# Engineering the speedup of quantum tunneling in Josephson systems via dissipation

Phys. Rev. B **106**, 045408 (2022)
Phys. Rev. Research **3**, 033019 (2021)

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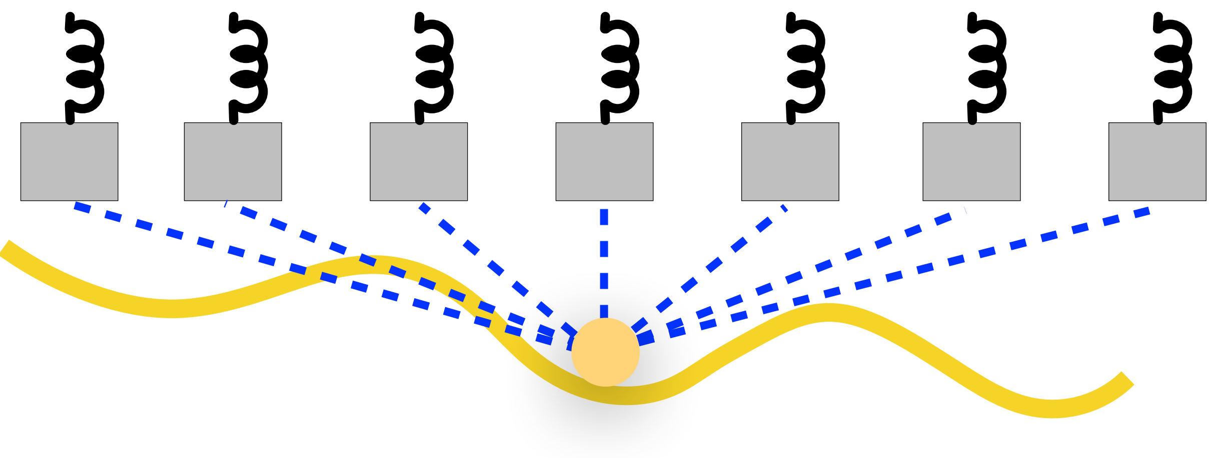
# Quantum dissipation

Quantum Brownian motion (Caldeira-Leggett model)



$$\hat{H} = \hat{H}_S + \hat{H}_{int} + \hat{H}_{bath}$$

$$\hat{H}_{int} \sim \hat{X}\,\hat{B}$$

particle's operator

bath's operator

conventional interaction $\hat{X} = \hat{q}$

unconventional interaction $\hat{X} = \hat{p}$

A. J. Leggett, *PRA* **30**, 1208 (1984)

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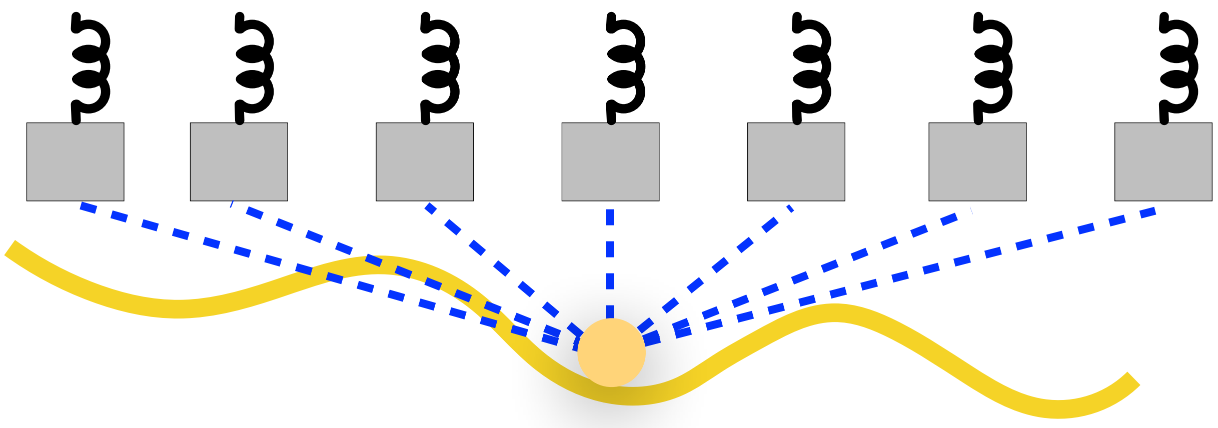

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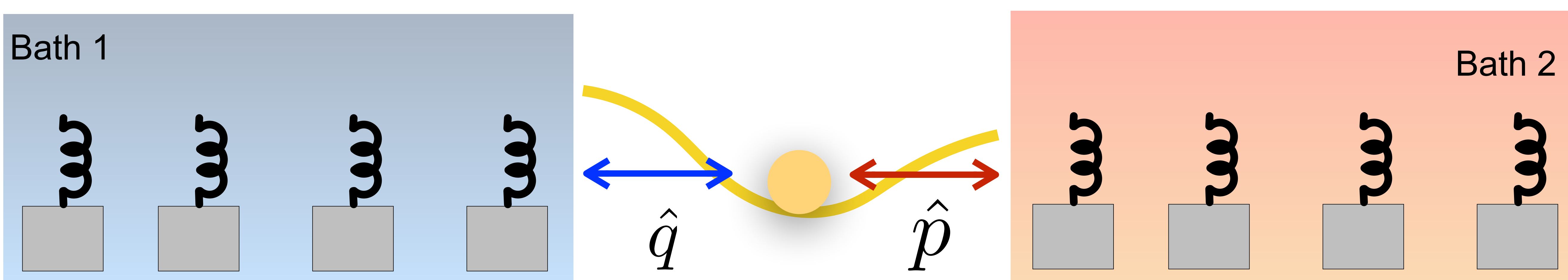
A. J. Leggett, *PRA* **30**, 1208 (1984)

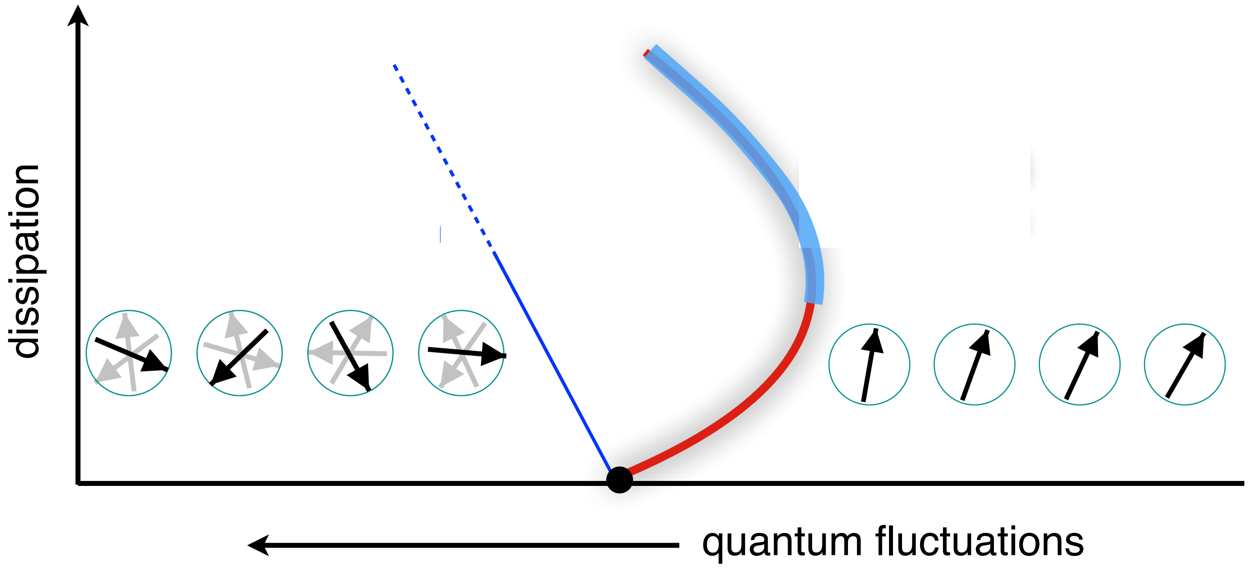
<u>Question</u>: coupling to two baths via two non-commuting observables?




[*see for instance*: New J. of Phys. **18**, 053033 (2016)]

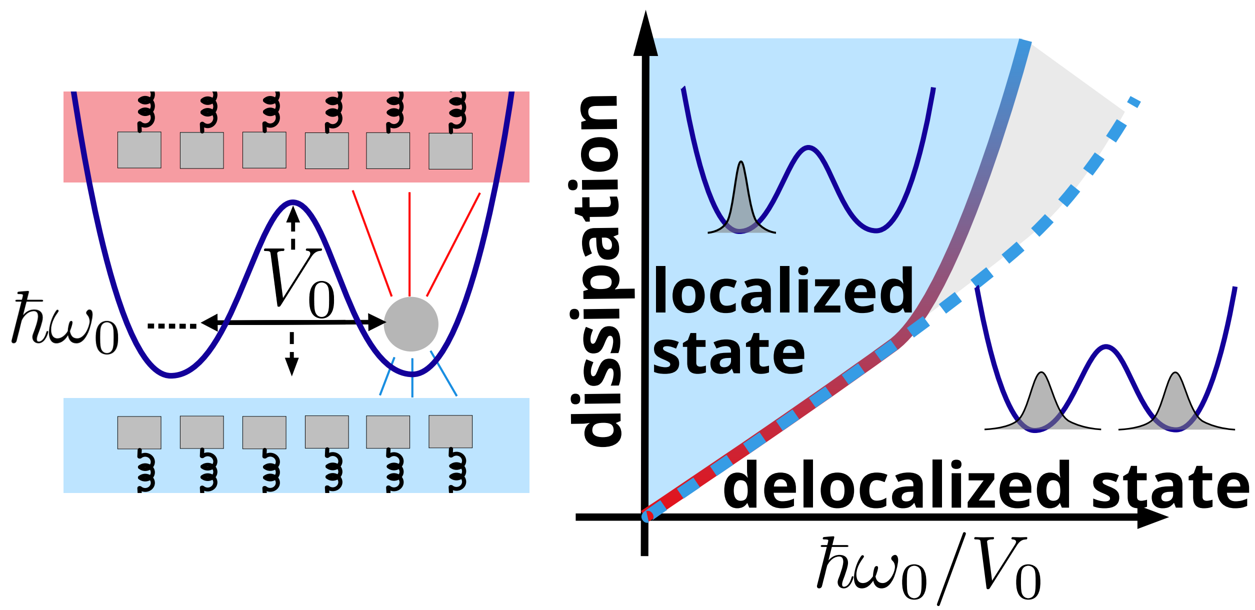
# Some results

Many-body systems (Josephson junction chains)



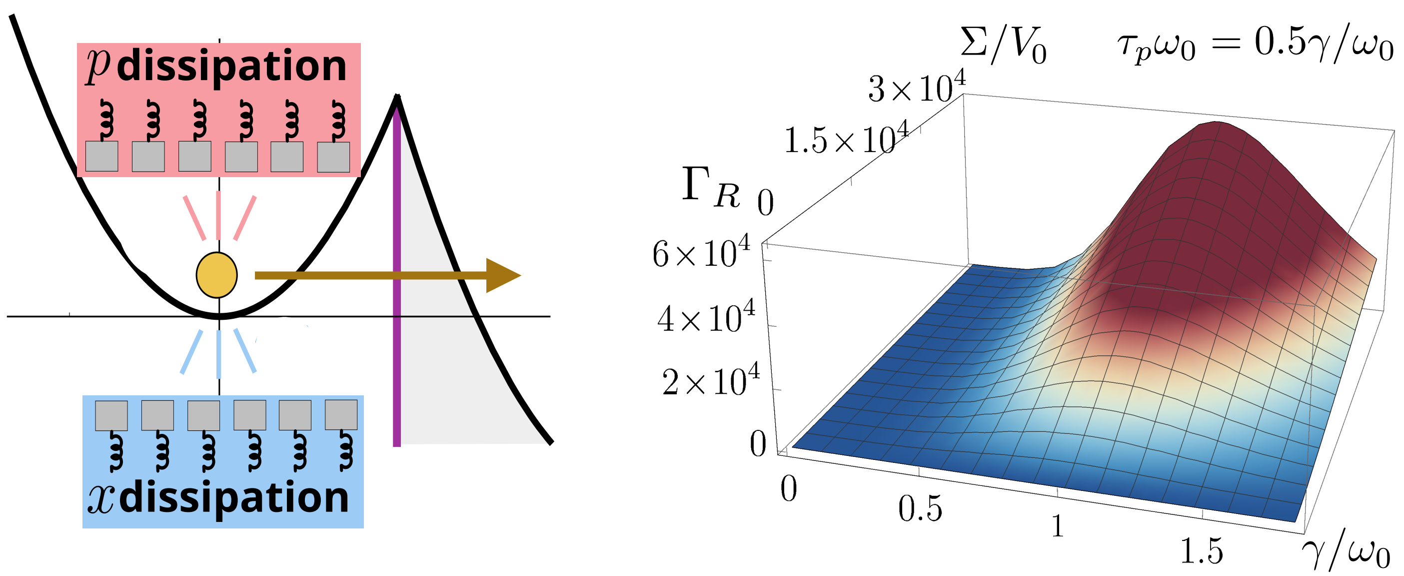

Phys. Rev. B **97**, 155427 (2018)

Double-well potential




Phys. Rev. Research **2**, 013226 (2020)

Metastable potential




Phys. Rev. Research **3**, 033019 (2021)

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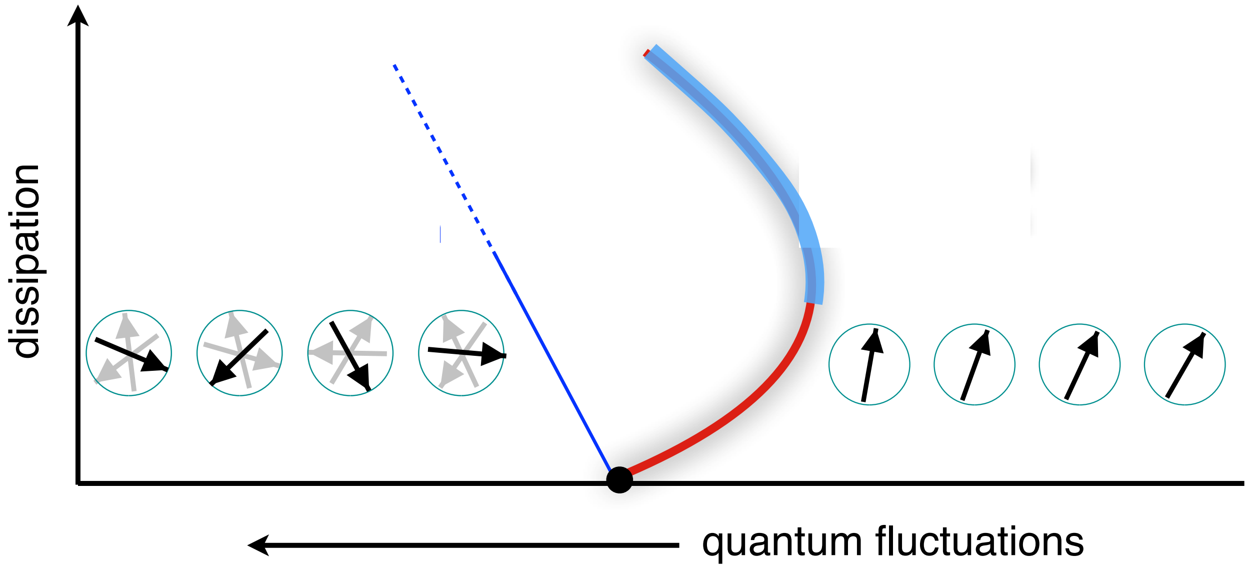


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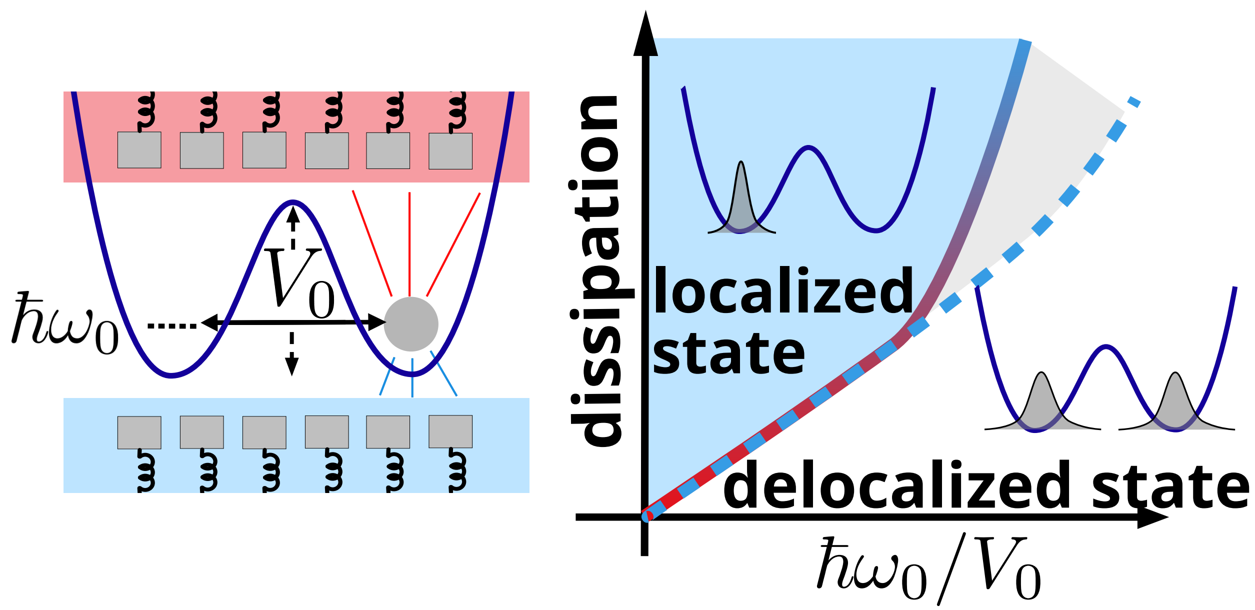


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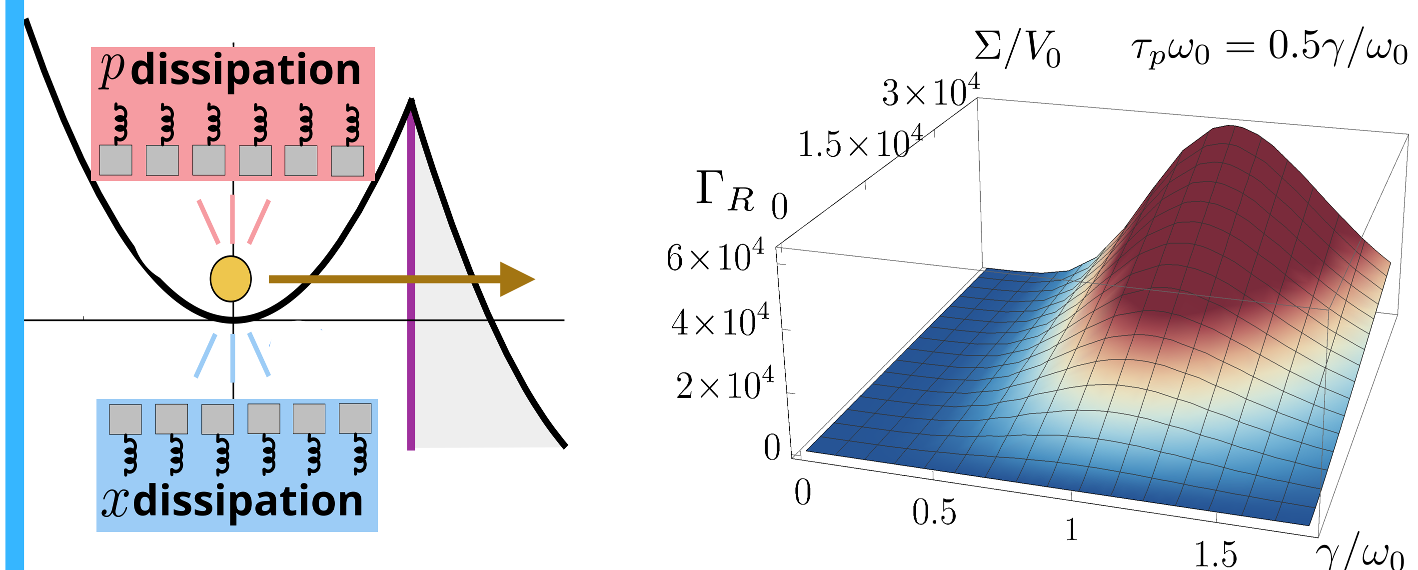


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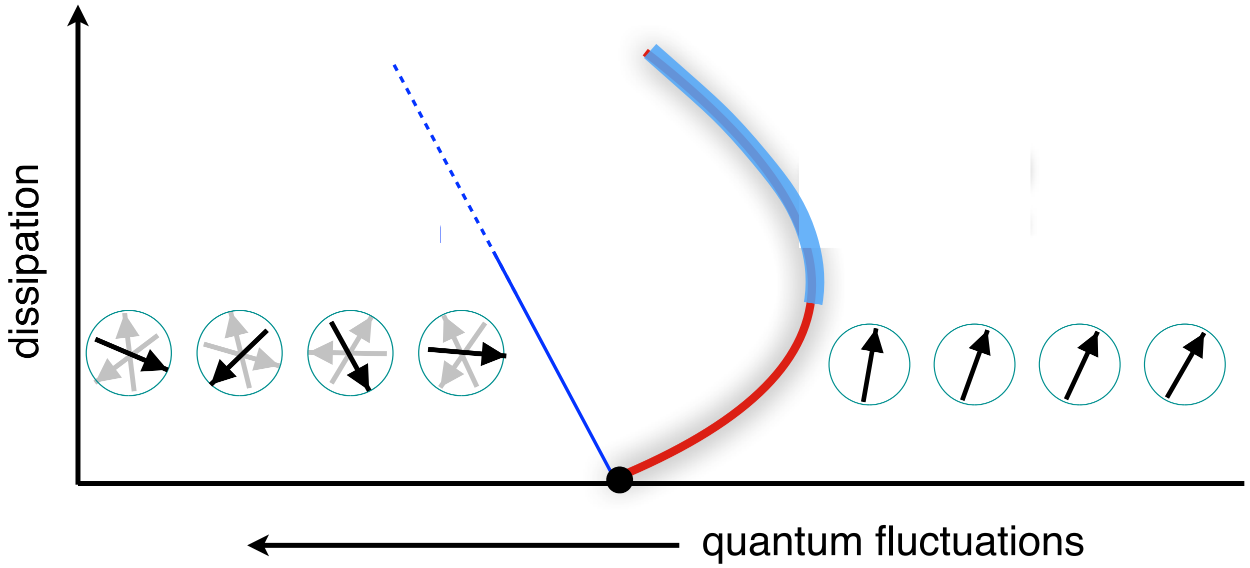


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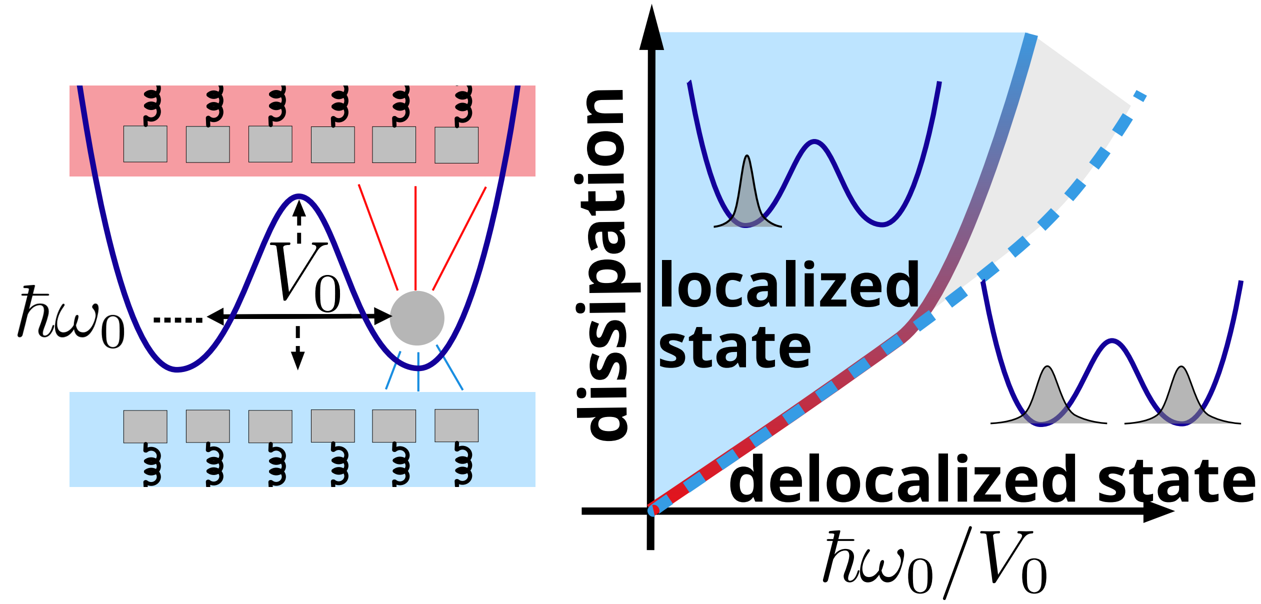


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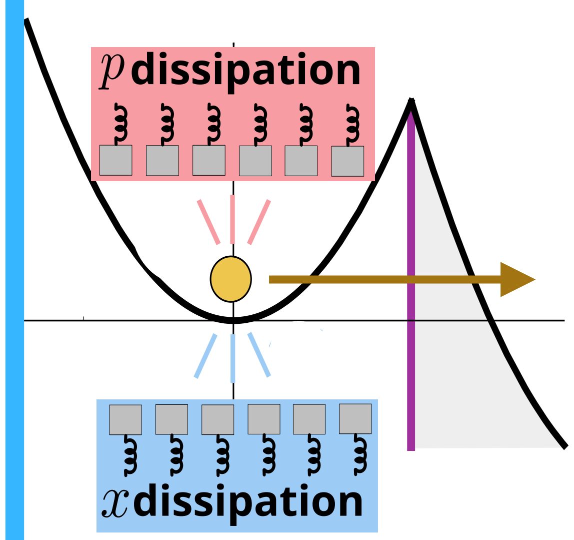


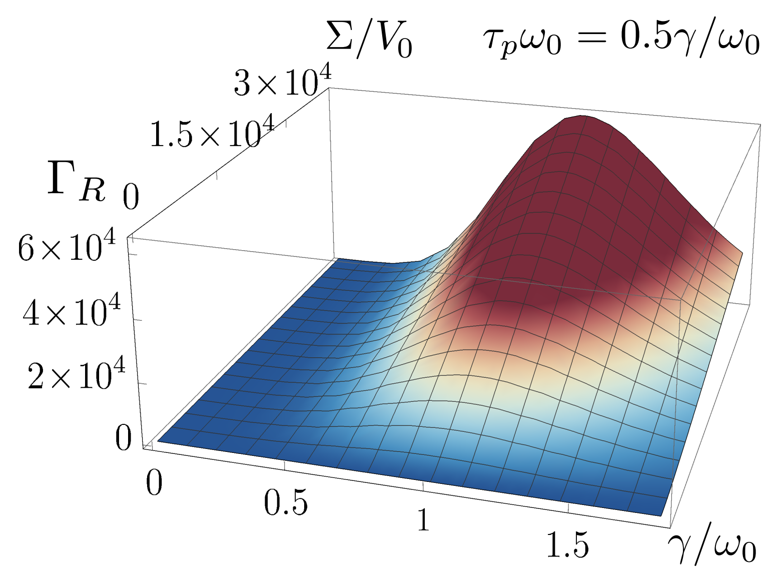


Phys. Rev. Research **3**, 033019 (2021)

How to realize it in an experimental system?

Josephson junction circuits

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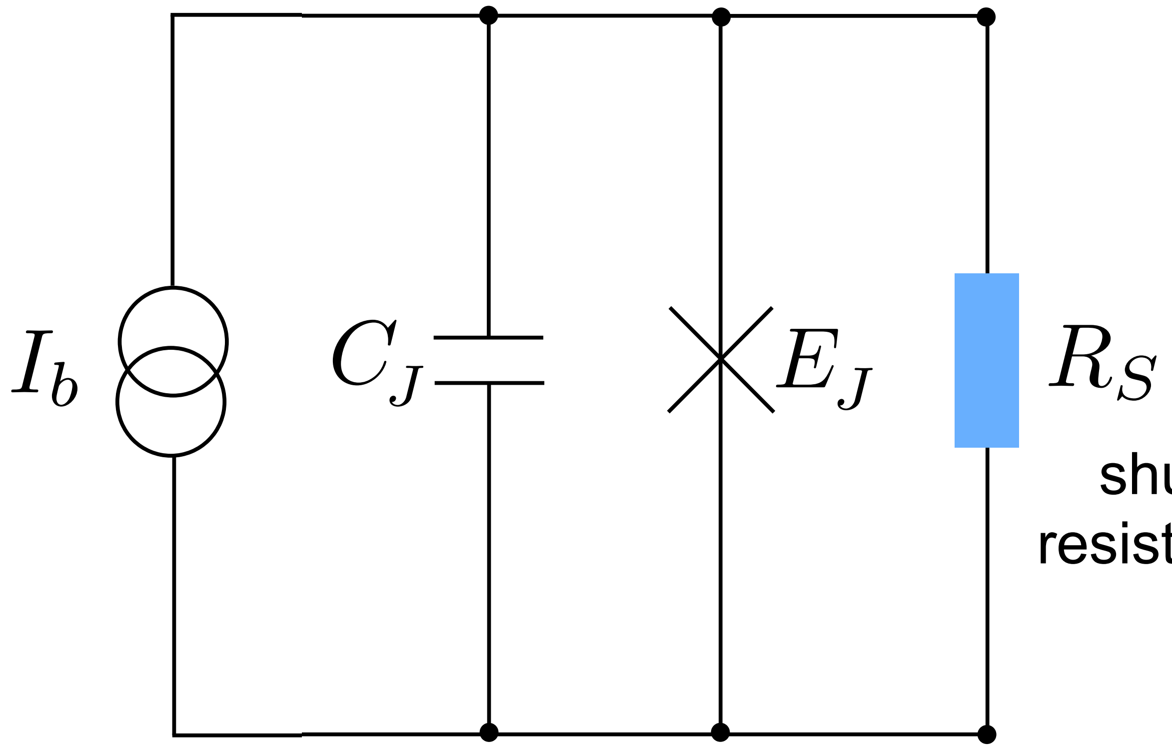
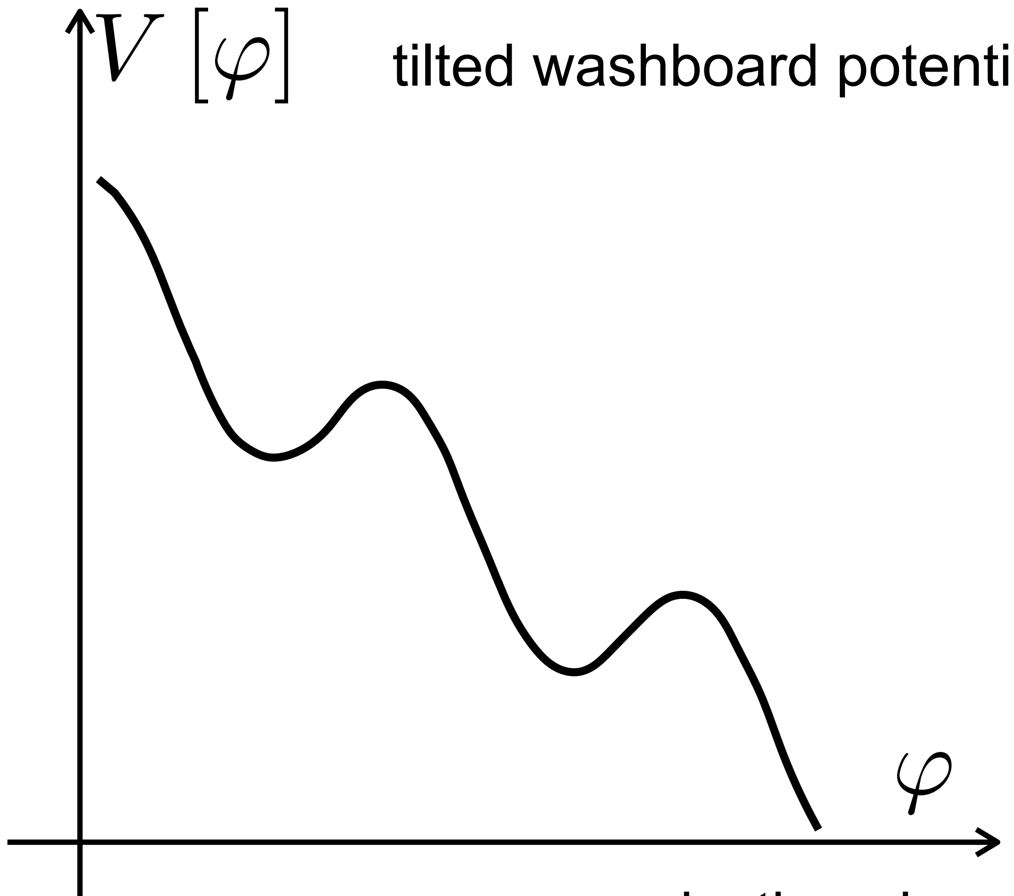
# Conventional dissipation in quantum tunneling

current bias Josephson junction



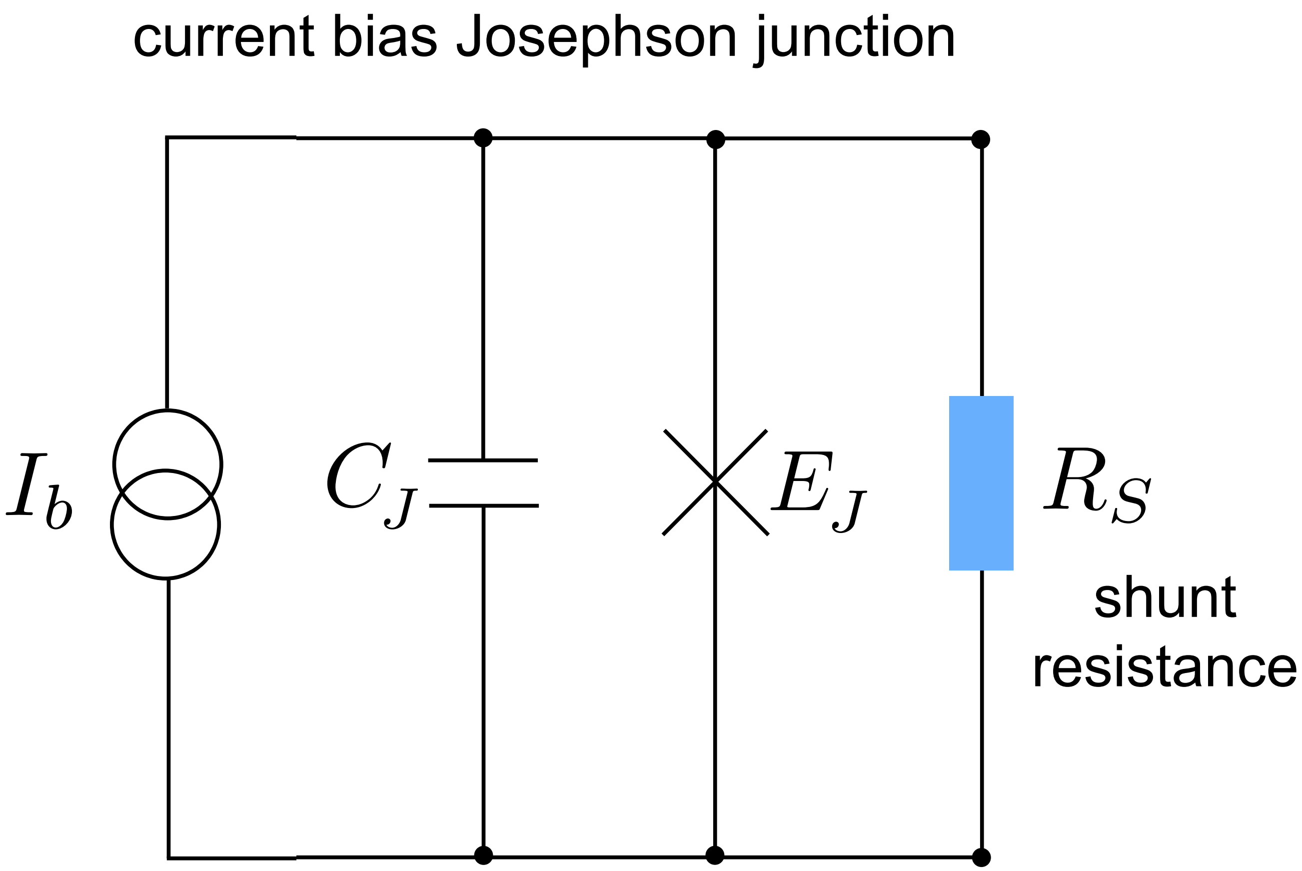

shunt
resistance

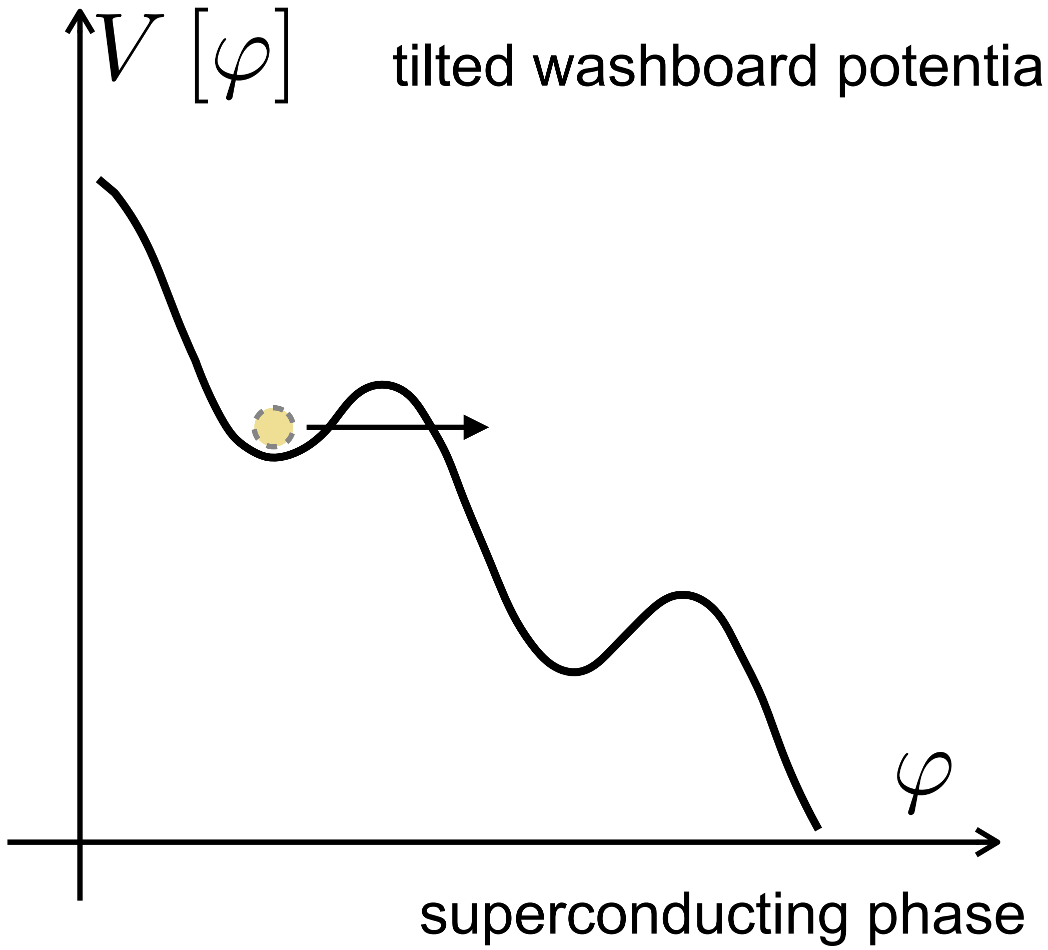
tilted washboard potential




superconducting phase

# Conventional dissipation in quantum tunneling




- first experimental observations M.H. Devoret, J.M. Martinis, J. Clarke, *PRL* **55**, 1908 (1985)

[see also: J.M. Martinis, M.H. Devoret, J. Clarke, *Nat. Physics* **16**, 234 (2020)]

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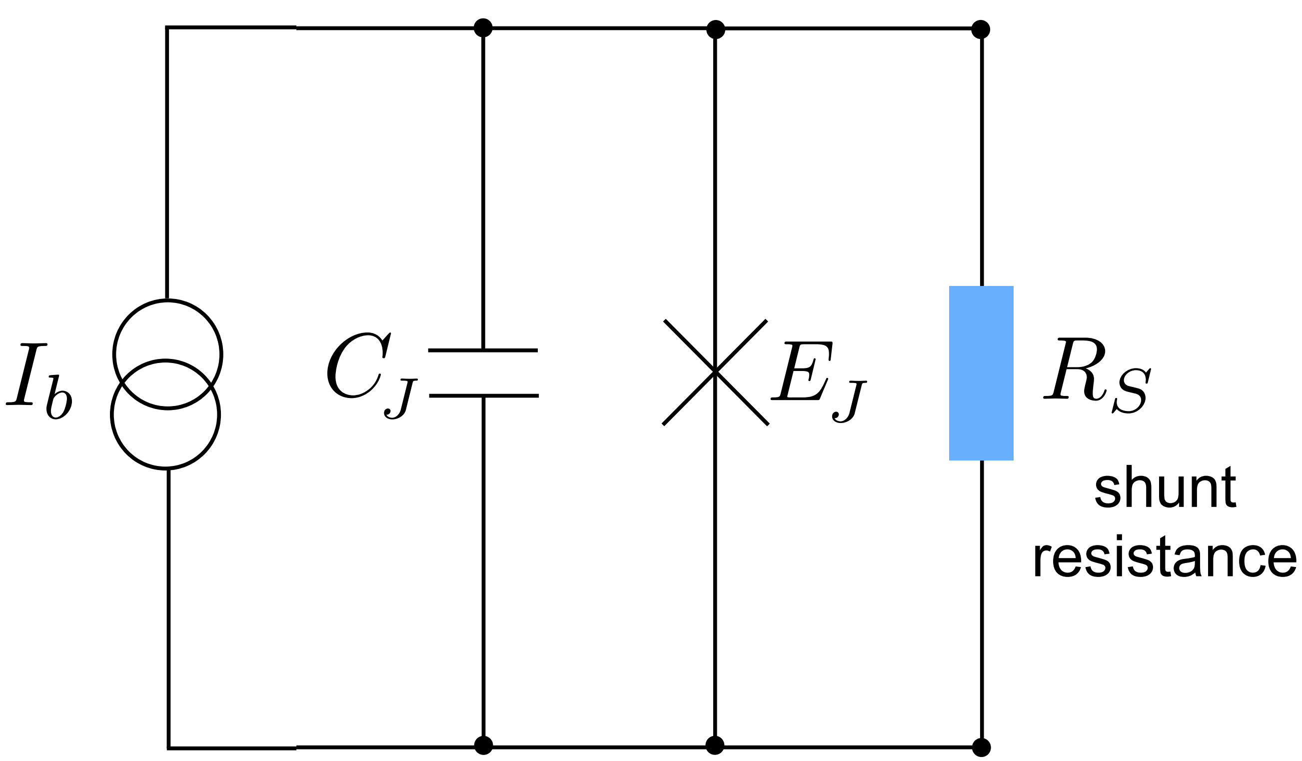


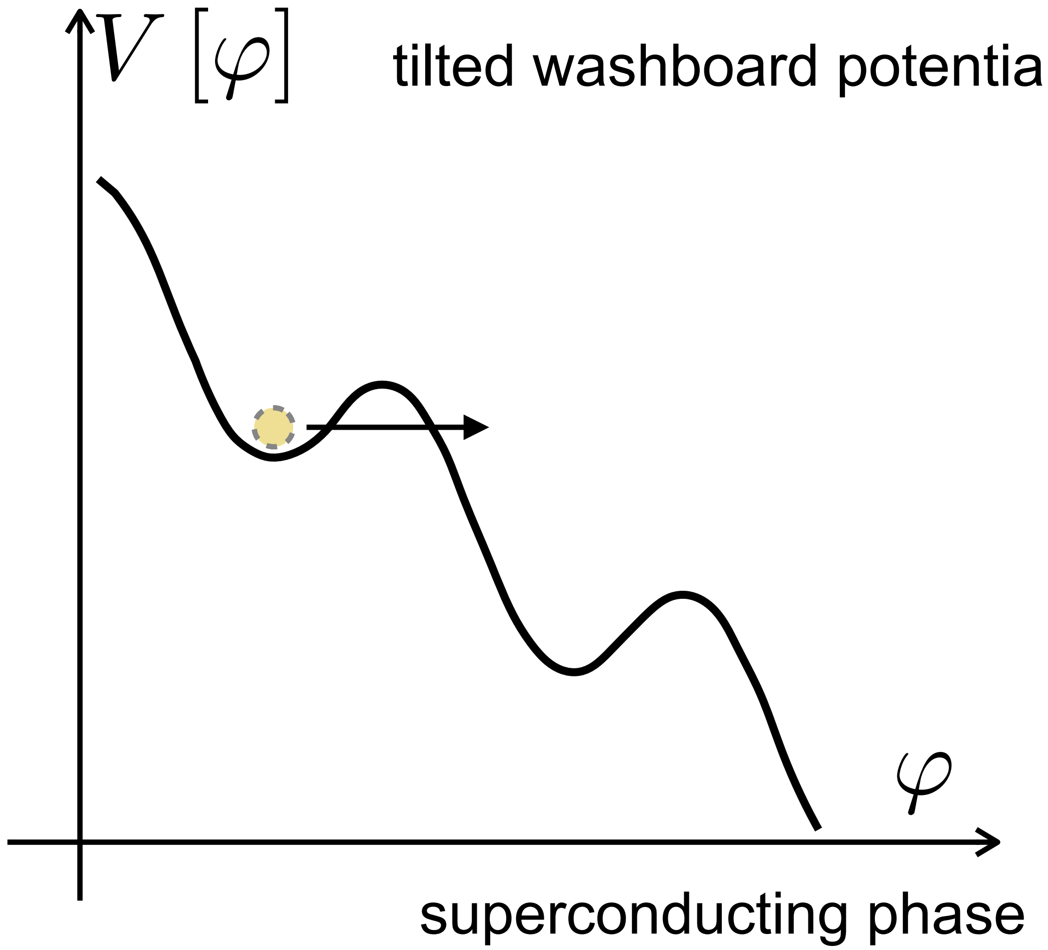


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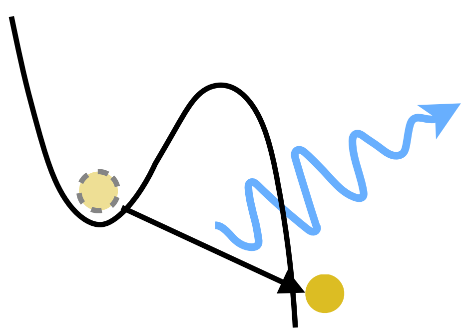

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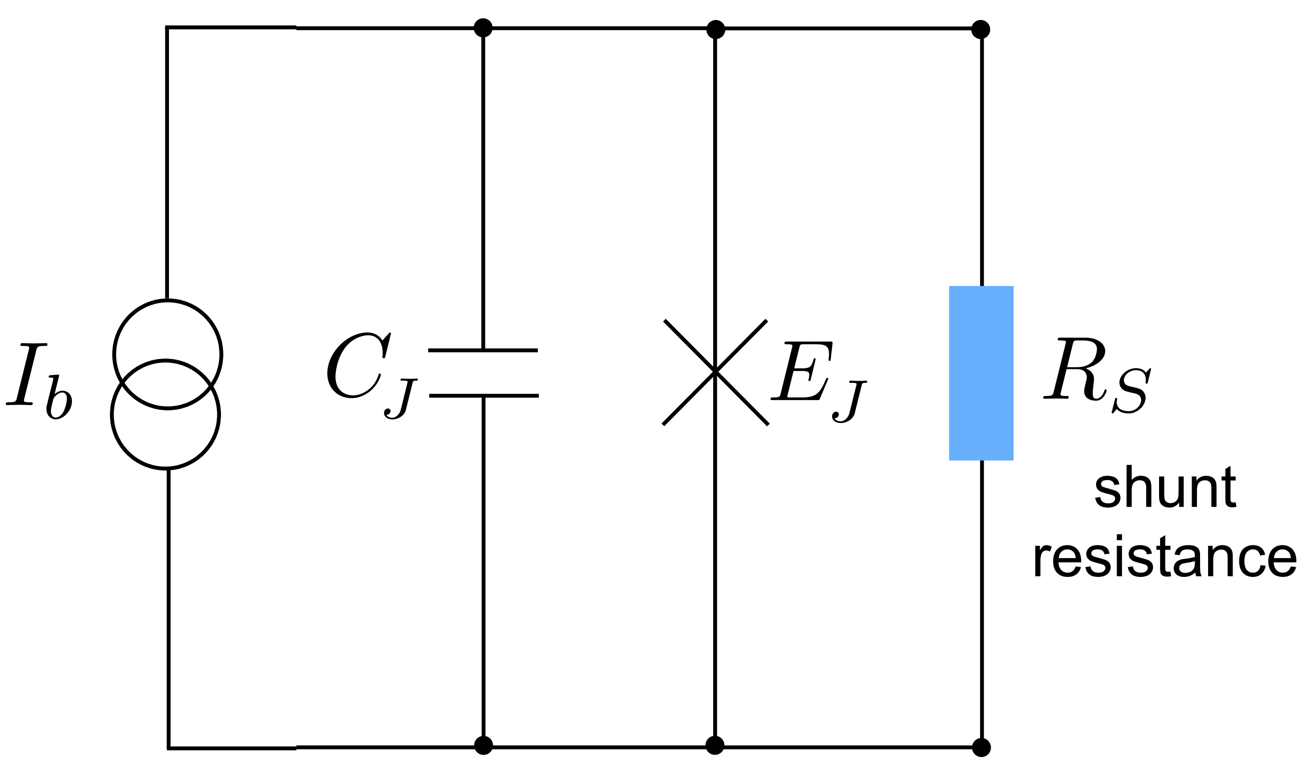


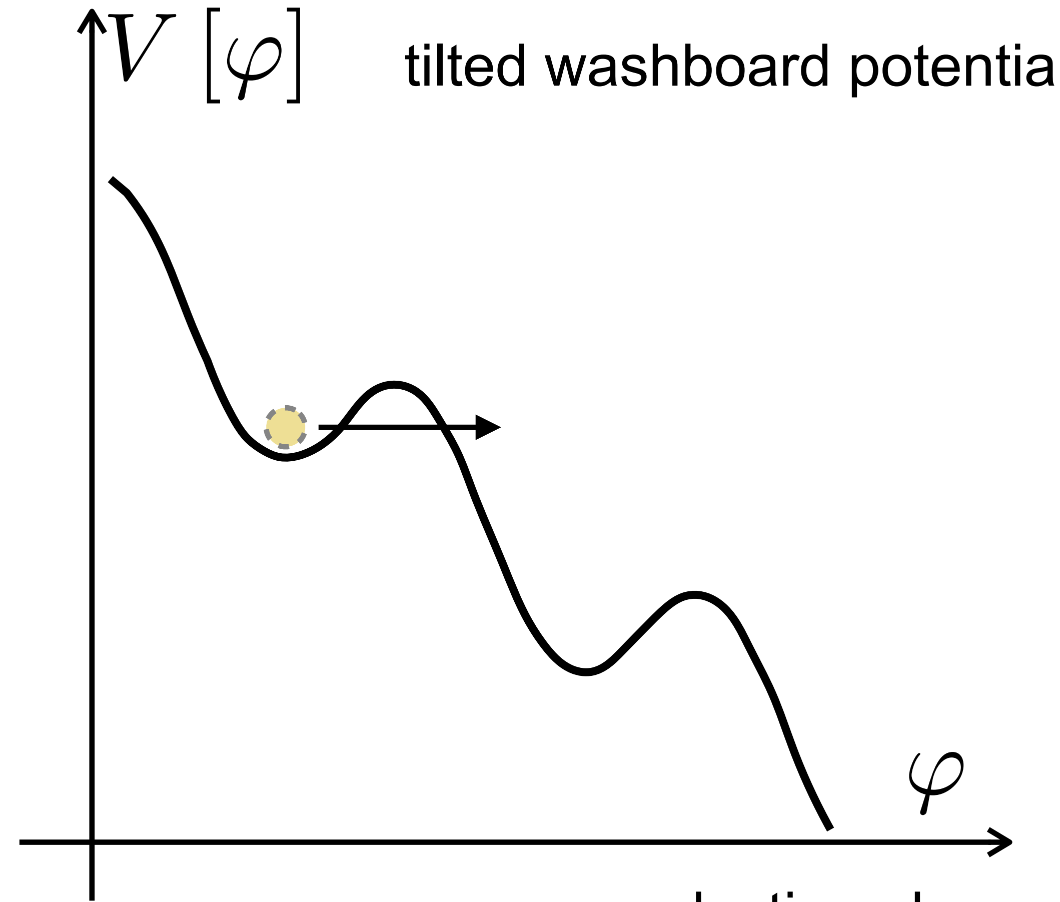


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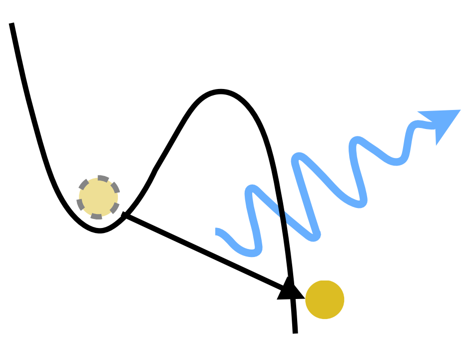
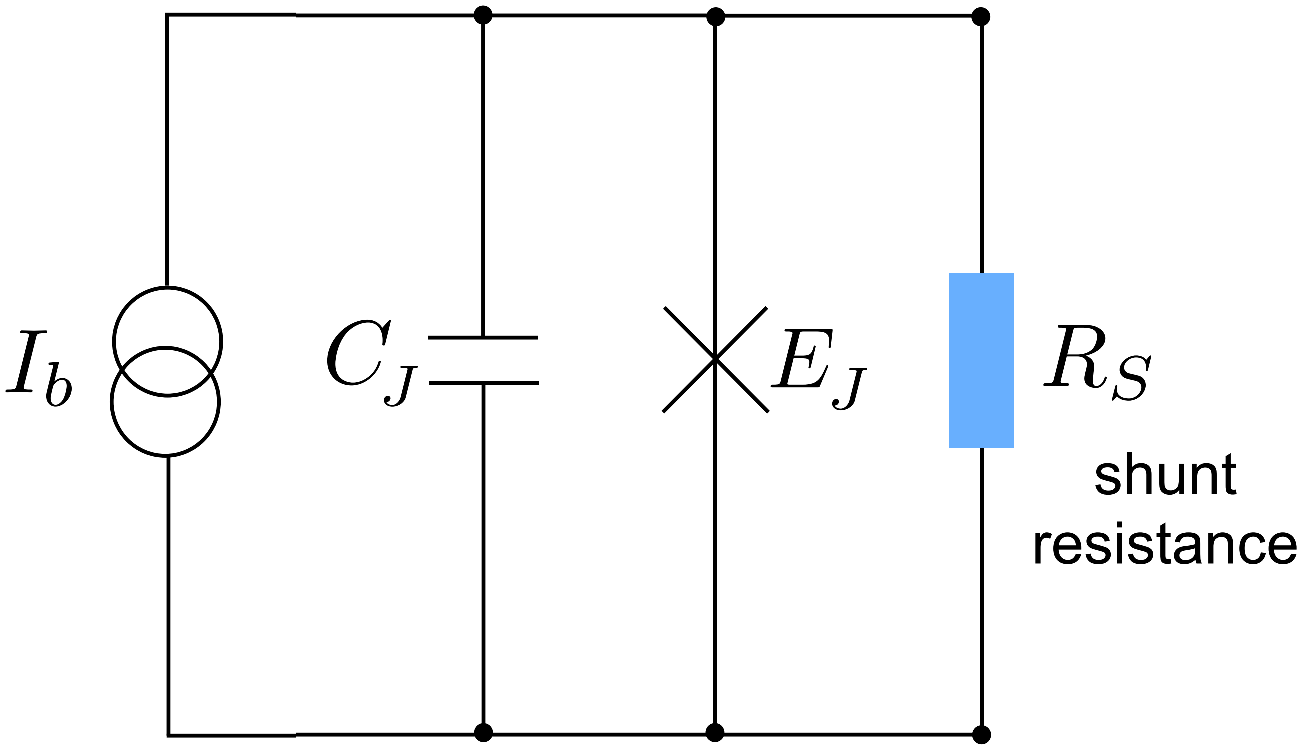


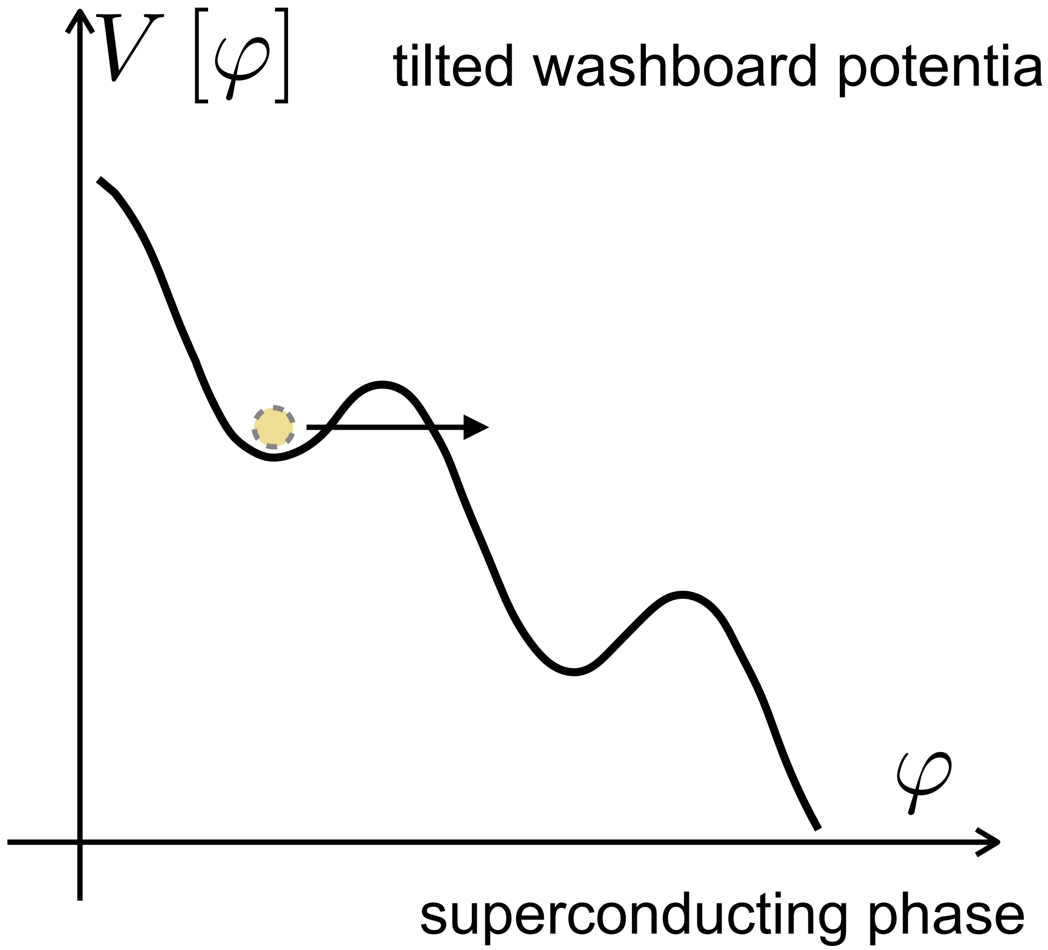
- exponential suppression of the escape rate

$$\sim \exp\left[-\frac{a\,\eta\,d^2}{\hbar}\right]$$

A.O. Caldeira, A.J. Leggett, *PRL* **46**, 211 (1981)

# Conventional dissipation in quantum tunneling

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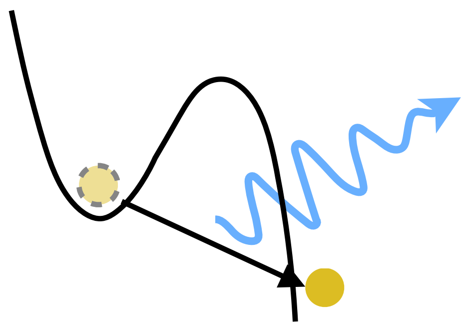

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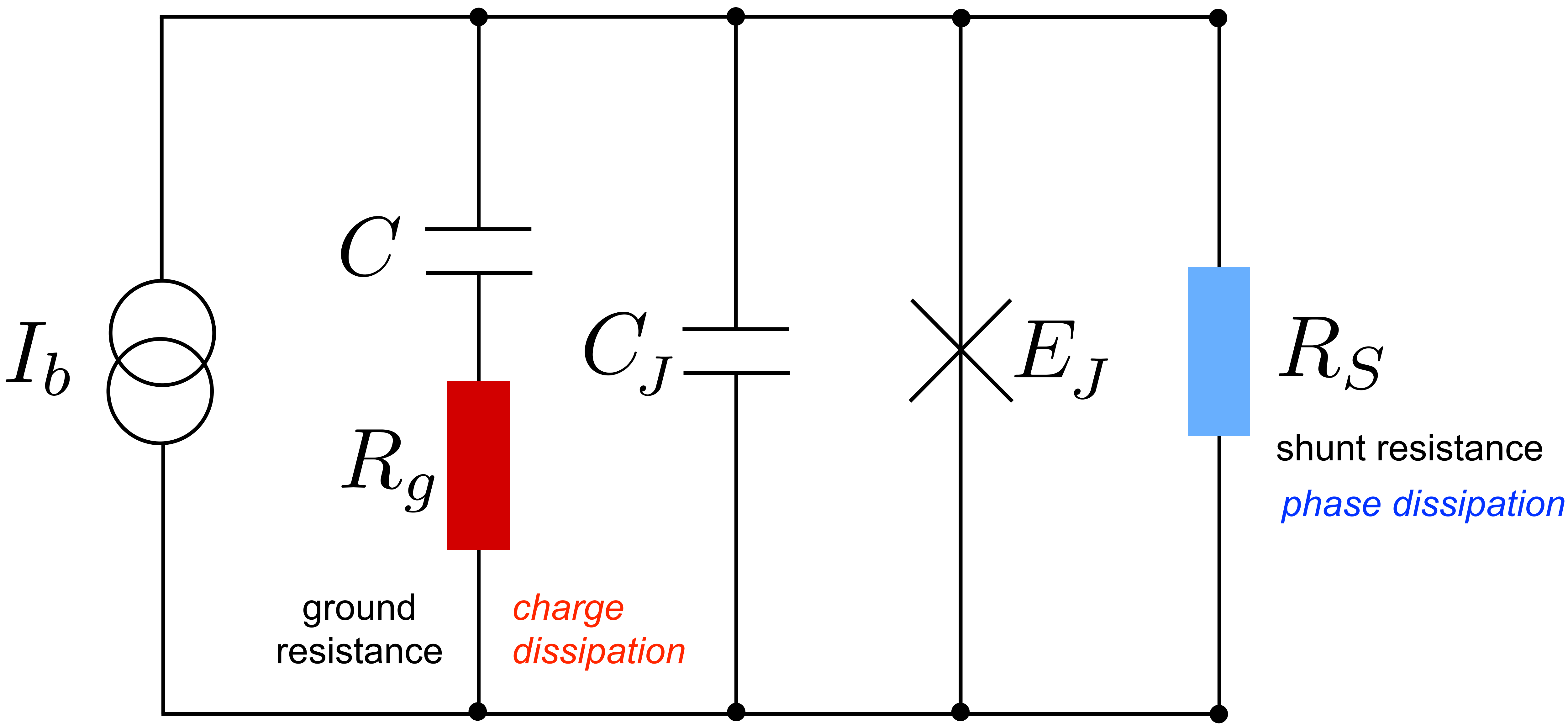
*phase dissipation*

A.O. Caldeira, A.J. Leggett, *PRL* **46**, 211 (1981)

# Engineered dissipation

phase and charge: non-commuting observables/operators



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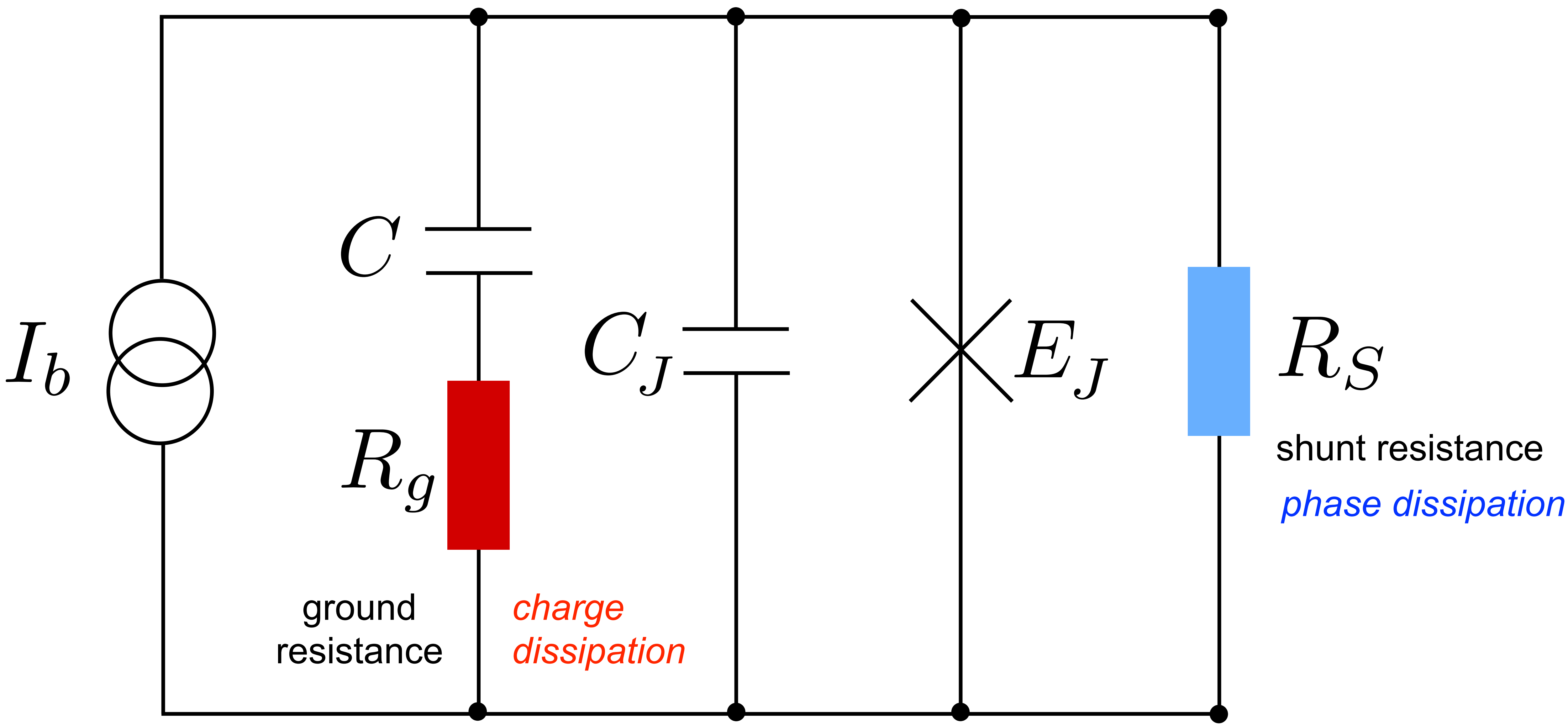
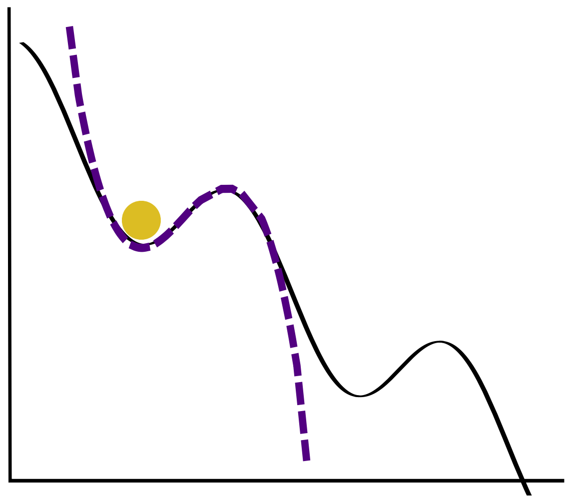



two non-commuting dissipative interactions

# Quantum escape rate without dissipation

$$\hat{H}_S = \frac{\hat{Q}^2}{2C_{tot}} + V\left[\hat{\varphi}\right]$$

$$V\left[\hat{\varphi}\right] = -\frac{\hbar I_C}{2e}\cos\left(\hat{\varphi}\right) - \frac{\Phi_0 I_b}{2\pi}\hat{\varphi} \;\simeq\; \frac{C_{tot}\Phi_0^2}{4\pi^2}\left(\frac{1}{2}\omega_I^2\hat{\varphi}^2 - \frac{1}{3}\omega_J^2\hat{\varphi}^3\right)$$



$(0)$

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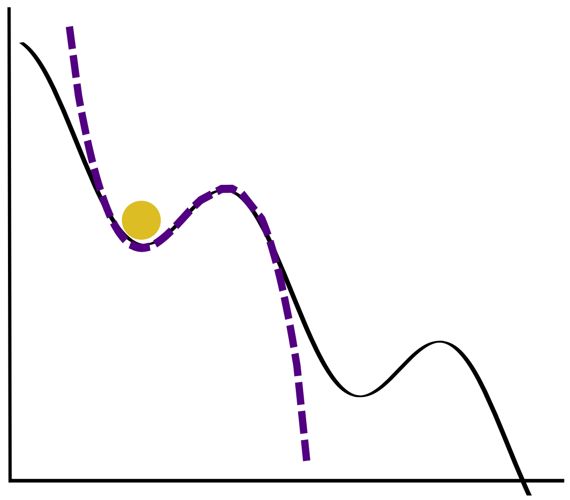


**Semiclassical regime (path integral method)** $V_0 \gg \hbar\omega_I$ $E_J \gg E_C$

escape rate $$\Gamma_0 = K_0 \exp\left[-S_B^{(0)}/\hbar\right]$$

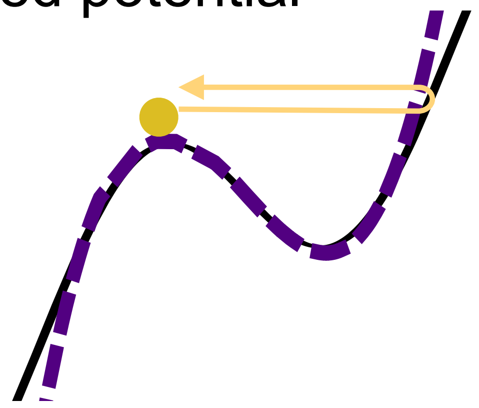
Euclidean action $$S_0\left[\varphi(\tau)\right] = \int_{-\frac{\beta}{2}}^{\frac{\beta}{2}} d\tau \left[\frac{C_{tot}\Phi_0^2}{8\pi^2}\dot{\varphi}^2(\tau) + V[\varphi(\tau)]\right]$$

$$\beta = \frac{\hbar}{k_B T} \longrightarrow \infty$$

(quantum limit)

bounce path in the inverted potential

$$S_B^{(0)} = S_0\left[\varphi_B^{(0)}(\tau)\right]$$



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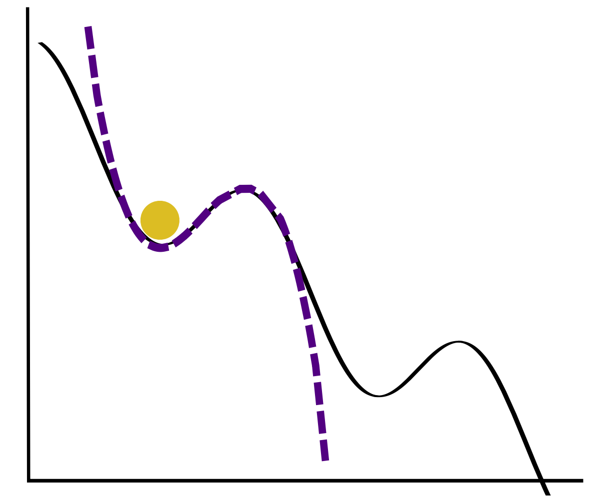

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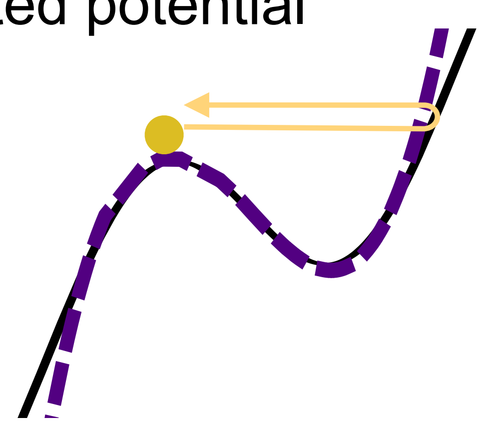


result:

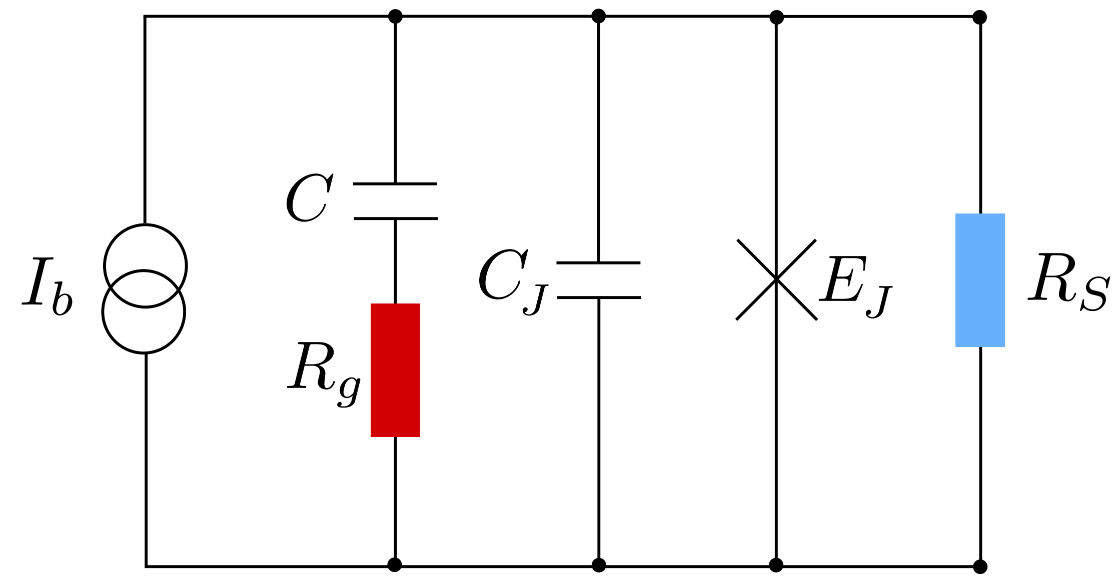
$$\boxed{S_B^{(0)} = \frac{108}{15}\frac{V_0}{\hbar\omega_I}}$$

# Electromagnetic environment

$$\Gamma = K \exp\Big[ - S_B/\hbar \Big]$$

$$S_B = S\Big[\varphi_B(\tau)\Big]$$

Euclidean action with the bounce




$$S\Big[\varphi(\tau)\Big] = S_0\Big[\varphi(\tau)\Big] + \frac{1}{2}\iint_{-\infty}^{\infty} d\tau d\tau' F^{(\varphi)}(\tau-\tau')\varphi(\tau)\varphi(\tau') + \frac{1}{2}\iint_{-\infty}^{\infty} d\tau d\tau' F^{(Q)}(\tau-\tau')\dot{\varphi}(\tau)\dot{\varphi}(\tau')$$

*phase dissipation*

$$F^{(\varphi)}(\omega) = \frac{\Phi_0^2}{4\pi^2 R_S}|\omega|$$

shunt resistance

*charge dissipation*

$$F^{(Q)}(\omega) = -\frac{C^2\Phi_0^2}{4\pi^2}\frac{R_g|\omega|}{1+R_g C|\omega|}$$

resistance to ground

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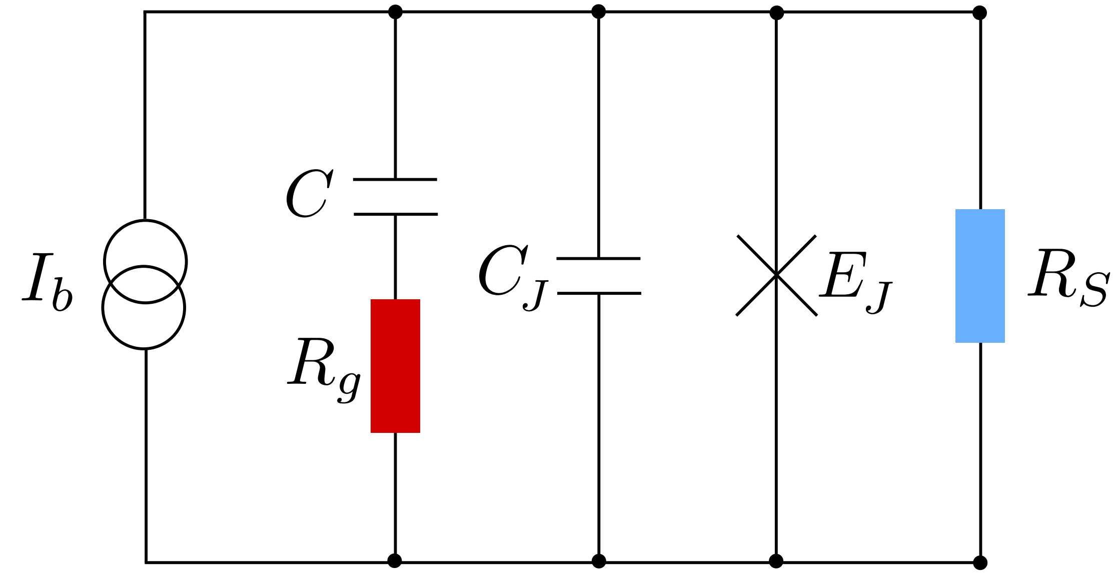


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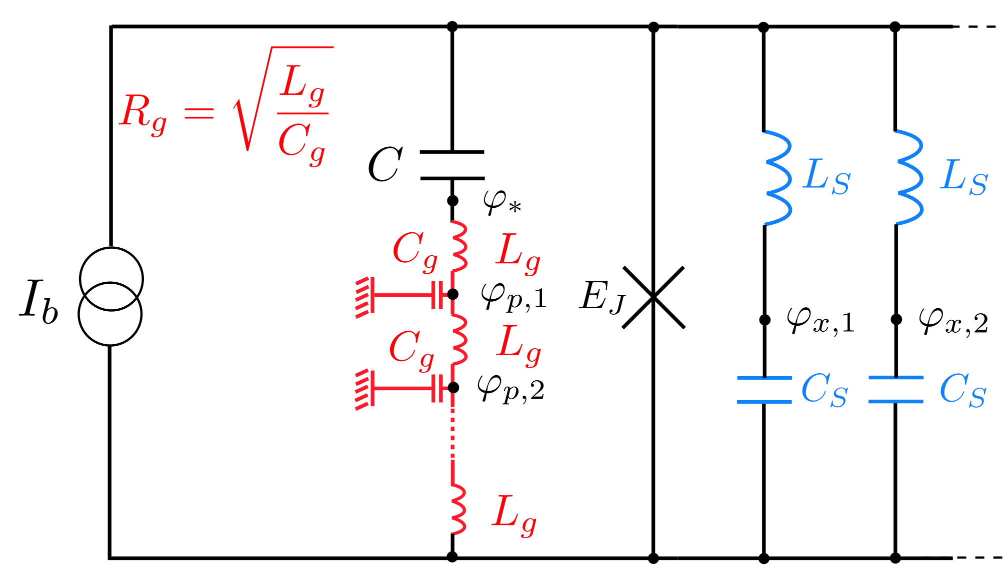
resistance to ground

theoretical model

(ensemble of LC oscillators)




$$R_g = \sqrt{\frac{L_g}{C_g}}$$

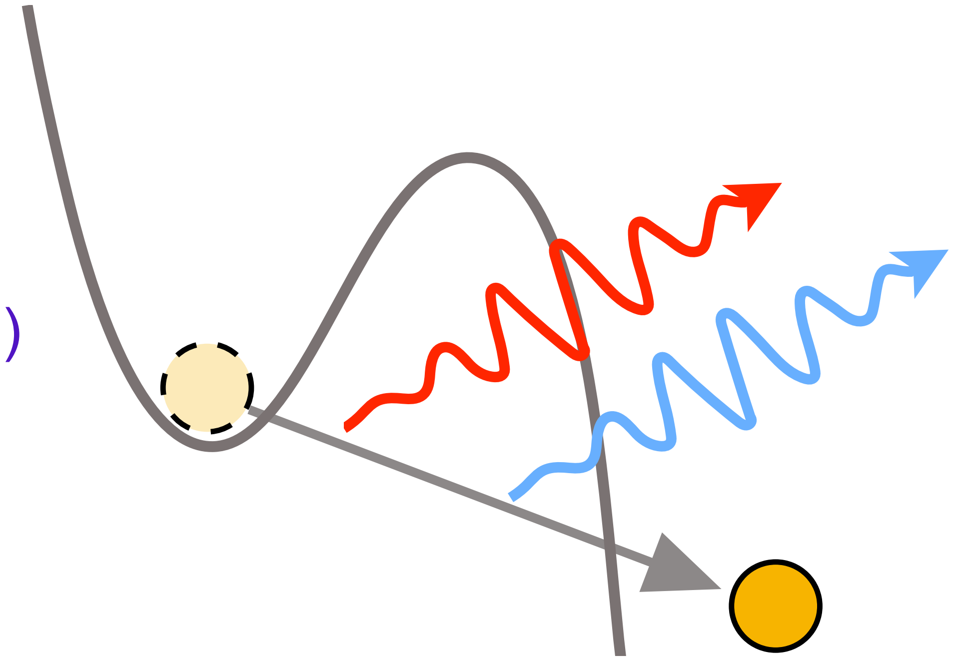
$$R_S = \sqrt{\frac{L_S}{C_S}}$$

# Environmental assisted tunneling

$$\Gamma = K \exp\left[ - S_B/\hbar \right]$$

- prefactor $K \sim K_0$ Phys. Rev. Research **3**, 033019 (2021)

- enhancement $\mathcal{E} = \exp\left[ -\frac{1}{\hbar}\left( S_B - S_B^{(0)} \right) \right]$



# Environmental assisted tunneling

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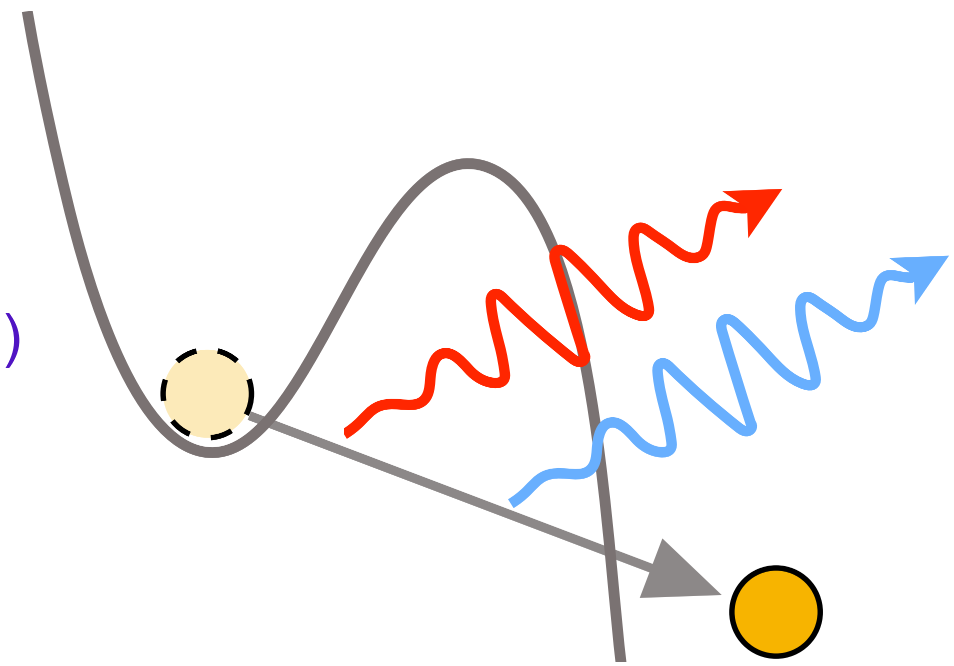

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**Methods**

- exact solution for the cubic potential only for $S_B^{(0)} = S_0\left[\varphi_B^{(0)}(\tau)\right]$

- perturbative approach $S_B \simeq S_0[\varphi_B^{(0)}(\tau)] + S_{env}[\varphi_B^{(0)}(\tau)]$ (undamped bounce)

A.O. Caldeira, A.J. Leggett, *Annals of Physics* **149**, 374 (1983)

- variational approach $\varphi_V(\tau)$ with variational parameters

E. Freidkin, P. Riseborough, P. Hänggi, *PRB* **34**, 1952 (1986)

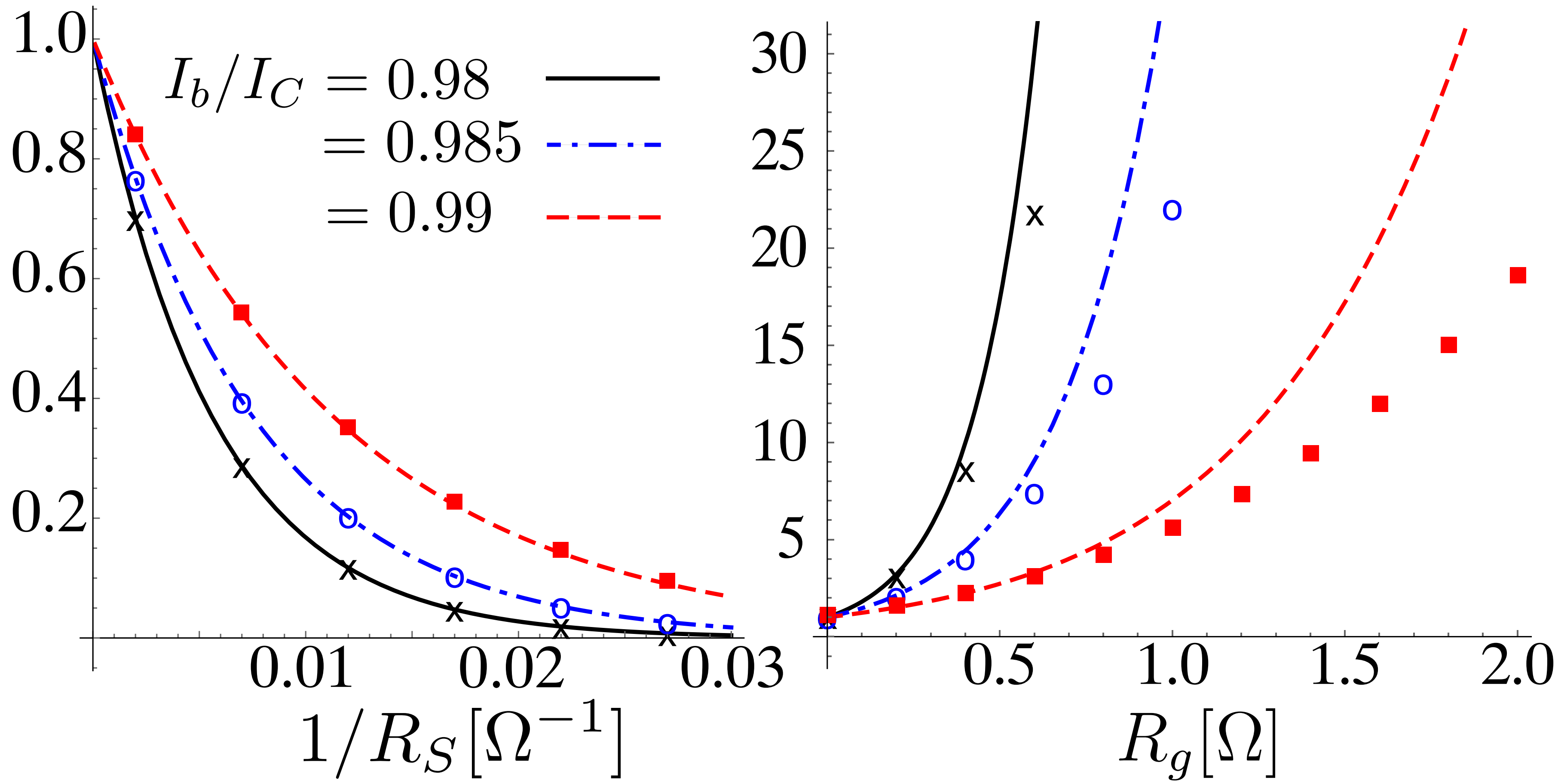
- numerical solution

L.D. Chang, S. Chakravarty, *PRB* **29**, 130 (1984)

# Results for different dissipations

$$\mathcal{E} = \exp\left[ -\frac{1}{\hbar}\left( S_B - S_B^{(0)} \right) \right]$$

*phase dissipation* *charge dissipation*



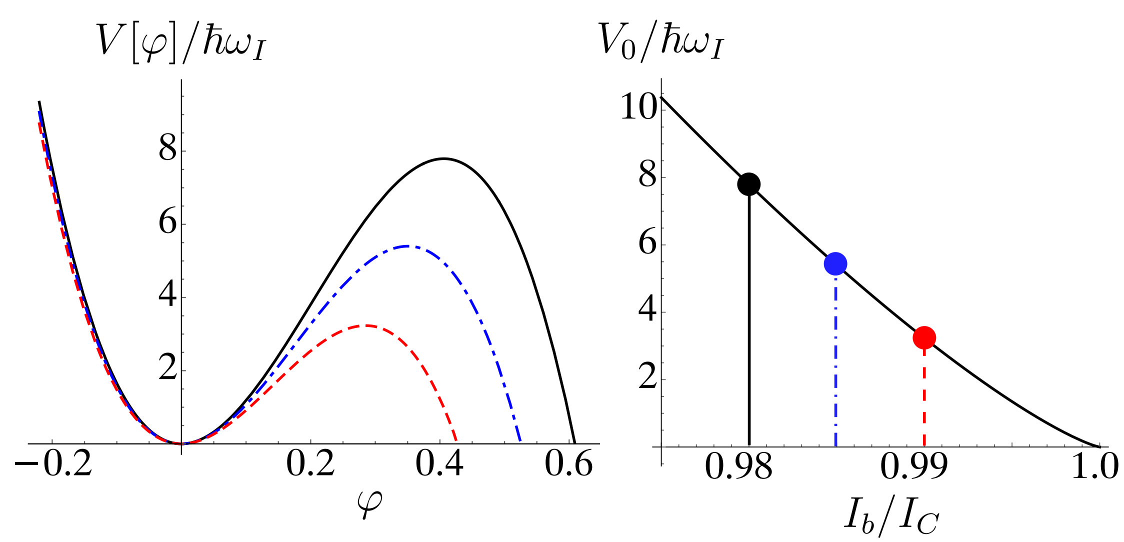

lines = variational method

points = perturbative approximation

Parameters

$I_C = 21\,\mu\mathrm{A}$

$C_{tot} = 6\,\mathrm{pF} \qquad C_J \ll C$

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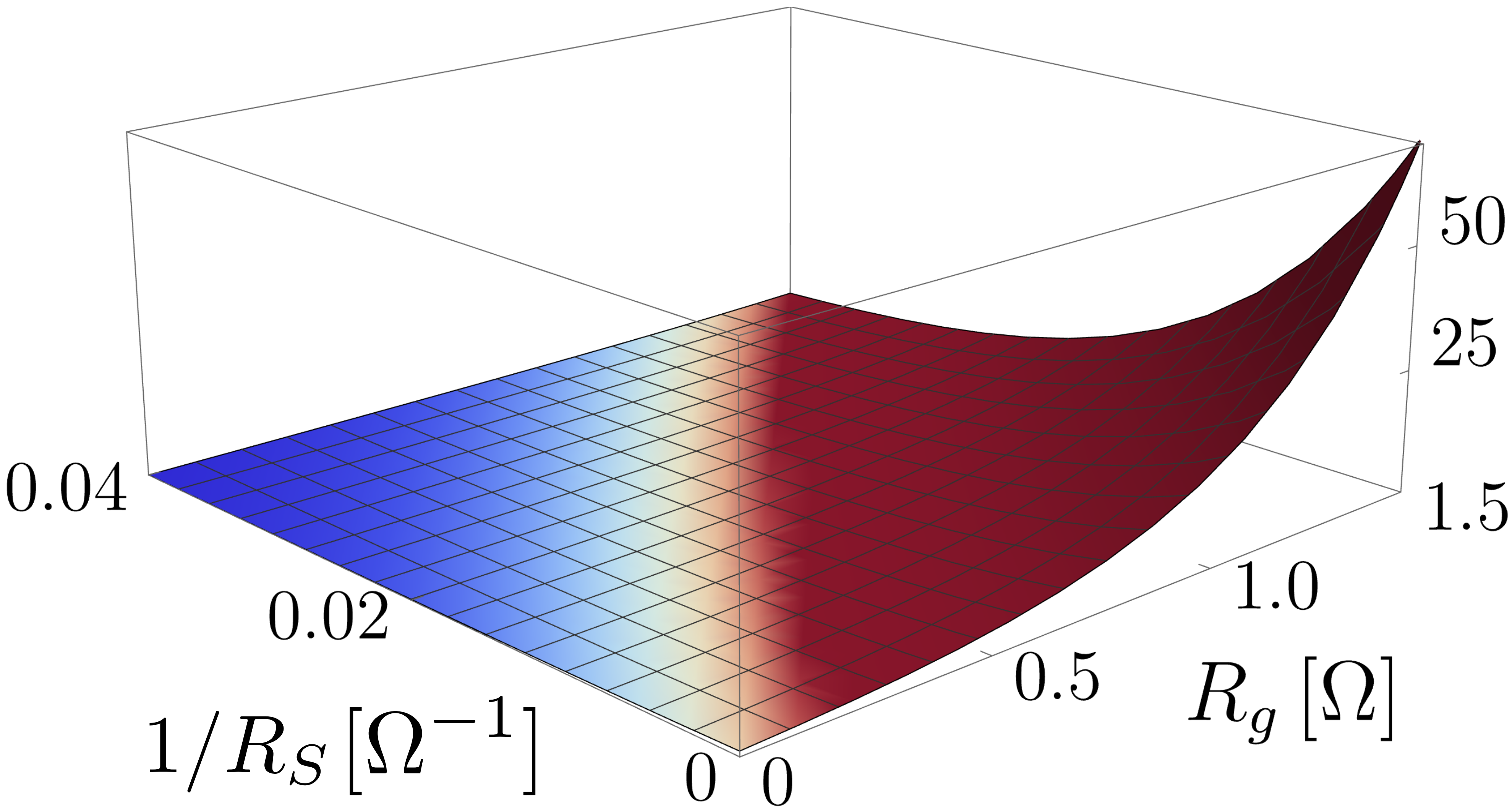
# General result

$$\mathcal{E} = \exp\left[ -\frac{1}{\hbar}\left( S_B - S_B^{(0)} \right) \right]$$




*phase dissipation* and *charge dissipation*

Parameters $I_C = 21\,\mu\mathrm{A}$ $V_0/\hbar\omega_I = 4$ $C_{tot} = 6\,\mathrm{pF}$ $C_J \ll C$

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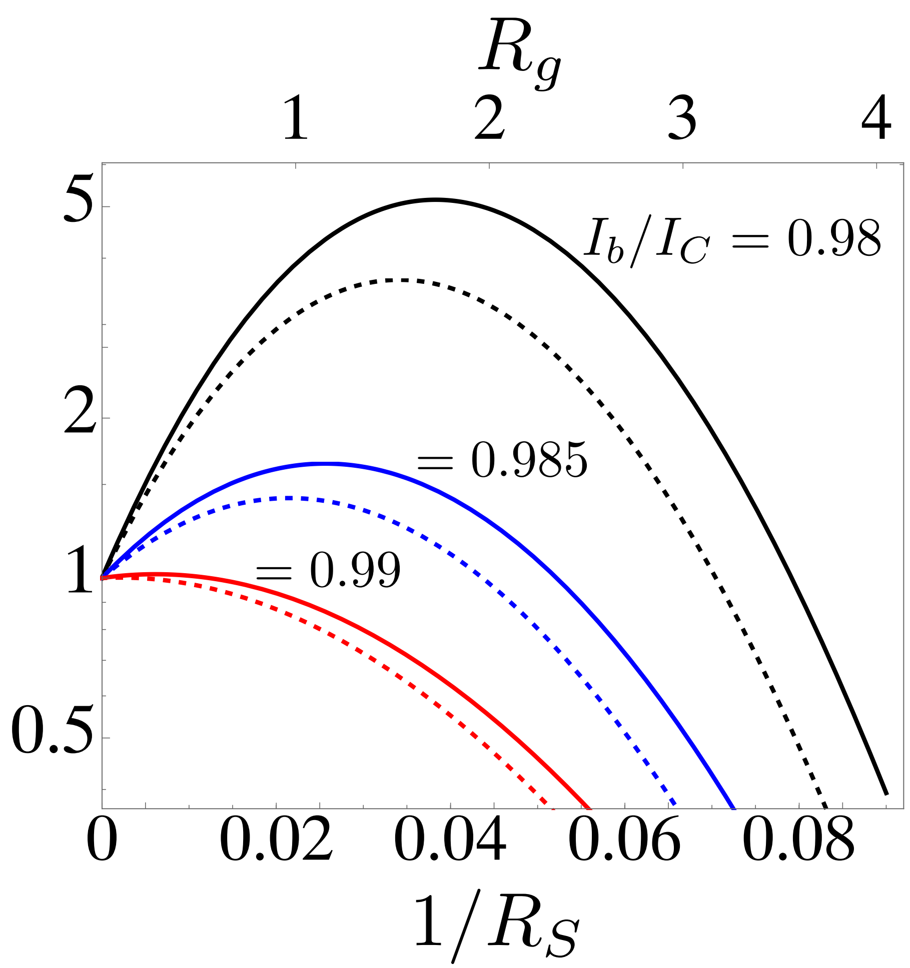
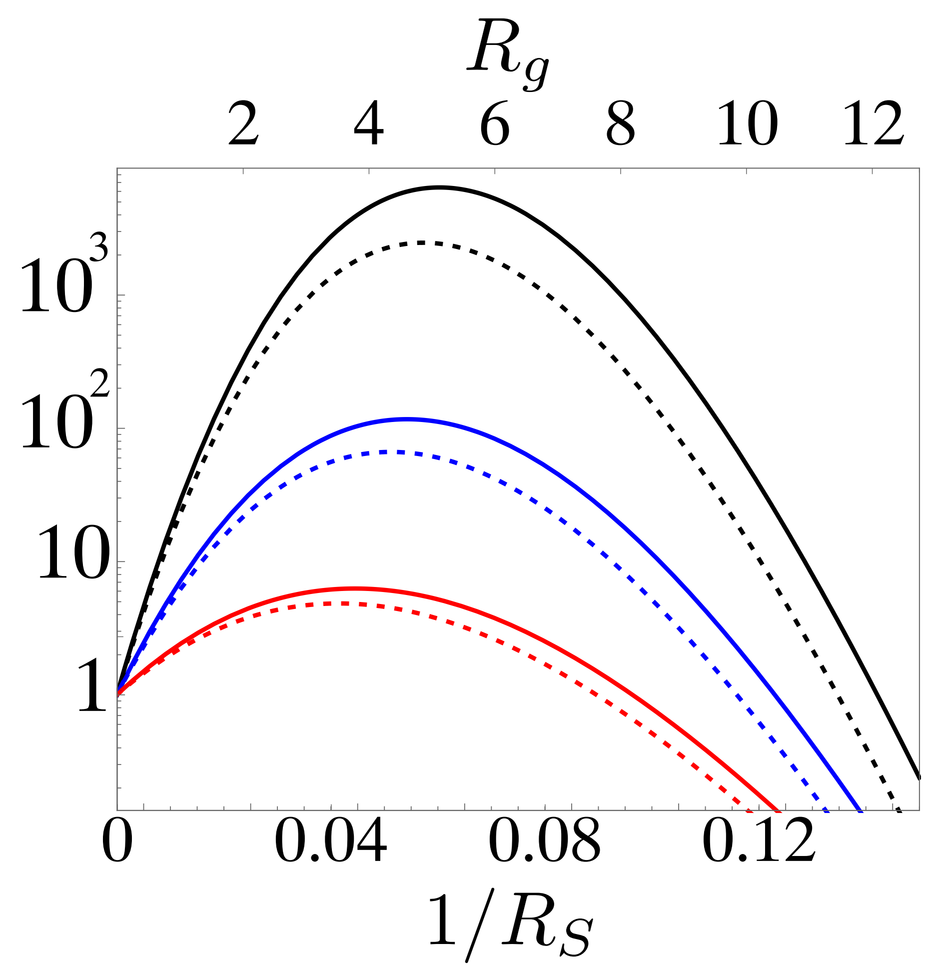
# Results

$$\mathcal{E} = \exp\left[ -\frac{1}{\hbar}\left(S_B - S_B^{(0)}\right)\right]$$




*phase dissipation*

and

*charge dissipation*

Parameters

$I_C = 21\,\mu\text{A}$

$C_{tot} = 6\,\text{pF}$

$R_g R_S \omega_I^2 =$ constant

$C_J \ll C$ lines

$C_J/C = 0.02$ dots

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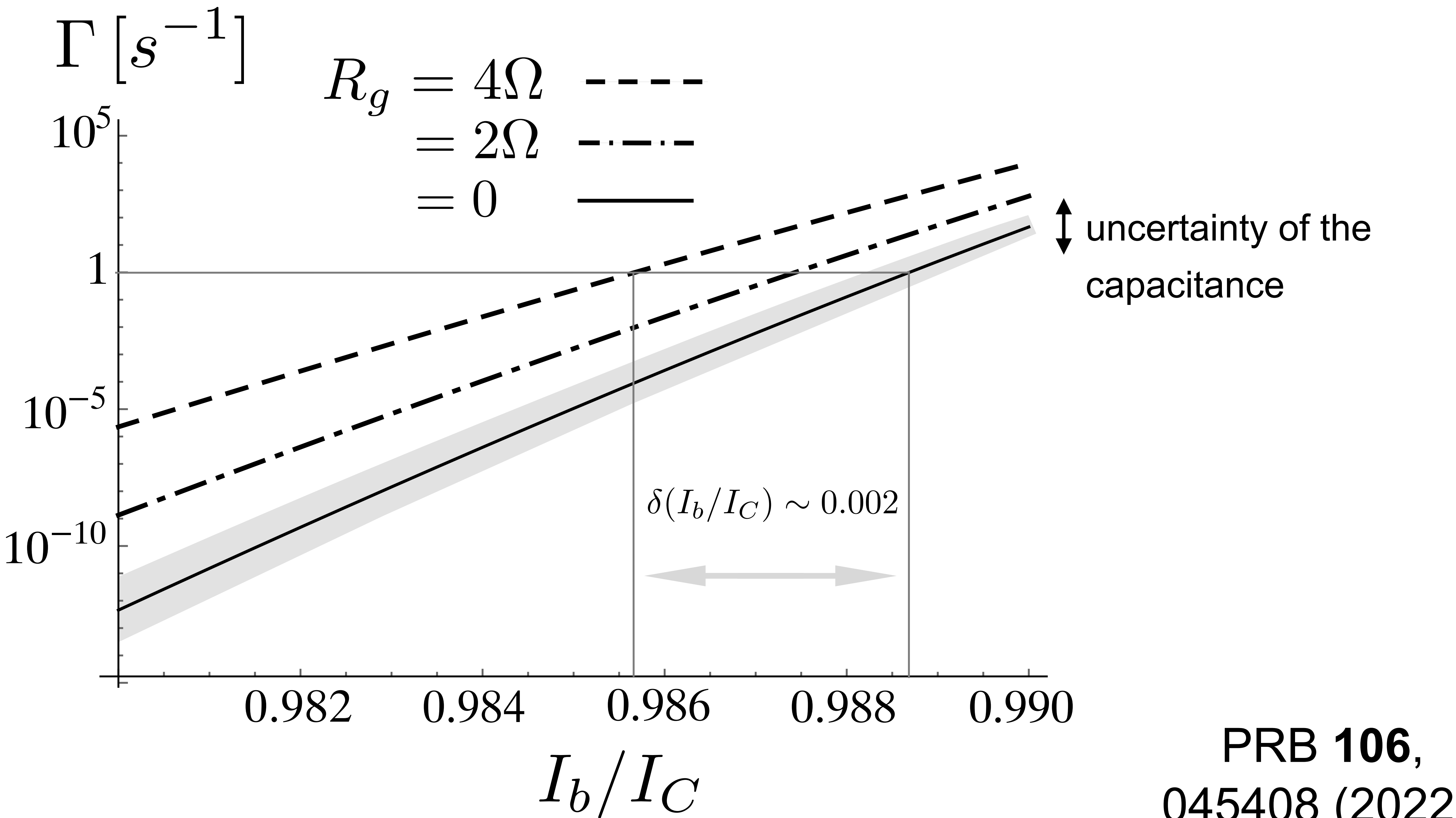
# Experimental detection

- characterisation of the electromagnetic environment

- tunability of the resistances

- accurate experimental control of the parameters

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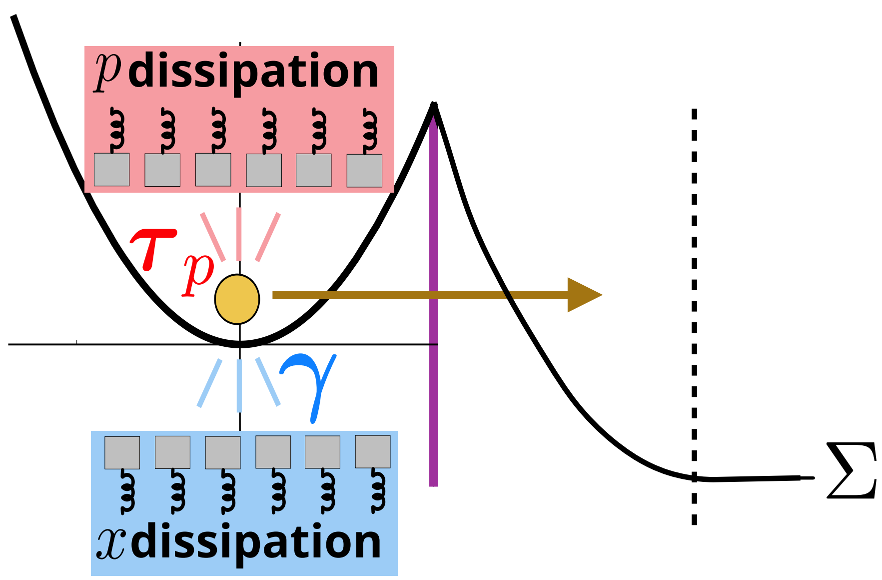
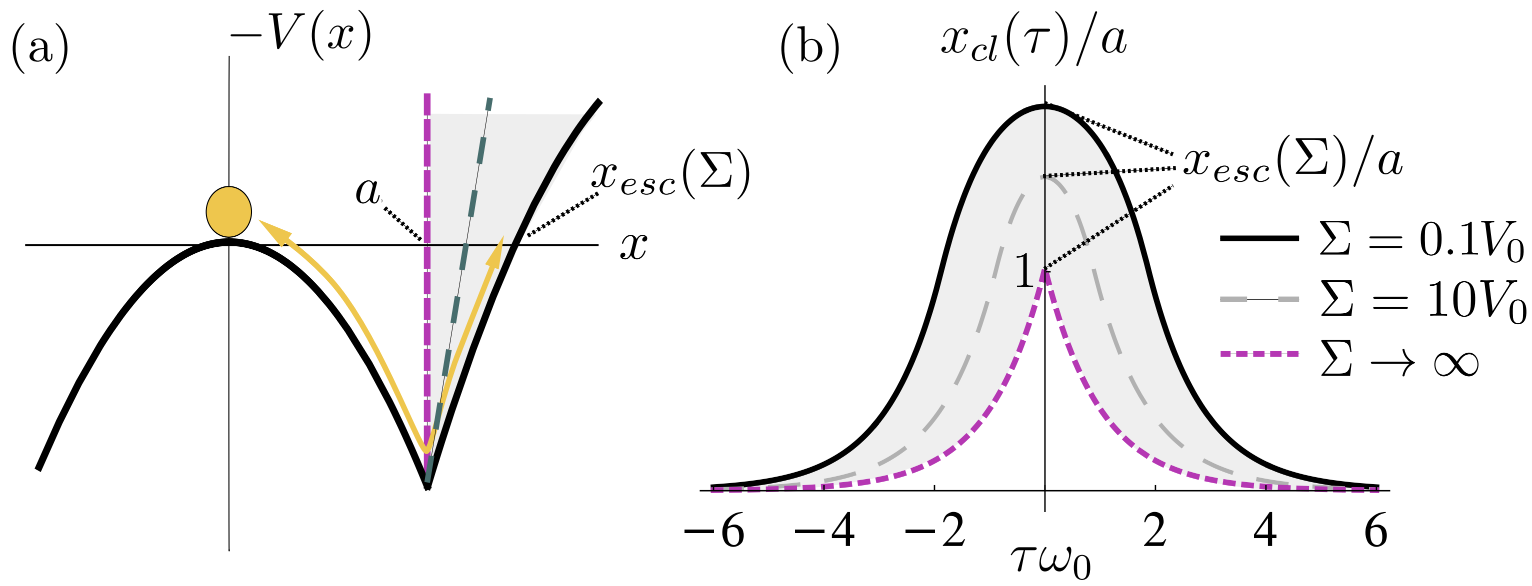
# Metastable systems with dissipation: a model with analytic results

Phys. Rev. Research **3**, 033019 (2021)

# Model potential




**Semiclassical regime (path integral method)** $\Gamma = K\, e^{-\frac{1}{\hbar}S_{cl}}$ escape rate

Euclidean action

$$S = S_0 + S_{dis}$$

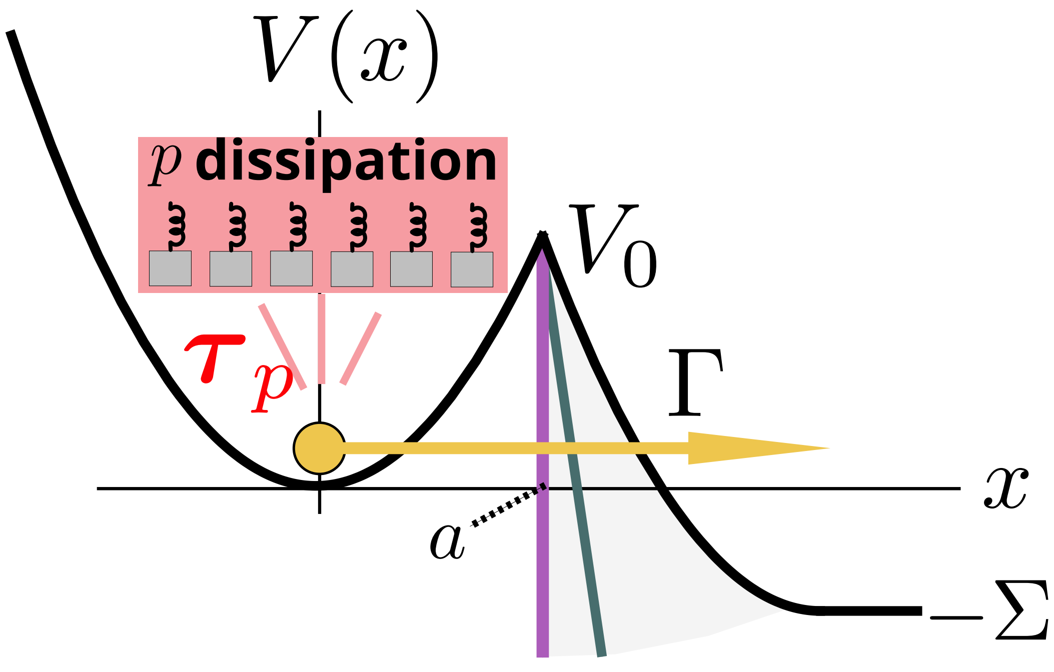
$$S_0[x(\tau)] = \int_{-\frac{\beta}{2}}^{\frac{\beta}{2}} d\tau \left[\frac{m}{2}\dot{x}^2(\tau) + V[x(\tau)]\right] \qquad \beta = \frac{\hbar}{k_B T} \longrightarrow \infty$$

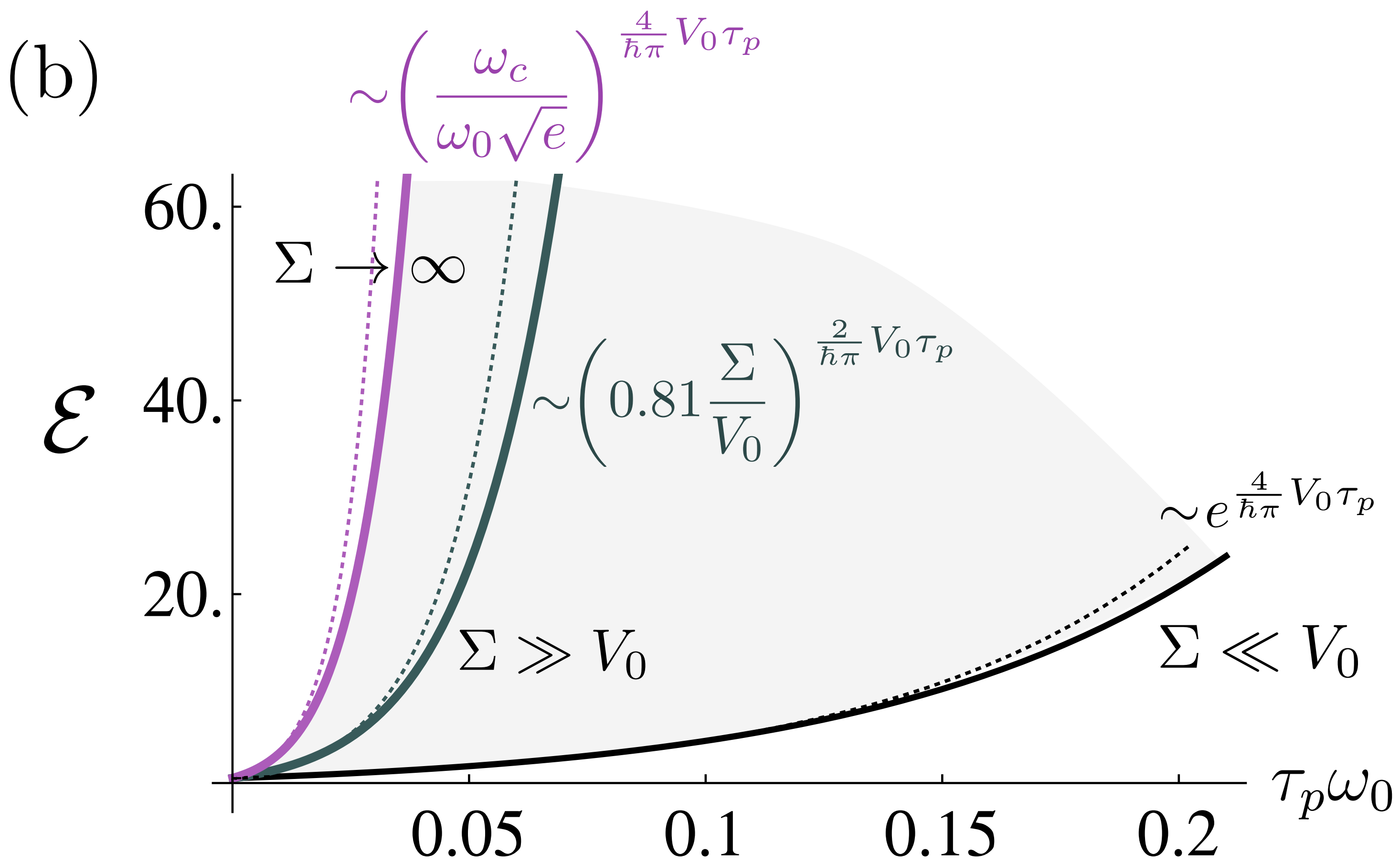
$$S_{dis}[x(\tau)] = \frac{1}{2}\iint_{-\frac{\beta}{2}}^{\frac{\beta}{2}} d\tau d\tau' F^{(x)}(\tau-\tau')x(\tau)x(\tau') + \frac{1}{2}\iint_{-\frac{\beta}{2}}^{\frac{\beta}{2}} d\tau d\tau' F^{(p)}(\tau-\tau')\dot{x}(\tau)\dot{x}(\tau')$$

$$F_l^{(x)} = \gamma m|\omega_l|\, f_c(\omega_l) \qquad F_l^{(p)} = m[-1 + (1+\tau_p|\omega_l|\, f_c(\omega_l))^{-1}] \qquad \omega_\ell = \frac{2\pi}{\beta}\ell$$

# Results with momentum dissipation

enhancement $\quad \mathcal{E} = \exp\left[ -\frac{1}{\hbar}\left( S_B - S_B^{(0)} \right) \right]$

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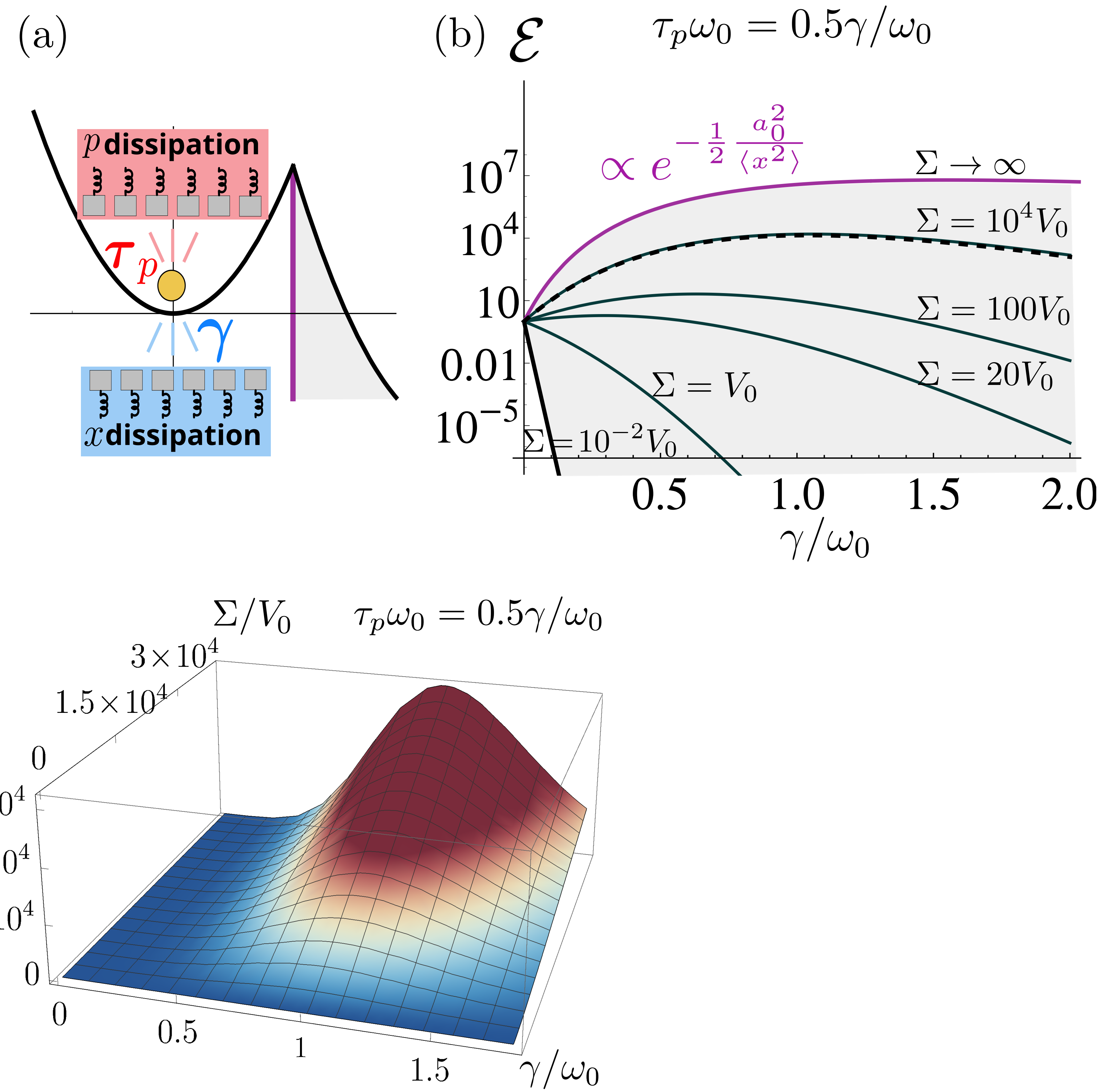
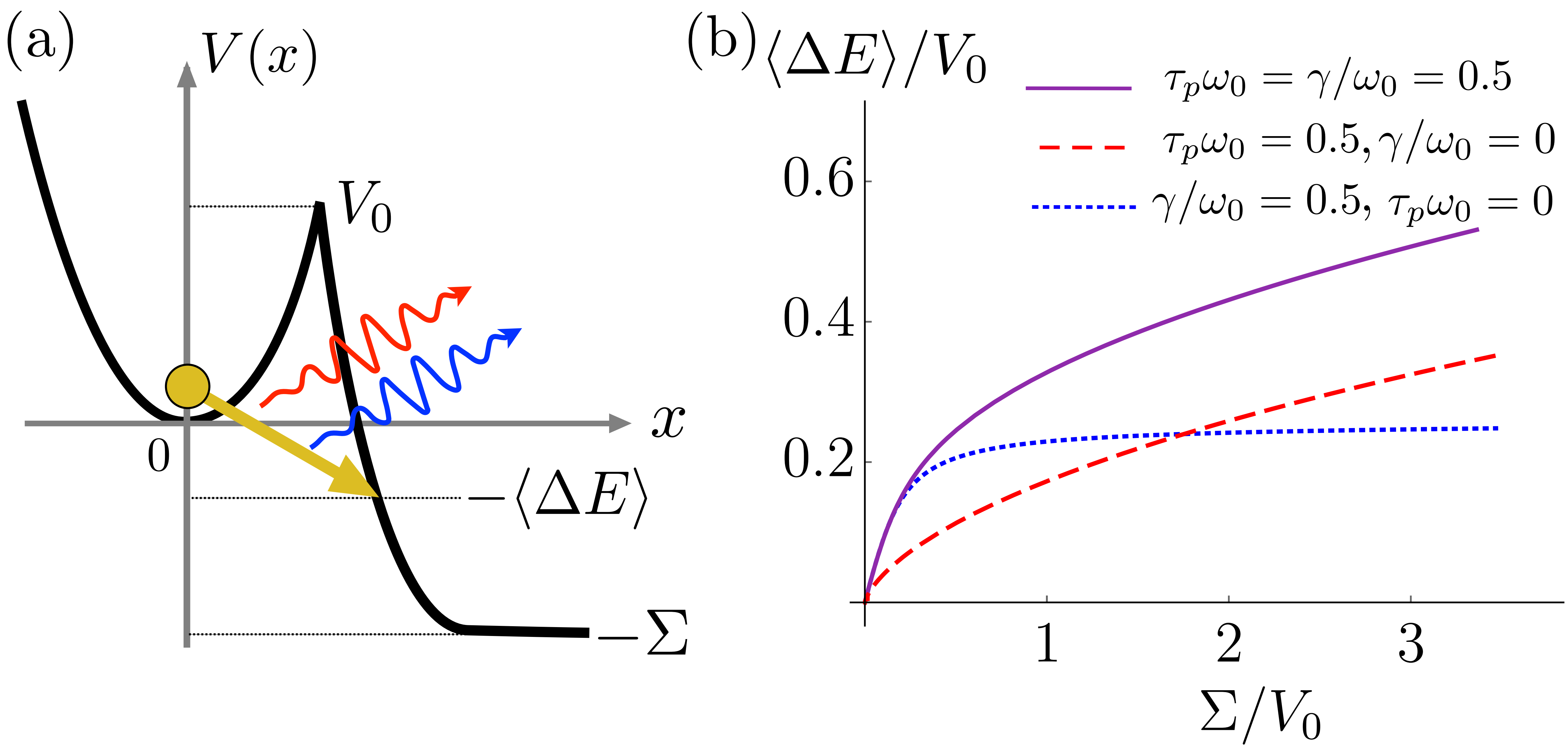
# General results

$$\mathcal{E} = \exp\left[-\frac{1}{\hbar}\left(S_B - S_B^{(0)}\right)\right]$$




Phys. Rev. Research **3**, 033019 (2021)

# Average energy loss

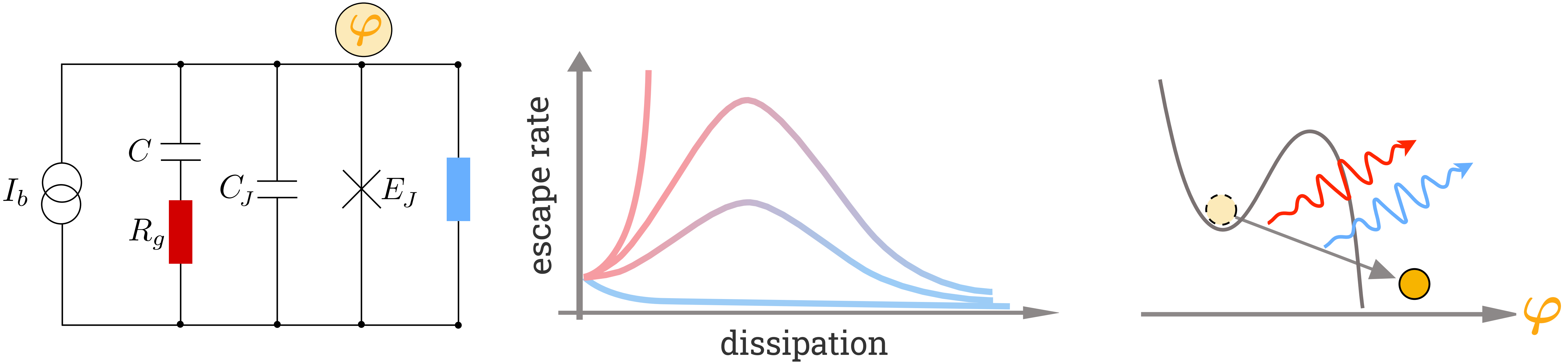


# Summary

- **enhancement of the escape rate** from a metastable state in a quantum Josephson circuit using a simple scheme

- possibly of speeding up the relaxation dynamics towards the energy minimum

- as proof of concept: **perspective of using quantum dissipative Josephson circuits as quantum simulators for optimization problems**




PRB **106**, 045408 (2022)

# Collaborators

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Universität Tübingen

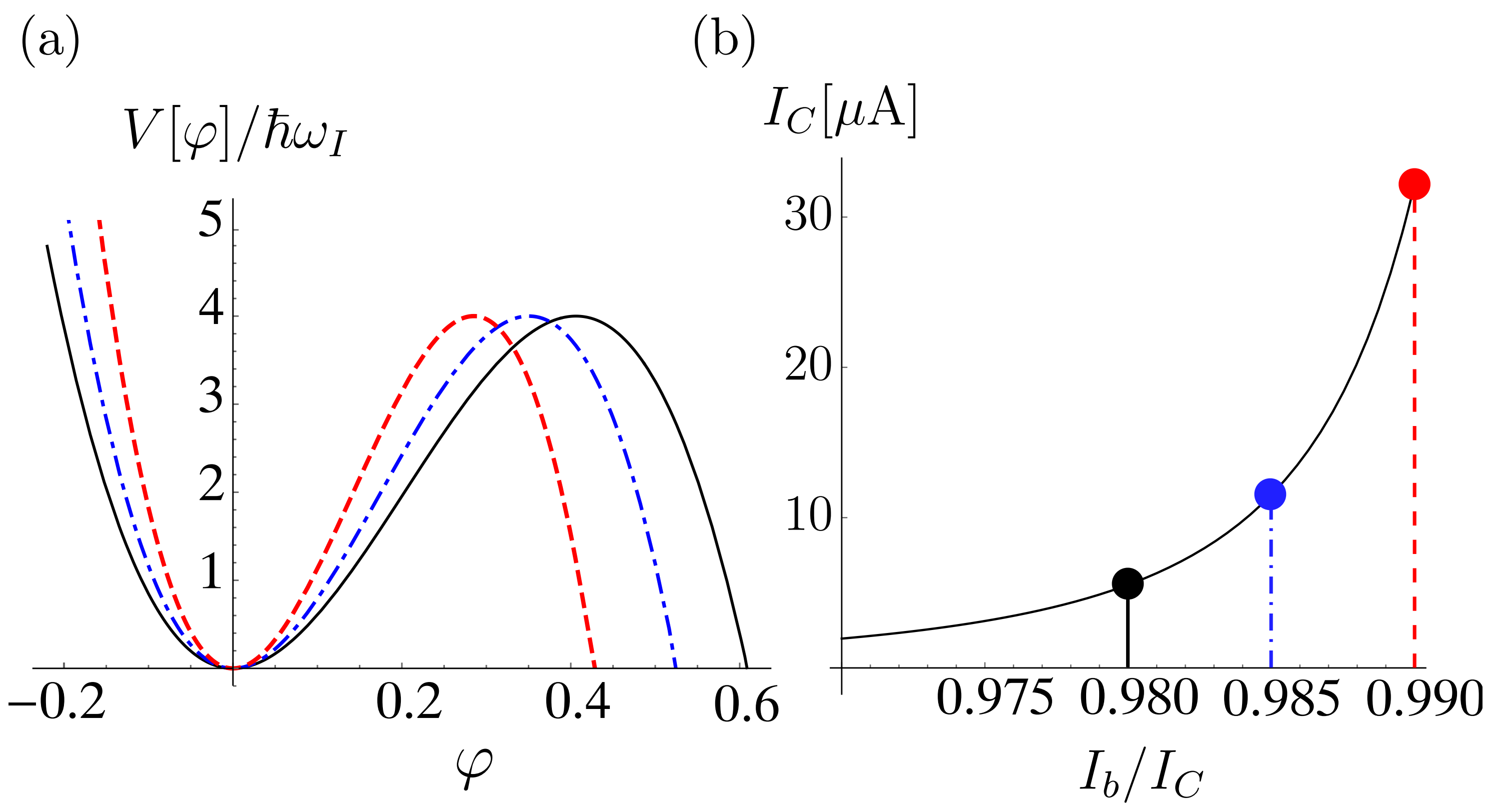
Wolfgang Belzig

Universität Konstanz

**Thank you for the attention**

# Additional slides

# Results: fixed barrier height

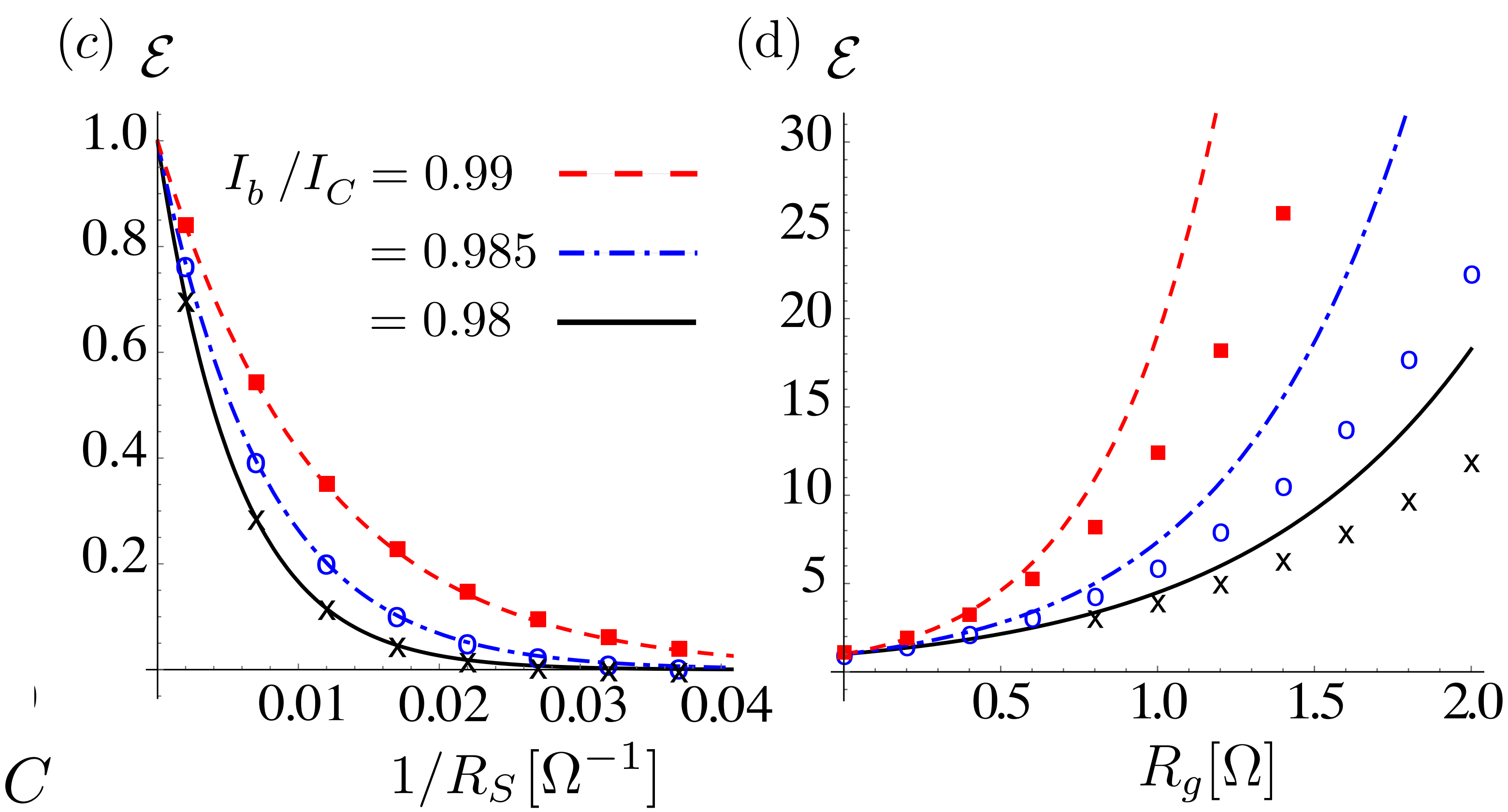



fixed barrier height

$$V_0/\hbar\omega_I = 4$$

*phase dissipation*

*charge dissipation*

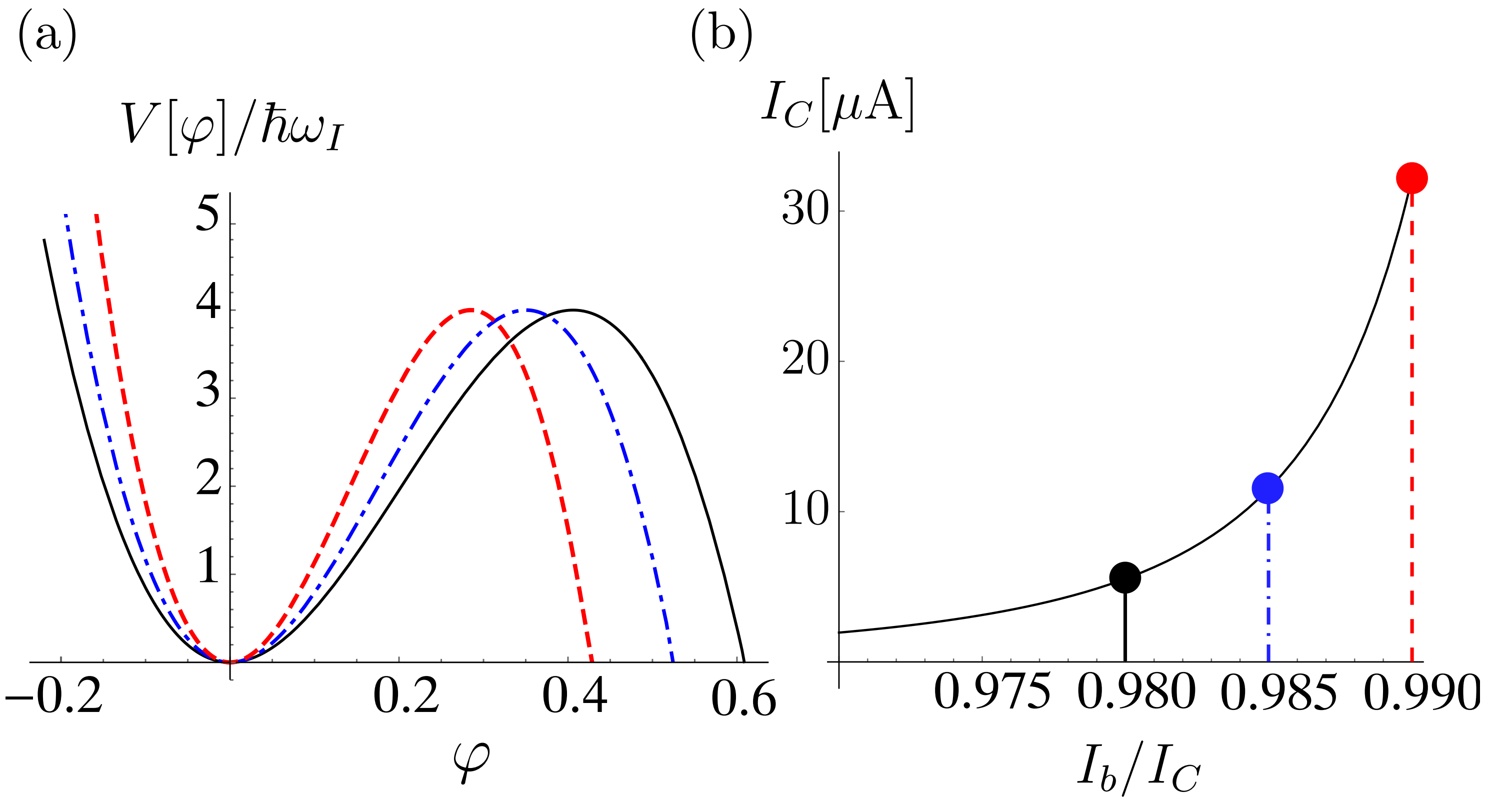



lines = variational method

points = perturbave approx.

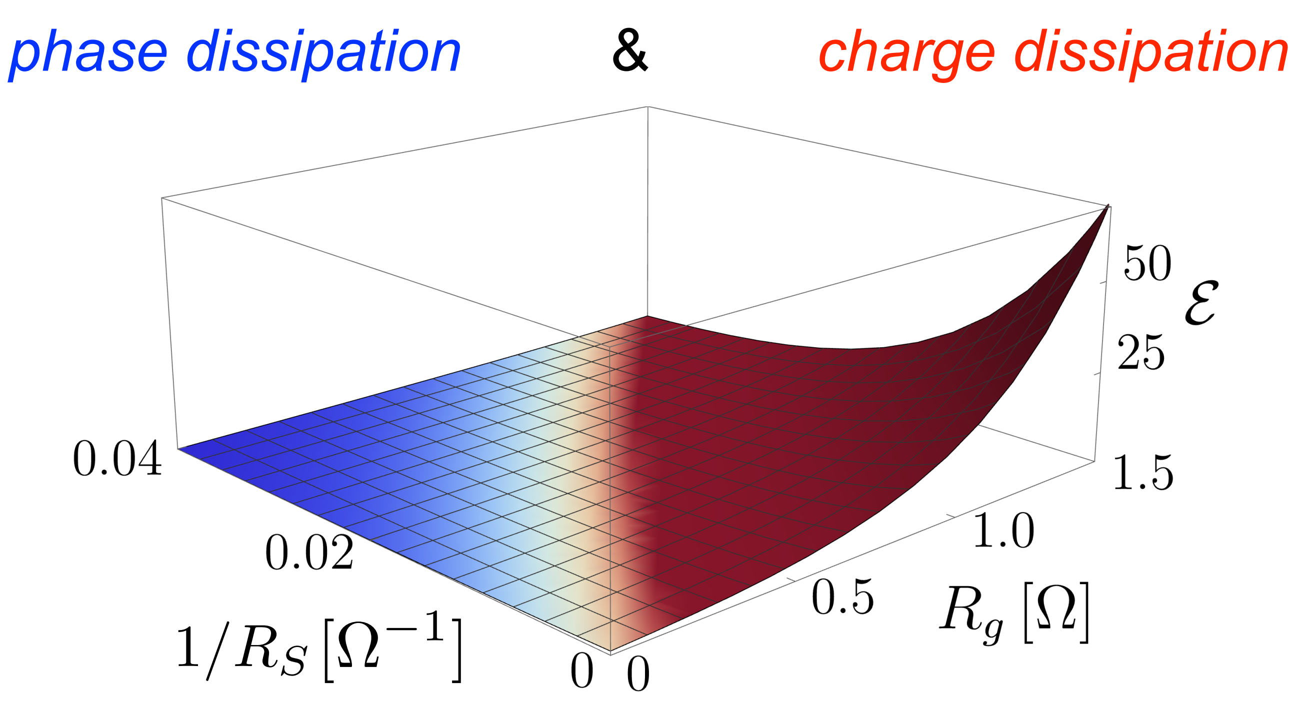
Parameters: $C_{tot} = 6\,\mathrm{pF}$ $\quad C_J \ll C$

# Results: fixed barrier height




fixed barrier height

$$V_0/\hbar\omega_I = 4$$

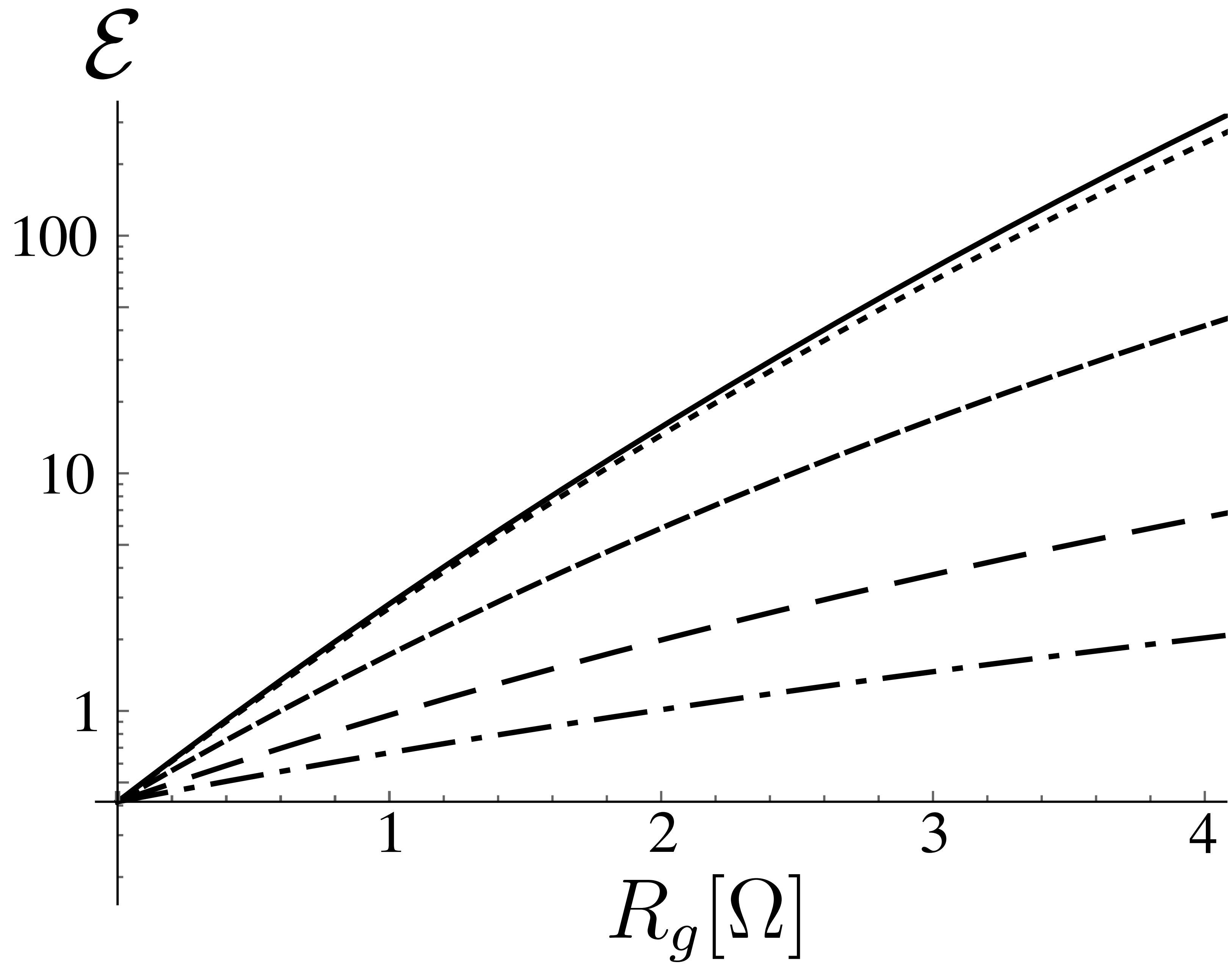



$$I_b/I_C = 0.99$$

variational method

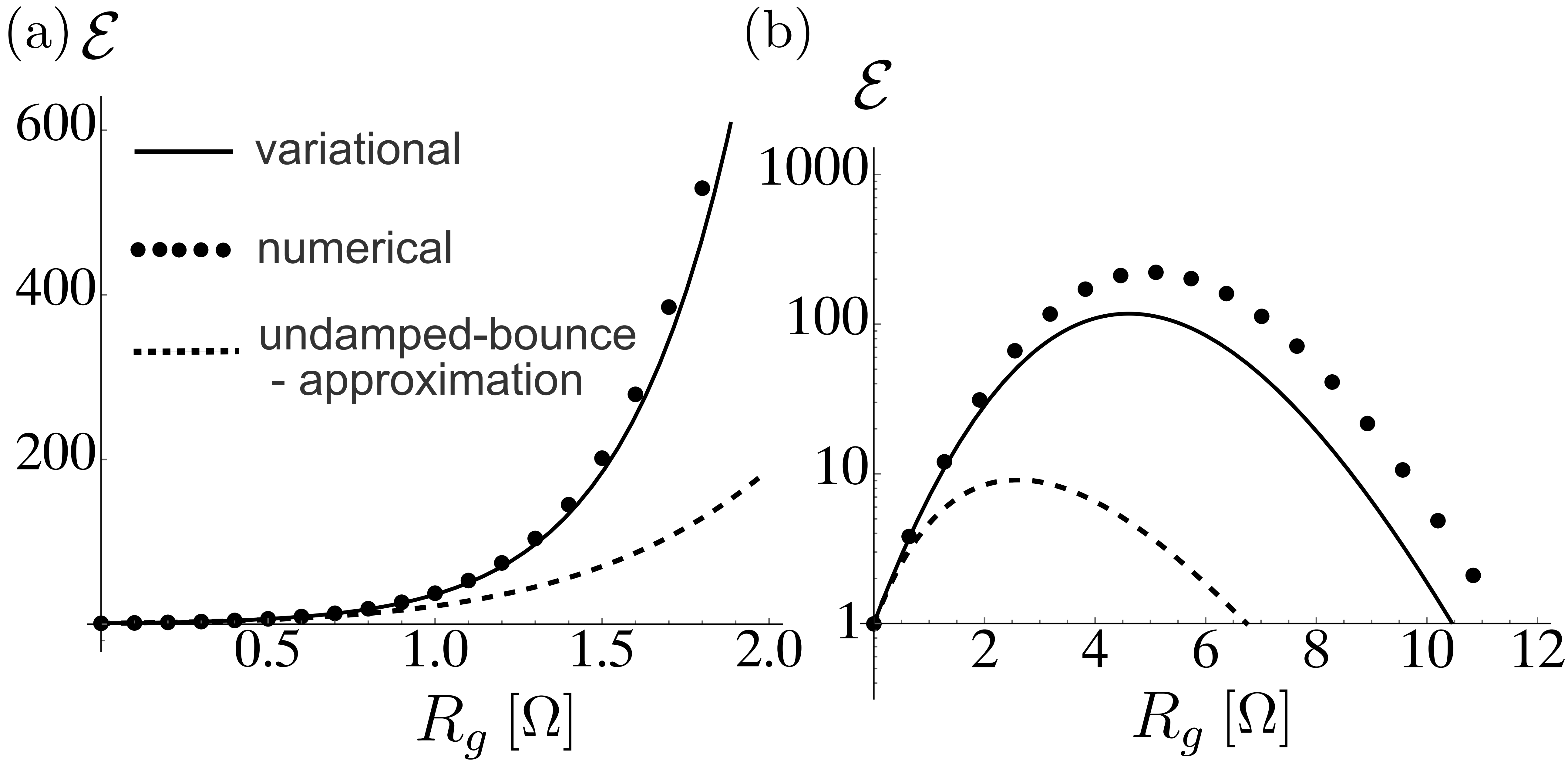
Parameters: $C_{tot} = 6\,\mathrm{pF}$ $\quad C_J \ll C$

# Results: effect of the capacitance

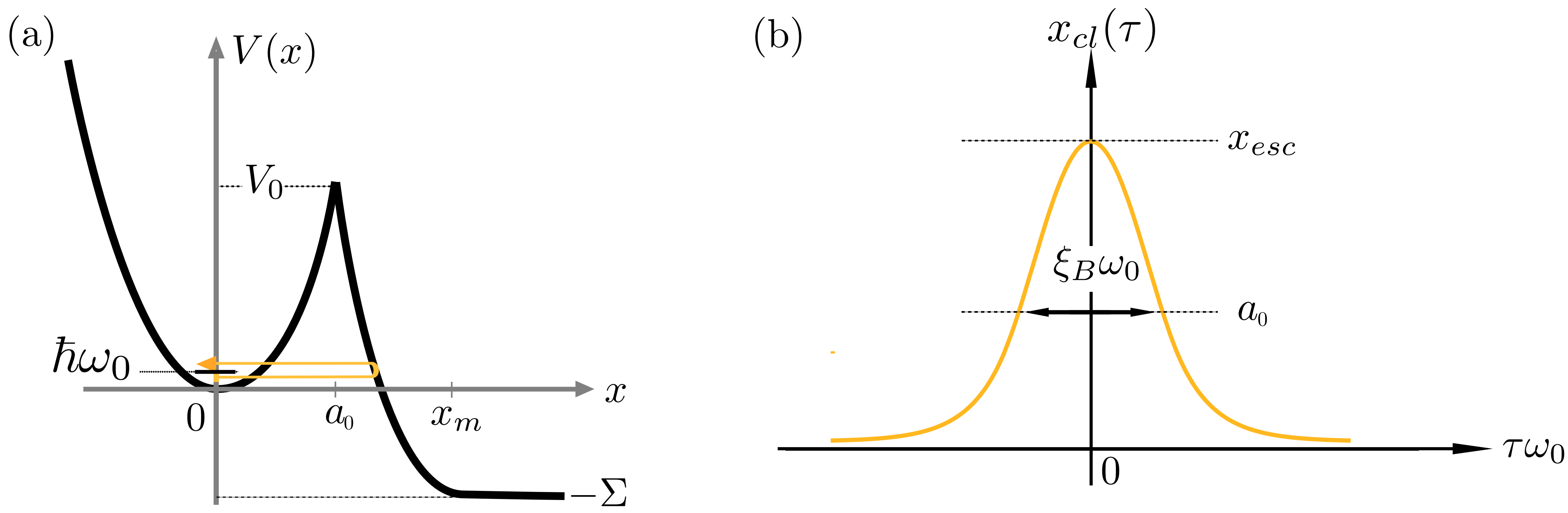



$$C_J/C = 0,\ 0.01,\ 0.2,\ 0.5,\ 1$$

# Numerics vs variational method



# General formula for the Action




$$-m\frac{d^2}{d\tau^2}x_{cl}(\tau,\xi)+\left.\frac{dV(x(\tau))}{dx}\right|_{x_{cl}}+\int_{-\frac{\beta}{2}}^{\frac{\beta}{2}}d\tau' F(\tau-\tau')\,x_{cl}(\tau',\xi)-\int_{-\frac{\beta}{2}}^{\frac{\beta}{2}}d\tau'\widetilde{F}(\tau-\tau')\frac{d^2}{d\tau'^2}x_{cl}(\tau',\xi)=0$$

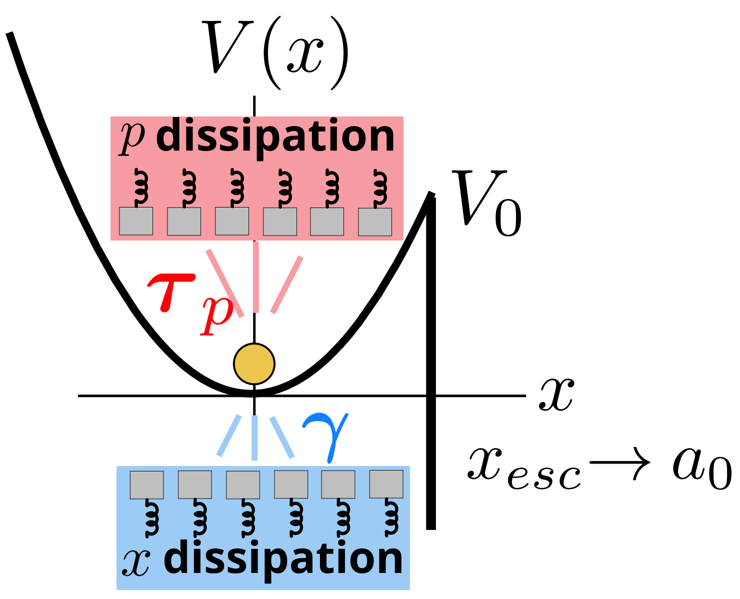
**bounce time:**

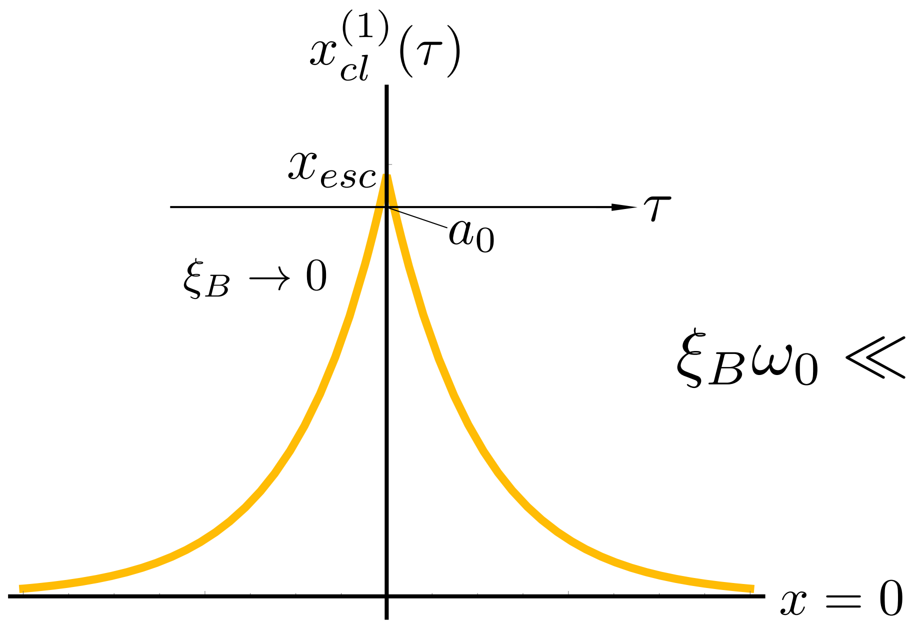
$$\frac{1}{\pi}\int_0^\infty d\omega\frac{\sin(\omega\xi_B)}{\omega\left(\frac{\omega^2}{1+\tau_p\omega f_c(|\omega|)}+\omega_0^2+\gamma\omega f_c(|\omega|)\right)}\overset{!}{=}\frac{1}{\omega_0^2}\frac{1}{\sqrt{1+\Sigma/V_0}+1}$$

**action on the saddle point path:**

$$S_{cl}=-\frac{2\omega_0^2V_0\left(\sqrt{1+\Sigma/V_0}+1\right)^2}{\pi}\int_0^\infty d\omega\frac{(1-\cos(\omega\xi_B))}{\omega^2\left(\frac{\omega^2}{1+\tau_p\omega f_c(|\omega|)}+\omega_0^2+\gamma\omega f_c(|\omega|)\right)}+2V_0\left(1+\sqrt{1+\Sigma/V_0}\right)\xi_B$$

# Limit case (1)




$$\xi_B \omega_0 \ll \xi_B \omega_c \ll 1$$

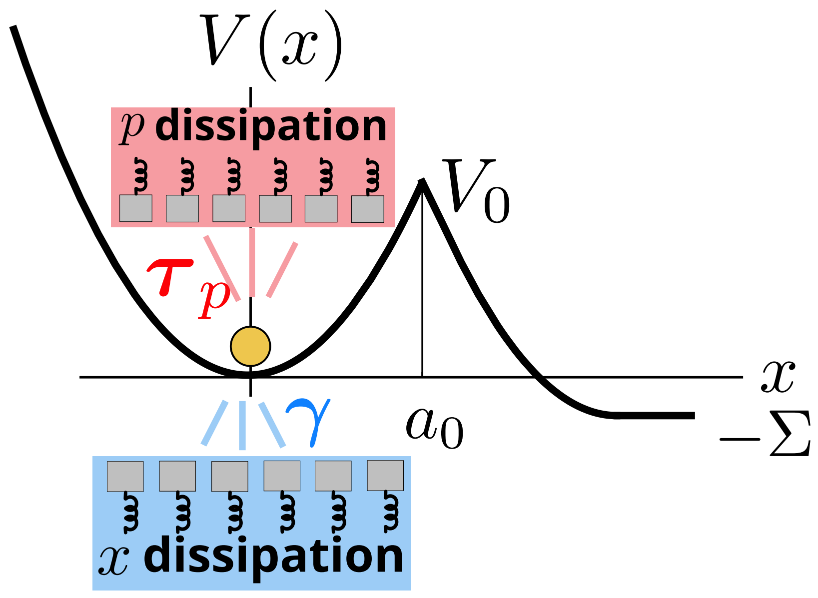
**bounce time:**

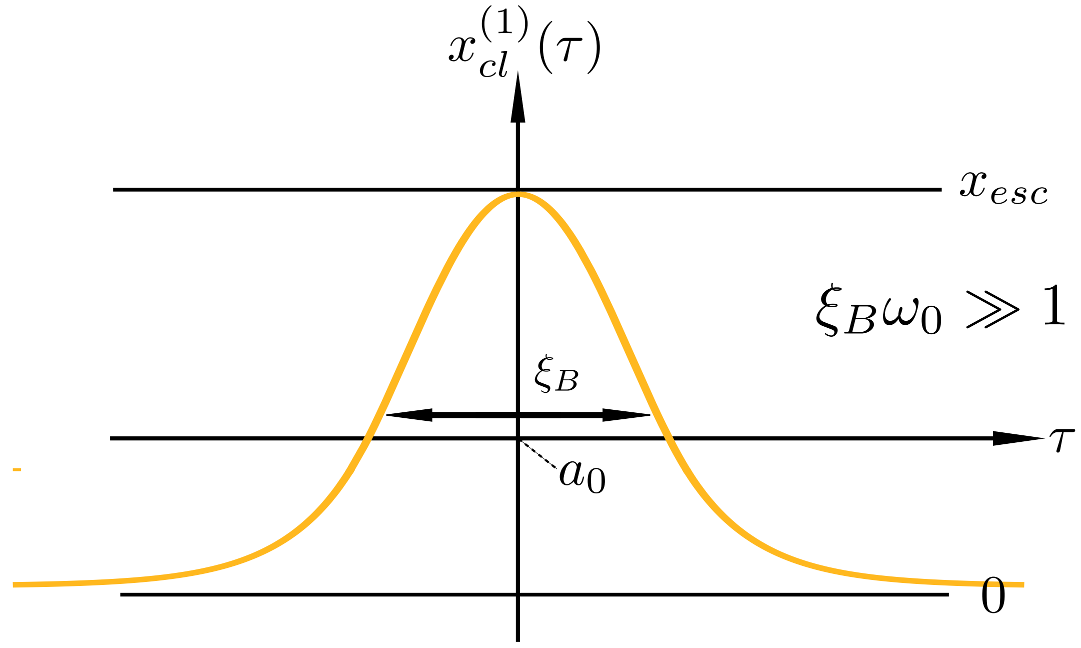
$$\xi_B \omega_0 = \frac{\hbar}{m\omega_0} \frac{1}{\sqrt{1 + \Sigma/V_0} + 1} \frac{1}{\langle x^2 \rangle}$$

**action on the saddle point path:**

$$S_{cl} = \frac{\hbar}{2} \frac{a_0^2}{\langle x^2 \rangle}$$

# Limit case (2)




**bounce time:**

$$\xi_B\omega_0 \approx \frac{8}{\pi}\frac{V_0}{\Sigma}\frac{\gamma}{\omega_0}$$

**action on the saddle point path:**

$$S_{cl} \approx \frac{\epsilon_0}{\omega_0} + \frac{8}{\pi}\frac{V_0}{\omega_0}\frac{\gamma}{\omega_0}\ln\left(\xi_B\omega_0\right) - \Sigma\xi_B$$

$$\frac{\epsilon_0}{\omega_0} = \frac{8}{\pi}\frac{V_0}{\omega_0}\left[\frac{\gamma}{\omega_0}\left(C - \frac{\ln\left(1+\sigma^2\right)}{2}\right) - \frac{\left(\frac{\gamma}{\omega_0}P_- - 1\right)}{2\sqrt{P_-^2 - 1}}\ln\left(\frac{\Lambda_1}{\Lambda_2}\right)\right]$$

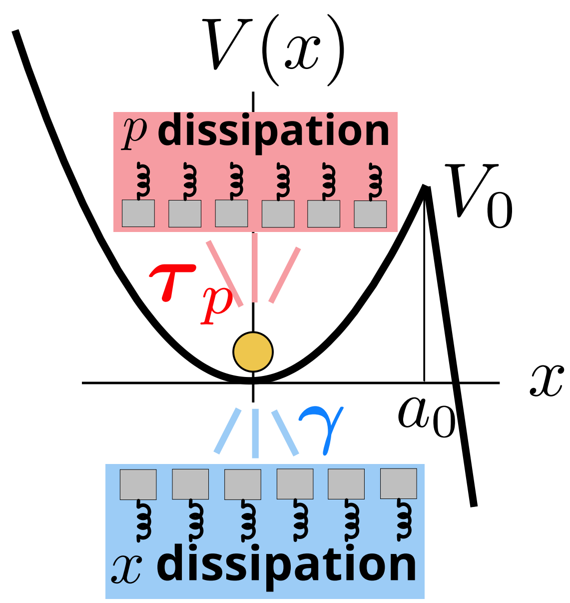
$$P_\pm = \left(\gamma \pm \tau_p\omega_0^2\right)/2\omega_0$$

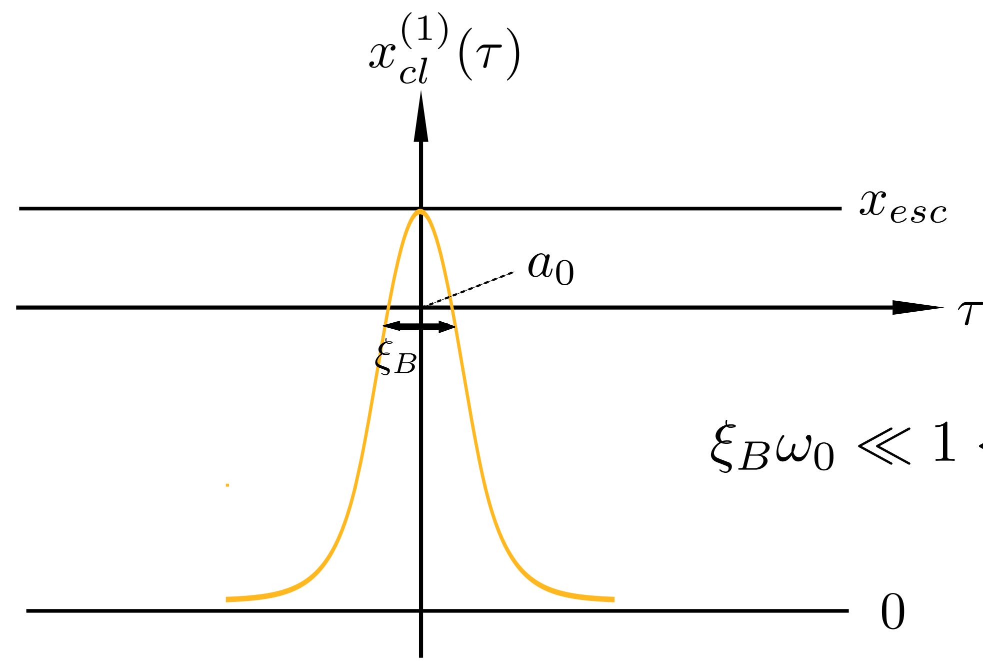
$$\sigma = \sqrt{\gamma\tau_p}$$

$$\Lambda_{1,2} = \frac{\omega_0}{1+\sigma^2}\left(P_+ \pm \sqrt{P_-^2 - 1}\right)$$

[U. Weiss et al. Phys. Rev. B 35, 9535 (1987)]

# Limit case (3)




$\xi_B\omega_0 \ll 1 \ll \xi_B\omega_c$

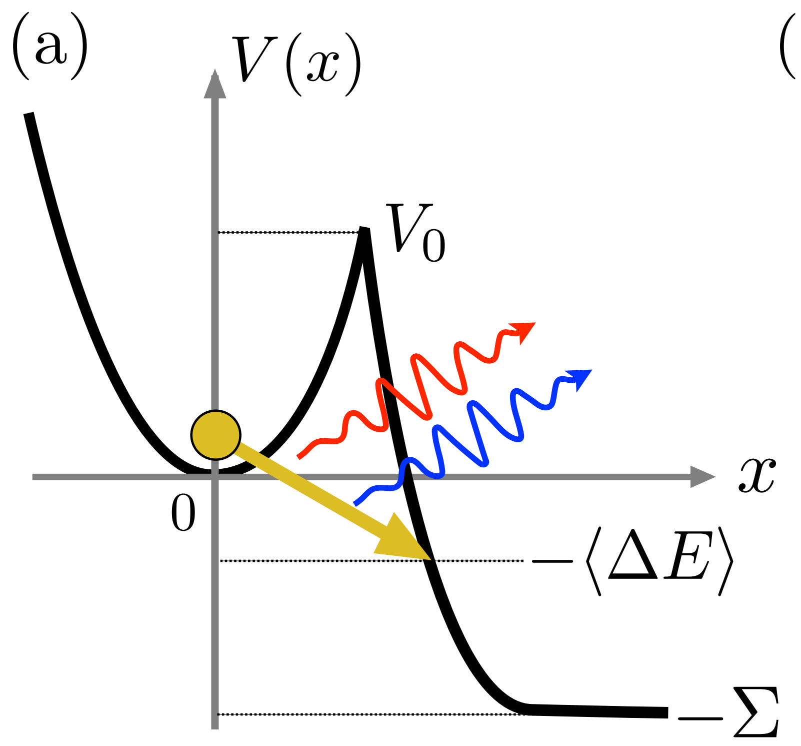
**bounce time:**

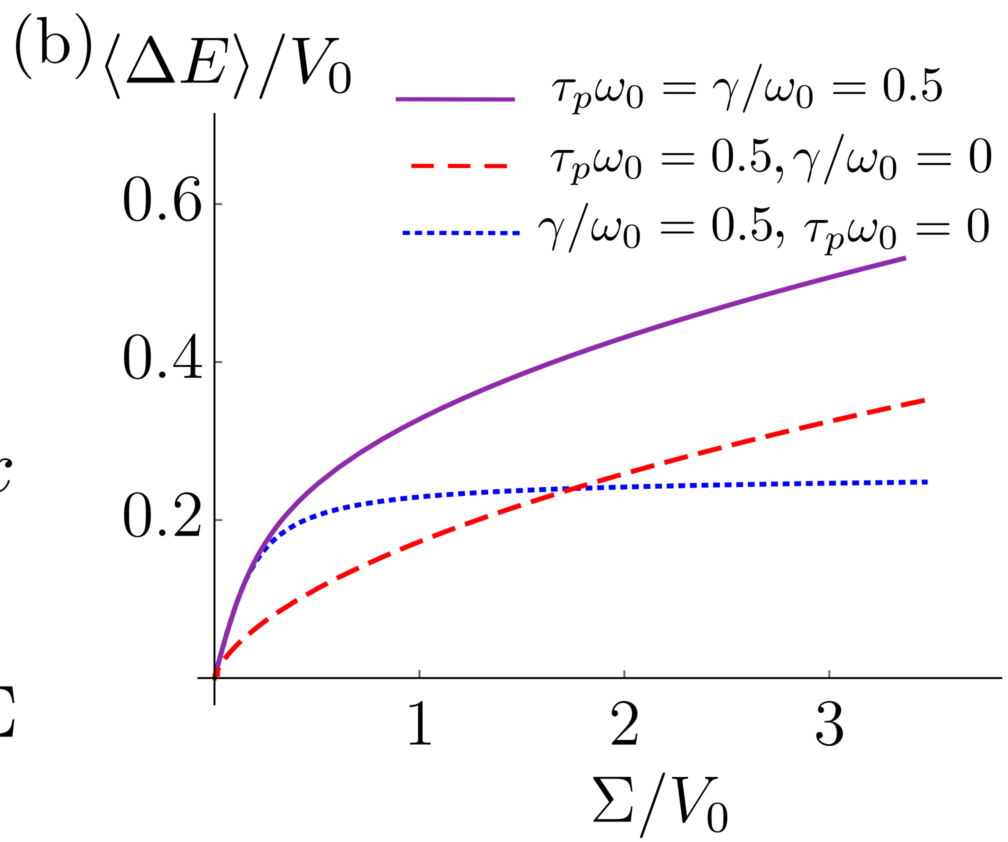
$$\frac{1}{\pi}\frac{\xi_B\omega_0}{(1+\sigma^2)}\left(\frac{(1+\tau_p\omega_0 P_-)}{2\sqrt{P_-^2-1}}\ln\left(\frac{\Lambda_1}{\Lambda_2}\right)-\tau_p\omega_0\left(\ln(\omega_0\xi_B)-\frac{1}{2}\ln\left(1+\sigma^2\right)+(C-1)\right)\right)\stackrel{!}{=}\frac{1}{\sqrt{1+\Sigma/V_0}+1}$$

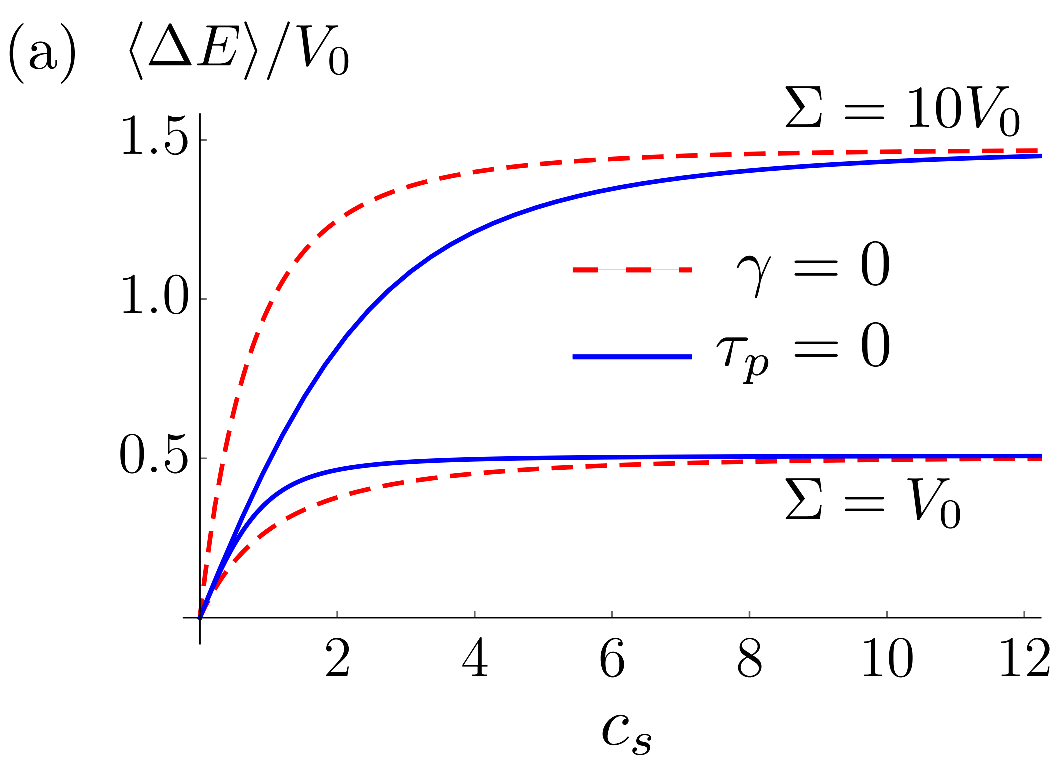
**action on the saddle point path:**

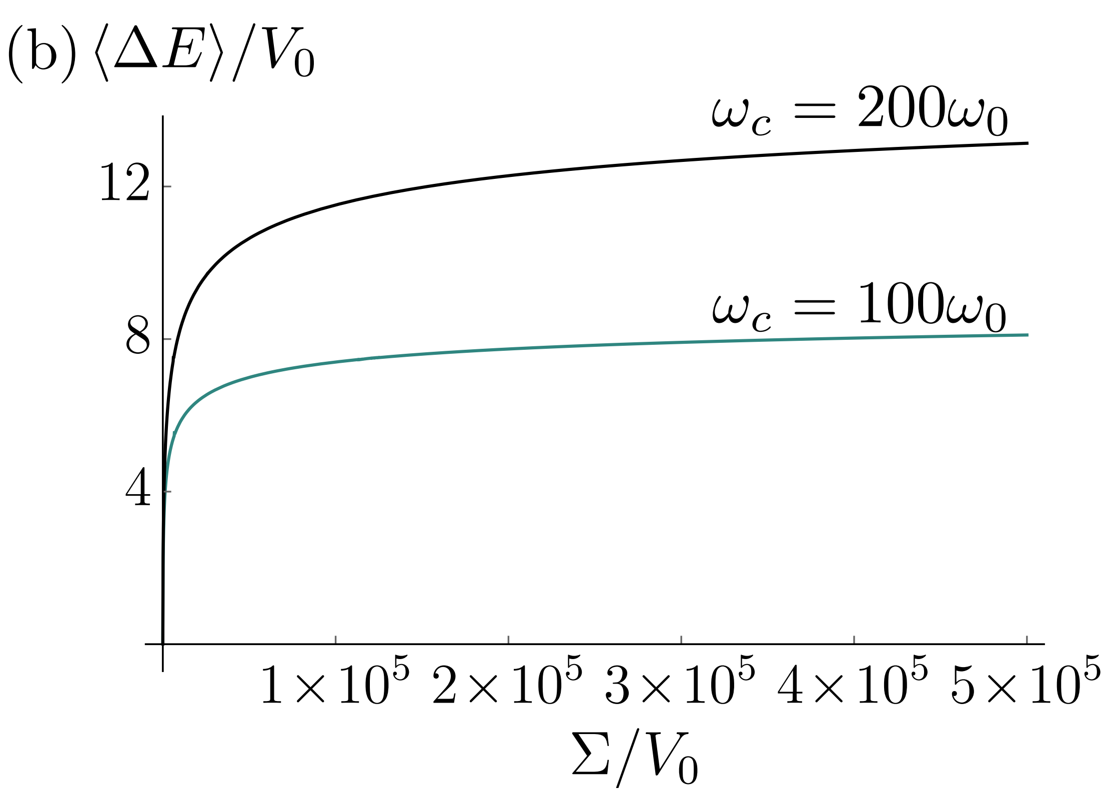
$$S_{cl}=\frac{V_0}{\omega_0}\left(\sqrt{1+\Sigma/V_0}+1\right)\xi_B\omega_0-\frac{V_0}{\omega_0}\frac{\left(\sqrt{\Sigma/V_0+1}+1\right)^2}{2\pi\left(1+\sigma^2\right)}\xi_B^2\omega_0^2\tau_p\omega_0$$

# Average energy loss

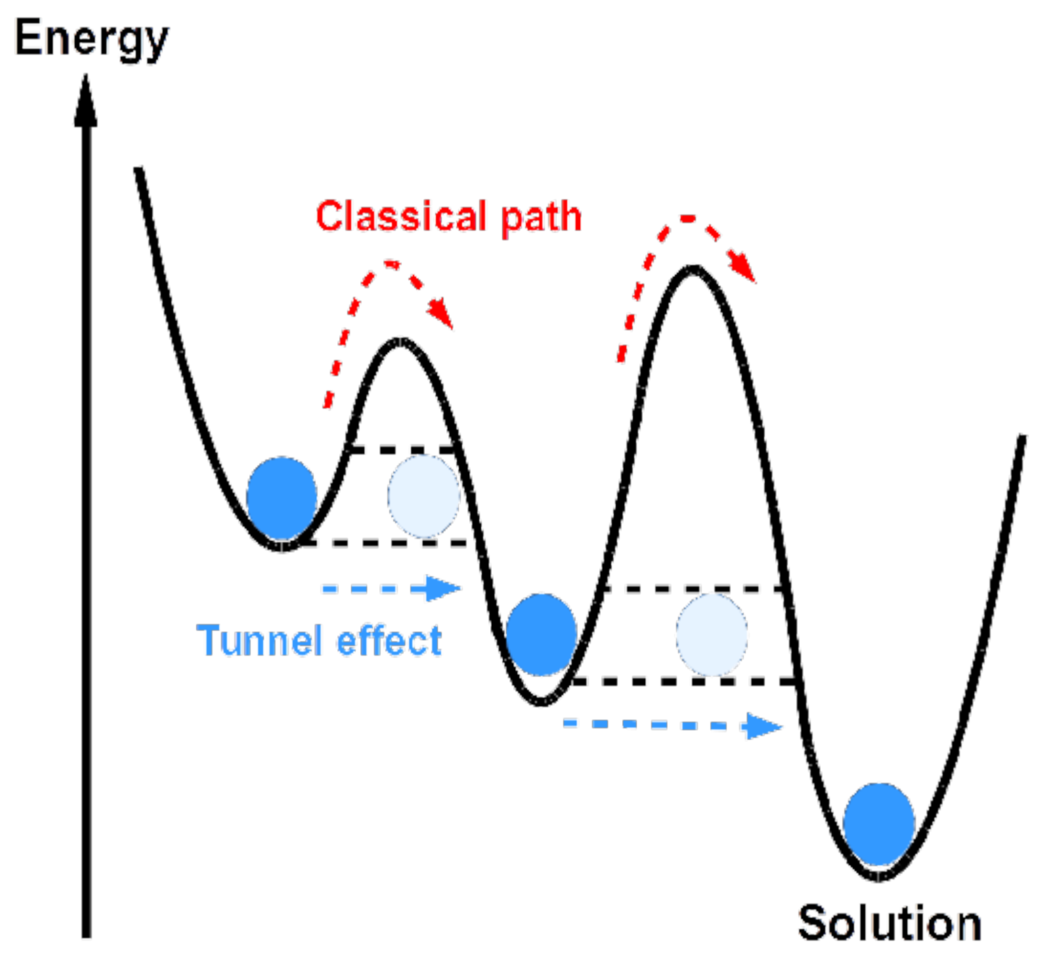


# Perspectives

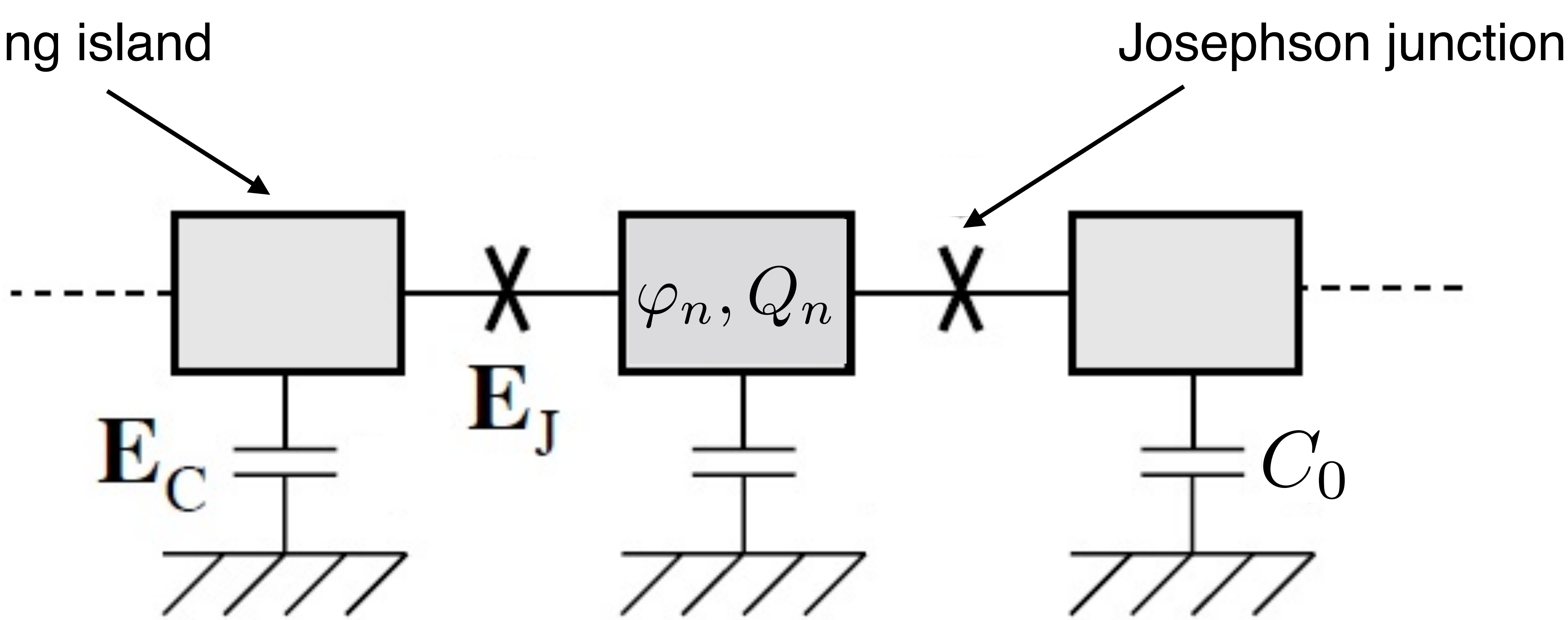



- **Classical Minimum= solution of an optimization problem**
- prepare the state in an arbitrary minimum
- switch on the *unconventional* interaction with the *engineered* environment
- the *incoherent* tunneling rate *increases* (environment assisted)

➡ adiabatic evolution is not required

➡ quantum coherence is not required

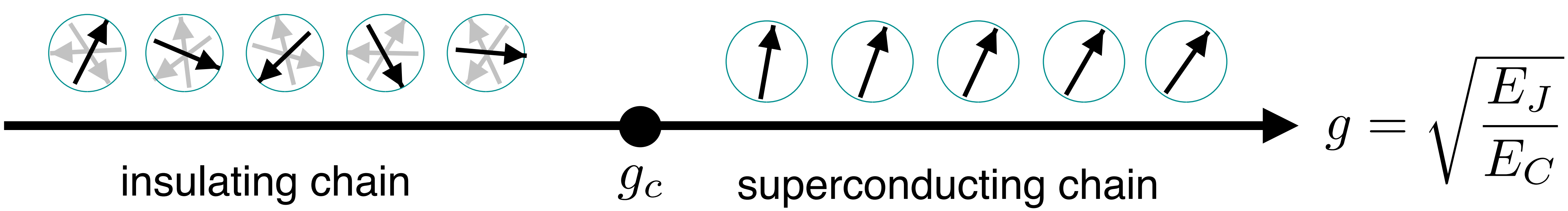
# Josephson junction chains




Fazio, van der Zant,Phys. Rep. 355,  235 ( 2001)

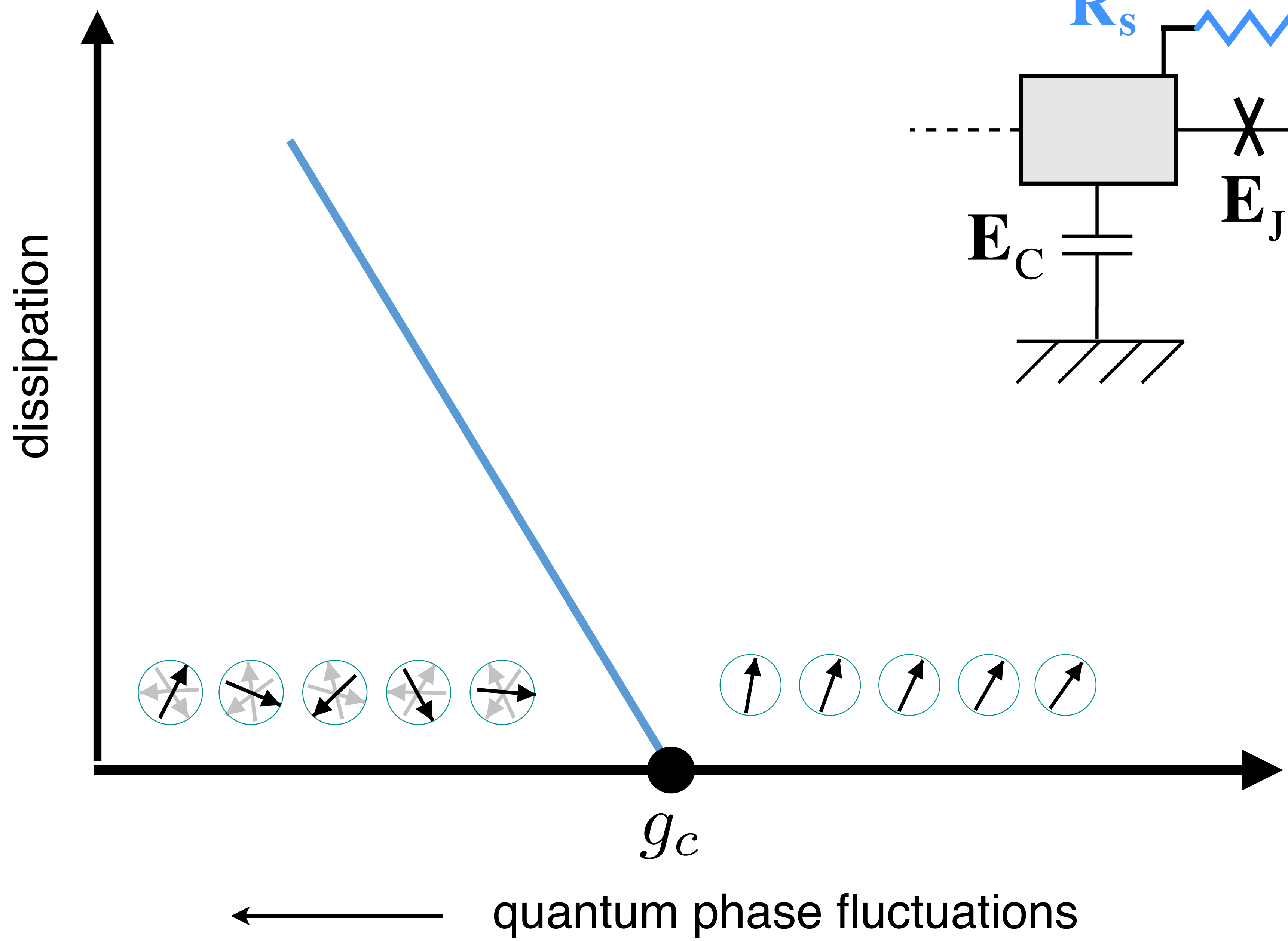
**Quantum phase transition**

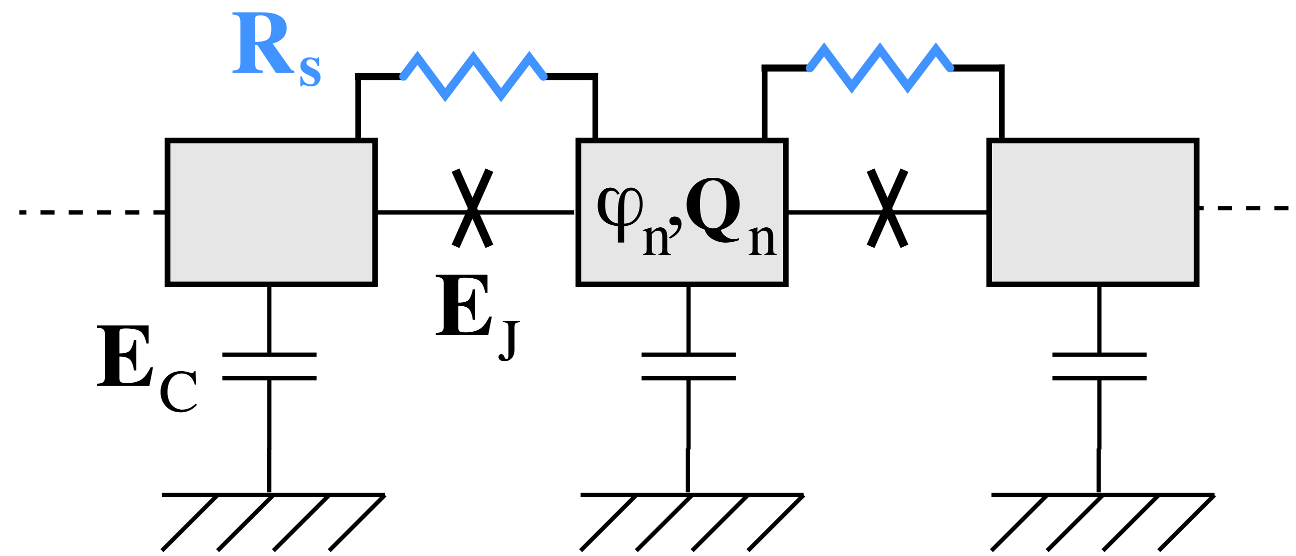
$T = 0K$




Bradley, Doniach,Phys. Rev. B **30**, 1138 (1984)

# Dissipative Quantum Phase Transition




Chakravarty et al., PRL **56**, 2303 (1986)

Panyukov,Zaikin, Phys.Lett.A **124**, 325 (1987)

Korshunov, EPL **9**, 107 (1989)

Chakravarty et al., PRB **37**, 3293 (1988)

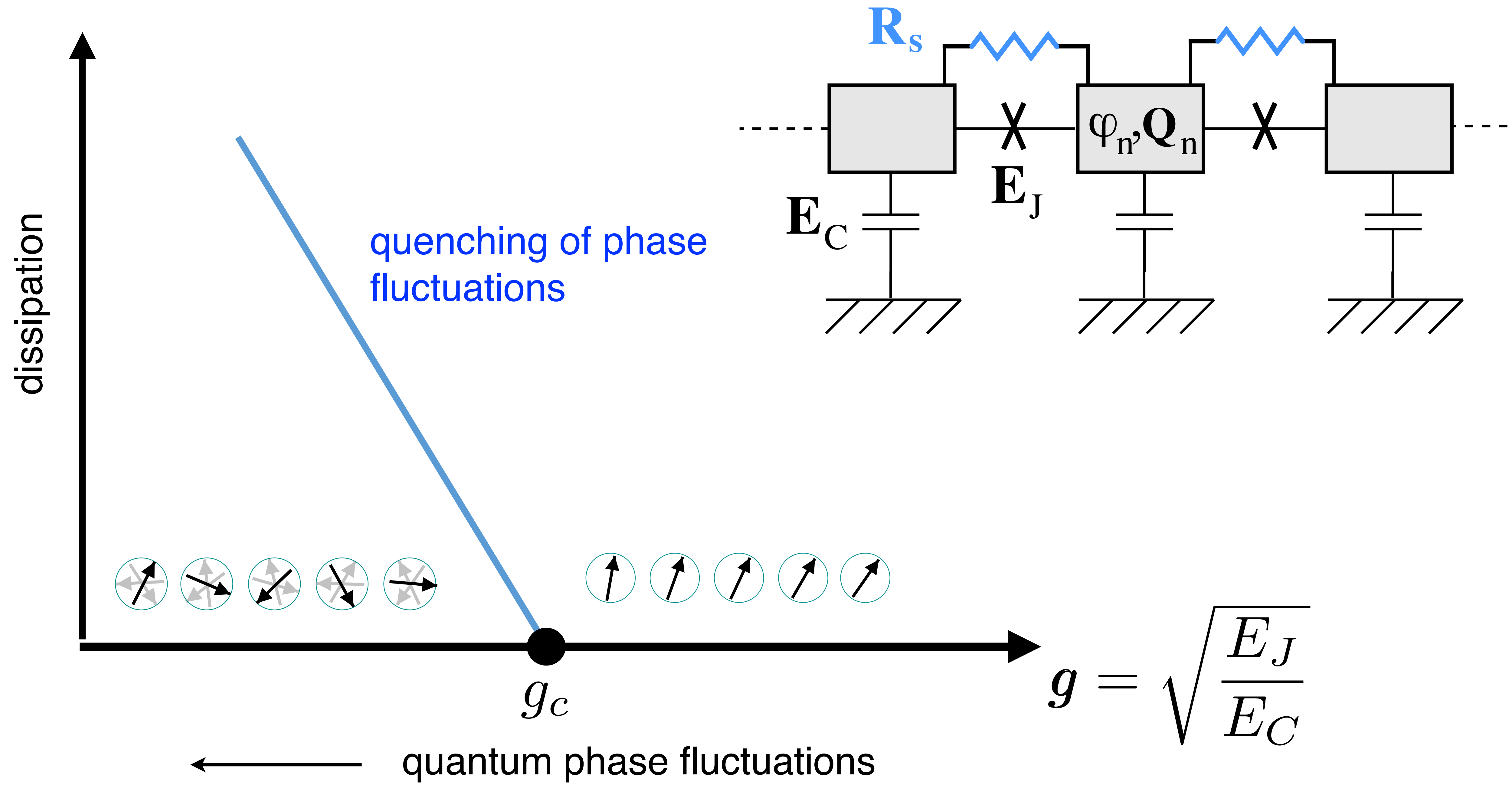
Bobbert et al., PRB **41**, 4009 (1990)

Bobbert et al., PRB **45**, 2294 (1992)

Wagenblast, PRL **79**, 2730 (1997)

Refael et al., PRB **75**, 014522 (2007)

# Dissipative Quantum Phase Transition





Chakravarty et al., PRL **56**, 2303 (1986)

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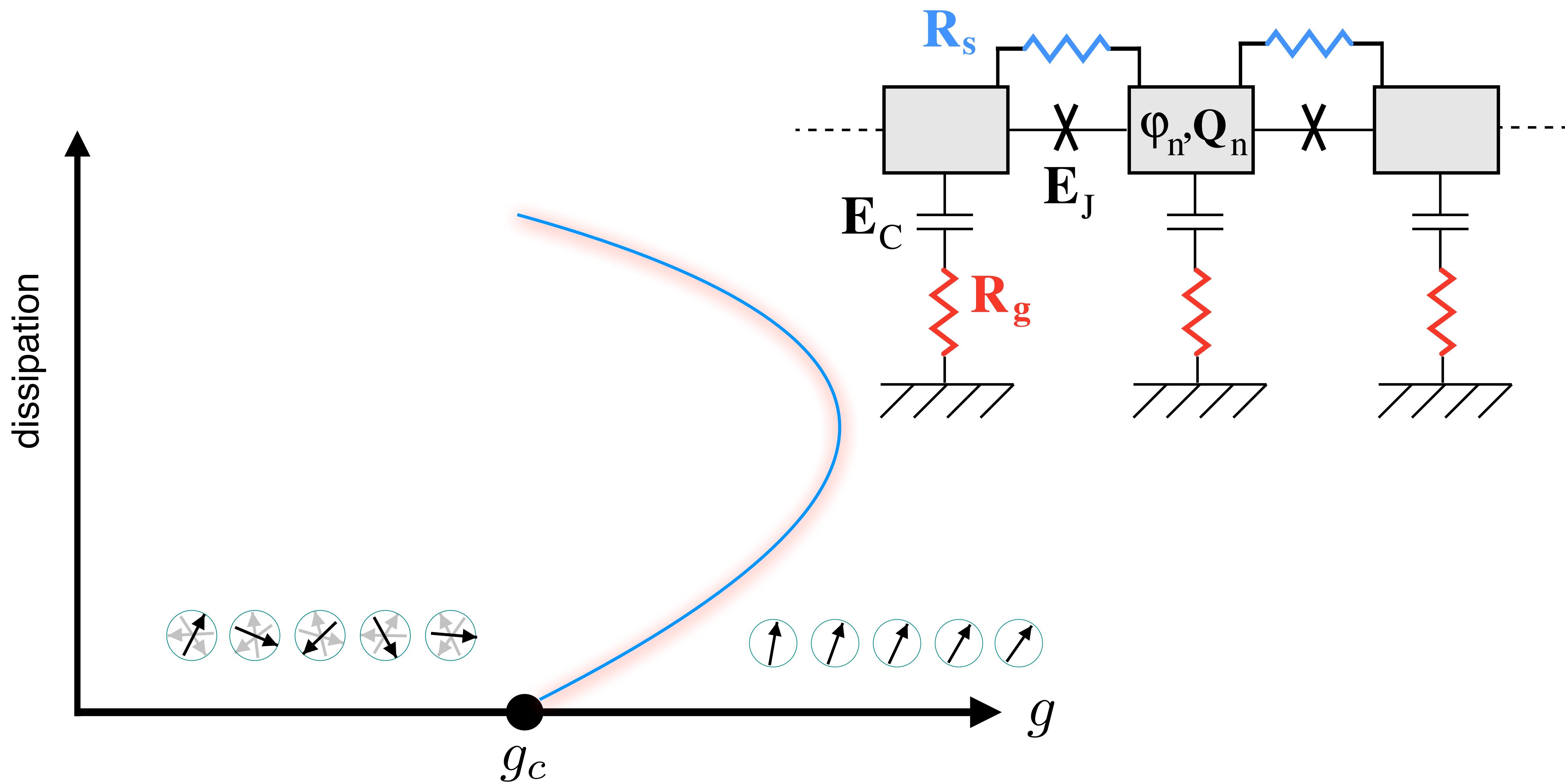
Wagenblast, PRL **79**, 2730 (1997)

Refael et al., PRB **75**, 014522 (2007)

# QPT with dissipative frustration




quantum phase fluctuations

D Maile, S. Andergassen. W. Belzig, G.Rastelli, PRB **97**, 155427 (2018).