GRAVITATIONAL WAVES AS PROBES OF STRONG GRAVITY

Lorenzo Pierini*

Supervisor: Leonardo Gualtieri

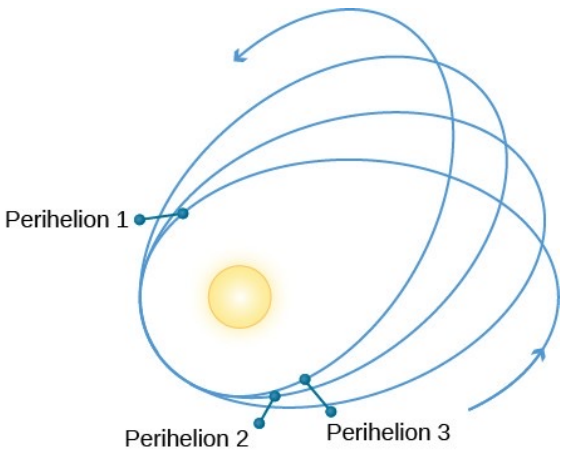
*not that one, the other

TESTS OF GENERAL RELATIVITY

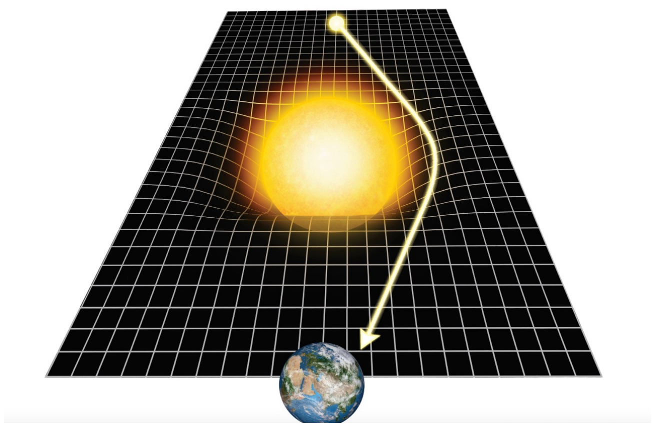
1916

The theory is so elegant and self-consistent:
do we even need experimental confirmation?

Advance of Mercury's Perihelion



Deflection of Light



Einstein: [If the measurement of the deflection of light disagrees with the theory I] «would feel sorry for the dear Lord, for the theory *is* correct!»

(The measurment agreed)

TESTS OF GENERAL RELATIVITY

1916

We actually need to test the theory (I guess)

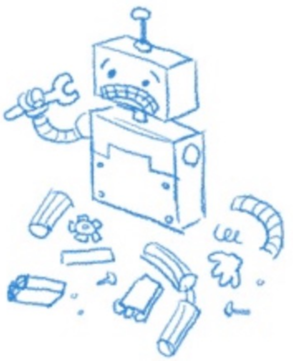
TESTS OF GENERAL RELATIVITY

1916

1920

1960

Theoretical work >> Technology and experiments



TESTS OF GENERAL RELATIVITY

1916

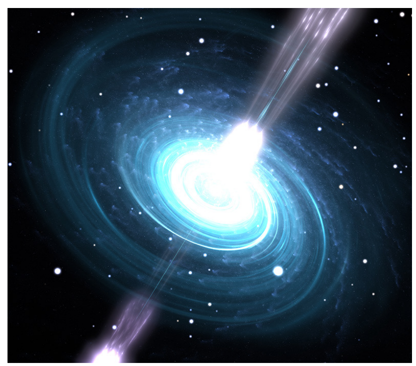
1920

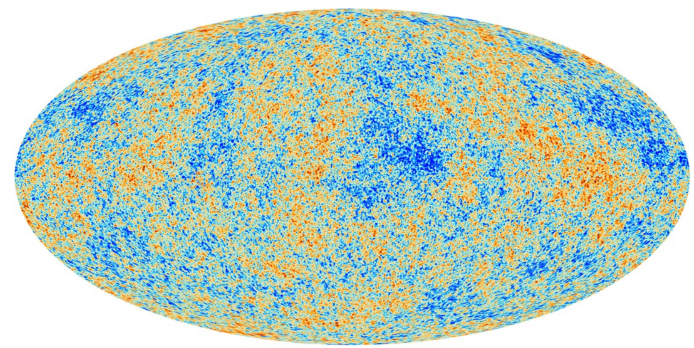
1960

1980

GOLDEN ERA

- New astrophysical discoveries (pulsars, quasars, cosmic background radiation)





- New technology (atomic clocks, radar, laser ranging, cryogenic capabilities, space probes…)

New tests of General Relativity (GR)

TESTS OF GENERAL RELATIVITY

1916

1920

1960

1980

2020s

QUEST FOR STRONG GRAVITY

Unprecedented opportunities to test the most *extreme events* in the universe.

What does «**strong**» mean?

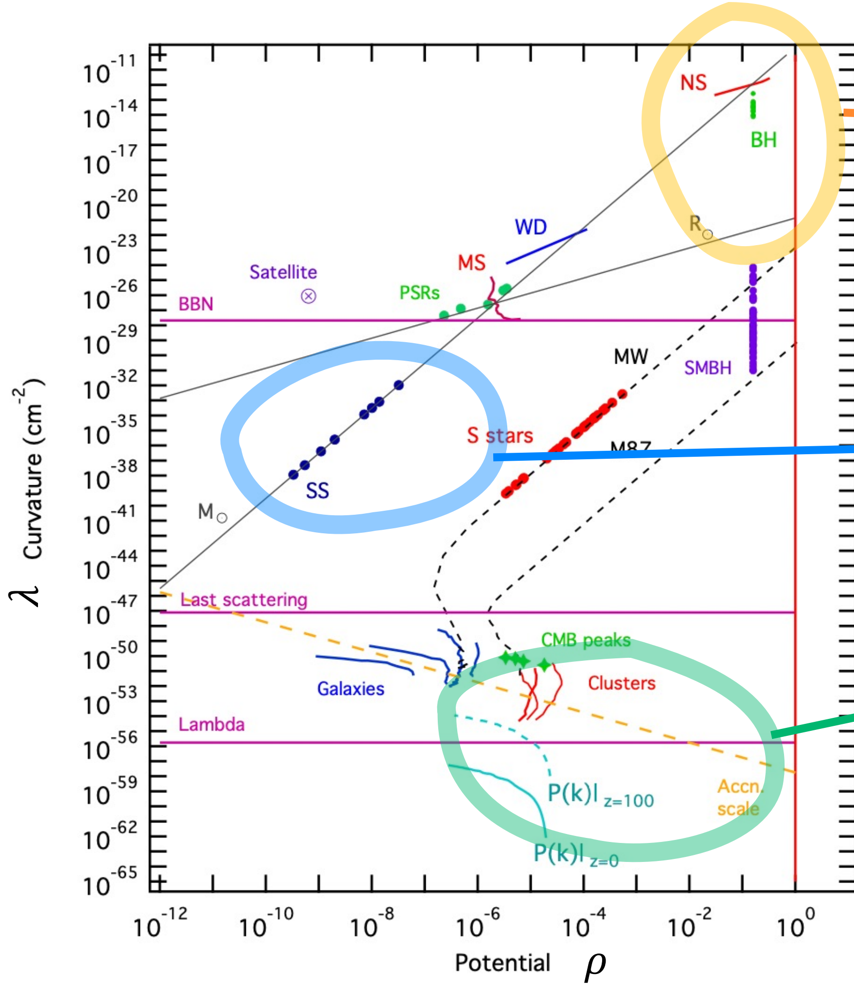
$$\rho \sim \frac{GM}{Rc^2}$$

gravitational potential

$$\lambda \sim \frac{GM}{R^3c^2}$$

curvature

TESTS OF GENERAL RELATIVITY



Black holes
Neutron stars

Strong gravity

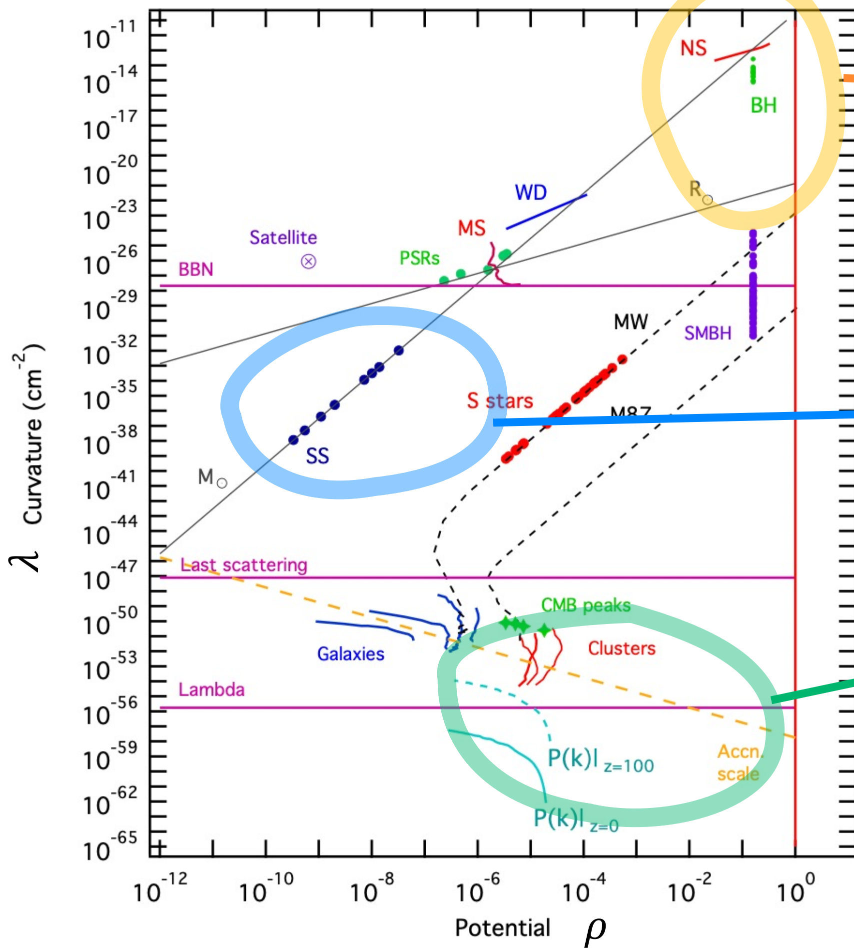
Solar system

Weak-intermediate gravity

Cosmological scale

Baker+, (2015) 1412.3455

TESTS OF GENERAL RELATIVITY



Black holes
Neutron stars Strong gravity

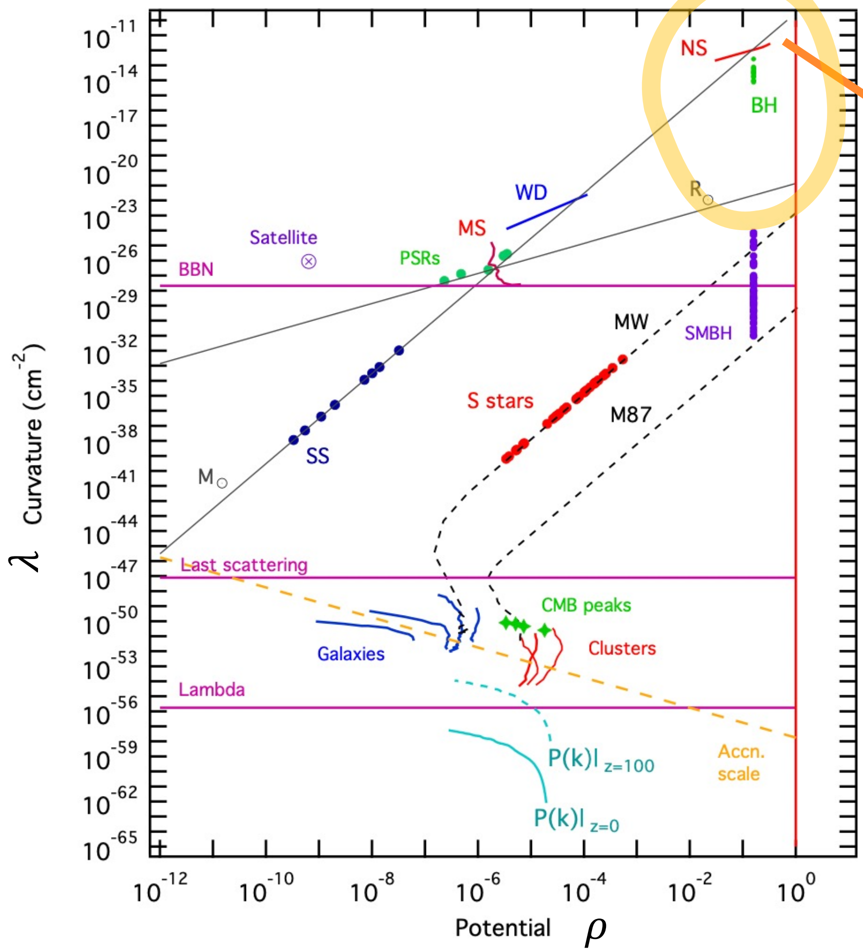
W ediate

Solar system

WELL TESTED

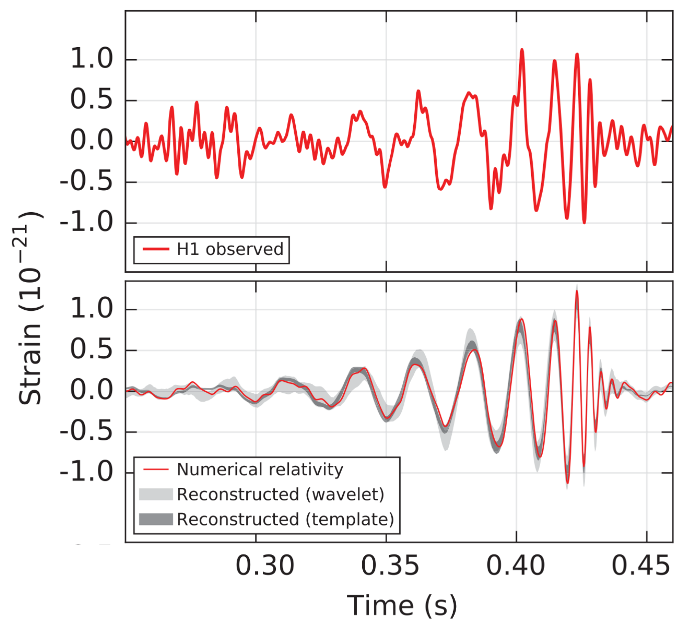
Cosmological scale

STRONG GRAVITY TESTS



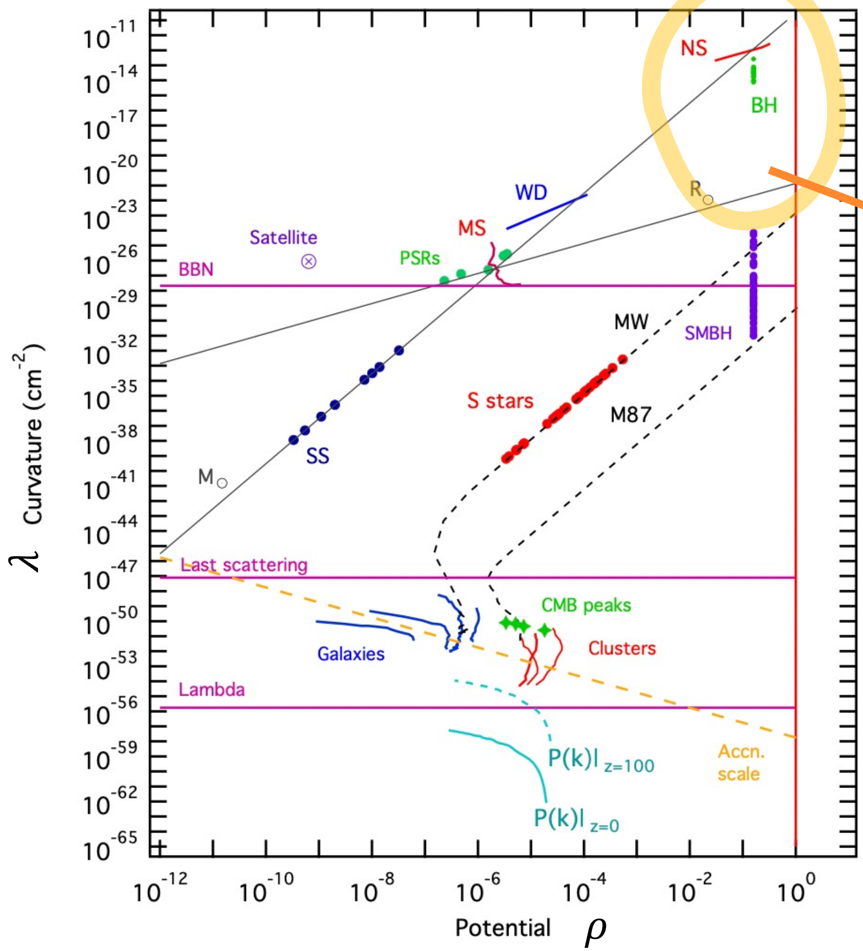
2016

Announcement of first **gravitational wave** detection GW150914

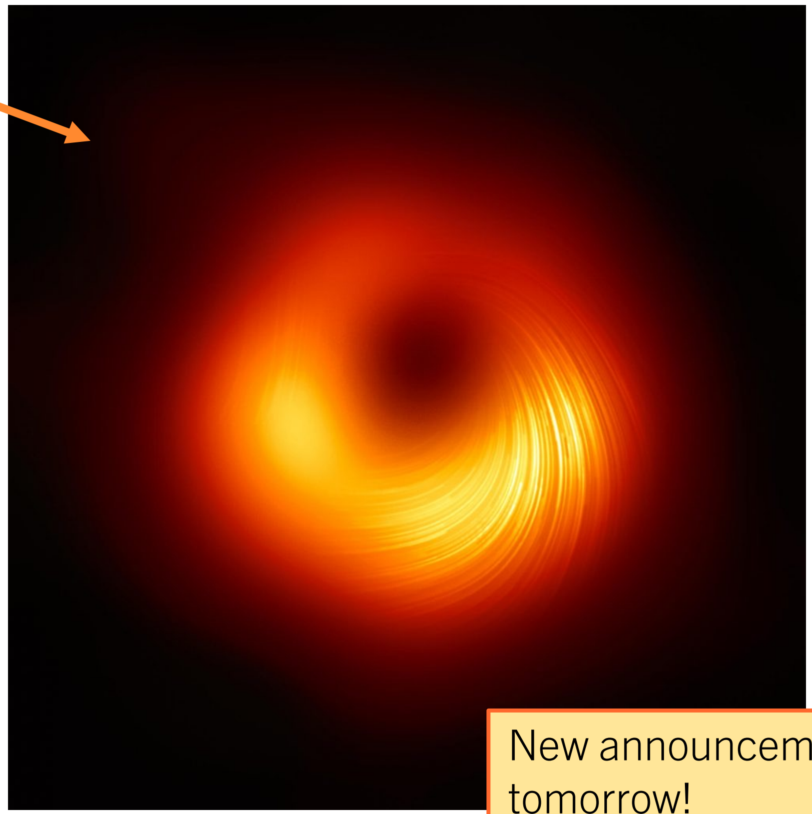


Abbott+, PRL 116 (2016) 061102

STRONG GRAVITY TESTS



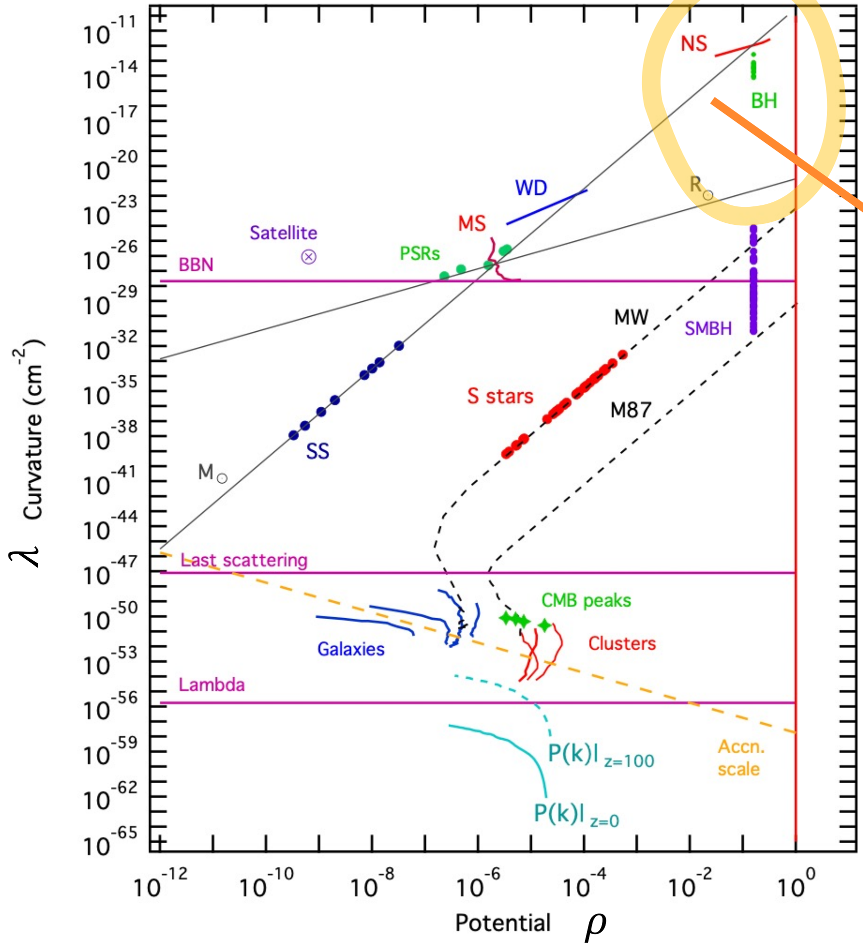
2019 First image of a Black Hole



New announcement tomorrow!

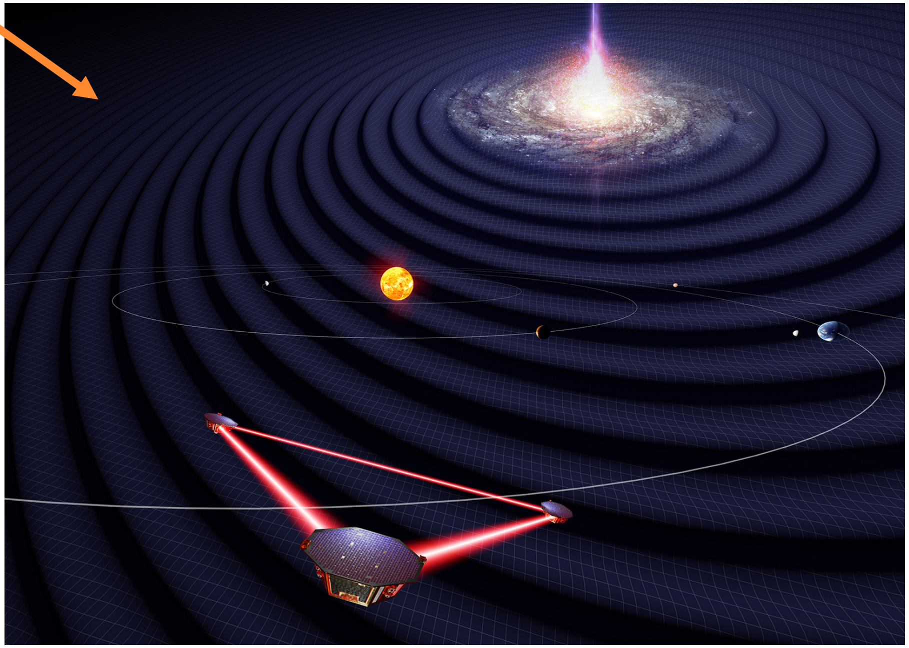
EHT collaboration (2021), ApJL 910 L12

STRONG GRAVITY TESTS



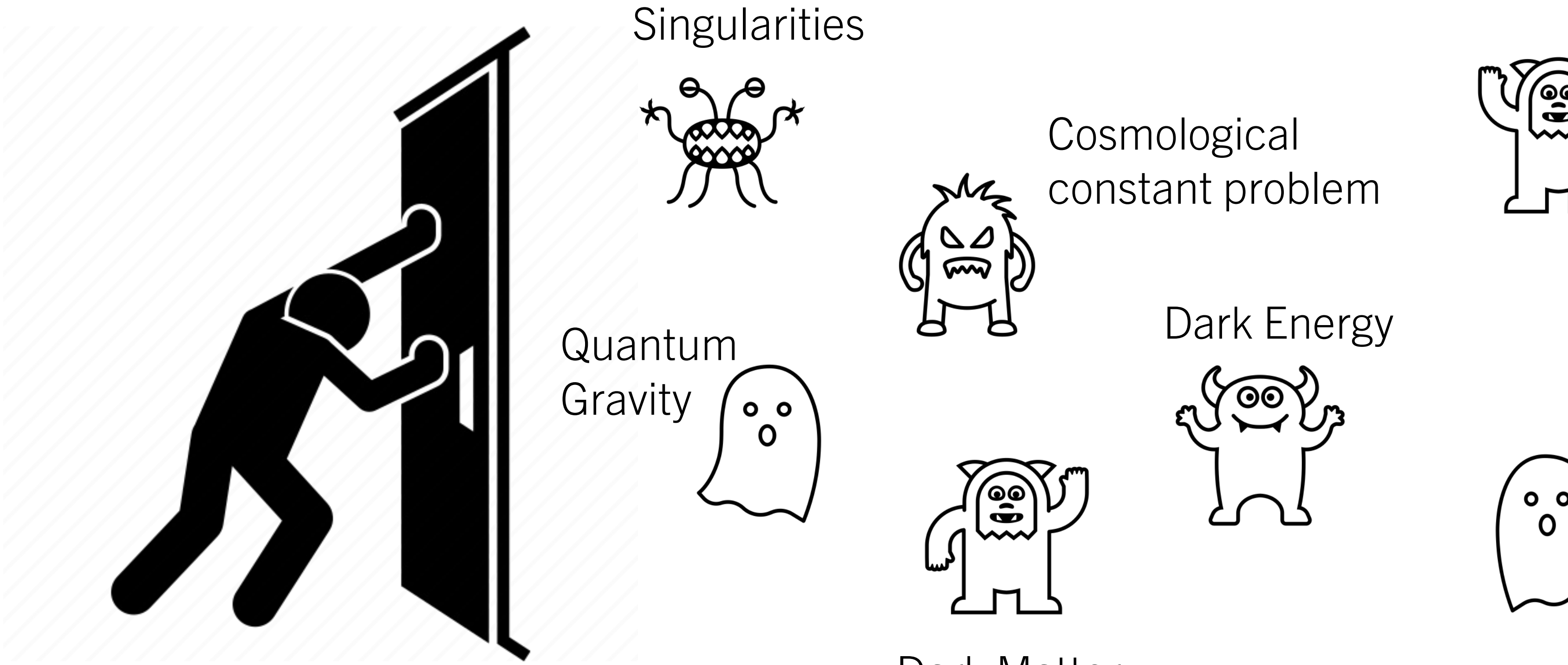
2037

Space-based gravitational wave observatory (LISA)



Amaro-Seoane+ (2017) 1702.00786

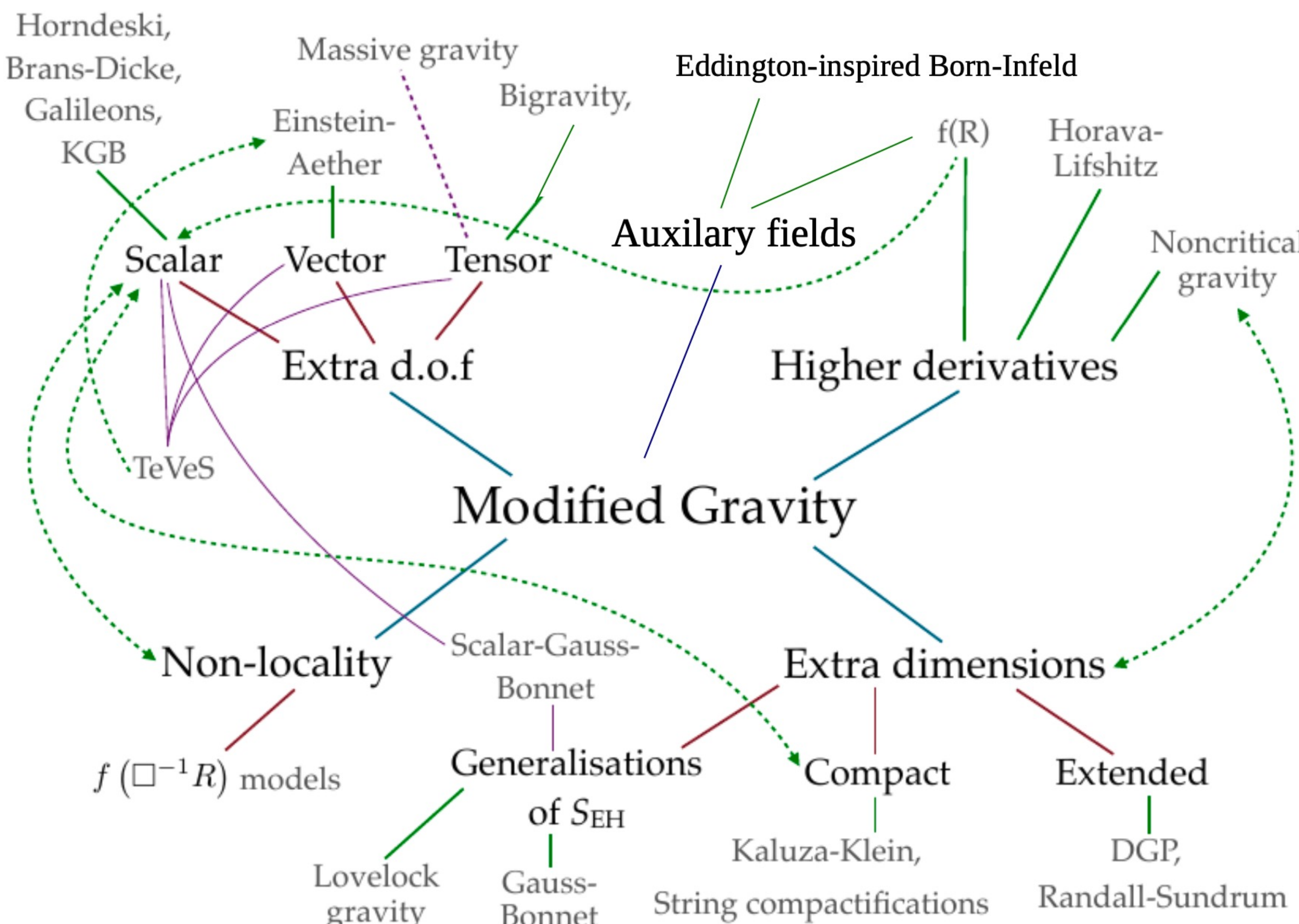
UNFINISHED BUSINESS



General Relativity (GR)

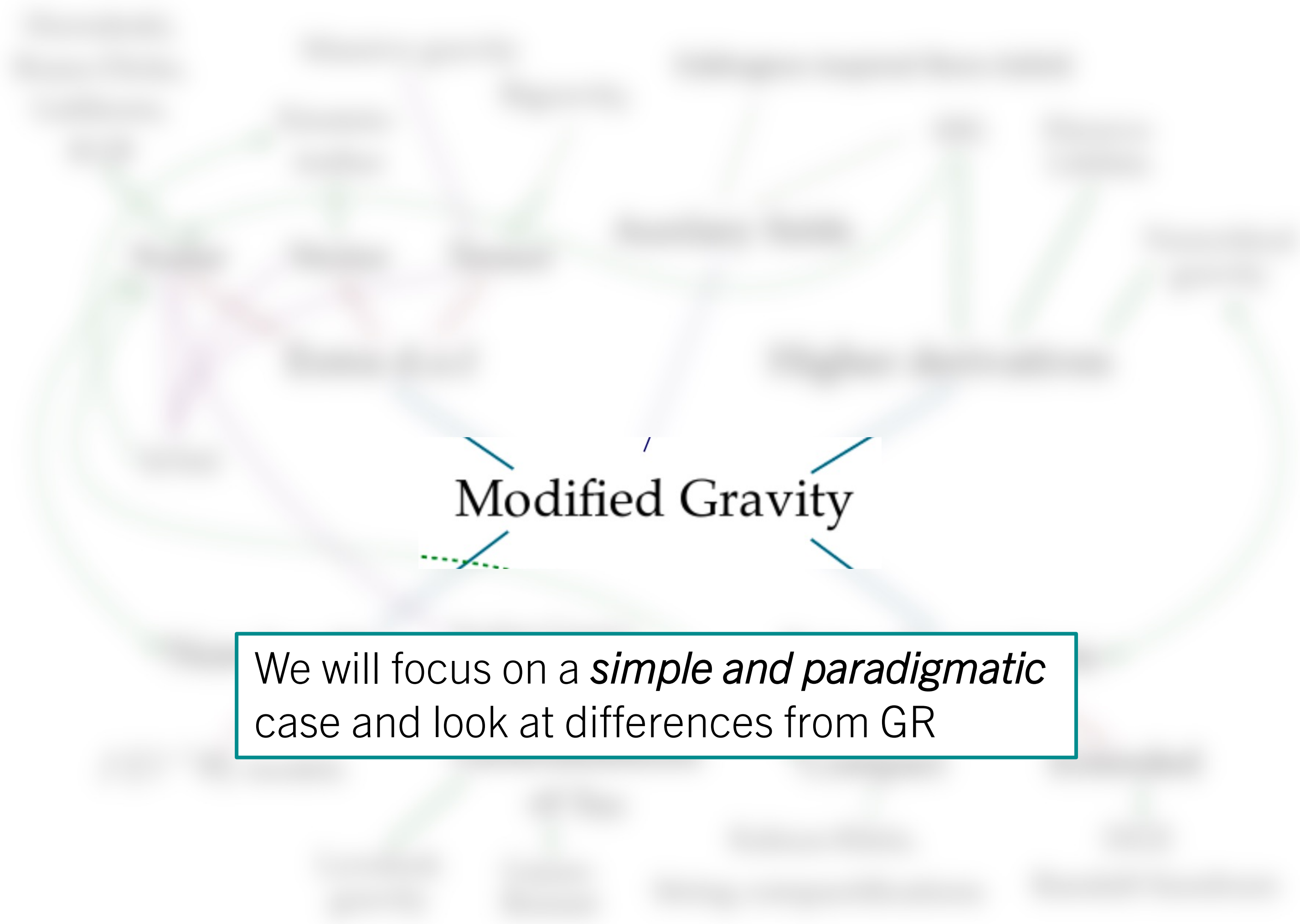
A very good door for intermediate-weak gravity:
Does it hold in the strong field regime? Let’s see if other theories don’t need a door

BEYOND GENERAL RELATIVITY



Clifton+ (2011) 1106.2476
Berti+ (2015) 1501.07274

BEYOND GENERAL RELATIVITY



We will focus on a ***simple and paradigmatic*** case and look at differences from GR

BEYOND GENERAL RELATIVITY

Starting from General Relativity

$$S = \int d^4x \, \frac{\sqrt{-g}}{16\,\pi} R[g_{\mu\nu}] + S_m$$

Einstein-Hilbert Action

$\delta g^{\mu\nu}$

Einstein's field equations

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 8\pi\, T_{\mu\nu}$$

EINSTEIN-DILATON GAUSS-BONNET GRAVITY (EdGB)

Simplest extension of GR that modifies the large-curvature regime

GR

$$S = \int d^4x \, \frac{\sqrt{-g}}{16\,\pi} \left(R - \frac{1}{2} \partial_\mu \phi \, \partial^\mu \phi + \frac{\alpha}{4} e^{\phi} \mathcal{R}^2_{GB} \right)$$

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- Low energy realization of string theories
- New dynamical field: ϕ (*dilaton field*) coupled with the Gauss-Bonnet scalar (quadratic in the curvature)
- Equations of motion still of *second order* in $g_{\mu\nu}$
- The *weak-field* is the *same as GR*

Kanti+, PRD 54 (1996), 5049

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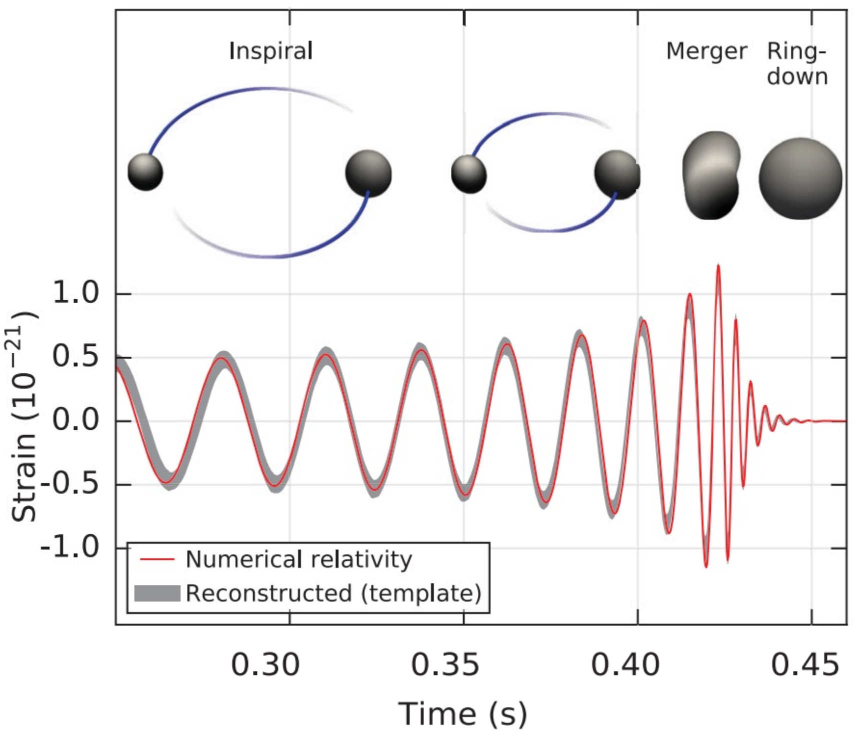
Kanti+, PRD 54 (1996), 5049

Where do we look for large curvature deviations from GR?

TESTING GR WITH GRAVITATIONAL WAVES

Gravitational Waves emitted from the coalescence of black holes or neutron stars are perfect laboratories to test the strong-field, large curvature regime of gravity.



Abbott+, PRL 116 (2016) 061102

TESTING GR WITH GRAVITATIONAL WAVES

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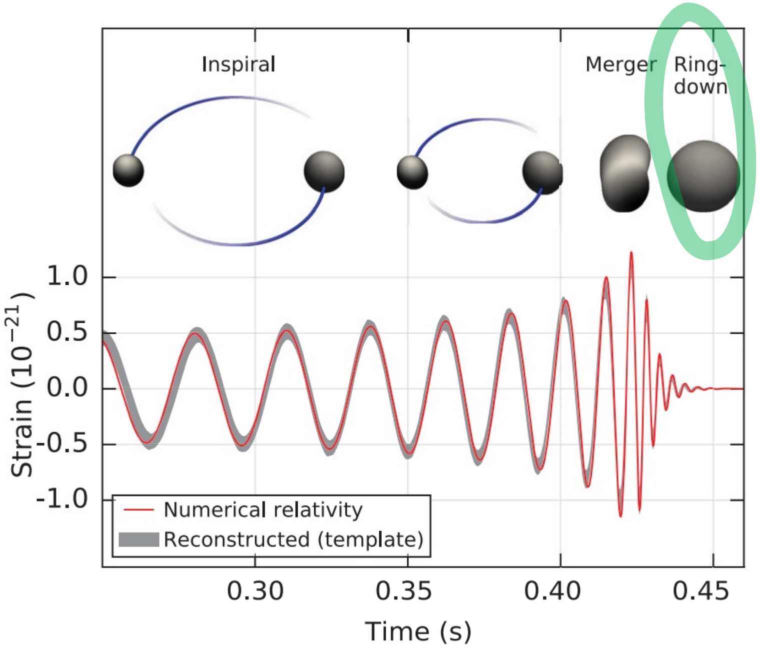
QUASINORMAL MODES

The relaxation phase of the perturbed remnant black hole through gravitational wave emission is called ringdown.

The (late) ringdown can be described as a superposition of damped sinusoids with specific frequencies, the *quasinormal modes*

Berti+, CQG 26 (2009) 163001



$$h_{rd} \sim \sum_{nlm} \mathcal{Y}^{lm} A_{nlm}\, e^{-i\omega^{nlm} t}$$

$$\omega^{nlm} = \omega_R^{nlm} + i\, \omega_I^{nlm}$$

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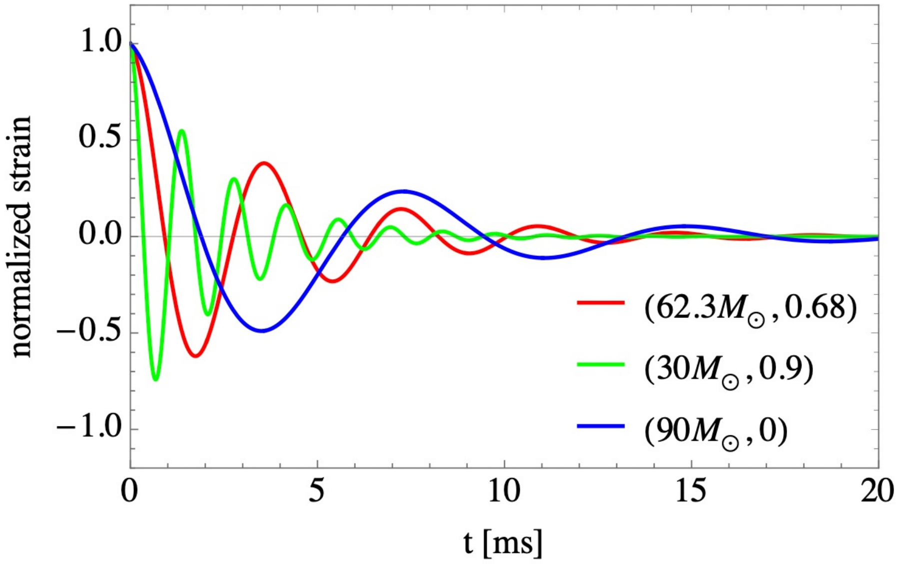


Yagi (2022), 2201.06186

$$h_{rd} \sim \sum_{nlm} \mathcal{Y}^{lm} A_{nlm}\, \underline{e^{-i\omega^{nlm} t}}$$

$$\boxed{\omega^{nlm} = \omega_R^{nlm} + i\,\omega_I^{nlm}}$$

$$\omega_R^{nlm} \equiv f^{nlm} \qquad |\omega_I^{nlm}| \equiv 1/\tau^{nlm}$$

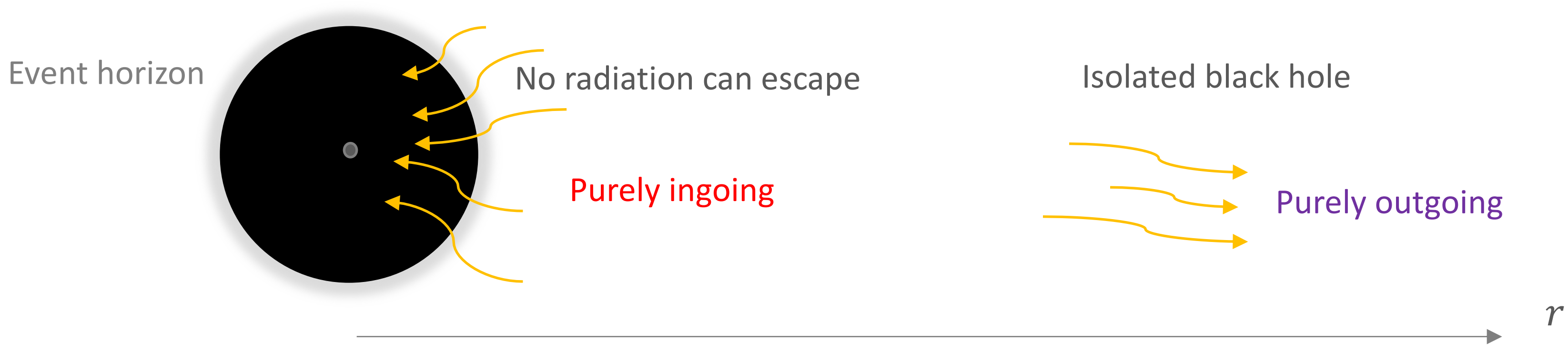
$$\underline{e^{-i\omega t}} = e^{-t/\tau^{nlm}} \cos(2\pi f^{nlm} t + \varphi^{nlm})$$

QUASINORMAL MODES

Quasinormal modes (QNMs) are eigenvalues of dissipative systems

$$\omega^{nlm} = \omega_R^{nlm} + \boxed{i\, \omega_I^{nlm}}$$

In the case of black holes, they are the eigenfunctions of the gravitational wave equation that satisfy the **boundary conditions**:

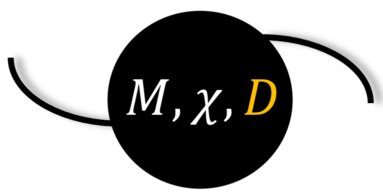


QUASINORMAL MODES

QNMs contain all the **information** about the underlying theory of gravity.

Black Holes in GR are described only by the **mass** M and **spin** χ

In EdGB $\quad \omega = \omega(M, \chi, \alpha)$



$D(\alpha, M)$ *scalar charge*

GOAL: Comparison of **QNM spectrum** of *rotating* black holes in EdGB with data and with GR to **look for possible deviations and their nature**

EDGB BLACK HOLES

$$S = \int d^4x \, \frac{\sqrt{-g}}{16\,\pi} \left(R - \frac{1}{2}\, \partial_\mu \phi \, \partial^\mu \phi + \frac{\alpha}{4} e^{\phi} \mathcal{R}^2_{GB} \right)$$

$\delta g^{\mu\nu}, \delta\phi$

Einstein's and scalar field equations → Black hole solutions

Perturbed EdGB black hole

$g^0_{\mu\nu} + \epsilon \, h_{\mu\nu}$

$\varphi_0 + \epsilon \, \delta\varphi \qquad \epsilon \ll 1$

Relaxed EdGB black hole

$g^0_{\mu\nu}$

φ_0

EDGB BLACK HOLES

Equilibrium solution ($\epsilon = 0$) describing a rotating black hole in EdGB gravity:

$$g_{\mu\nu} = g^0_{\mu\nu} + \epsilon\, h_{\mu\nu}$$
$$\phi = \varphi_0 + \epsilon\, \delta\varphi$$

Let's start with what we know and build up the solution **perturbatively**

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Non rotating, GR black hole, i.e. Schwarzschild

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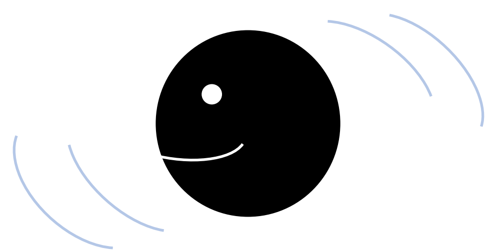

Slowly rotating, GR black hole

$\chi = J/M^2 \ll 1$

EDGB BLACK HOLES

Equilibrium solution ($\epsilon = 0$) describing a rotating black hole in EdGB gravity:

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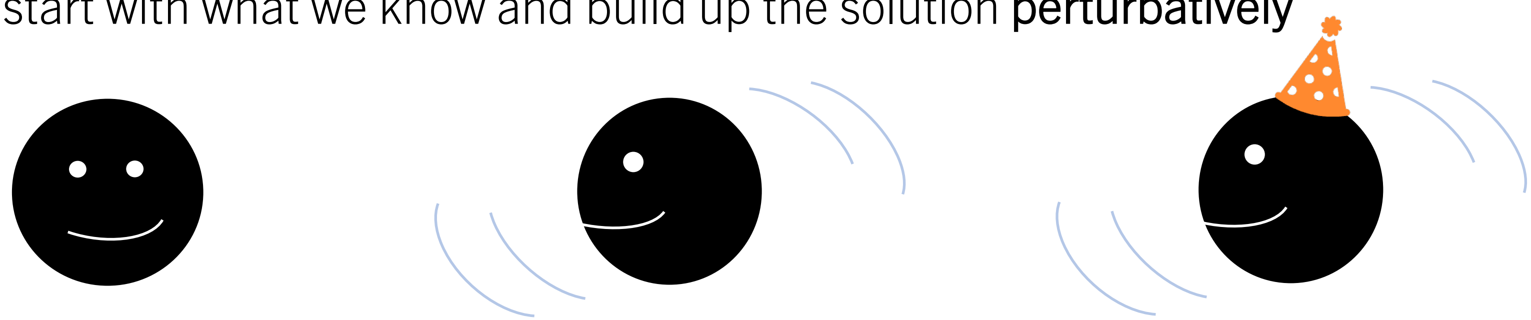


Non rotating, GR black hole, i.e. Schwarzschild

Slowly rotating, GR black hole

Slowly rotating, EdGB black hole

$\chi = J/M^2 \ll 1$

$\zeta = \alpha/M^2 \leq 0.691$

EDGB BLACK HOLES

$$g_{\mu\nu} = g^0_{\mu\nu} + \epsilon\, h_{\mu\nu}$$
$$\phi = \varphi_0 + \epsilon\, \delta\varphi$$

Metric and scalar field of a slowly rotating EdGB black hole in the small coupling limit

$$g_{\mu\nu} = g^0_{\mu\nu} + \epsilon\, h_{\mu\nu}$$
$$\Phi = \varphi_0 + \epsilon\, \delta\varphi$$
$$\epsilon \ll 1$$

We **perturb** it to find its **characteristic oscillation frequencies** (QNMs)

PERTURBATIONS OF EDGB BLACK HOLES

$$g_{\mu\nu} = g^0_{\mu\nu} + \epsilon\, h_{\mu\nu}$$
$$\Phi = \varphi_0 + \epsilon\, \delta\varphi$$
$$\epsilon \ll 1$$

Black hole perturbation theory

Regge, Wheeler Phys.Rev. 108(1957) 1063-1069
Zerilli PRD 2 (1970) 2141-2160

Scalar, vector, tensor spherical harmonics decomposition

$$\delta\varphi \longrightarrow \sum_{\ell m} \varphi_{1,\ell m}(r) Y^{\ell m}(\theta,\phi) e^{-i\omega t}$$

radial *angular*

$$h_{\mu\nu} \longrightarrow K_{\ell m}(r),\ H_{1,\ell m}(r),\ h_{0,\ell m}(r),\ h_{1,\ell m}(r)$$

Two families of solutions:

- Polar (even parity)
- Axial (odd parity)

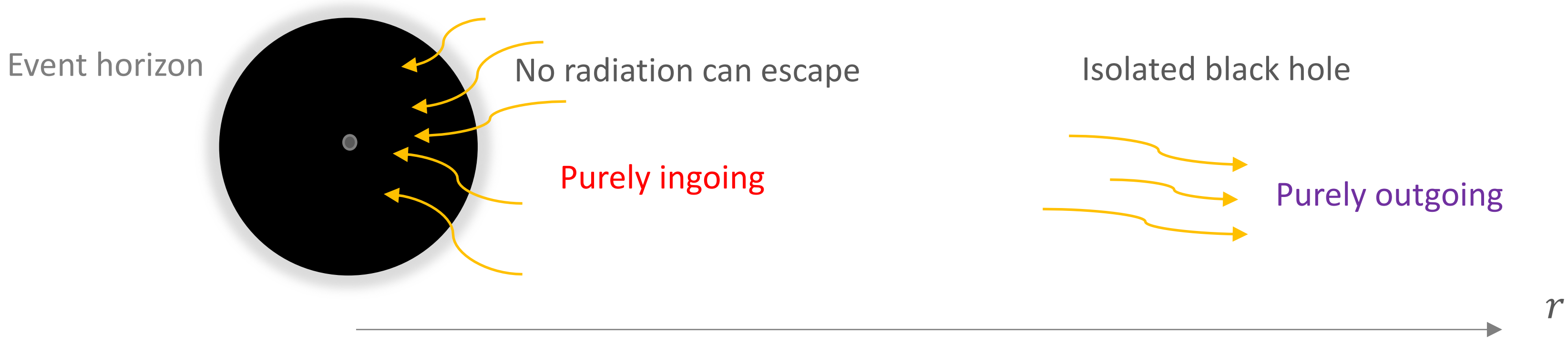
2nd ORDER IN THE SPIN: POLAR

We can recast the polar set of equations as

$$\frac{d}{dr}\boldsymbol{\Psi}_{\ell m} + \hat{P}_{\ell m}\boldsymbol{\Psi}_{\ell m} = 0$$

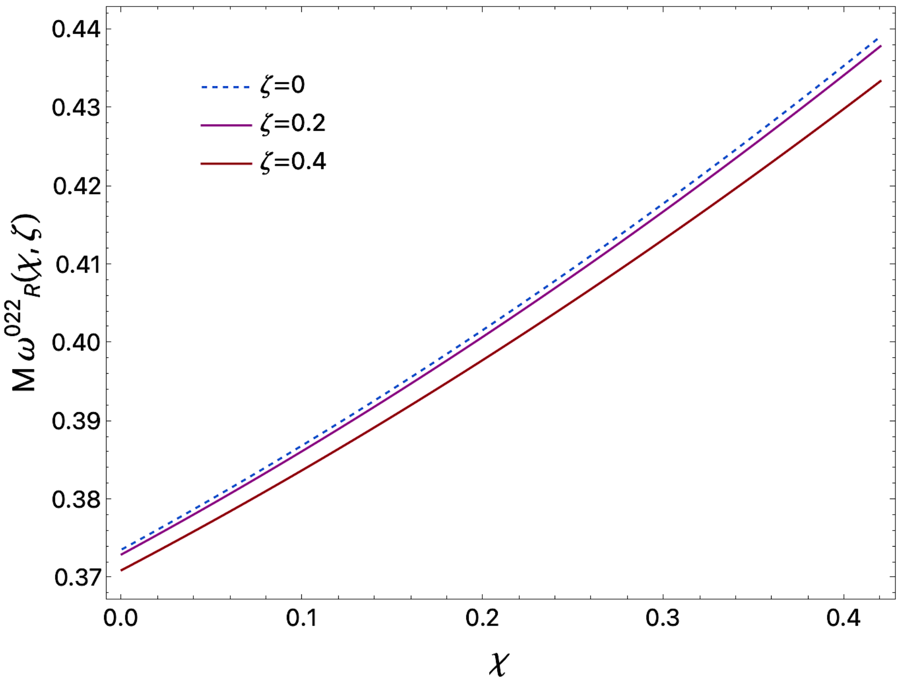
with $\boldsymbol{\Psi}_{\ell m} = \{H_{1\,\ell m}, K_{\ell m}, \varphi_{1\,\ell m}, \varphi'_{1\,\ell m}, h_{0\,\ell-1 m}, h_{1\,\ell-1 m}\, h_{0\,\ell+1 m}, h_{1\,\ell+1 m}\}$

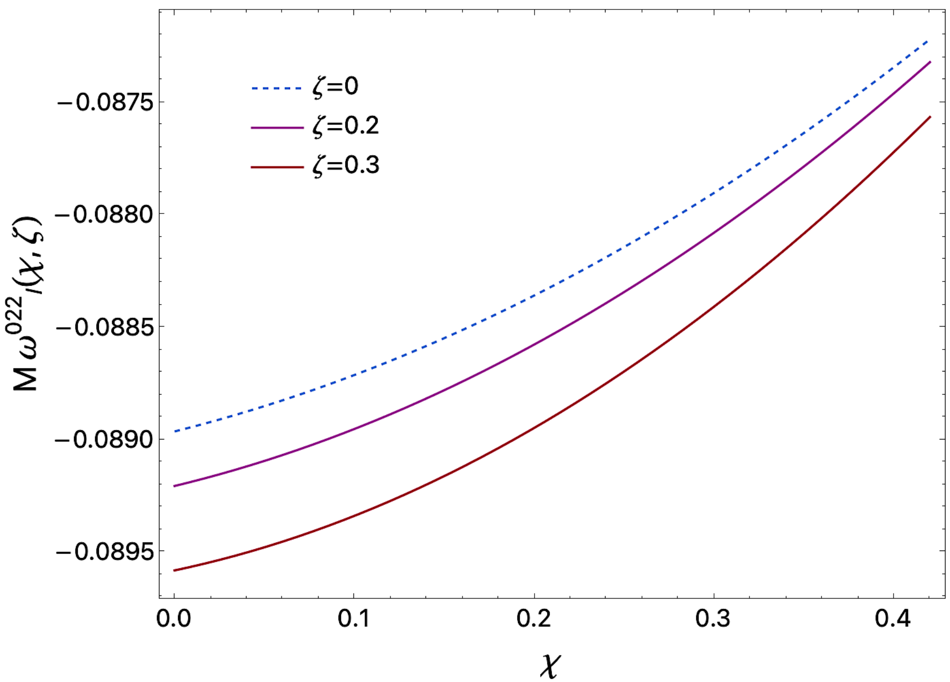
QNMs: solutions of the perturbation equations that satisfy purely ingoing wave condition at the horizon and purely outgoing condition at infinity.



QNM SPECTRUM: 2nd SPIN ORDER

$$\omega^{n\ell m}(\chi,\zeta) = \omega_0^{n\ell}(\zeta) + m\,\chi\,\omega_1^{n\ell}(\zeta) + \chi^2\left[\omega_{2a}^{n\ell}(\zeta) + m^2\omega_{2b}^{n\ell}(\zeta)\right] + O(\chi^3)$$





Pierini, Gualtieri PRD103 (2021) 124017

Pierini, Gualtieri (2022, TA)

QNM SPECTRUM: 2nd SPIN ORDER

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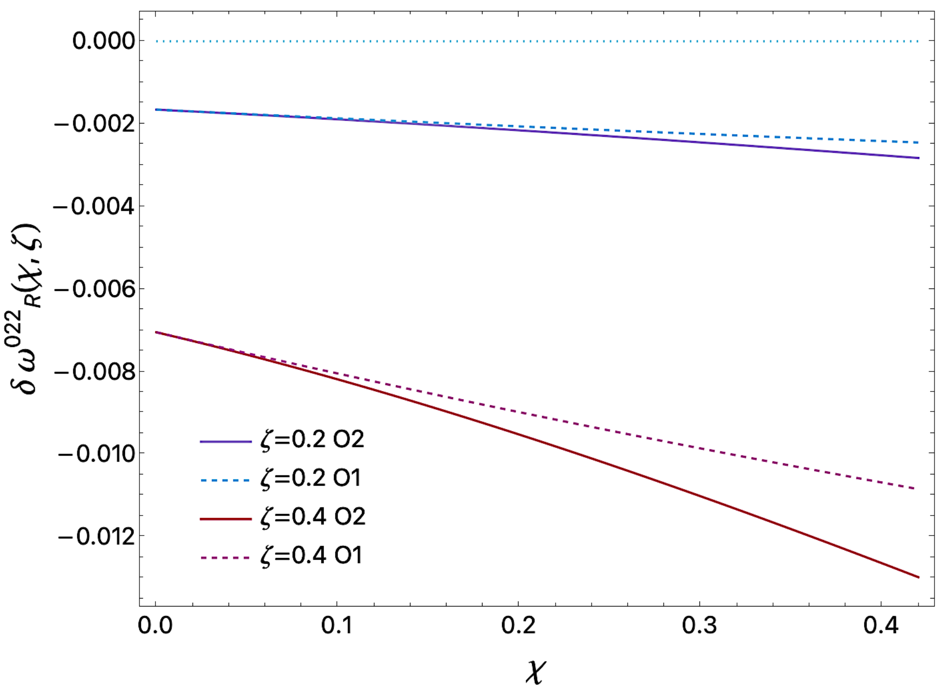
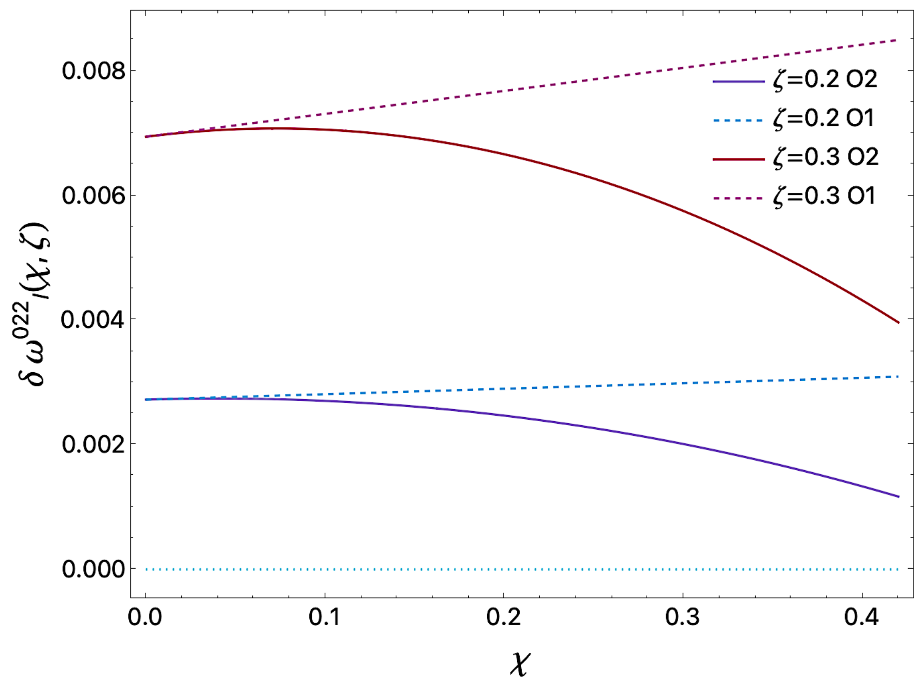
$$\delta\omega_{R,I}(\chi,\zeta) \equiv \frac{\omega_{R,I}(\chi,\zeta) - \omega_{R,I}(\chi,0)}{\omega_{R,I}(\chi,0)}$$





Pierini, Gualtieri (2022, TA)

CONCLUSIONS AND FUTURE PROSPECTS

- First analytic computation of QNMs of rotating black holes in an extension of General Relativity!

- Effect of rotation computed **up to second order in the spin**

- Is the shift from GR detectable by future GW detectors (LISA, ET)?

$$\omega = \omega^{GR} + \delta\omega$$
$$\tau = \tau^{GR} + \delta\tau$$

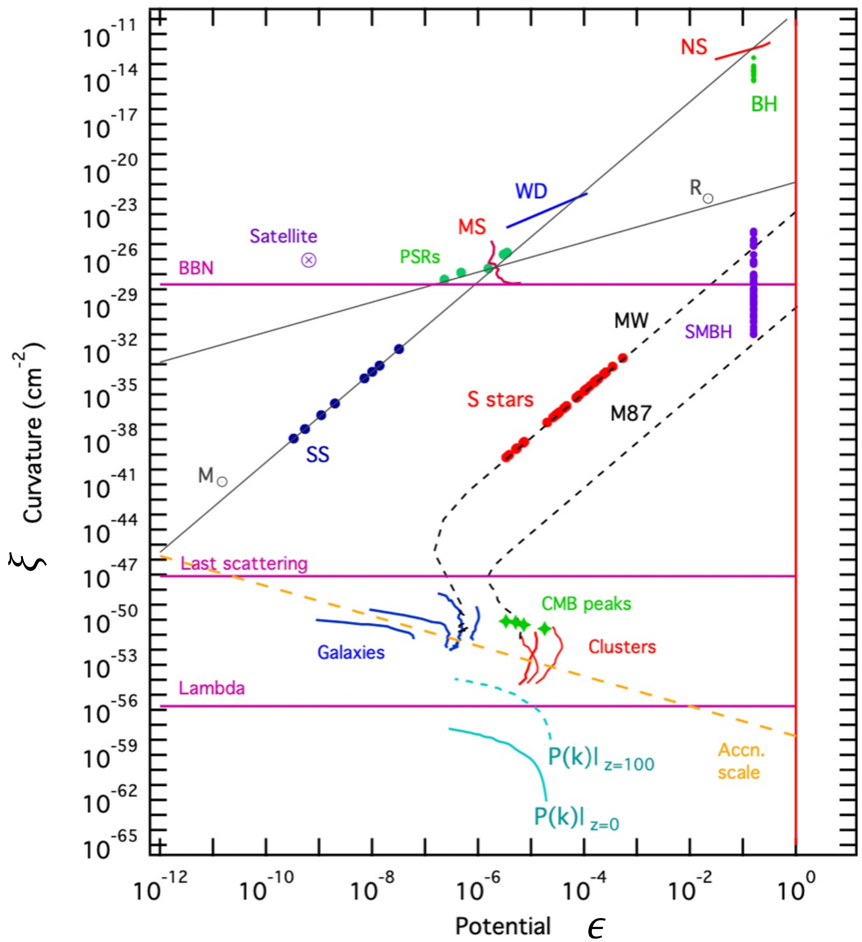
$\longrightarrow$ $P(\theta|d)$

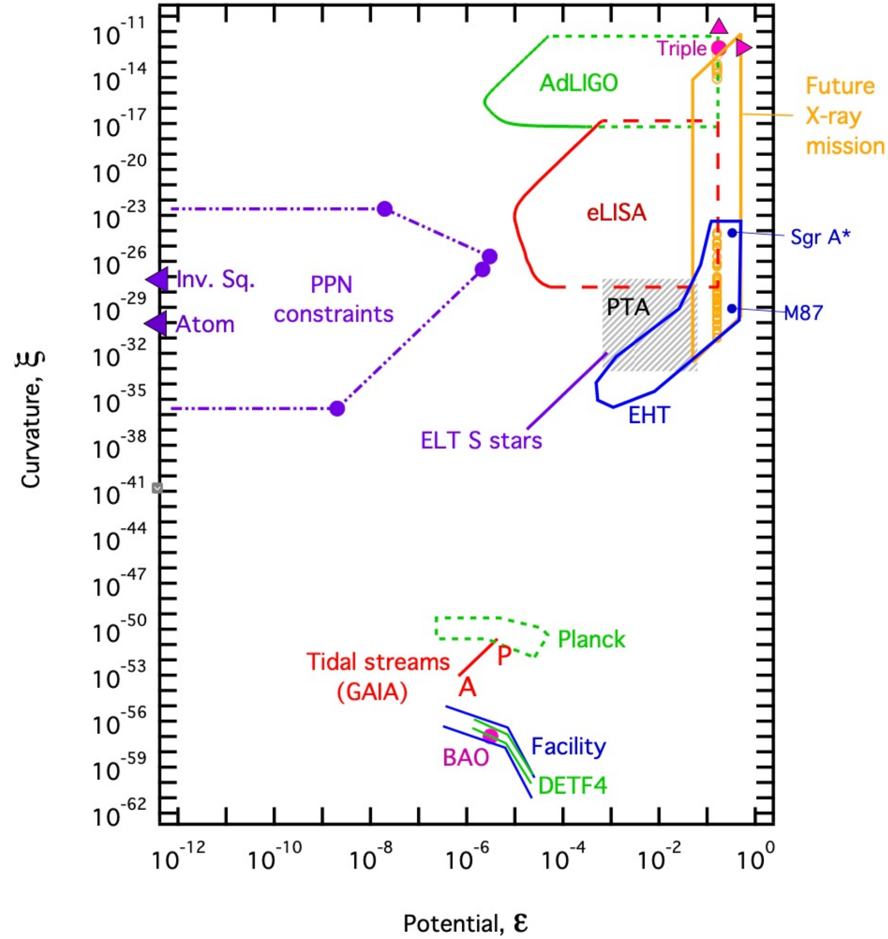
θ beyond-GR parameters
d ringdown observations

Maselli+ PRD 101 (2020) 2, 024043

BACK UP SLIDES

STRONG GRAVITY TESTS

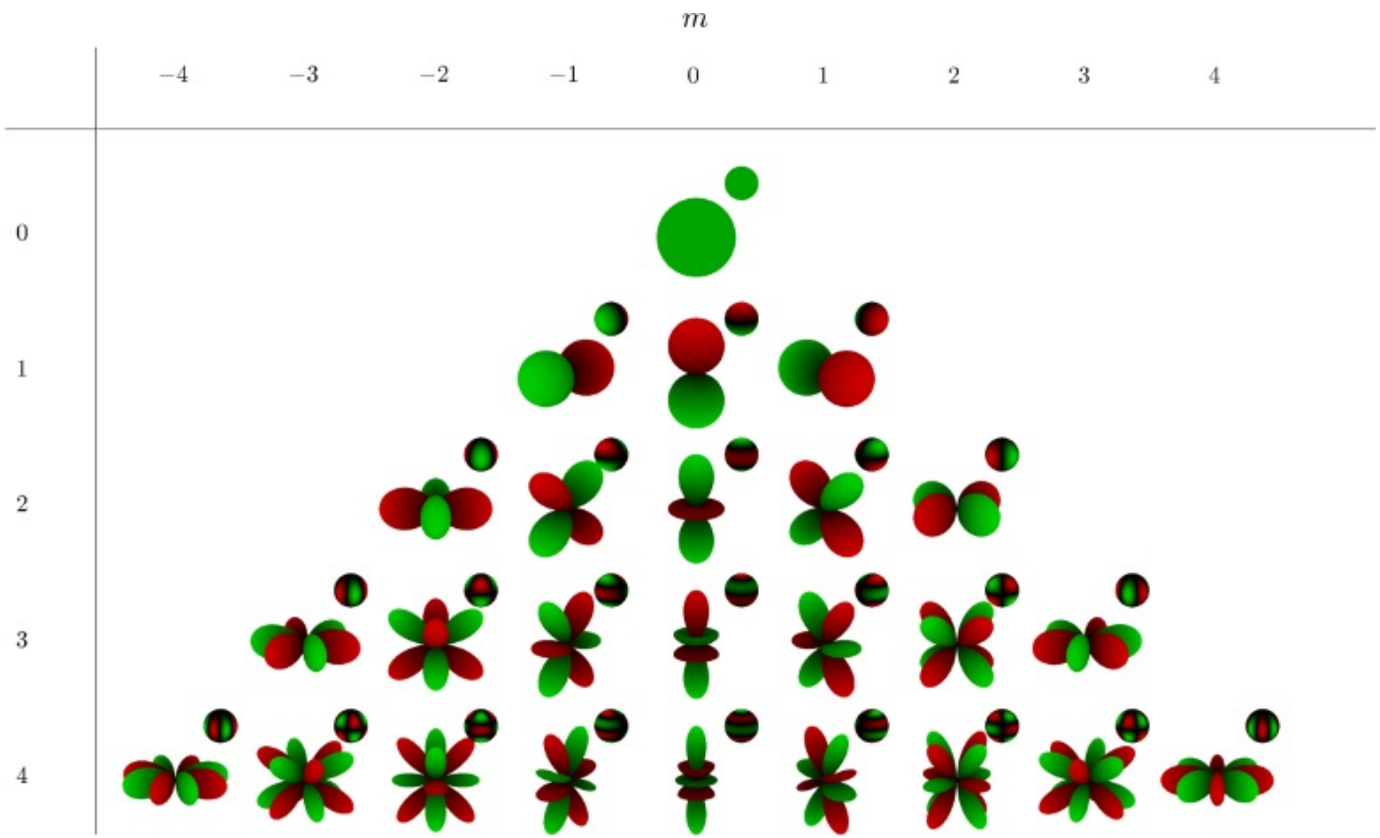




Psaltis (2018) 1806.09740

GW EMISSION

$Y^{\ell m}$



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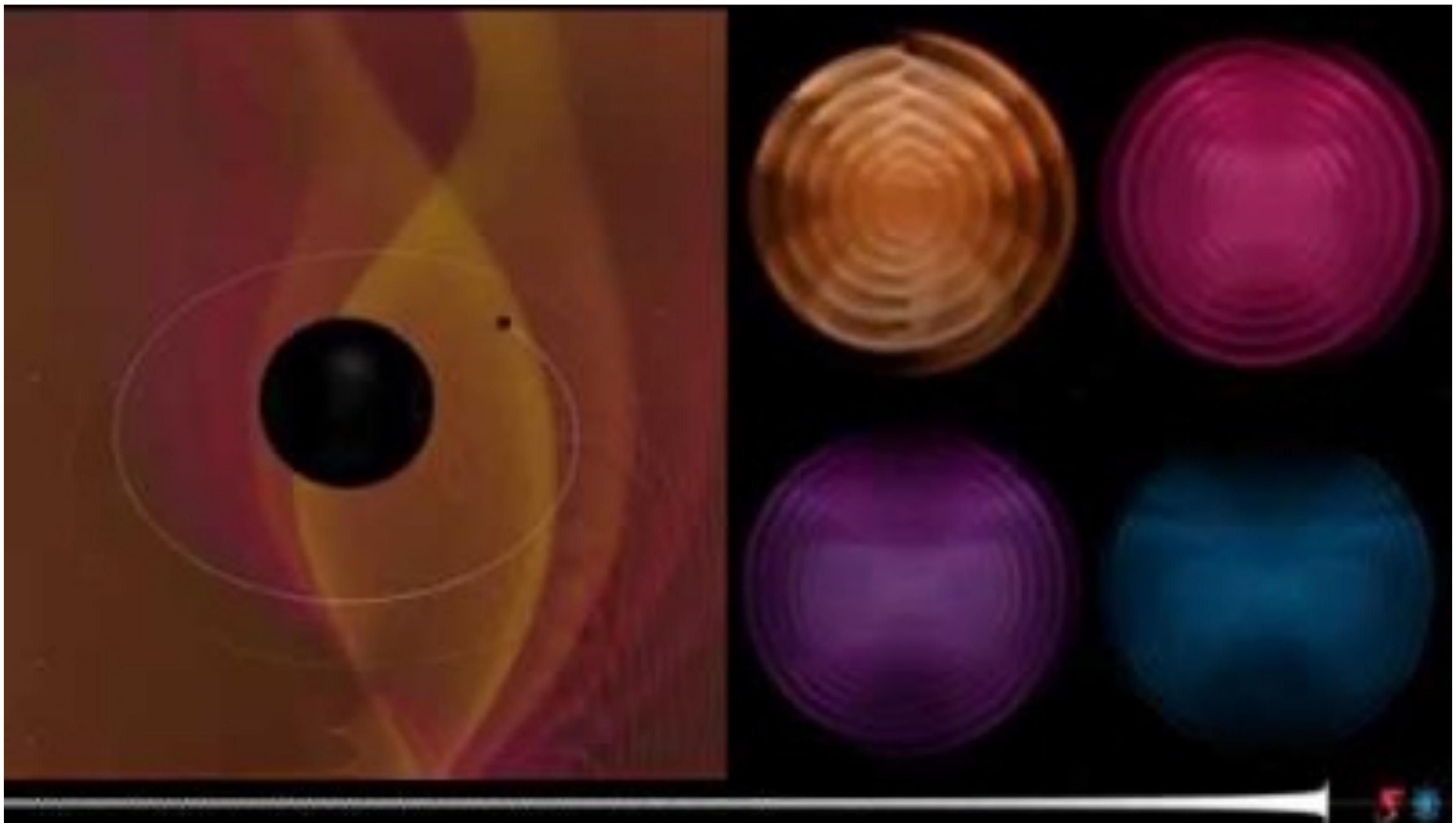

QNM SPECTRUM: SLOW ROTATION RANGE

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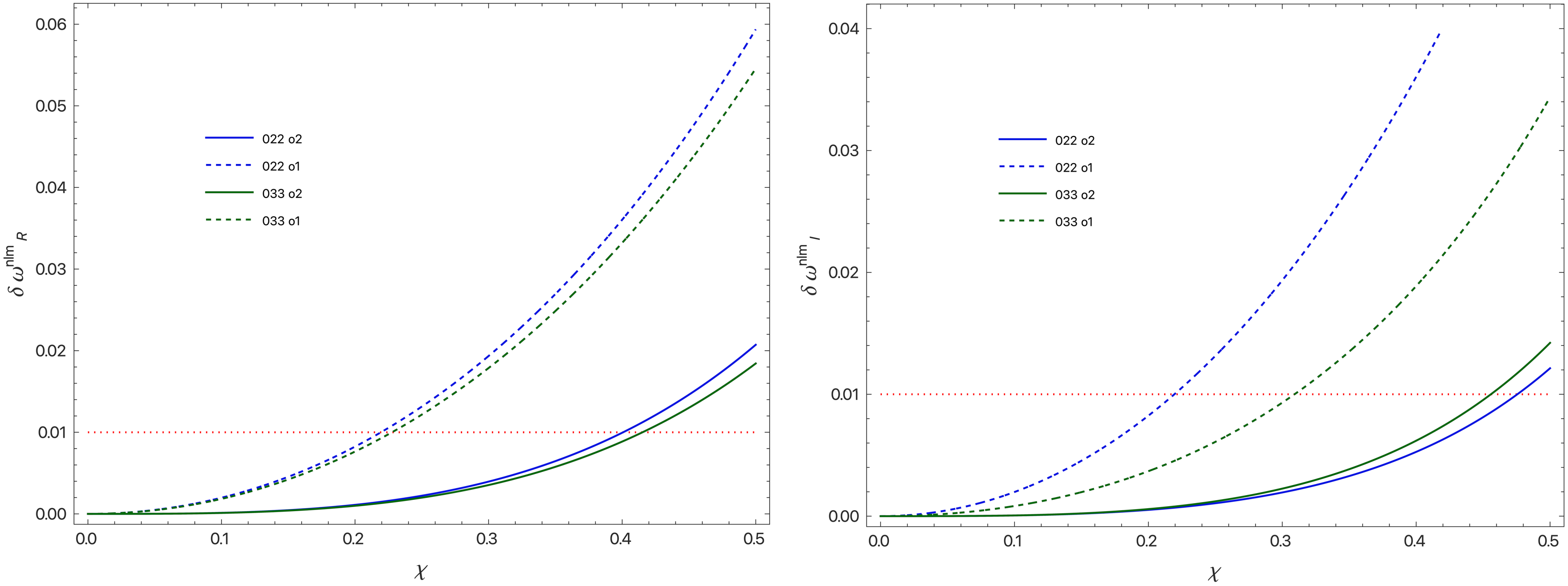
$$\delta\omega_{R,I} \equiv \frac{\omega_{R,I} - \omega_{R,I}^{kerr}}{\omega_{R,I}^{kerr}}$$

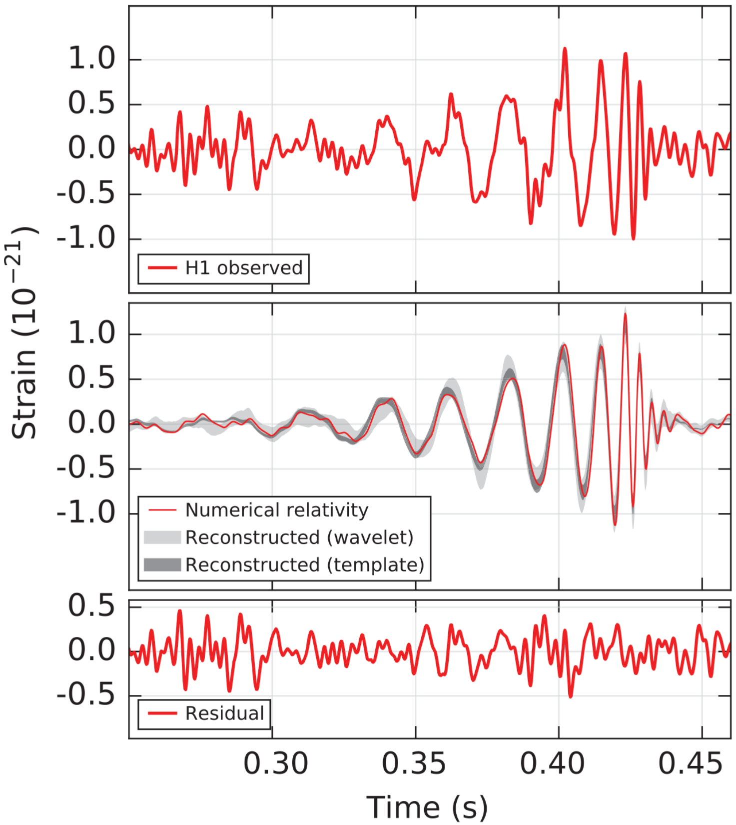
https://pages.jh.edu/eberti2/ringdown/



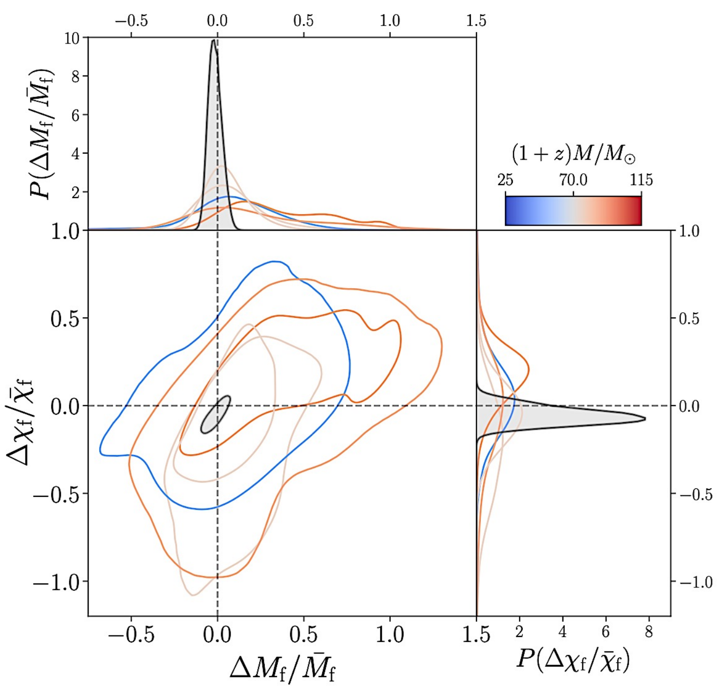
TESTING GENERAL RELATIVITY

Null tests of GR from GW events: *Consistency tests*

Abbott+, PRL 116 (2016) 061102

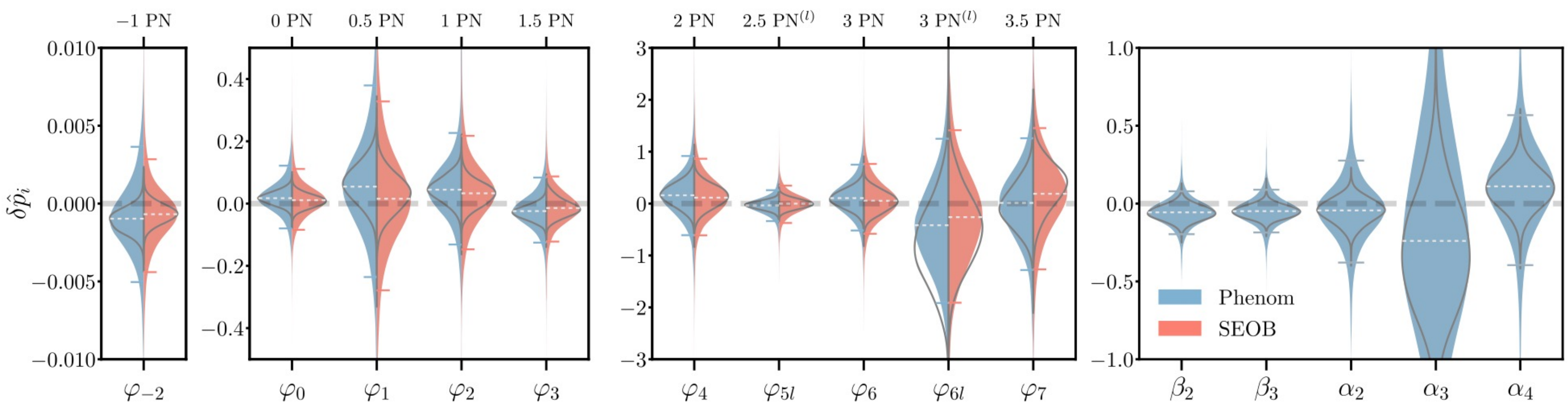


LIGO (2021) arXiv:2112.06861



TESTING GENERAL RELATIVITY

Null tests of GR from GW events : *Parametrised tests*



Abbott+, PRD 103 (2021) 12, 122002

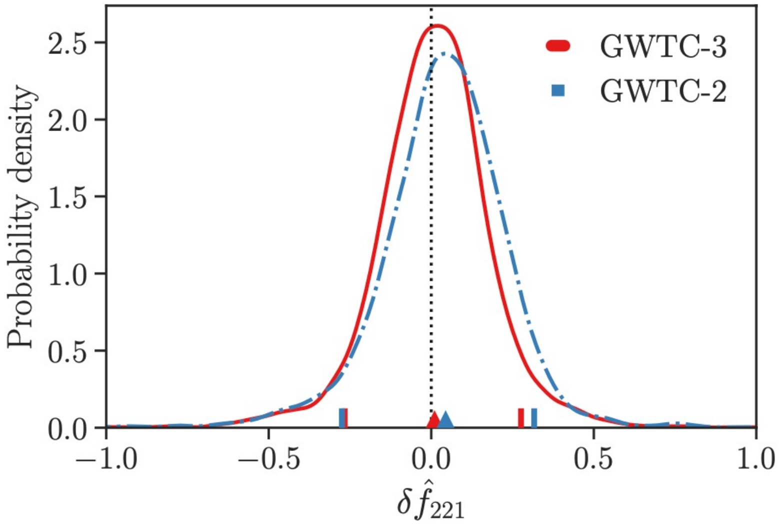
TESTING GENERAL RELATIVITY

Null tests of GR from GW events : *Ringdown tests*

The (late) ringdown can be described as a superposition of damped sinusoids with specific frequencies, the *quasinormal modes*

Berti+, CQG 26 (2009) 163001

LIGO (2021) arXiv:2112.06861



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$$e^{i\omega t} = e^{-t/\tau^{nlm}} \cos(2\pi f^{nlm} t + \varphi^{nlm})$$

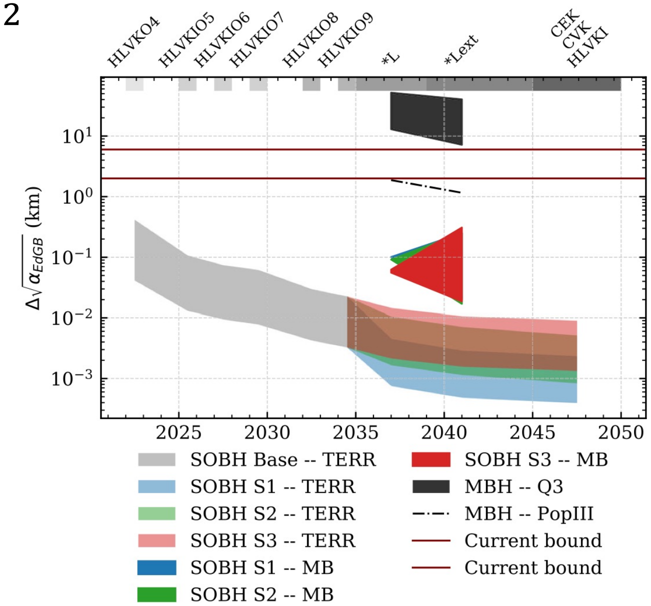
CONSTRAINTS ON EDGB

$\zeta = \alpha / M^2 \leq 0.691 \longrightarrow \alpha \leq 0.691\, M^2$

Lightest BHs detected:
J1655-40 $M \sim 5.4\, M_\odot \Rightarrow \sqrt{\alpha} < 6.6\, km$
(GW190814 $M \sim 2.6\, M_\odot \Rightarrow \sqrt{\alpha} < 3.3\, km$)

-1PN effect, SNR scaling with chirp mass:
Multiband and terrestrial observations of SOBH more constraining than MBHs

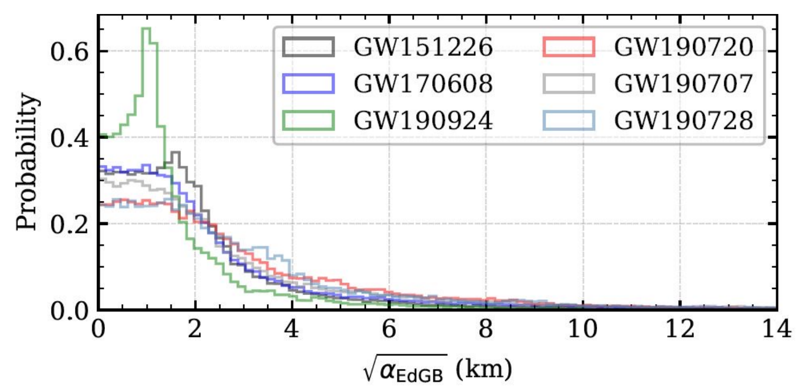
Perkins+, PRD 103 (2021) 4, 044024



Experimental constraints:
ppE formalism + Bayesian inference on stacked events

$$\sqrt{\alpha} < 1.7\, km$$

Perkins+, PRD 104 (2021) 2, 024060



DILATONIC BLACK HOLES

$$D \equiv -\frac{1}{4\pi}\int d^2\Sigma^\mu \, \nabla_\mu \varphi_0$$

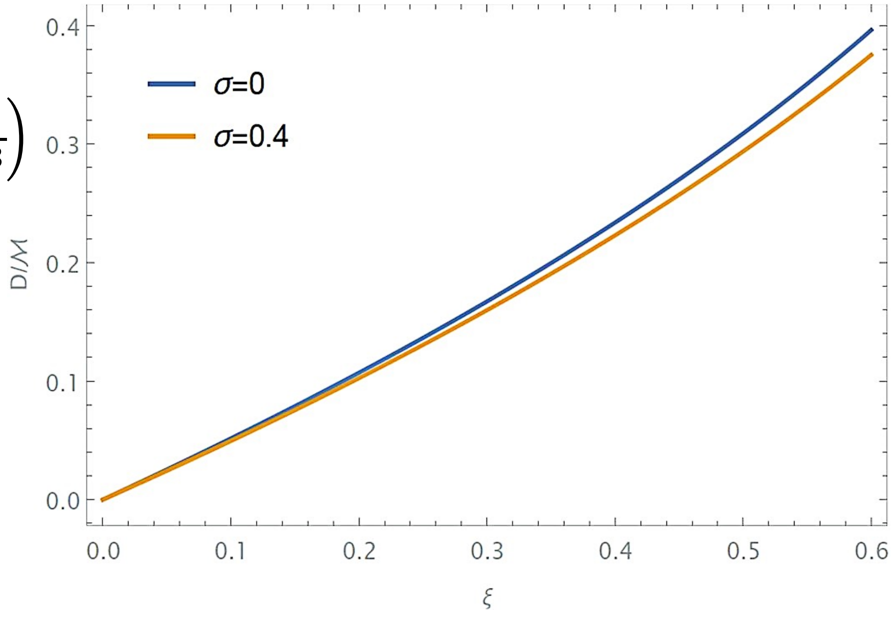
The mass and scalar charge can be read from the asymptotic behaviour of

$$\varphi_0(r \to +\infty) = \varphi_{0,\infty} + \frac{D}{r} - \frac{2MD}{r^2} + O\left(\frac{1}{r^3}\right)$$

$$g_{tt}(r \to +\infty) = -1 + \frac{2M}{r} + O\left(\frac{1}{r^2}\right)$$

The horizon is the largest root of

$$g_{\phi\phi} g_{tt} - g_{t\phi}^2 = 0$$



BEYOND GENERAL RELATIVITY

Observational point of view: some of the most outstanding open questions in physics can be explained by modifying the gravitational sector

Theoretical point of view: there is no a priori reason why the General Relativity assumptions should be true.
It is reasonable to question each of them and explore alternative theories of gravity

- Gravitational interaction mediated ONLY by the metric tensor
- The metric tensor is massless
- Spacetime is four-dimensional
- Position-invariant, Lorentz-invarian, and parity-invariant theory

The deviations from GR must be negligible in the weak field regime

BEYOND GENERAL RELATIVITY

Starting from General Relativity

$$S = \int d^4x \, \frac{\sqrt{-g}}{16\,\pi} R(g_{\mu\nu})$$

Einstein-Hilbert Action

Not renormalizable
(necessary for quantum gravity)

General Relativity becomes renormalizable if we assume that the Einstein-Hilbert action is only the first term of an **expansion in curvature invariants**

$$S = \int d^4x \, \frac{\sqrt{-g}}{16\,\pi} \left(R + \epsilon^2\, \mathcal{R}^2 \,\middle|\, + \epsilon^3 \mathcal{R}^3 + \cdots \right)$$

Effective field theory approach: we consider a *quadratic theory* only as a truncation of a more general theory

SLOWLY ROTATING EDGB BLACK HOLES

$$g_{\mu\nu} = g^0_{\mu\nu} + \epsilon\, h_{\mu\nu}$$
$$\Phi = \varphi_0 + \epsilon\, \delta\varphi$$

Metric and scalar field of a slowly rotating EdGB black hole in the small coupling limit

Axially symmetric spacetime expanded around static solution

Hartle, Thorne Astrophys. J. 153 (1968) 807

$$ds^2 = g^0_{\mu\nu} dx^\mu dx^\nu = -A(r)[1 + 2h(r,\theta)]dt^2 + \frac{1}{B(r)}[1 + 2m(r,\theta)]dr^2$$
$$+r^2[1 + 2k(r,\theta)][d\theta^2 + \sin^2\theta(d\phi - \hat{\omega}(r,\theta)dt)^2]$$

$$\chi = J/M^2 \ll 1$$

Slow-rotation expansion of $\hat{\omega}, h, m, k, \varphi_0 \longrightarrow \omega^n_l(r), h^n_l(r), m^n_l(r), k^n_l(r), \phi^n_l(r)$

$n =$ spin order

SLOWLY ROTATING EDGB BLACK HOLES

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$$+r^2[1 + 2k(r,\theta)][d\theta^2 + \sin^2\theta(d\phi - \hat{\omega}(r,\theta)dt)^2]$$

$$\zeta \equiv \alpha/M^2 \leq 0.691 < 1$$

Kanti+, PRD 54 (1996), 5049

Maselli+, PRD 92 (2015) 8, 083014

Small-coupling expansion of $\boldsymbol{f} = \{A, B, \omega^n_l, m^n_l, h^n_l, k^n_l, \phi^n_l\}$

$$f_i = \sum_{j=0}^{N_\zeta} \zeta^i f_i^{(j)}$$ → Into the field eqs → Integration order by order in ζ to find each $f_i^{(j)}$

PERTURBATIONS OF EDGB BLACK HOLES

$$g_{\mu\nu} = g^0_{\mu\nu} + \epsilon\, h_{\mu\nu}$$
$$\Phi = \varphi_0 + \epsilon\, \delta\varphi$$
$$\epsilon \ll 1$$

Black hole perturbation theory

Regge, Wheeler Phys.Rev. 108(1957) 1063-1069
Zerilli PRD 2 (1970) 2141-2160

Scalar, vector, tensor spherical harmonics decomposition

$$\delta X_{\mu_1\ldots}(t,r,\theta,\phi) = \sum_{\ell m} \delta X^{(i)}_{\ell m}(r)\, \mathcal{Y}^{\ell m(i)}_{\mu_1\ldots}(\theta,\phi)\, e^{-i\omega t}$$

$\delta X^{(i)}_{\ell m}(r)$ → Polar (even) and Axial (odd)

$\delta\varphi$ → $\sum_{\ell m} \varphi_{1,\ell m}(r) Y^{\ell m}(\theta,\phi) e^{-i\omega t}$

$h_{\mu\nu}$ → 10 radial functions → $K_{\ell m}$, $H_{1,\ell m}$, $h_{0,\ell m}$, $h_{1,\ell m}$

PERTURBATIONS OF EDGB BLACK HOLES

$$g_{\mu\nu} = g^0_{\mu\nu} + \epsilon\, h_{\mu\nu}$$
$$\Phi = \varphi_0 + \epsilon\, \delta\varphi$$

$$h_{tt} = A(r) H_0(t,r) Y^{\ell m}$$
$$h_{tr} = H_1(t,r) Y^{\ell m}$$
$$h_{rr} = 1/B(r)\, H_2(t,r) Y^{\ell m}$$
$$h_{tA} = q_0(t,r) Y_A^{\ell m} + h_0(t,r) X_A^{\ell m}$$
$$h_{rA} = q_1(t,r) Y_A^{\ell m} + h_1(t,r) X_A^{\ell m}$$
$$h_{AB} = r^2 \left[K(t,r) U_{AB}^{\ell m} + G(t,r) Y_{AB}^{\ell m} \right] + h_2(t,r) X_{AB}^{\ell m}$$

$$\delta\varphi = \varphi_1(t,r) Y^{\ell m}$$

EVEN/POLAR

$$Y_A^{\ell m} \equiv \nabla_A Y^{\ell m}$$
$$U_{AB}^{\ell m} \equiv \gamma_{AB} Y^{\ell m}$$
$$Y_{AB}^{\ell m} \equiv \left[\nabla_A \nabla_B + \frac{1}{2} \ell(\ell+1) \gamma_{AB} \right] Y^{\ell m}$$

ODD/AXIAL

$$X_A^{\ell m} \equiv \epsilon_{AC} \gamma^{BC} \nabla_B Y^{\ell m} = \epsilon_A{}^B \nabla_B Y^{\ell m}$$
$$X_{AB}^{\ell m} \equiv \frac{1}{2} \left(\epsilon_A{}^C \nabla_B + \epsilon_B{}^C \nabla_A \right) Y^{\ell m}$$

Diffeomorphism invariance:
Regge-Wheeler gauge

PERTURBATION EQUATIONS

10 components of linearized Einstein's equations + scalar field equation

$tt, tr, rr, \theta\theta + \phi\phi, eq\varphi$

$$\sum_{\ell m}\left[\left(A^I_{\ell m} + \tilde{A}^I_{\ell m} cos\theta\right) Y_{\ell m} + B^I_{\ell m} sin\theta\, \partial_\theta Y_{\ell m} + C^I_{\ell m} \partial_\phi Y_{\ell m}\right] = 0 \qquad (I = 0,1,2,3,4)$$

$t\theta, r\theta$

$$\sum_{\ell m}\left[(\alpha^J_{\ell m} + \tilde{\alpha}^J_{\ell m} cos\theta)\partial_\theta Y_{\ell m} - (\beta^J_{\ell m} + \tilde{\beta}^J_{\ell m} cos\theta)\frac{\partial_\phi Y_{\ell m}}{sin\theta} + \eta^J_{\ell m}(sin\theta\, Y_{\ell m}) + \xi^J_{\ell m} X_{\ell m} + \gamma^J_{\ell m}(sin\theta\, W_{\ell m})\right] = 0$$

$$(J = 0,1)$$

$\theta\phi$

$$\sum_{\ell m}\left[f_{\ell m}\, \partial_\theta Y_{\ell m} + g_{\ell m}\frac{\partial_\phi Y_{\ell m}}{sin\theta} + s_{\ell m}\frac{X_{\ell m}}{\sin^2\theta} + t_{\ell m}\frac{W_{\ell m}}{sin\theta}\right] = 0$$

Angular integration:
Single ℓ, m mode

$$X_{\ell m} \equiv 2\,\partial_\phi(\partial_\theta - cot\theta) Y_{\ell m}$$

$$W_{\ell m} \equiv \left(\partial^2_\theta - cot\theta\,\partial_\theta - \frac{\partial^2_\phi}{\sin^2\theta}\right) Y_{\ell m}$$

Kojima PRD 46 (1992) 4289
Pani+ PRD 92 (2015) 2, 024010

ANGULAR INTEGRATION

Sum over ℓ, m → we want a single mode ℓ, m

$$\int d\Omega \, Y^*_{\ell' m'} Y_{\ell m} = \delta_{\ell' \ell} \delta_{m' m}$$

We multiply the equations by certain combinations of $Y^*_{\ell' m'}$ and we integrate over the solid angle

$$cos\theta \, Y_{\ell m} = Q_{\ell+1\,m} Y_{\ell+1\,m} + Q_{\ell m} Y_{\ell-1\,m}$$
$$sin\theta \, \partial_\theta Y_{\ell m} = \ell Q_{\ell+1\,m} Y_{\ell+1\,m} - (\ell+1) Q_{\ell m} Y_{\ell-1\,m}$$

$$Q_{\ell m} \equiv \sqrt{\frac{\ell^2 - m^2}{4\,\ell^2 - 1}}$$

$$\sum_{\ell' m'} A_{\ell' m'} \int d\Omega \, Y^*_{\ell m} \, sin\theta \, \partial_\theta Y_{\ell' m'} = (\ell - 1) Q_{\ell m} A_{\ell-1\,m} - (\ell + 2) Q_{\ell+1\,m} A_{\ell+1\,m}$$

We introduce couplings:
At 1st order in rotation we have $\ell \pm 1$
At 2nd order in rotation we have $\ell \pm 2, \ell \pm 1$

1st ORDER IN THE SPIN

At first order in the spin we get two families of equations:

$$0 = \mathcal{P}_\ell + \chi m\, \bar{\mathcal{P}}_\ell + \chi[Q_{\ell m}\tilde{\mathcal{A}}_{\ell-1} + Q_{\ell+1\, m}\tilde{\mathcal{A}}_{\ell+1}]$$

$$0 = \mathcal{A}_\ell + \chi m\, \bar{\mathcal{A}}_\ell + \chi[Q_{\ell m}\tilde{\mathcal{P}}_{\ell-1} + Q_{\ell+1\, m}\tilde{\mathcal{P}}_{\ell+1}]$$

Pani IJMPA 28 (2013) 1340018

For symmetry reasons, the QNM spectrum is

$$\omega^{n\ell m}(\chi,\zeta) = \omega_0^{n\ell}(\zeta) + m\,\chi\,\omega_1^{n\ell}(\zeta) + O(\chi^2)$$

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Pani IJMPA 28 (2013) 1340018

For symmetry reasons, the QNM spectrum is

$$\omega^{n\ell m}(\chi,\zeta) = \omega_0^{n\ell}(\zeta) + m\,\chi\,\omega_1^{n\ell}(\zeta) + O(\chi^2)$$

The m dependence in the spectrum can only arise from $\chi m\, \bar{\mathcal{P}}_\ell$ and $\chi m\, \bar{\mathcal{A}}_\ell$

$$\mathcal{P}_\ell + \chi m\, \bar{\mathcal{P}}_\ell = 0$$
$$\mathcal{A}_\ell + \chi m\, \bar{\mathcal{A}}_\ell = 0$$

No couplings with different ℓs!

1st ORDER IN THE SPIN: POLAR SECTOR

We can recast the polar set of equations as

$$\frac{d}{dr}\boldsymbol{\Psi}_{\ell m} + \hat{V}_{\ell m}\boldsymbol{\Psi}_{\ell m} + \chi\, m\, \hat{U}_{\ell m}\boldsymbol{\Psi}_{\ell m} = 0 \qquad \text{with } \boldsymbol{\Psi} \equiv \begin{pmatrix} H_1 \\ K \\ \varphi_1 \\ \varphi_1' \end{pmatrix}$$

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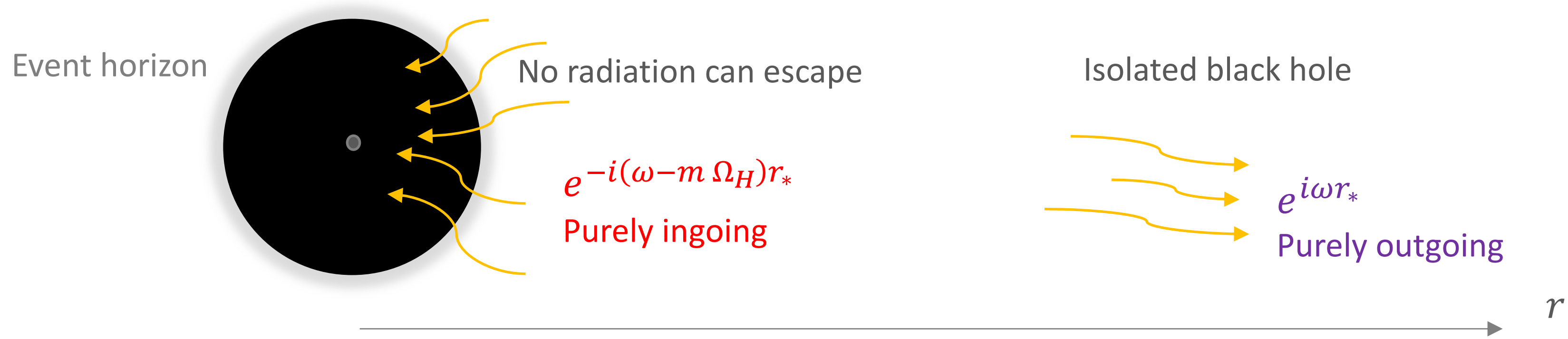

$\hat{V}, \hat{U}$ are 4x4 matrices $\longrightarrow$ $\hat{X} \equiv (\mathbf{\Psi}_1^{in} \quad \mathbf{\Psi}_2^{in} \quad \mathbf{\Psi}_1^{out} \quad \mathbf{\Psi}_2^{out})$

QNMs

$$\det \hat{X}\Big|_{r_m}(\omega_{n\ell m}) = 0$$

2nd ORDER IN THE SPIN

At second order in the spin the equations assume the schematic form

$$0 = \mathcal{P}_\ell + \chi m\, \bar{\mathcal{P}}_\ell + \chi^2\, \hat{\mathcal{P}}_\ell + m^2\chi^2 \bar{\bar{\mathcal{P}}}_\ell + \chi[Q_{\ell m}\tilde{\mathcal{A}}_{\ell-1} + Q_{\ell+1\, m}\tilde{\mathcal{A}}_{\ell+1}]$$

$$+m\,\chi^2[Q_{\ell m}\check{\mathcal{A}}_{\ell-1} + Q_{\ell+1\, m}\check{\mathcal{A}}_{\ell+1}] + \chi^2[Q_{\ell-1 m}Q_{\ell m}\breve{\mathcal{P}}_{\ell-2} + Q_{\ell+1\, m}Q_{\ell+2\, m}\breve{\mathcal{P}}_{\ell+2}]$$

Similarly for axial perturbations

Two subsets of solutions:

- Polar-led: only polar perturbations at 0-th order in the spin
- Axial-led: only axial perturbations at 0-th order in the spin

2nd ORDER IN THE SPIN

At second order in the spin we have

$$0 = \mathcal{P}_\ell + \chi m\, \bar{\mathcal{P}}_\ell + \chi^2\, \hat{\mathcal{P}}_\ell + m^2\chi^2 \bar{\bar{\mathcal{P}}}_\ell + \chi[Q_{\ell m}\tilde{\mathcal{A}}_{\ell-1} + Q_{\ell+1\,m}\tilde{\mathcal{A}}_{\ell+1}]$$
$$+m\,\chi^2[Q_{\ell m}\check{\mathcal{A}}_{\ell-1} + Q_{\ell+1\,m}\check{\mathcal{A}}_{\ell+1}] + \chi^2[Q_{\ell-1 m}Q_{\ell m}\breve{\mathcal{P}}_{\ell-2} + Q_{\ell+1\,m}Q_{\ell+2\,m}\breve{\mathcal{P}}_{\ell+2}]$$

$$0 = \mathcal{A}_\ell + \chi m\, \bar{\mathcal{A}}_\ell + \chi^2\, \hat{\mathcal{A}}_\ell + m^2\chi^2 \bar{\bar{\mathcal{A}}}_\ell + \chi[Q_{\ell m}\tilde{\mathcal{P}}_{\ell-1} + Q_{\ell+1\,m}\tilde{\mathcal{P}}_{\ell+1}]$$
$$+m\,\chi^2[Q_{\ell m}\check{\mathcal{P}}_{\ell-1} + Q_{\ell+1\,m}\check{\mathcal{P}}_{\ell+1}] + \chi^2[Q_{\ell-1 m}Q_{\ell m}\breve{\mathcal{A}}_{\ell-2} + Q_{\ell+1\,m}Q_{\ell+2\,m}\breve{\mathcal{A}}_{\ell+2}]$$

The **QNM spectrum** is

$$\omega^{n\ell m}(\chi,\zeta) = \omega_0^{n\ell}(\zeta) + m\,\chi\,\omega_1^{n\ell}(\zeta) + \chi^2[\omega_{2a}^{n\ell}(\zeta) + m^2\omega_{2b}^{n\ell}(\zeta)] + O(\chi^3)$$

2nd ORDER IN THE SPIN

At zero-th order in the spin, a perturbation with index ℓ will not excite other perturbations with different ℓs

Let us assume I excite $p_\ell = p_\ell^{(0)} + \chi\, p_\ell^{(1)} + \chi^2\, p_\ell^{(2)}$, then:

$$a_{\ell\pm1} = a_{\ell\pm1}^{(0)} + \chi\, a_{\ell\pm1}^{(1)} + \chi^2\, a_{\ell\pm1}^{(2)} + O(\chi^3)$$

$$p_{\ell\pm2} = p_{\ell\pm2}^{(0)} + \chi\, p_{\ell\pm2}^{(1)} + \chi^2\, p_{\ell\pm2}^{(2)} + O(\chi^3)$$

$$0 = \mathcal{P}_\ell + \chi m\, \bar{\mathcal{P}}_\ell + \chi^2\, \hat{\mathcal{P}}_\ell + m^2\chi^2 \bar{\bar{\mathcal{P}}}_\ell + \chi[Q_{\ell m}\tilde{\mathcal{A}}_{\ell-1} + Q_{\ell+1\,m}\tilde{\mathcal{A}}_{\ell+1}]$$

$$+m\,\chi^2\left[Q_{\ell m}\cancel{\breve{\mathcal{A}}^{(0)}_{\ell-1}} + Q_{\ell+1\,m}\cancel{\breve{\mathcal{A}}^{(0)}_{\ell+1}}\right] + \chi^2\left[Q_{\ell-1m}Q_{\ell m}\cancel{\breve{\mathcal{P}}^{(0)}_{\ell-2}} + Q_{\ell+1\,m}Q_{\ell+2\,m}\cancel{\breve{\mathcal{P}}^{(0)}_{\ell+2}}\right]$$

$$0 = \mathcal{A}_{\ell+1} + \chi m\, \bar{\mathcal{A}}_{\ell+1} + \chi^2\, \cancel{\hat{\mathcal{A}}^{(0)}_{\ell+1}} + m^2\chi^2 \cancel{\bar{\bar{\mathcal{A}}}^{(0)}_{\ell+1}} + \chi\left[Q_{\ell+1\,m}\tilde{\mathcal{P}}_\ell + Q_{\ell+2\,m}\cancel{\tilde{\mathcal{P}}^{(0,1)}_{\ell+2}}\right]$$

$$+m\,\chi^2\left[Q_{\ell+1m}\check{\mathcal{P}}_\ell + Q_{\ell+2\,m}\cancel{\check{\mathcal{P}}^{(0)}_{\ell+2}}\right] + \chi^2\left[Q_{\ell m}Q_{\ell+1m}\cancel{\check{\mathcal{A}}^{(0)}_{\ell-1}} + Q_{\ell+2\,m}Q_{\ell+3\,m}\cancel{\check{\mathcal{A}}^{(0)}_{\ell+3}}\right]$$

2nd ORDER IN THE SPIN

Two subsets of solutions:

- Polar-led: polar perturbation not vanishing at zero-th order in the spin
- Axial-led: axial perturbation not vanishing at zero-th order in the spin

Focusing on the polar-led sector

$$0 = \mathcal{P}_\ell + \chi m\, \bar{\mathcal{P}}_\ell + \chi^2\, \hat{\mathcal{P}}_\ell + m^2\chi^2 \bar{\bar{\mathcal{P}}}_\ell + \chi[Q_{\ell m}\tilde{\mathcal{A}}_{\ell-1} + Q_{\ell+1\, m}\tilde{\mathcal{A}}_{\ell+1}]$$

$$0 = \mathcal{A}_{\ell+1} + \chi m\, \bar{\mathcal{A}}_{\ell+1} + \chi Q_{\ell+1\, m}\tilde{\mathcal{P}}_\ell + m\, \chi^2 Q_{\ell+1 m}\check{\mathcal{P}}_\ell$$

$$0 = \mathcal{A}_{\ell-1} + \chi m\, \bar{\mathcal{A}}_{\ell-1} + \chi Q_{\ell\, m}\tilde{\mathcal{P}}_\ell + m\, \chi^2 Q_{\ell m}\check{\mathcal{P}}_\ell$$

2nd ORDER IN THE SPIN: POLAR-LED

We can recast the system as

$$\frac{d}{dr}\mathbf{\Psi}_{\ell m} + \hat{P}_{\ell m}\mathbf{\Psi}_{\ell m} = 0$$

with $\mathbf{\Psi}_{\ell m} = \{H_{1\ \ell m}, K_{\ell m}, \varphi_{1\ \ell m}, \varphi'_{1\ \ell m}, h_{0\ \ell-1 m}, h_{1\ \ell-1 m}\, h_{0\ \ell+1 m}, h_{1\ \ell+1 m}\}$

In order to find the QNM spectrum we build an 8x8 matrix $\hat{X}$ containing a basis of solutions

$$\hat{X} \equiv (\mathbf{\Psi}_1^{in} \quad \mathbf{\Psi}_2^{in} \quad \mathbf{\Psi}_3^{in} \quad \mathbf{\Psi}_4^{in} \quad \mathbf{\Psi}_1^{out} \quad \mathbf{\Psi}_2^{out} \quad \mathbf{\Psi}_3^{out} \quad \mathbf{\Psi}_4^{out})$$

QNMs

$$\det \hat{X}(\omega_{n\ell m}) = 0$$

QNM SPECTRUM

$$\omega^{n\ell m}(\chi,\zeta) = \omega_0^{n\ell}(\zeta) + m\,\chi\,\omega_1^{n\ell}(\zeta) + \chi^2\left[\omega_{2a}^{n\ell}(\zeta) + m^2\omega_{2b}^{n\ell}(\zeta)\right] + O(\chi^3)$$

How do we find the coefficients of the expansion?

QNM SPECTRUM

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How do we find the coefficients of the expansion?

Data points for different values of χ and ζ —interpolation→ $\omega^{n\ell m}(\chi,\zeta)$

$$\omega_0^{n\ell}(\zeta) = \lim_{\chi\to 0}\ \omega^{n\ell m}(\chi,\zeta)$$

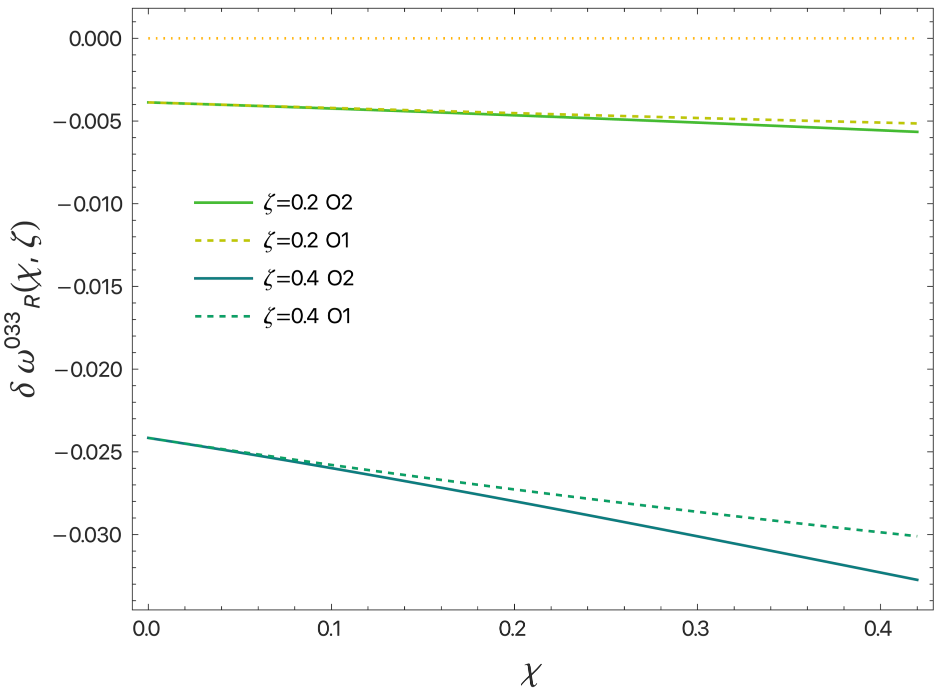
$$\omega_1^{n\ell}(\zeta) = \lim_{\chi\to 0} \frac{\partial_\chi[\omega^{n\ell m}(\chi,\zeta)]}{m}$$

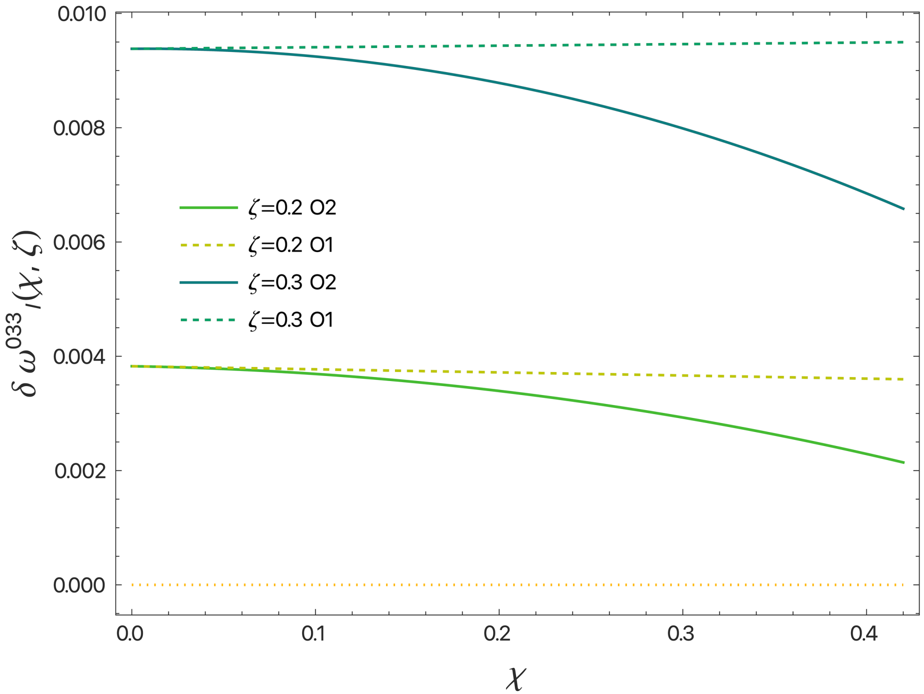
$$\omega_2^{n\ell m}(\zeta) \equiv \left[\omega_{2a}^{n\ell}(\zeta) + m^2\omega_{2b}^{n\ell}(\zeta)\right] = \frac{1}{2}\lim_{\chi\to 0}\partial_\chi^2[\omega^{n\ell m}(\chi,\zeta)]$$

QNM SPECTRUM: 2nd SPIN ORDER

$$\omega^{n\ell m}(\chi,\zeta) = \omega_0^{n\ell}(\zeta) + m\,\chi\,\omega_1^{n\ell}(\zeta) + \chi^2\left[\omega_{2a}^{n\ell}(\zeta) + m^2\omega_{2b}^{n\ell}(\zeta)\right] + O(\chi^3)$$

$$\delta\omega_{R,I}(\chi,\zeta) \equiv \frac{\omega_{R,I}(\chi,\zeta) - \omega_{R,I}(\chi,0)}{\omega_{R,I}(\chi,0)}$$





Pierini, Gualtieri (2022)

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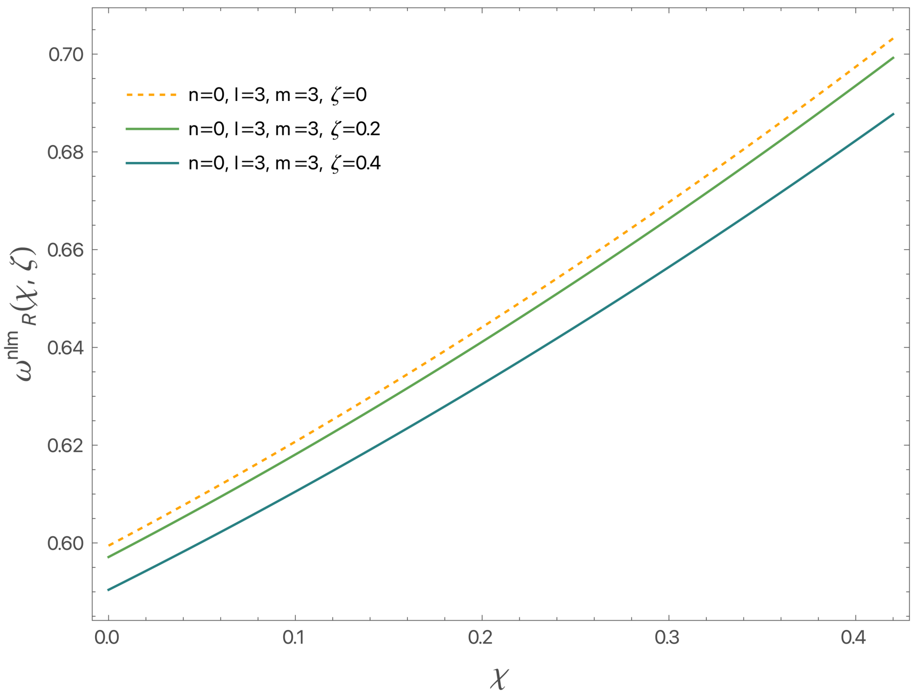

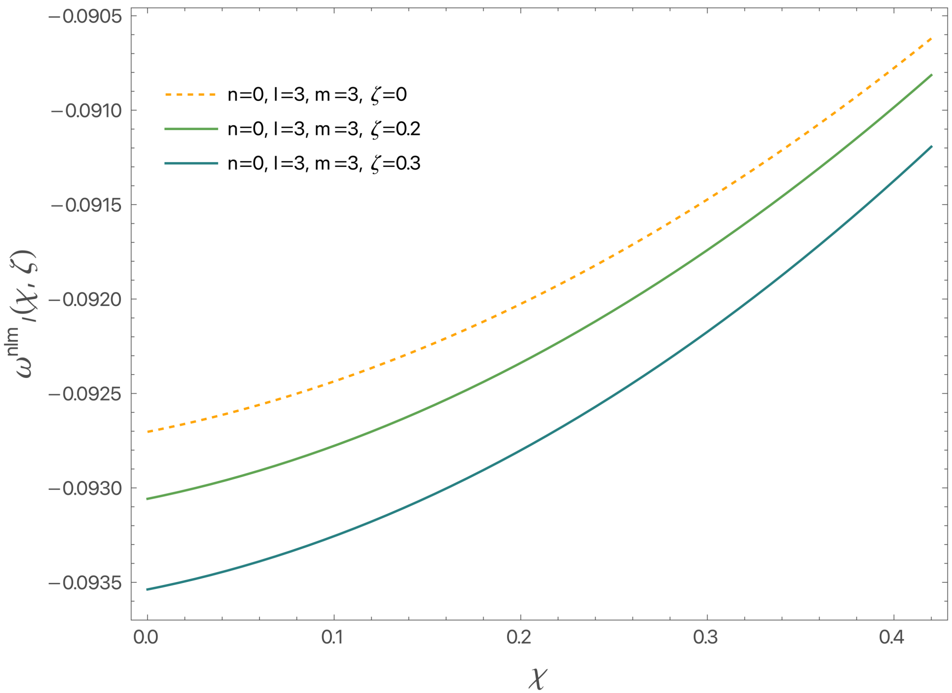

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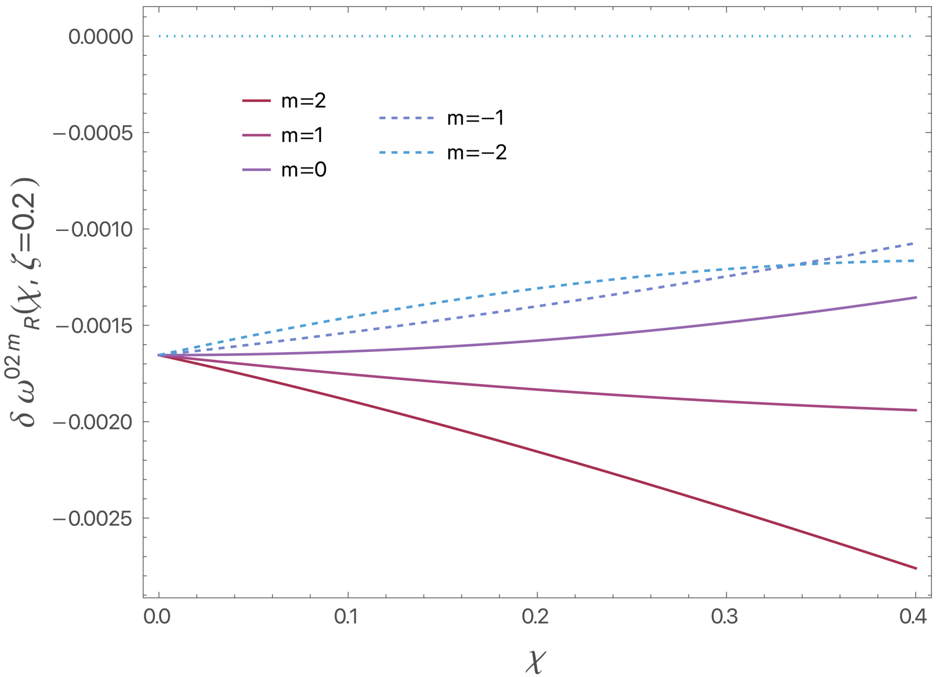

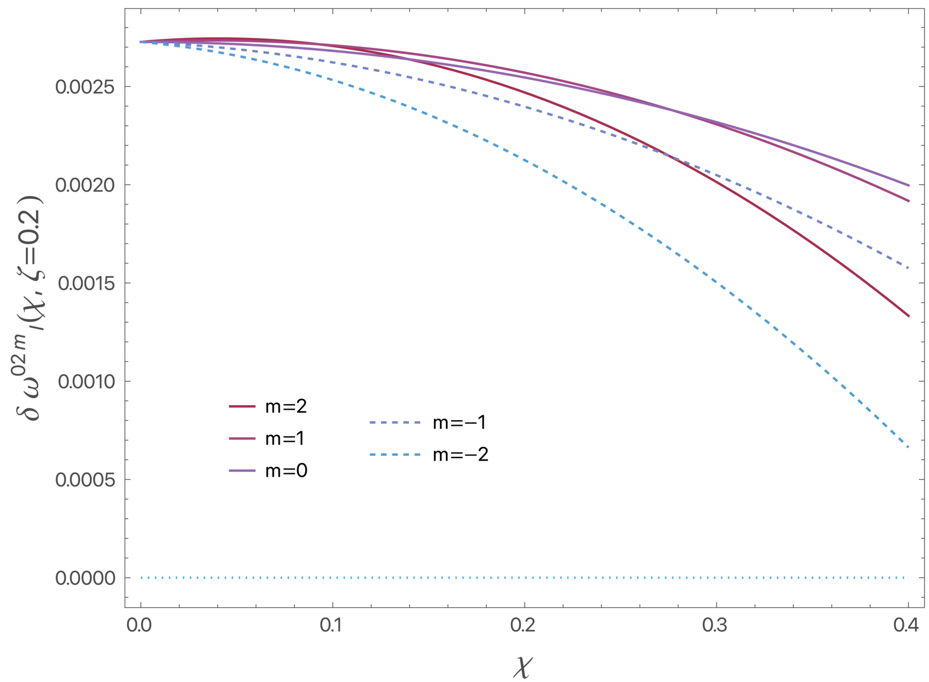

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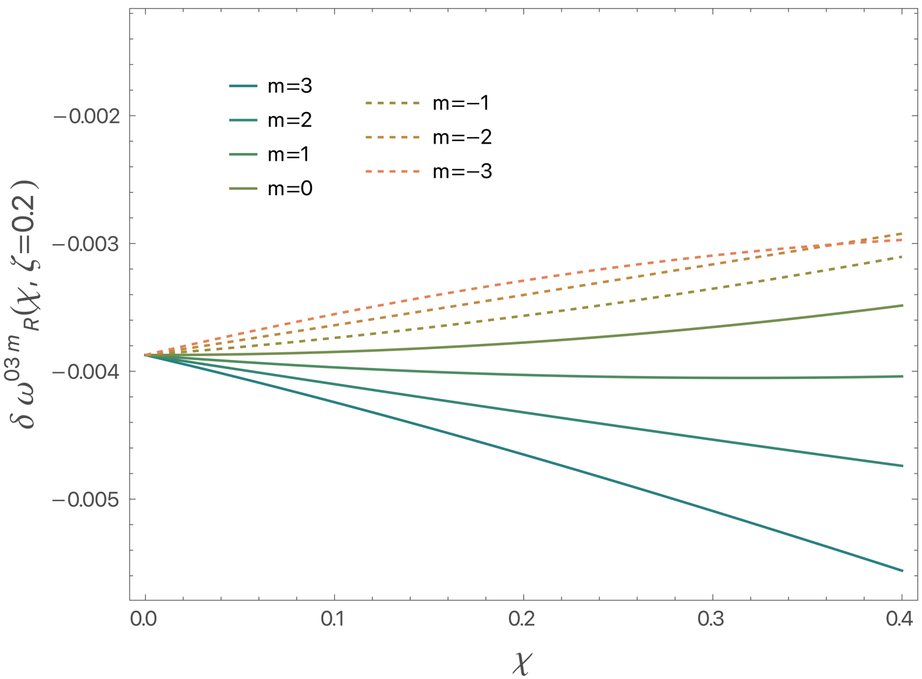

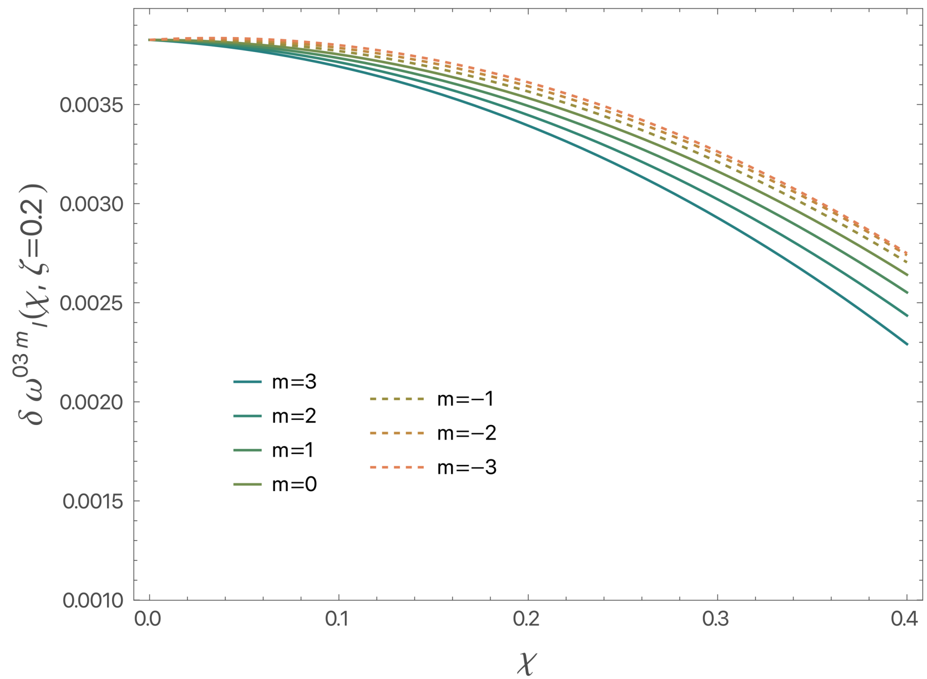

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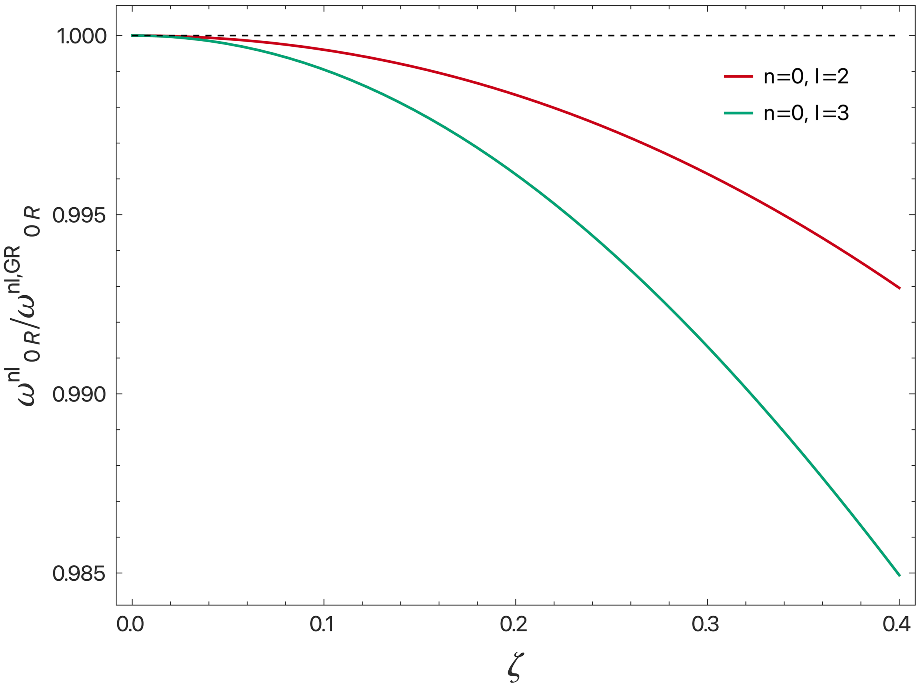
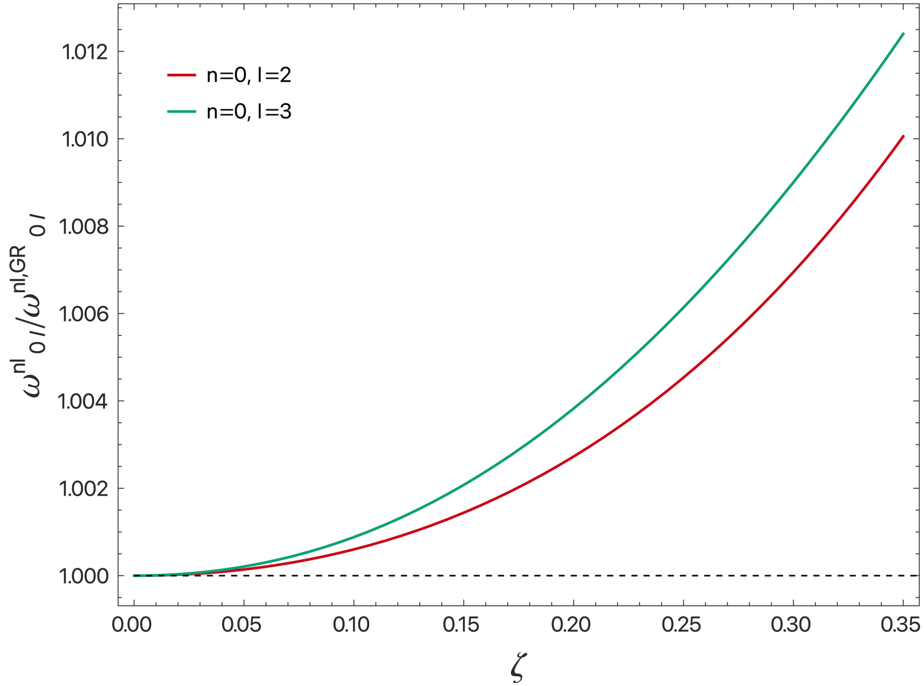
QNM SPECTRUM: 0th SPIN ORDER

$$\omega^{n\ell m}(\chi,\zeta) = \omega_0^{n\ell}(\zeta) + m\,\chi\,\omega_1^{n\ell}(\zeta) + \chi^2\left[\omega_{2a}^{n\ell}(\zeta) + m^2\omega_{2b}^{n\ell}(\zeta)\right] + O(\chi^3)$$

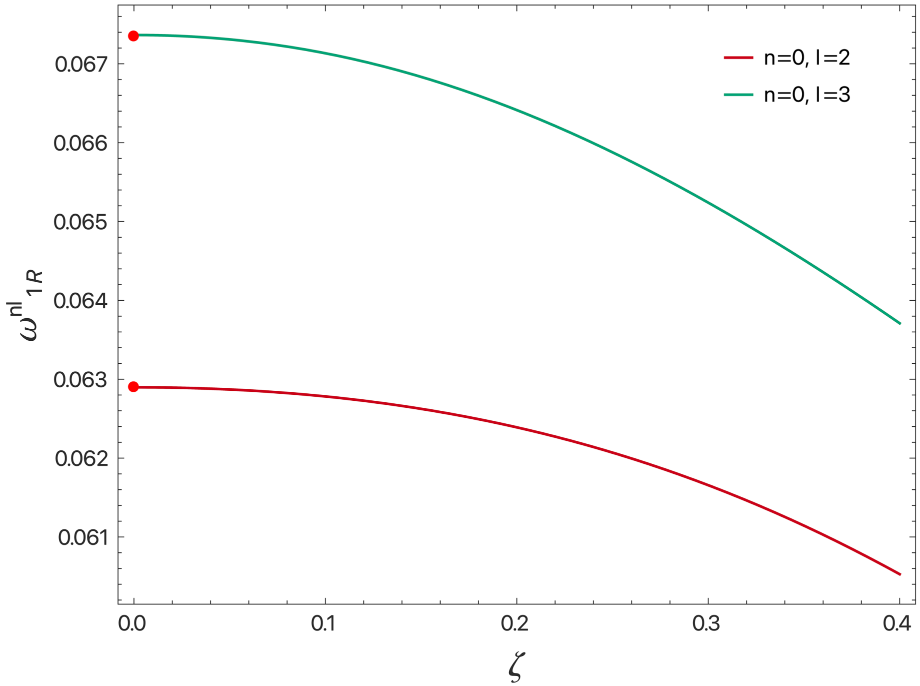


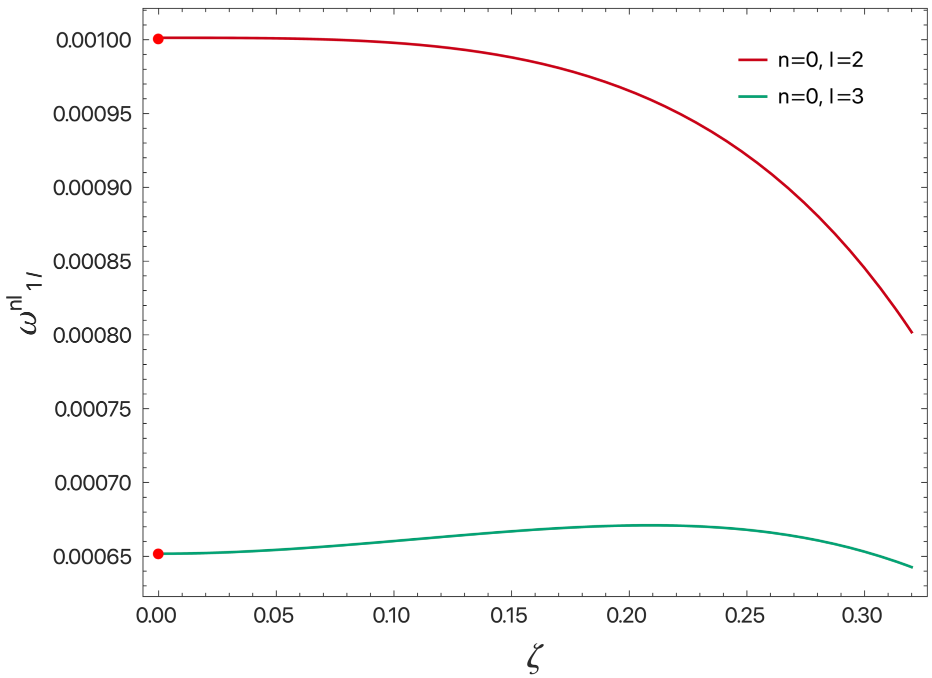


Checked with Blázquez Salcedo+ PRD 94 (2016) 10, 104024

QNM SPECTRUM: 1st SPIN ORDER

$$\omega^{n\ell m}(\chi,\zeta) = \omega_0^{n\ell}(\zeta) + m\,\chi\,\omega_1^{n\ell}(\zeta) + \chi^2\left[\omega_{2a}^{n\ell}(\zeta) + m^2\omega_{2b}^{n\ell}(\zeta)\right] + O(\chi^3)$$





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QNM SPECTRUM: 2nd SPIN ORDER

$$\omega^{n\ell m}(\chi,\zeta) = \omega_0^{n\ell}(\zeta) + m\,\chi\,\omega_1^{n\ell}(\zeta) + \chi^2\left[\omega_{2a}^{n\ell}(\zeta) + m^2\omega_{2b}^{n\ell}(\zeta)\right] + O(\chi^3)$$

Using different values of m for

$$\omega_2^{n\ell m}(\zeta) \equiv \left[\omega_{2a}^{n\ell}(\zeta) + m^2\omega_{2b}^{n\ell}(\zeta)\right]$$

we get a set of simple equations for ω_{2a}, ω_{2b}.

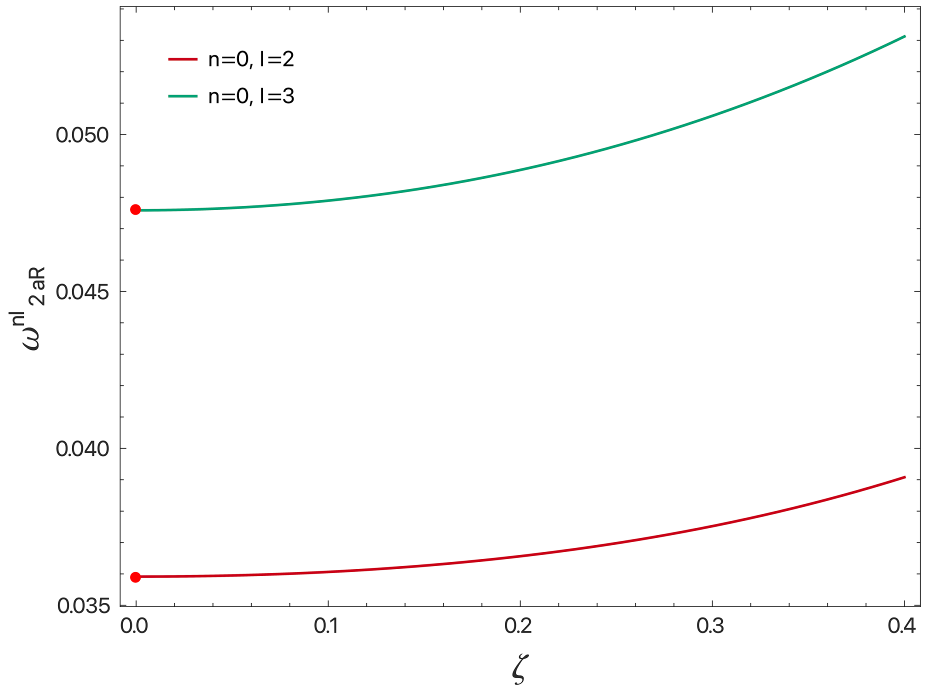
For example for $\ell = 2$:

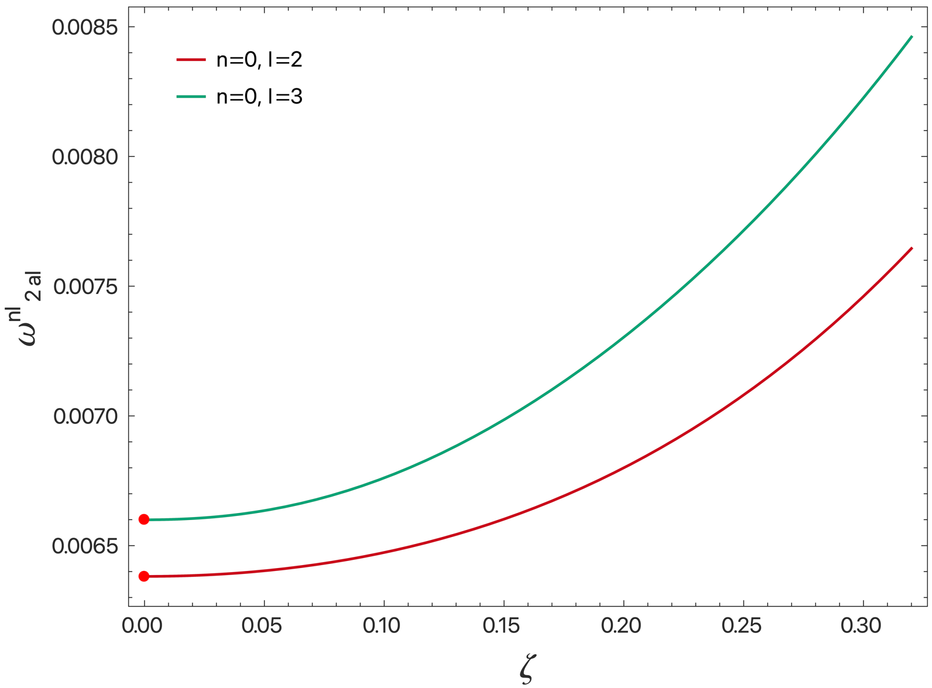
$$\omega_{2a}^{n2} = \omega_2^{n20} = \frac{4\,\omega_2^{n21} - \omega_2^{n22}}{3}$$

$$\omega_{2b}^{n2} = \omega_2^{n21} - \omega_2^{n20} = \frac{\omega_2^{n22} - \omega_2^{n20}}{4} = \frac{\omega_2^{n22} - \omega_2^{n21}}{3}$$

QNM SPECTRUM: 2nd SPIN ORDER

$$\omega^{n\ell m}(\chi,\zeta) = \omega_0^{n\ell}(\zeta) + m\,\chi\,\omega_1^{n\ell}(\zeta) + \chi^2\left[\omega_{2a}^{n\ell}(\zeta) + m^2\omega_{2b}^{n\ell}(\zeta)\right] + O(\chi^3)$$





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GW EMISSION

$$\boldsymbol{p} = \sum_i q_i(\boldsymbol{r_i} - \boldsymbol{R})$$

$$\sum_i m_i(\boldsymbol{r_i} - \boldsymbol{R_{CM}}) = 0$$

$$\frac{d^2\boldsymbol{R}_{CM}}{dt^2} = \sum_i m_i\boldsymbol{a_i} - Ma_{CM} = 0$$