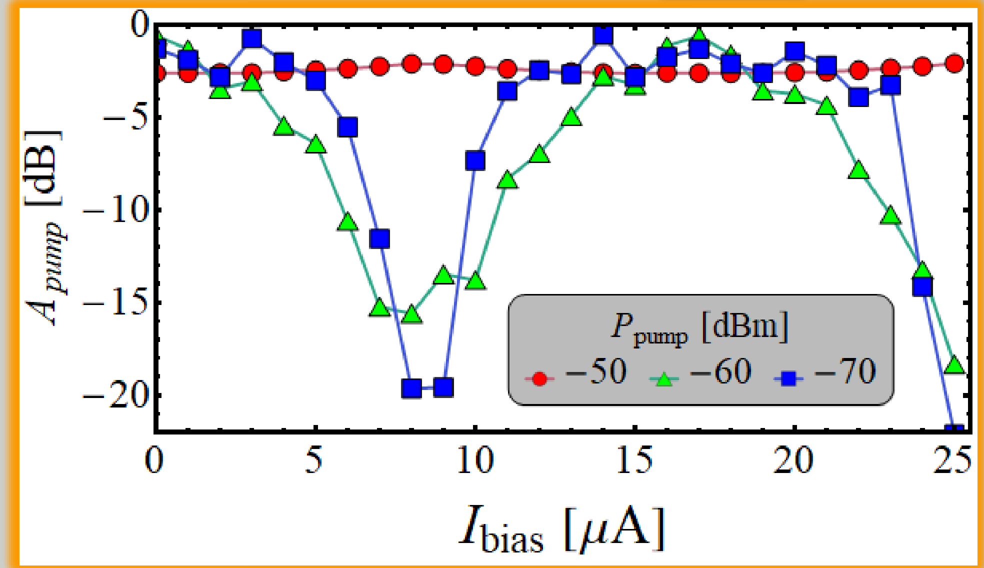
$$A_{pump} = 20\log_{10}\left(\frac{V_{out}}{V_{pump}}\right)\ \mathrm{dB}$$

The limit response vs I_{bias}

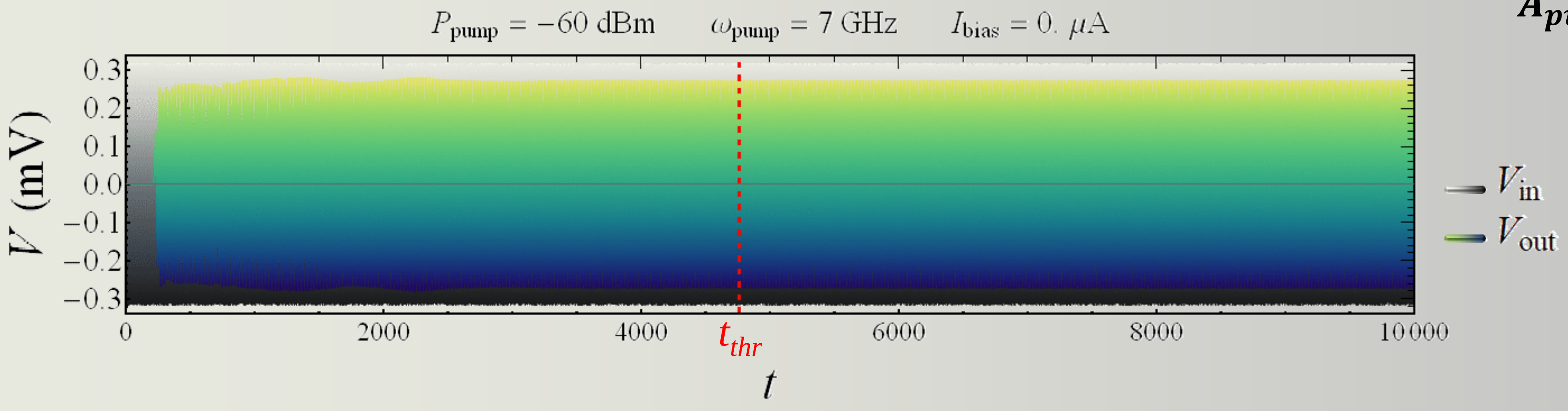
	ω_{pump} [GHz]	P_{pump} [dBm]	I_{bias} [μA]
2)	7	−60	[0 ÷ 25]





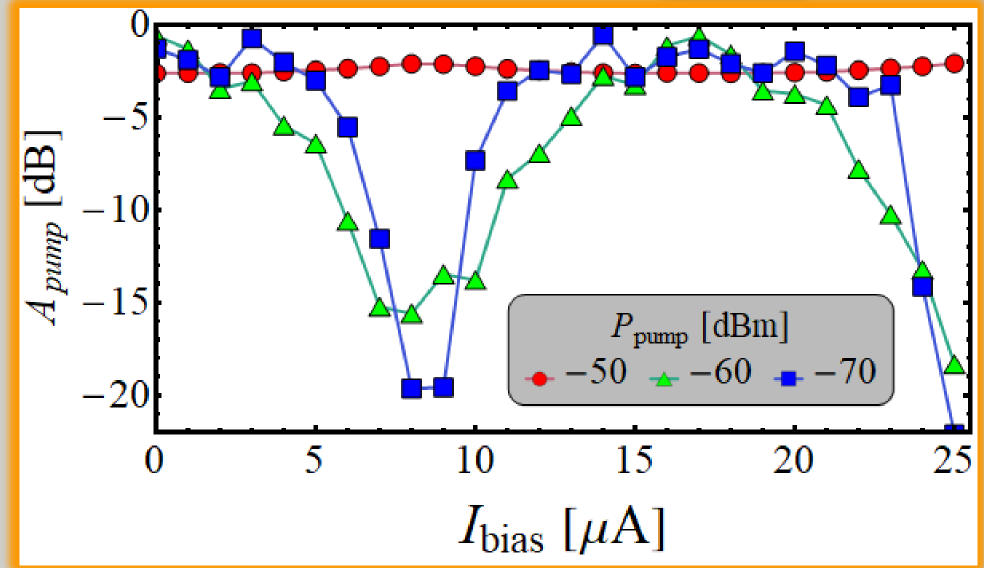


$$A_{pump} = 20 \log_{10}\left(\frac{V_{out}}{V_{pump}}\right)\ \text{dB}$$

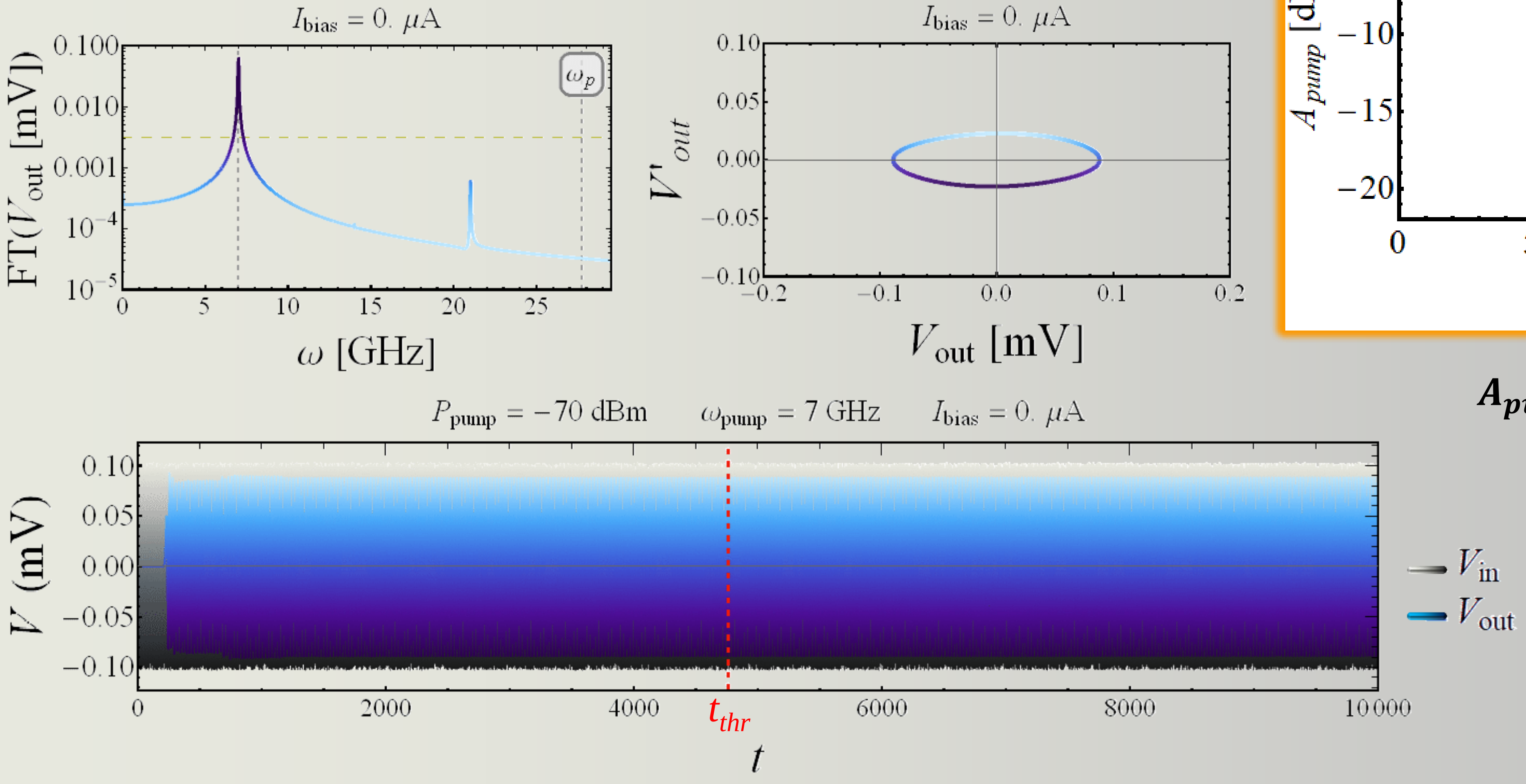


The limit response vs I_{bias}

	ω_{pump} [GHz]	P_{pump} [dBm]	I_{bias} [μA]
2)	7	-70	$[0 \div 25]$



$$A_{pump} = 20\log_{10}\left(\frac{V_{out}}{V_{pump}}\right)\ \text{dB}$$



JTWPA

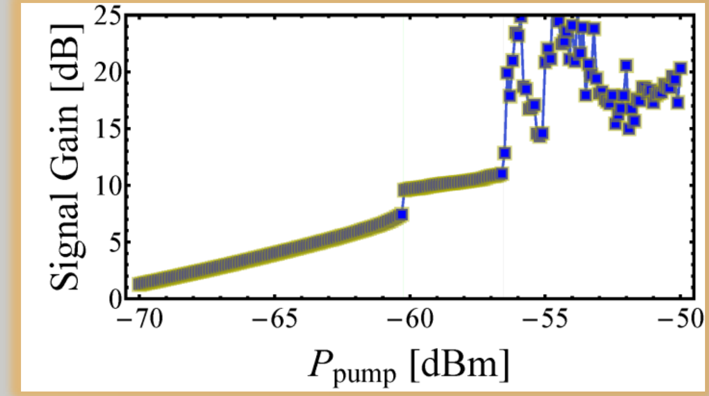
	P_{sign} [dBm]	ω_{pump} [GHz]	P_{pump} [dBm]	ω_{sign} [GHz]	I_{bias} [μA]
1)	−100	7	[−70 ÷ −50]	6	0

Here, we assume an input voltage equal to

$$V_i = V_{pump}\sin(2\pi\omega_{pump}t) + V_{sign}\sin(2\pi\omega_{sign}t)$$

where $V = \sqrt{2R_i 0.001} \times 10^{\frac{P}{20}}.$

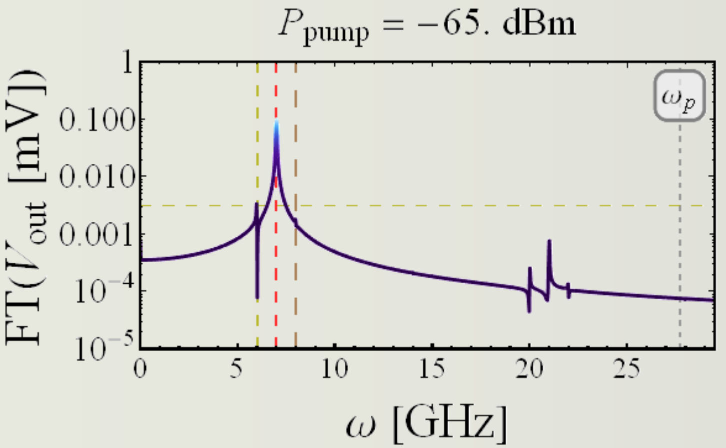
The **gain** is calculated as $\boldsymbol{Gain} = \mathbf{20}\log_{10}\left(\frac{V_{out}}{V_{sign}}\right)$ dB.

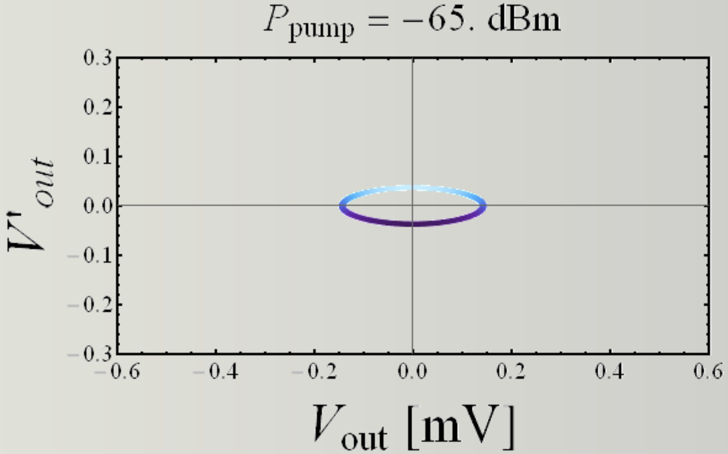


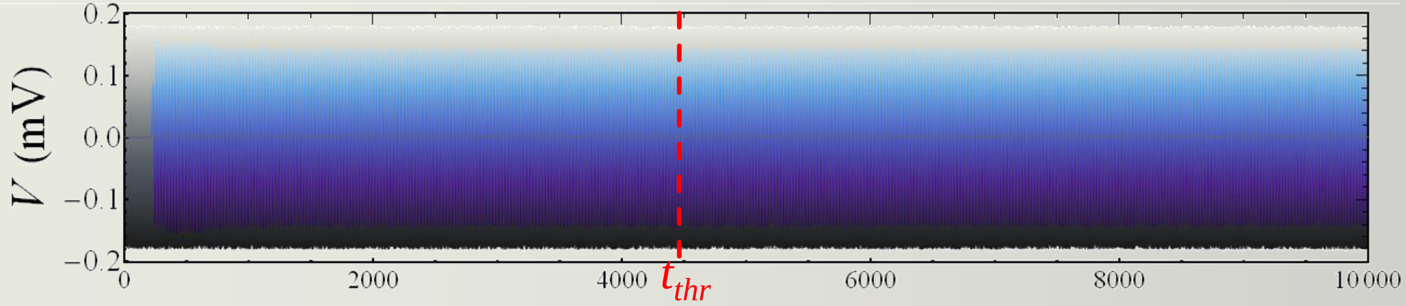
	P_{sign} [dBm]	ω_{pump} [GHz]	P_{pump} [dBm]	ω_{sign} [GHz]	I_{bias} [μA]
1)	−100	7	[−70 ÷ −50]	6	0
2)	−100	7	−60	[1 ÷ 27]	0
3)	−100	7	−60	6	[0 ÷ 25]

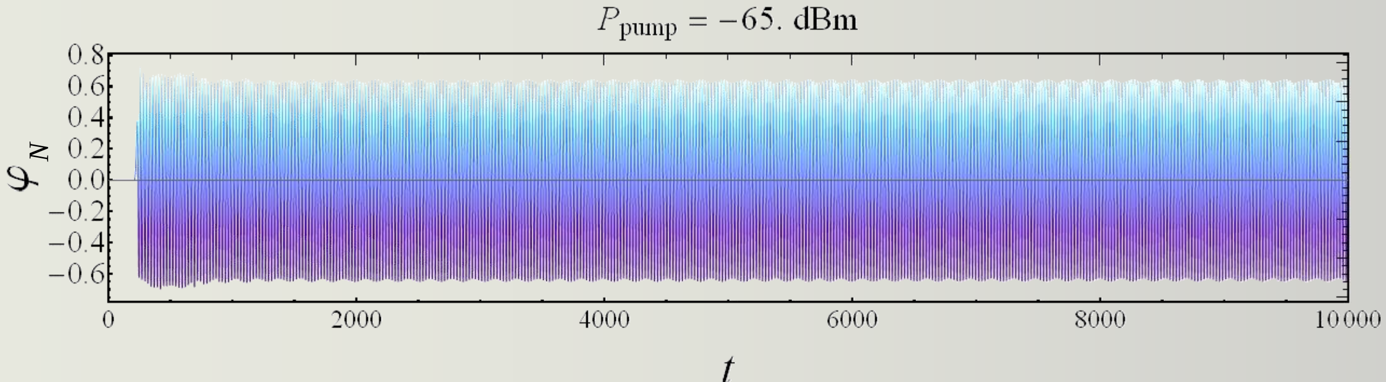
Gain vs P_{pump}

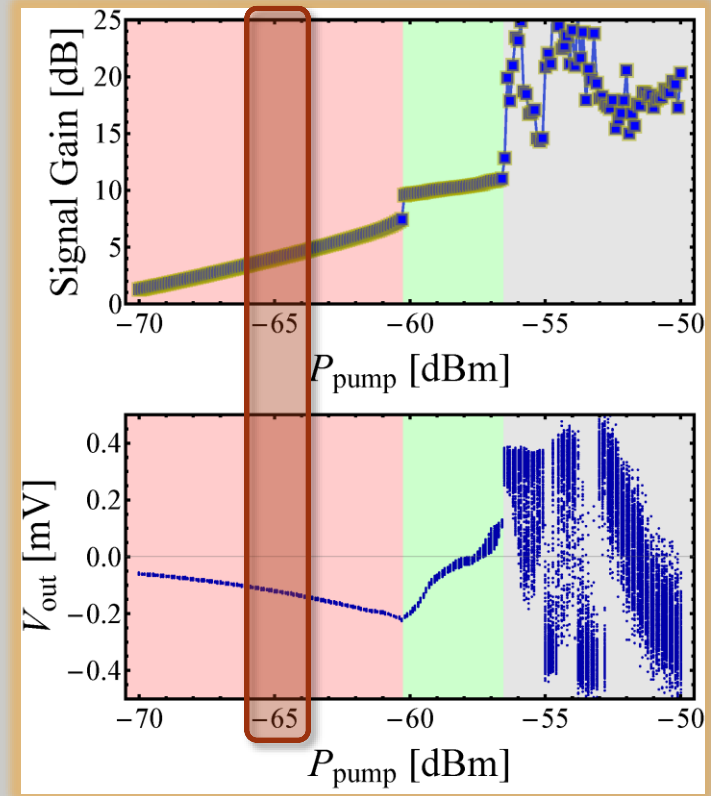
	P_{sign} [dBm]	ω_{pump} [GHz]	P_{pump} [dBm]	ω_{sign} [GHz]	I_{bias} [μA]
1)	−100	7	−65	6	0





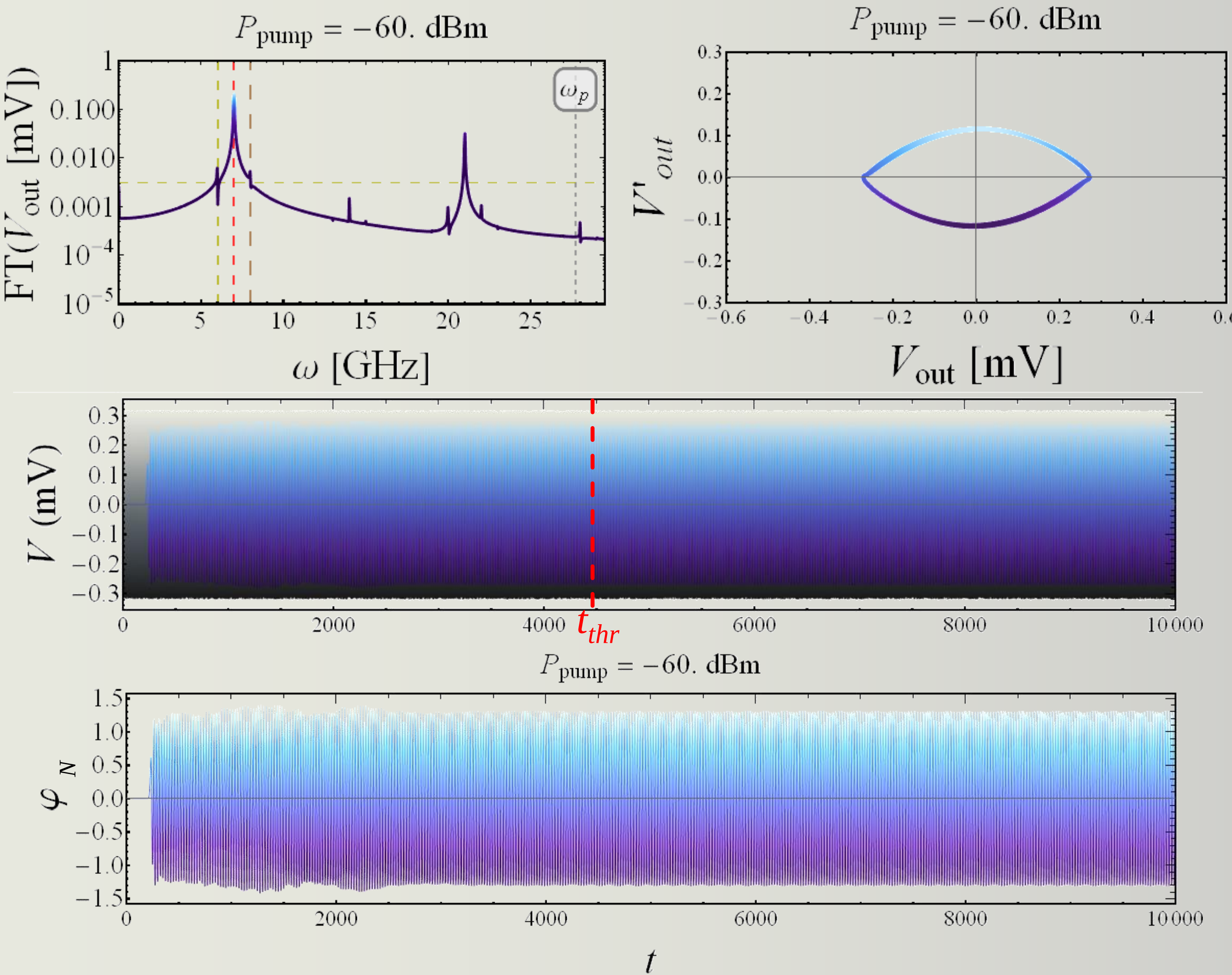


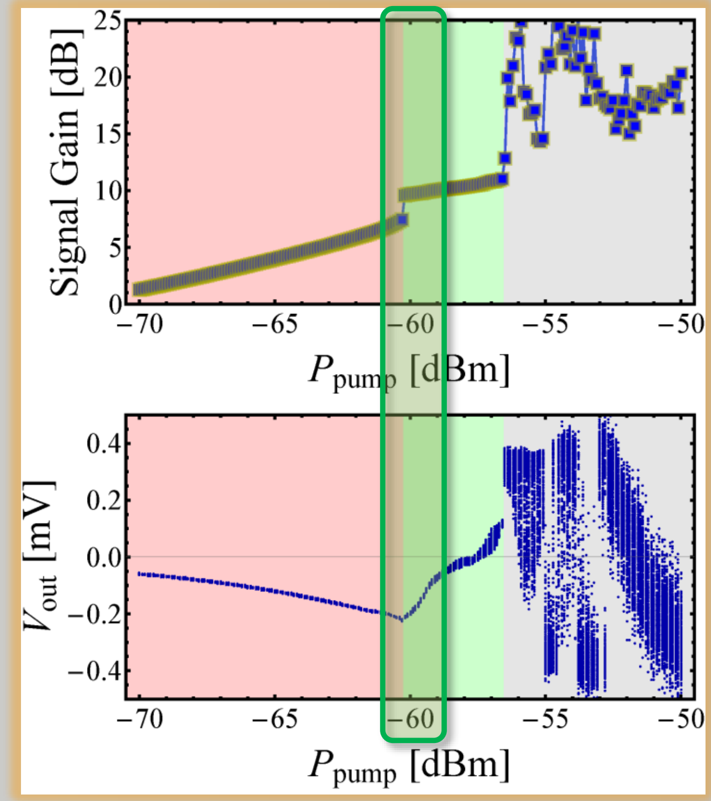




Gain vs P_{pump}

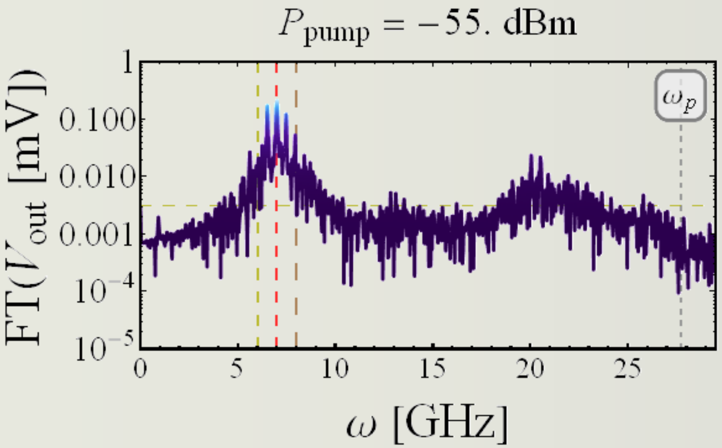
	P_{sign} [dBm]	ω_{pump} [GHz]	P_{pump} [dBm]	ω_{sign} [GHz]	I_{bias} [μA]
1)	−100	7	−60	6	0

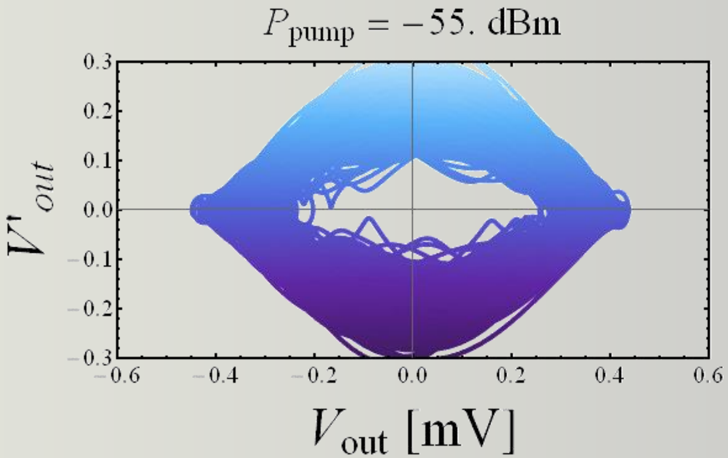


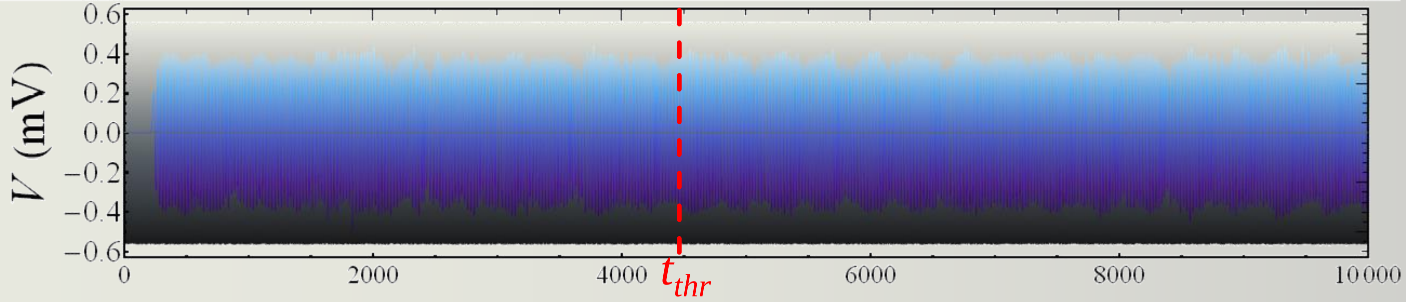


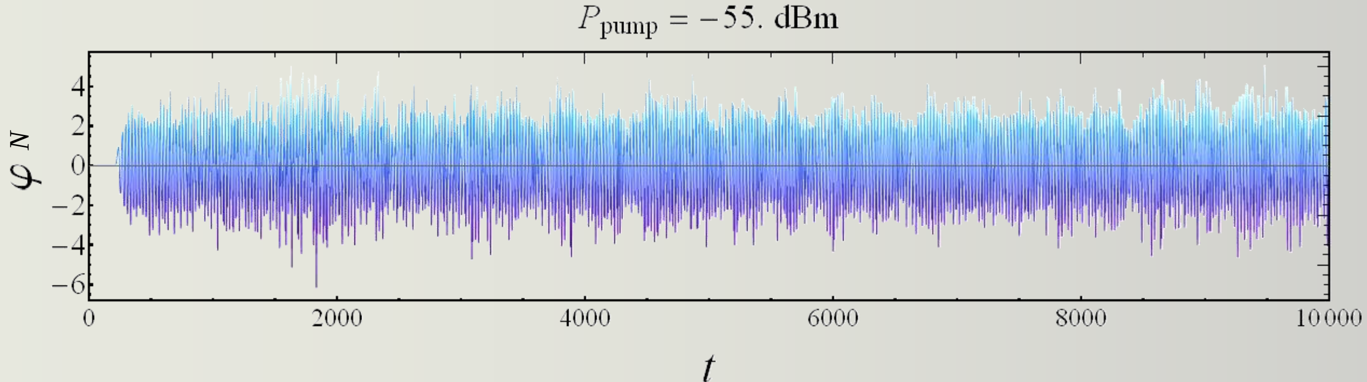
Gain vs P_{pump}

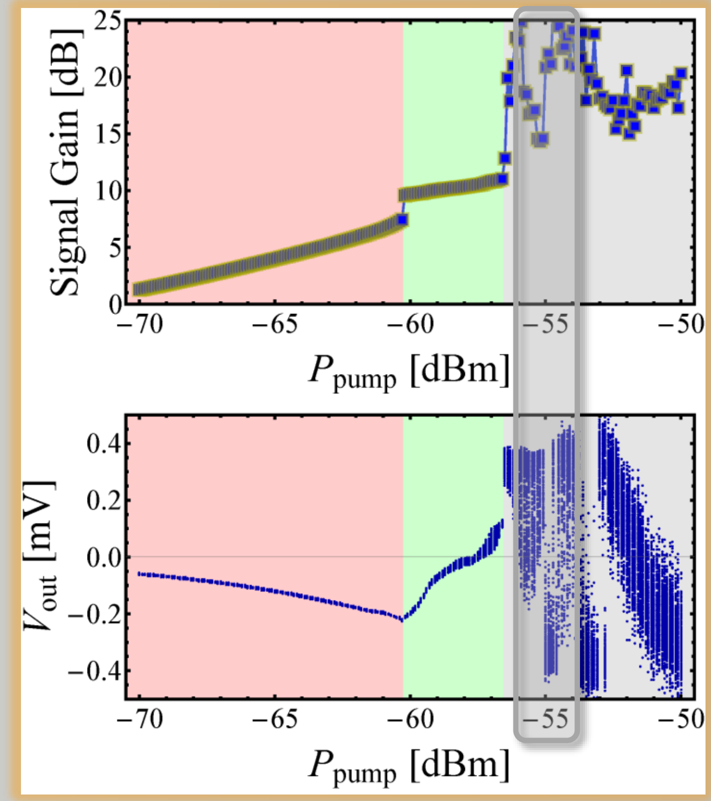
	P_{sign} [dBm]	ω_{pump} [GHz]	P_{pump} [dBm]	ω_{sign} [GHz]	I_{bias} [μA]
1)	−100	7	−55	6	0





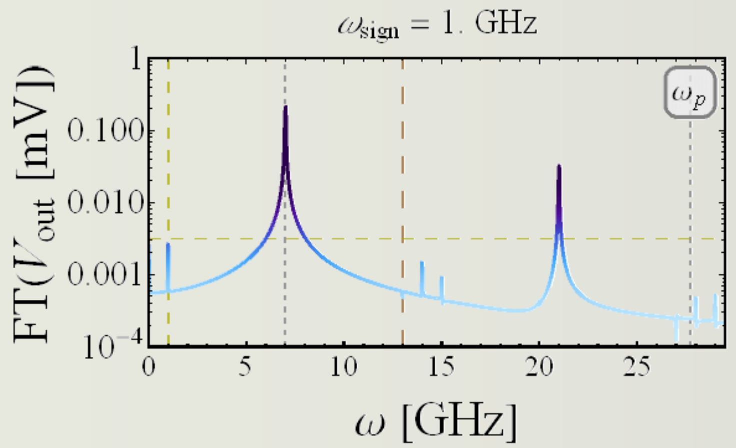


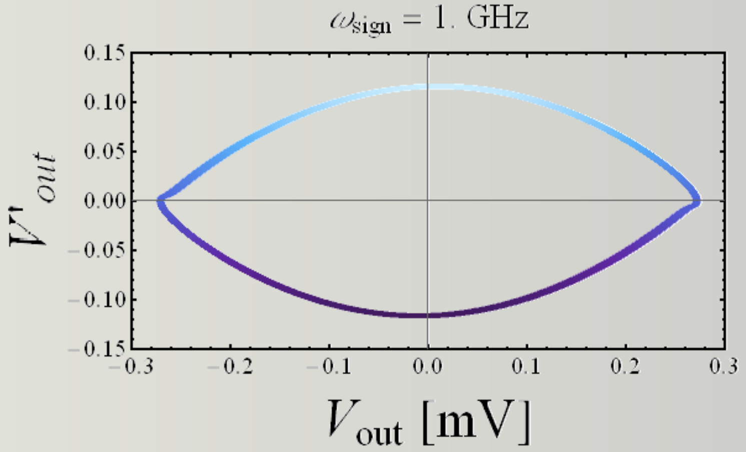


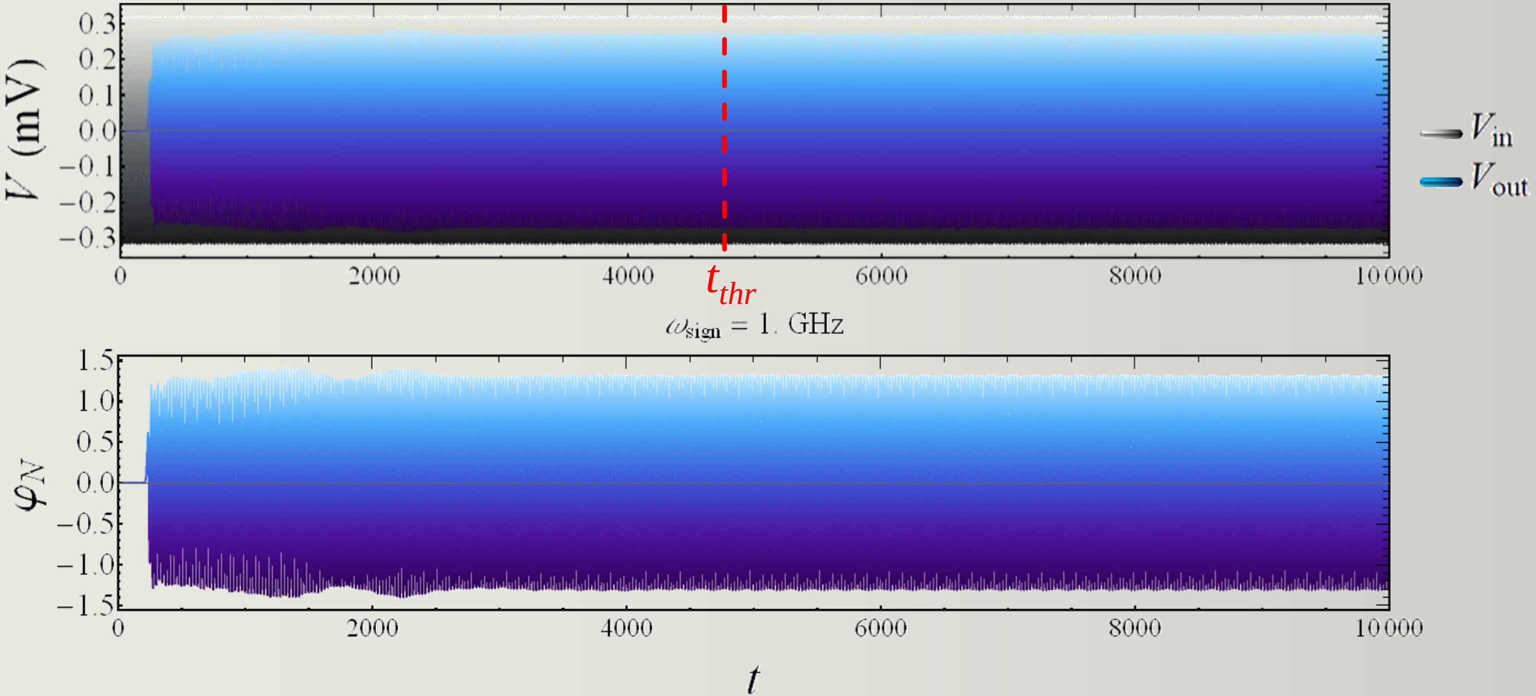


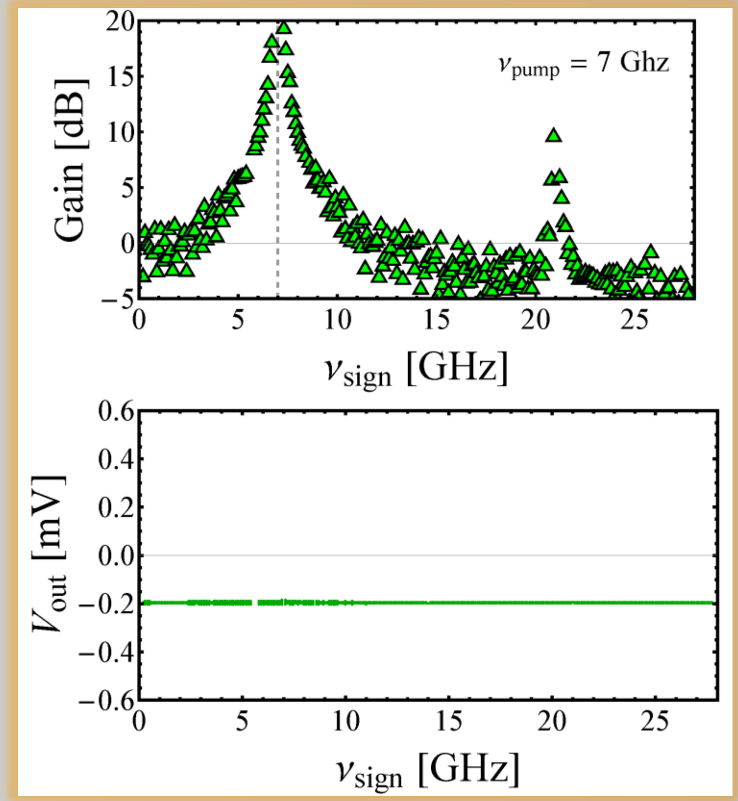
Gain vs ω_{sign}

	P_{sign} [dBm]	ω_{pump} [GHz]	P_{pump} [dBm]	ω_{sign} [GHz]	I_{bias} [μA]
2)	−100	7	−60	[1 ÷ 27]	0



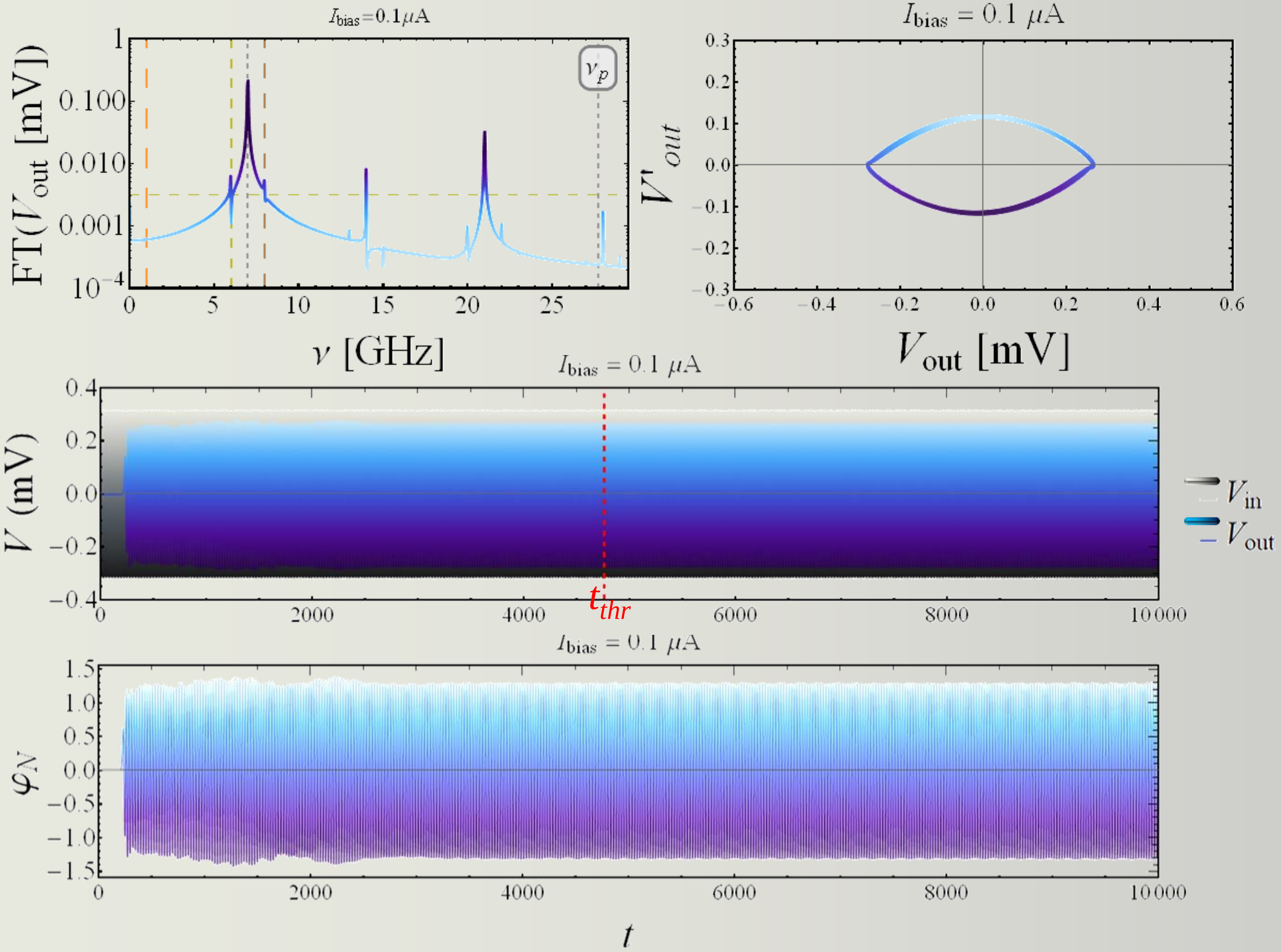


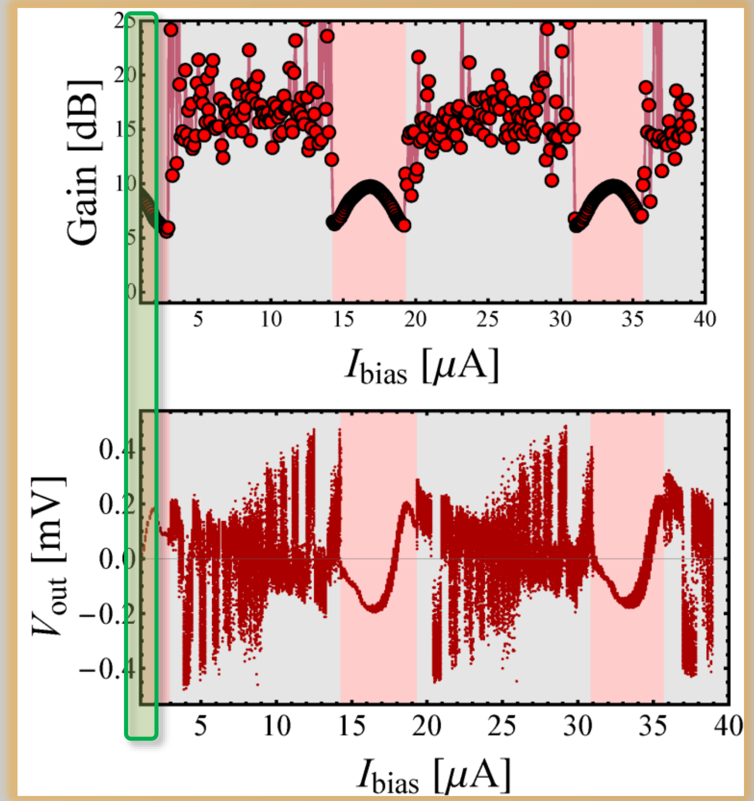




Gain vs I_{bias}

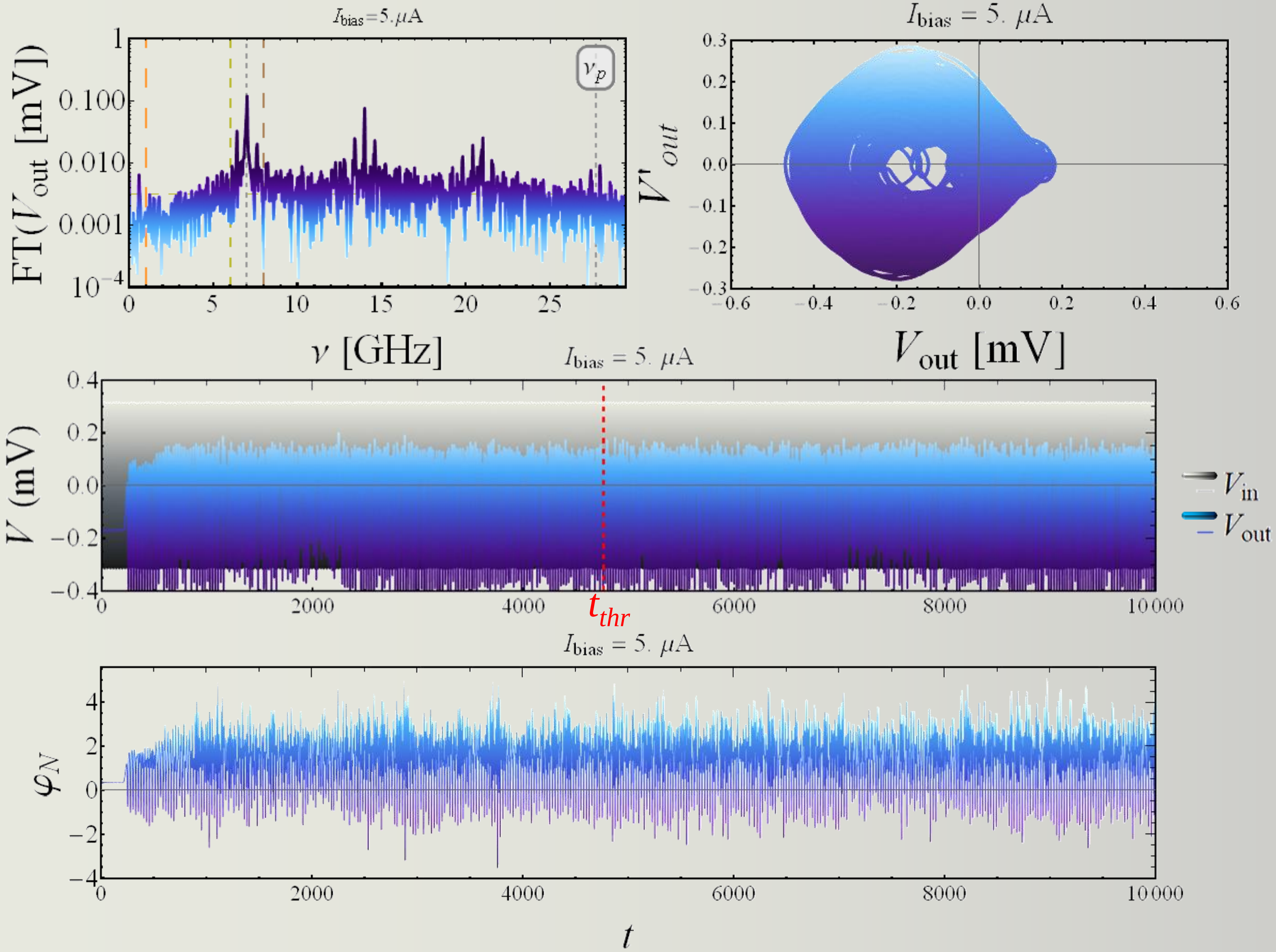
	P_{sign} [dBm]	ω_{pump} [GHz]	P_{pump} [dBm]	ω_{sign} [GHz]	I_{bias} [μA]
3)	−100	7	−60	6	[0 ÷ 25]

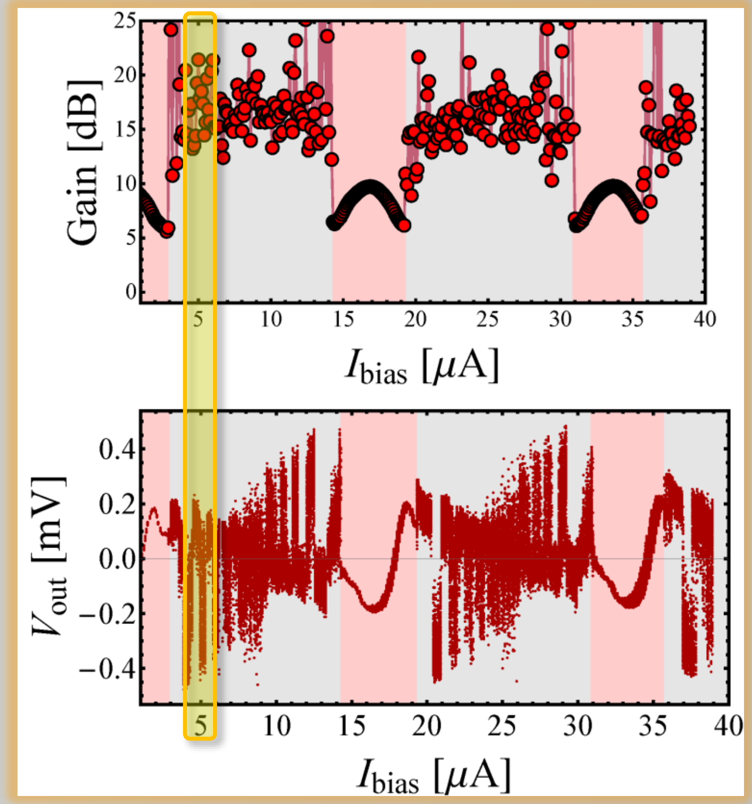




Gain vs I_{bias}

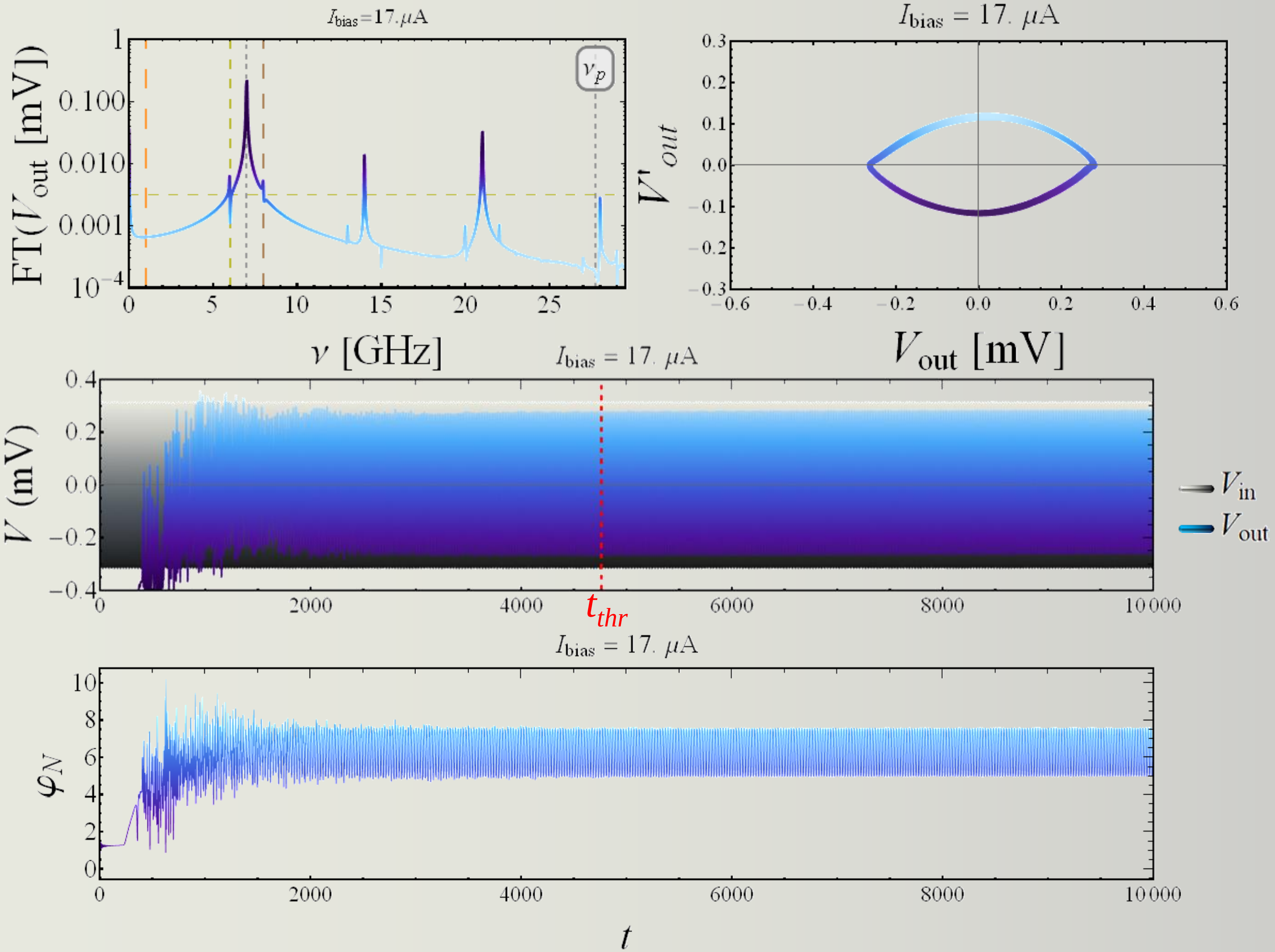
	P_{sign} [dBm]	ω_{pump} [GHz]	P_{pump} [dBm]	ω_{sign} [GHz]	I_{bias} [μA]
3)	-100	7	-60	6	[0 ÷ 25]

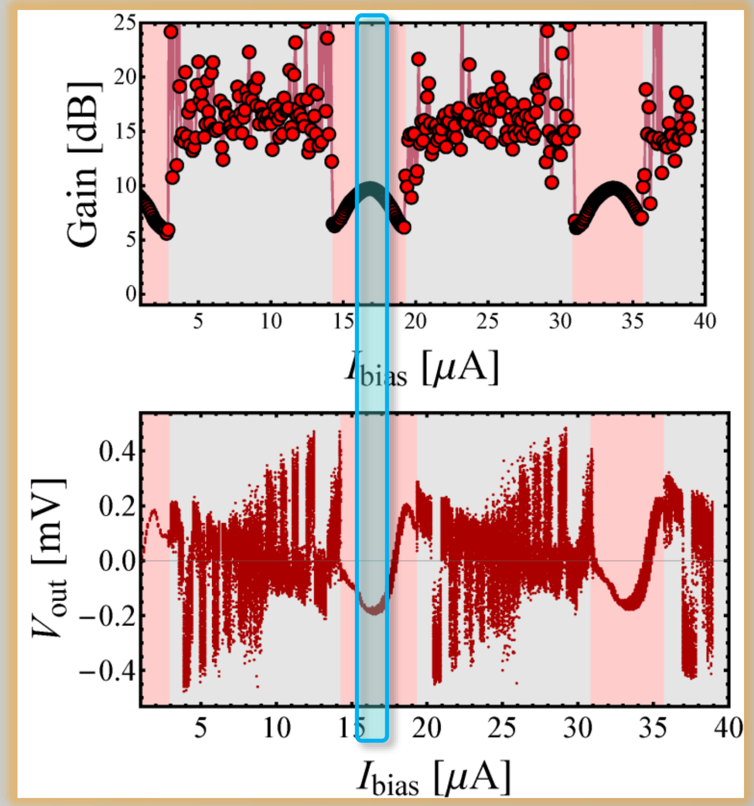




Gain vs I_{bias}

	P_{sign} [dBm]	ω_{pump} [GHz]	P_{pump} [dBm]	ω_{sign} [GHz]	I_{bias} [μA]
3)	-100	7	-60	6	[0 ÷ 25]



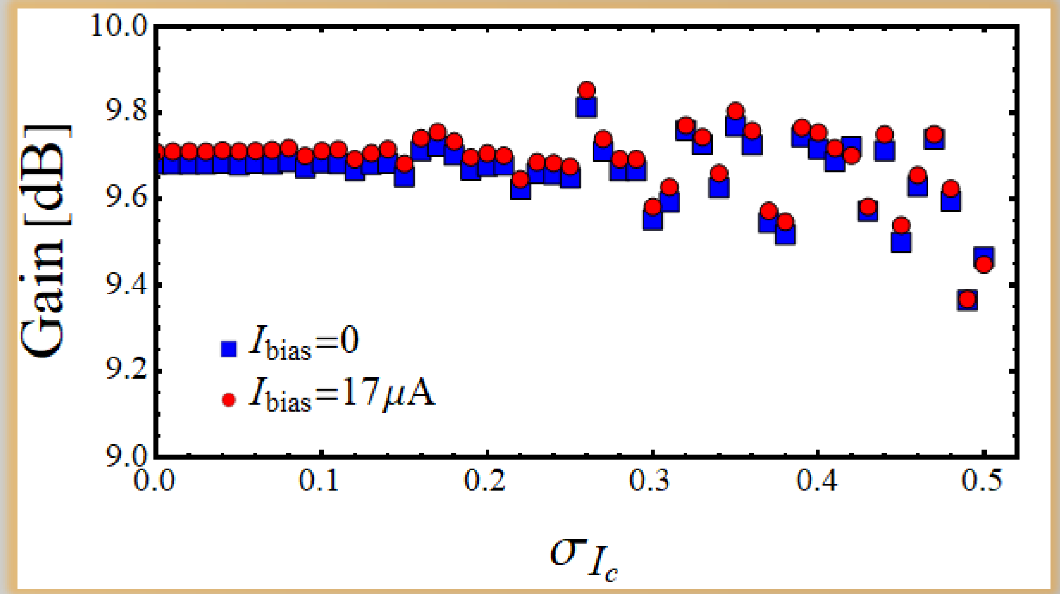


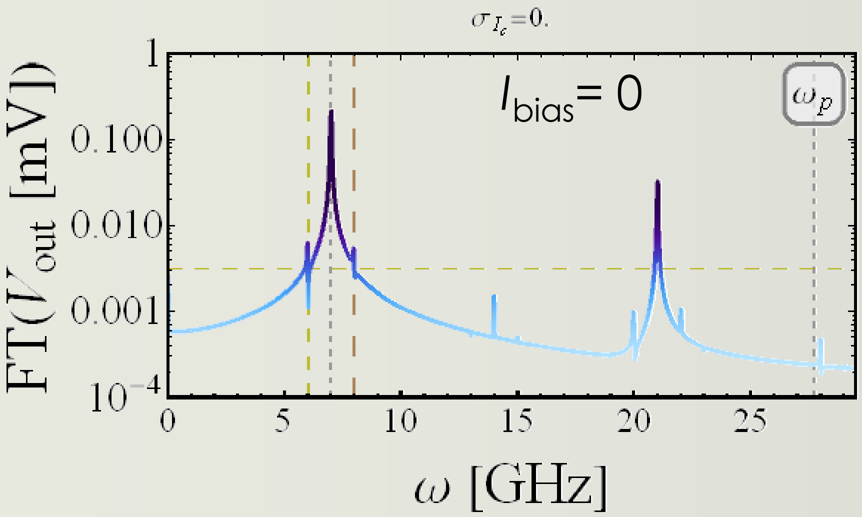
TWJPA – Gain vs δI_c

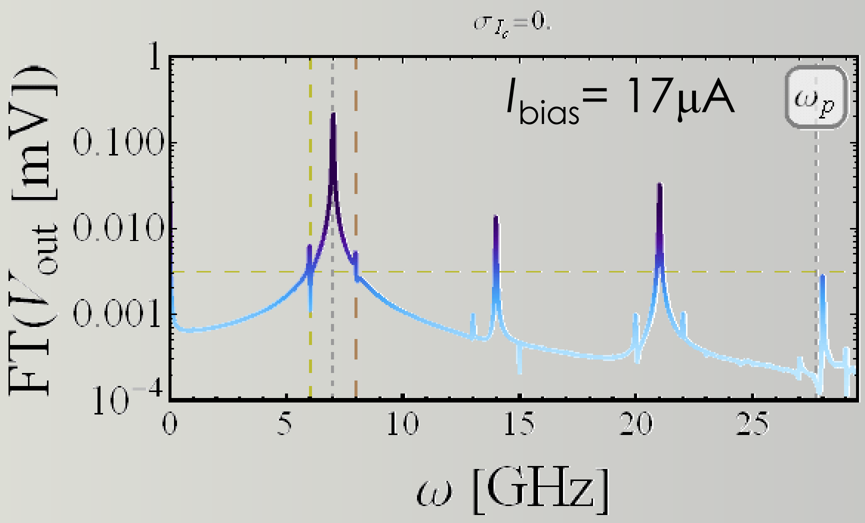
Here, we assume the critical currents to be distributed according to

$$I_{c,n} = \overline{I_c}(1 + \delta I_c)$$

with δI_c being a Gaussianly distributed number with zero average and variance $\sigma_{I_c}^2$.





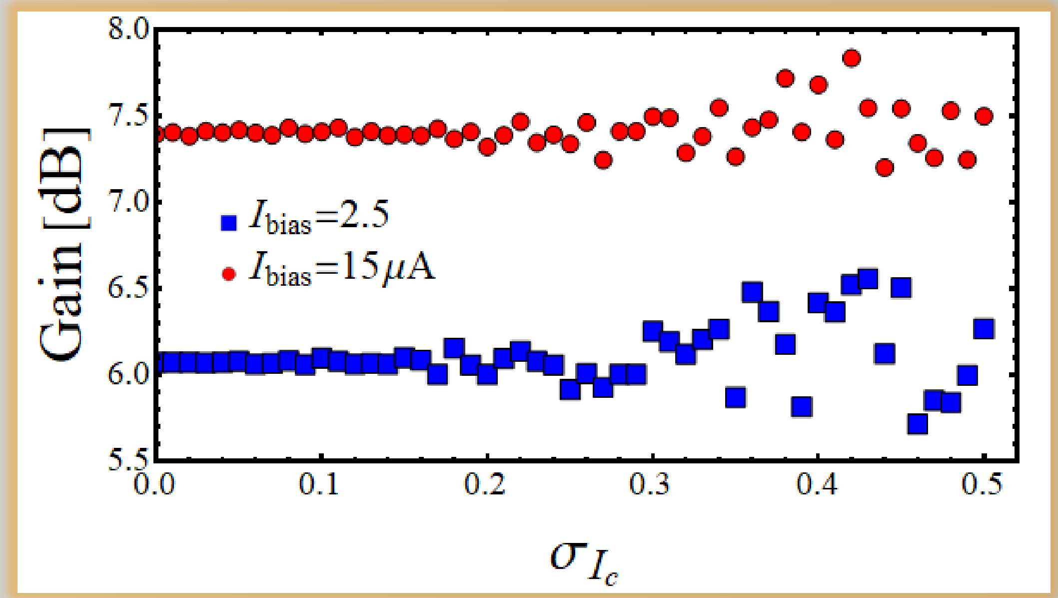


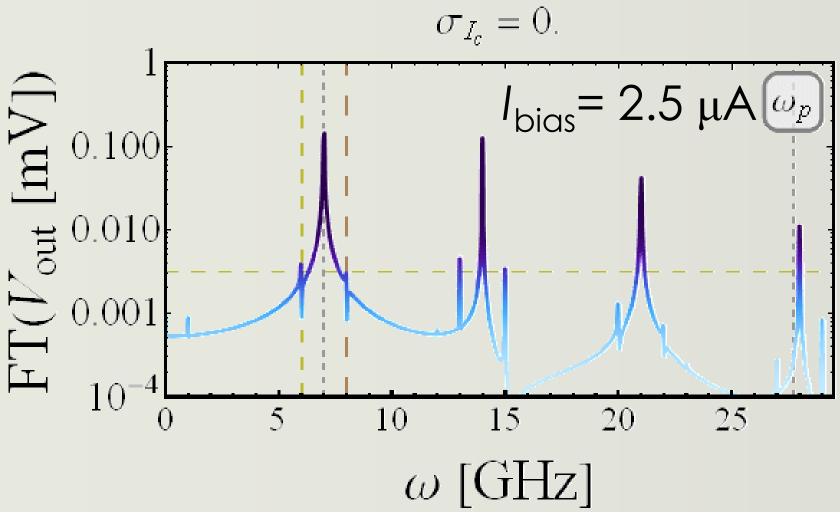
TWJPA – Gain vs δI_c

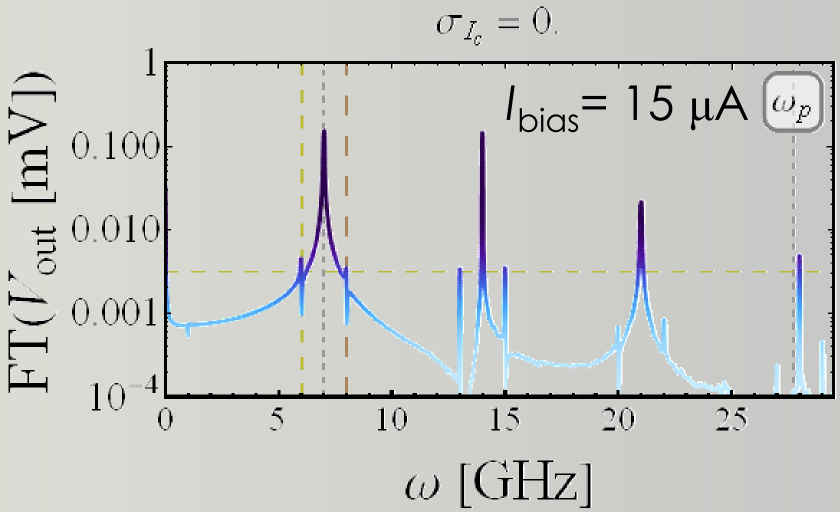
Here, we assume the critical currents to be distributed according to

$$I_{c,n} = \overline{I_c}(1 + \delta I_c)$$

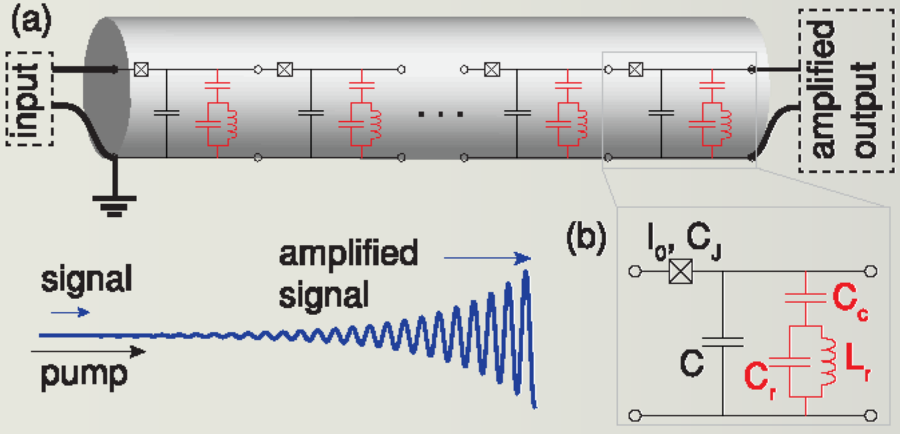
with δI_c being a Gaussianly distributed number with zero average and variance $\sigma_{I_c}^2$.







Perspectives: Resonant Phase Matching strategy



This new design is specifically made to reduce higher harmonics generation and unwanted mixing products, by lowering the Josephson plasma frequency and making the dispersion relation highly nonlinear with the introduction of elements to induce resonant phase matching

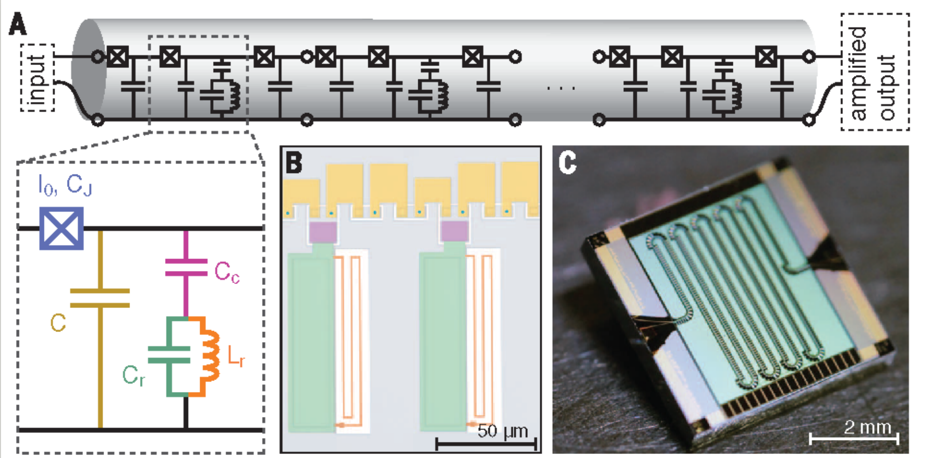
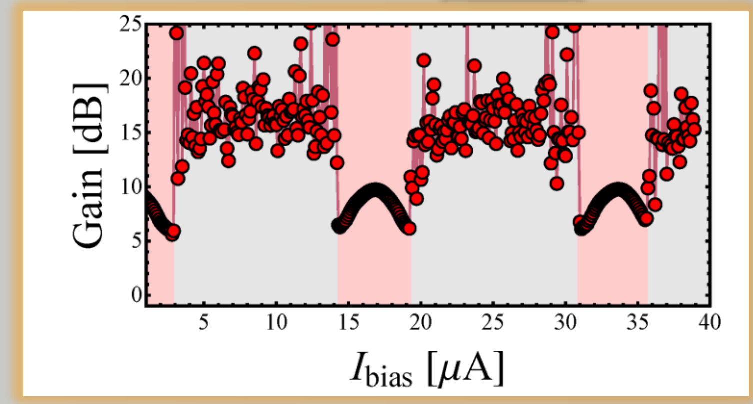


Fig. 1. Josephson traveling-wave parametric amplifier. **(A)** Circuit diagram. The JTWPA is implemented as a nonlinear lumped-element transmission line; one unit cell consists of a Josephson junction with critical current $I_0 = 4.6\ \mu A$ and intrinsic capacitance $C_J = 55$ fF with a capacitive shunt to ground $C = 45$ fF. Every third unit cell includes a lumped-element resonator designed with capacitance $C_r = 6$ pF and inductance $L_r = 120$ pH, with coupling strength set by a capacitor $C_c = 20$ fF. The value of C in the resonator-loaded cell is reduced to compensate for the addition of C_c. **(B)** False-color optical micrograph. The coloring corresponds to the inset in (A), with the lower metal layer shown in gray. **(C)** Photograph of a 2037 junction JTWPA. The line is meandered several times on the 5 mm by 5 mm chip to achieve the desired amplifier gain.

K. O'Brien, *et al.*, Resonant Phase Matching of JJ TWPAs, PRL **113**, 157001 (2014)
C. Macklin, *et al.*, A near-quantum-limited JTWPA, Science **350**, 307 (2015)

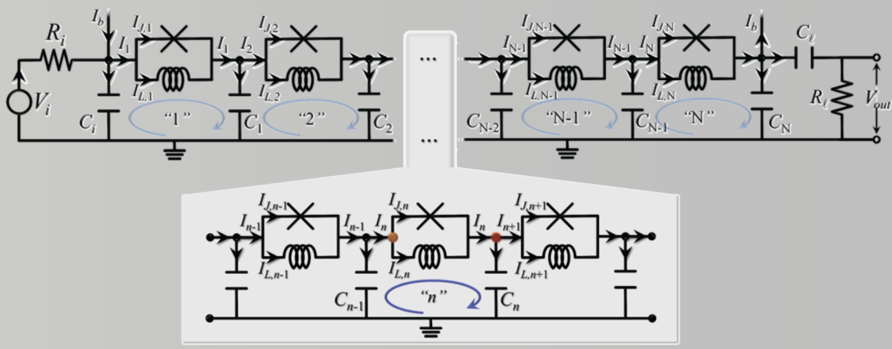
DART WARS

Conclusions



We demonstrated:

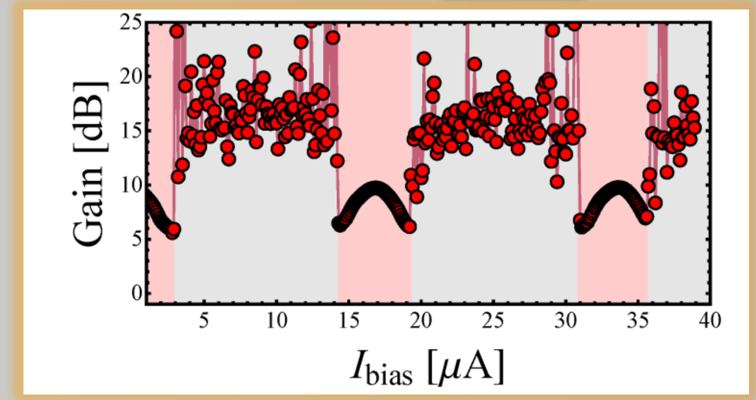
- the amplification of a weak signal in the presence of a strong pump tone in a TWJPA, obtaining a Gain up to ~10 dB for a specific device configuration;
- the robustness of the effect against unavoidable fluctuations in the main Josephson parameter, i.e., the critical current.



Perspectives:

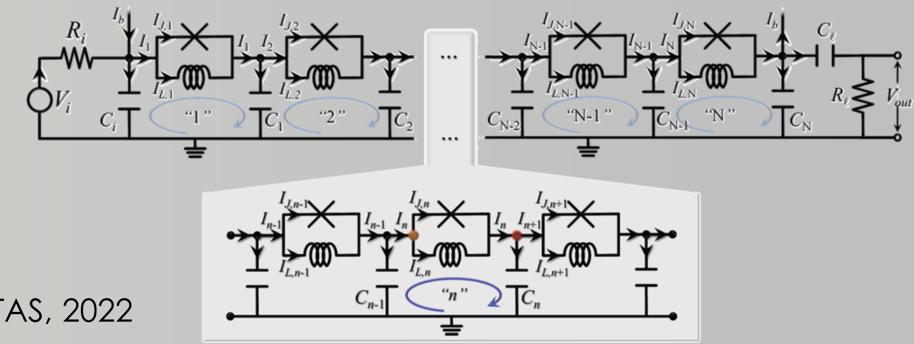
- Optimization of the system parameter to maximize the amplification;
- Study of impact of fluctuations of the various system parameters;
- Study of the impact of thermal noise;
- Implementing the "resonant phase matching strategy";
- Change the specifics of the devices forming the transmission lines, e.g., the CPR of the junction or including dc-SQUID.

Conclusions



We demonstrated:

- the amplification of a weak signal in the presence of a strong pump tone in a TWJPA, obtaining a Gain up to ~10 dB for a specific device configuration;
- the robustness of the effect against unavoidable fluctuations in the main Josephson parameter, i.e., the critical current.



DARTWARS publications

- 🕮 C. Guarcello, *et al.*, *Modeling of Josephson Traveling Wave Parametric Amplifiers*, accepted on IEEE TAS, 2022
- 🕮 M. Borghesi, *et al.*, *Progress in the development of a KITWPA for the DARTWARS project*, under review on NIMA, 2022
- 🕮 V. Granata, *et al.*, *Characterization of Traveling-Wave Josephson Parametric Amplifiers at T = 0.3 K*, under review on IEEE TAS, 2022
- 🕮 A. Rettaroli, *et al.*, *Ultra low noise readout with travelling wave parametric amplifiers: the DARTWARS project*, under review on NIMA, 2022
- 🕮 A. Giachero, *et al.*, *Detector Array Readout with Traveling Wave Amplifiers*, J Low Temp Phys, 2022
- 🕮 S. Pagano, *et al.*, *Development of Quantum Limited Superconducting Amplifiers for Advanced Detection*, IEEE TAS, **32**, 4, 1-5, 2022