# Advances on clusters and correlations in nuclear structure and reactions

## Selected Topics in Nuclear and Atomic Physics 2022

September 25, 2022 to October 1, 2022
Fiera di Primiero (TN),

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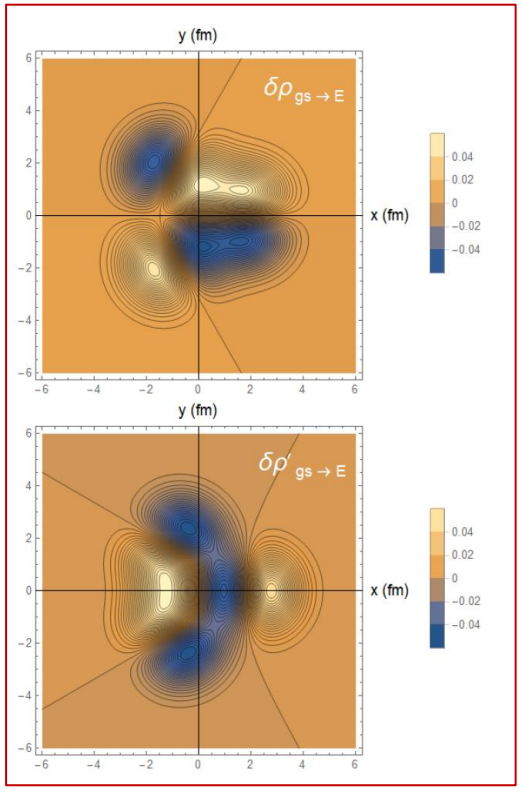
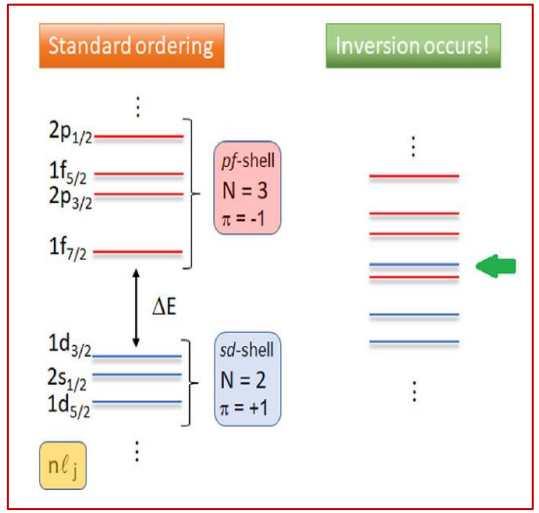
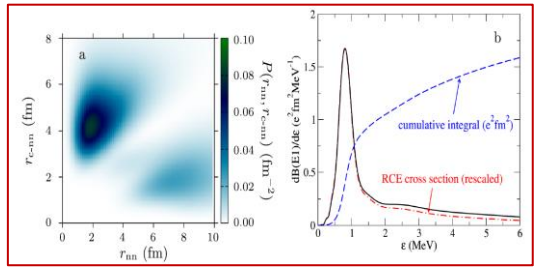
| name | shape | group | $\Gamma_{vib}$ | Patterns |
|---|---|---|---|---|
| linear = |  | $\mathcal{D}_{\infty h}$ | $A_{1g} + A_{1u} + E_{1u}$ |  |
| linear $\neq$ |  | $\mathcal{C}_{\infty\nu}$ | $2A_1 + E_1$ |  |
| equilateral |  | $\mathcal{D}_{3h}$ | $A_1' + E'$ |  |
| isosceles |  | $\mathcal{C}_{2\nu}$ | $2A_1 + B_1$ |  |
| scalene |  | $\mathcal{C}_s$ | $3A'$ |  |

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Oct. 2021

# Outline of the presentation

- A brief introduction/summary about the main group theory ideas that are useful in the business of cluster nuclei
- Polarized gamma beams: scenario for future measurements on the depolarization ratio for the case of 12C with triangular and more exotic structures
- Transition densities in action for 12C and 16O, recent results on alpha-transfer form factors
- New insight on the structure of 29F at the border of the island of inversion

# A word of caution

DISCLAIMER for the first topic:

- It is perfectly clear to us that molecular models of nuclei are FUNDAMENTALLY DIFFERENT from molecular physics, where the Born-Oppenheimer approximation is valid and one can think of nuclear motion as a small vibration, happening only close to the minimum of a very deep potential energy surface (in molecular energy scales).

- Nuclei have large kinetic energy <T>, comparable to the potential energy <V> and the zero point motion inside the P.E.S. is a large fraction of the well depth, therefore there are LARGE FLUCTUATIONS around the equilibrium points and we SHOULD NOT EXPECT that the vibrational levels are deeply lying in the potential well, at most they can be weakly bound states, close to threshold, or more probably resonances in the continuum!

- Dispite this, it is instructive to look at
  1. the normal modes, i.e. the «best» internal coordinates
  2. symmetry-adapted vibrational orbitals
  3. the energy scale and structure of the vibrational levels

# Do you remember what a group is in mathematics ?

A group is a set of elements $\{A, B, C, \cdots\}$ (finite or infinite) endowed with a binary operation of composition $\circ$ (also called multiplication ) that satisfies certain requirements, called Group Axioms:

- the operation takes two elements of the set and gives back one element that is still in the set $A \circ B = C$ (CLOSURE)
- $A \circ (B \circ C) = (A \circ B) \circ C$ (ASSOCIATIVITY)
- the inverse element is defined for each element of the set $A \circ A^{-1} = E$ (INVERTIBILITY)
- the identity element is defined $A \circ E = E \circ A = A$ (IDENTITY)

Groups are powerful mathematical structures that have been invented (by Galois, Cauchy, Cauley, and others) to prove theorems about the solution of fourth order equations of the type

$$ax^4 + bx^3 + cx^2 + dx + e = 0$$

and then found applications in other branches of mathematics: permutation theory, number theory, geometry, topology, and in sciences like physics, quantum physics, chemistry, cristallography, **nuclear physics**, particle physics, etc.

# Examples

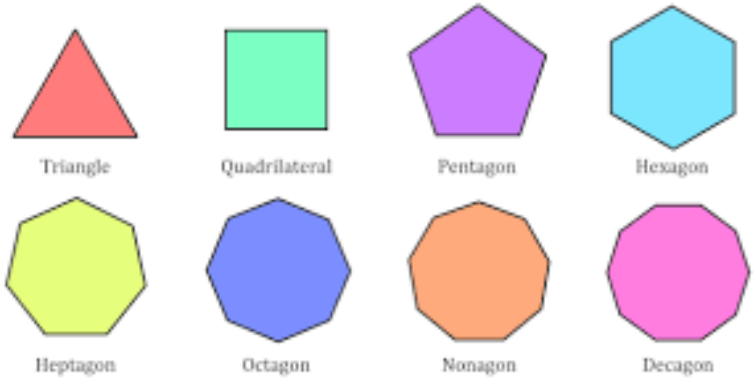
There are so many example of this particular abstract structure !

In number theory:

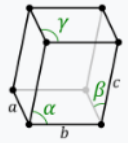
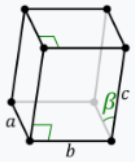
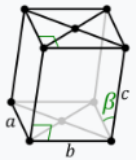
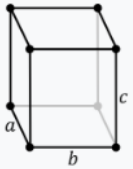
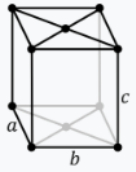
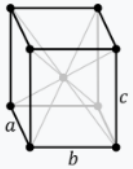
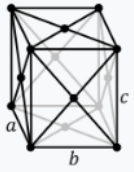
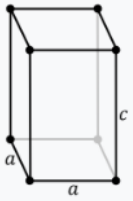
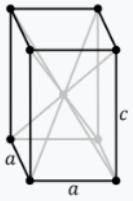
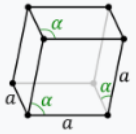
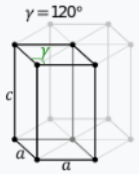
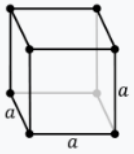
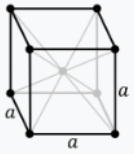
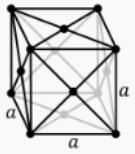
- The integer numbers $\{1,2,3,\cdots\}$ with the operation of multiplication x (is anything missing here?)
- The rational numbers with respect to multiplication
- …

In geometry:

- The sphere with respect to all possible rotations around an axis passing through the center
- The circle with respect to all possible rotations around an axis passing through the center and perpendicular to the diameter
- Regular polygons (exagon, square, triangle, ⋯ ) and polyhedra



# Examples

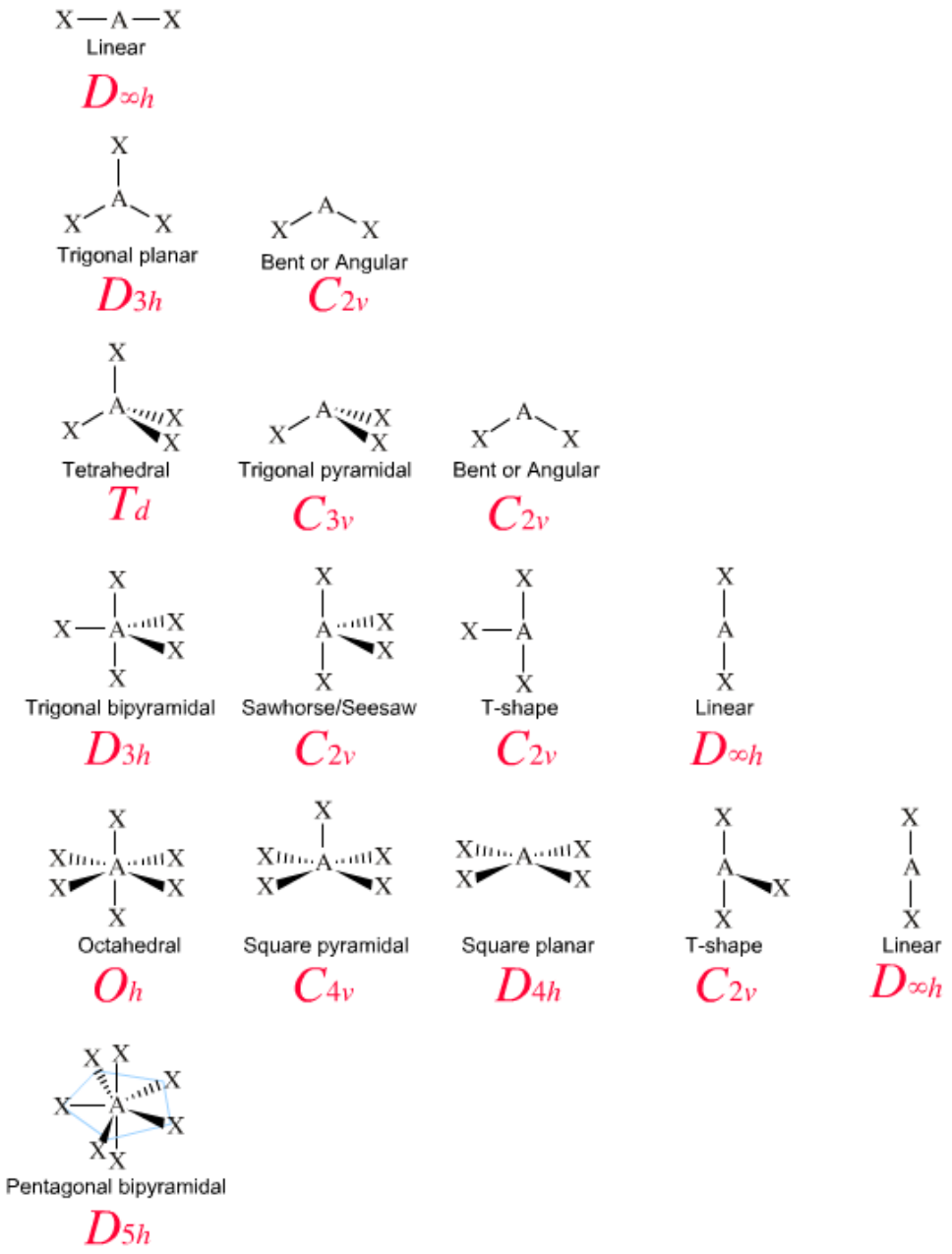
| Crystal family | Lattice system | Schönflies | 14 Bravais Lattices | | | |
|---|---|---|---|---|---|---|
| | | | Primitive | Base-centered | Body-centered | Face-centered |
| triclinic | | $C_i$ |  | | | |
| monoclinic | | $C_{2h}$ |  |  | | |
| orthorhombic | | $D_{2h}$ |  |  |  |  |
| tetragonal | | $D_{4h}$ |  | |  | |
| hexagonal | rhombohedral | $D_{3d}$ |  | | | |
| | hexagonal | $D_{6h}$ |  | | | |
| cubic | | $O_h$ |  | |  |  |

There are applications to natural sciences that are very important:

… for example there are only 32 classes of crystals that you can find in minerals (1830 by Johann Friedrich Christian Hessel) that are arranged in 7 crystal systems that give rise to 14 Bravais lattices with translational symmetry.

… correspondigly there are also only 32 classes of different molecular symmetries

# Point groups (molecules, crystals ... and nuclei)




| type | point groups |
|---|---|
| nonaxial | $C_i$, $C_s$ |
| cyclic | $C_1$, $C_2$, $C_3$, $C_4$, $C_6$ |
| cyclic with horizontal planes | $C_{2h}$, $C_{3h}$, $C_{4h}$, $C_{6h}$ |
| cyclic with vertical planes | $C_{2v}$, $C_{3v}$, $C_{4v}$, $C_{6v}$ |
| dihedral | $D_2$, $D_3$, $D_4$, $D_6$ |
| dihedral with horizontal planes | $D_{2h}$, $D_{3h}$, $D_{4h}$, $D_{6h}$ |
| dihedral with planes between axes | $D_{2d}$, $D_{3d}$ |
| improper rotation | $S_4$, $S_6$ |
| cubic groups | $T$, $T_h$, $T_d$, $O$, $O_h$ |

Without entering too much into mathematical details there are a finite number of classified groups (32) that correspond to all possible symmetry elements of a «molecular» structure.

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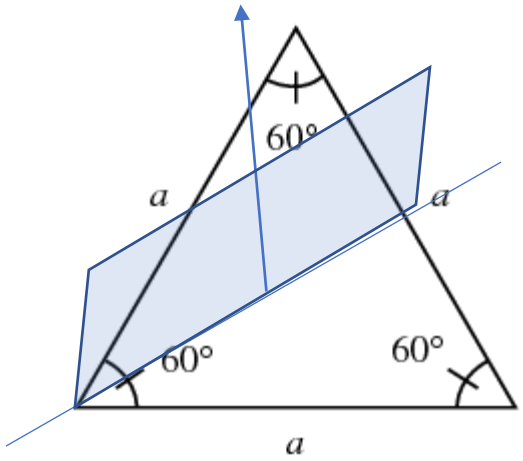
# Common Symmetry Point Groups

Schönflies notation

| | |
|---|---|
| **Nonrotational Groups** | |
| $C_1$ | $E$ (asymmetric) |
| $C_s$ | $E, \sigma_h$ |
| $C_i$ | $E, i$ |
| **Single-Axis Groups ($n = 2, 3, \ldots, \infty$)** | |
| $C_n$ | $E, C_n, \ldots, C_n^{n-1}$ |
| $C_{nv}$ | $E, C_n, \ldots, C_n^{n-1}, n\sigma_v$ ($n/2$ $\sigma_v$ and $n/2$ $\sigma_d$ if $n$ even) |
| $C_{nh}$ | $E, C_n, \ldots, C_n^{n-1}, \sigma_h$ |
| $S_{2n}$ | $E, S_{2n}, \ldots, S_{2n}^{2n-1}$ |
| $C_{\infty v}$ | $E, C_\infty, \infty\sigma_v$ (noncentrosymmetric linear) |
| **Dihedral Groups ($n = 2, 3, \ldots, \infty$)** | |
| $D_n$ | $E, C_n, \ldots, C_n^{n-1}, nC_2(\perp C_n)$ |
| $D_{nd}$ | $E, C_n, \ldots, C_n^{n-1}, S_{2n}, \ldots, S_{2n}^{2n-1}, nC_2(\perp C_n), n\sigma_d$ |
| $D_{nh}$ | $E, C_n, \ldots, C_n^{n-1}, nC_2(\perp C_n), \sigma_h, n\sigma_v$ |
| $D_{\infty h}$ | $E, C_\infty, S_\infty, \infty C_2(\perp C_\infty), \sigma_h, \infty\sigma_v, i$ (centrosymmetric linear) |
| **Cubic Groups** | |
| $T_d$ | $E, 4C_3, 4C_3^2, 3C_2, 3S_4, 3S_4^3, 6\sigma_d$ (tetrahedron) |
| $O_h$ | $E, 4C_3, 4C_3^2, 6C_2, 3C_4, 3C_4^3, 3C_2(= C_4^2), i, 3S_4, 3S_4^3, 4S_6, 4S_6^5, 3\sigma_h, 6\sigma_d$ (octahedron) |
| $I_h$ | $E, 6C_5, 6C_5^2, 6C_5^3, 6C_5^4, 10C_3, 10C_3^2, 15C_2, i, 6S_{10}, 6S_{10}^3, 6S_{10}^7, 6S_{10}^9, 10S_6, 10S_6^5, 15\sigma$ (icosahedron, dodecahedron) |

BOOKS: Hamermesh, *Group theory* Carter, *Introduction to molecular symmetry*

# In the case of an equilateral triangle



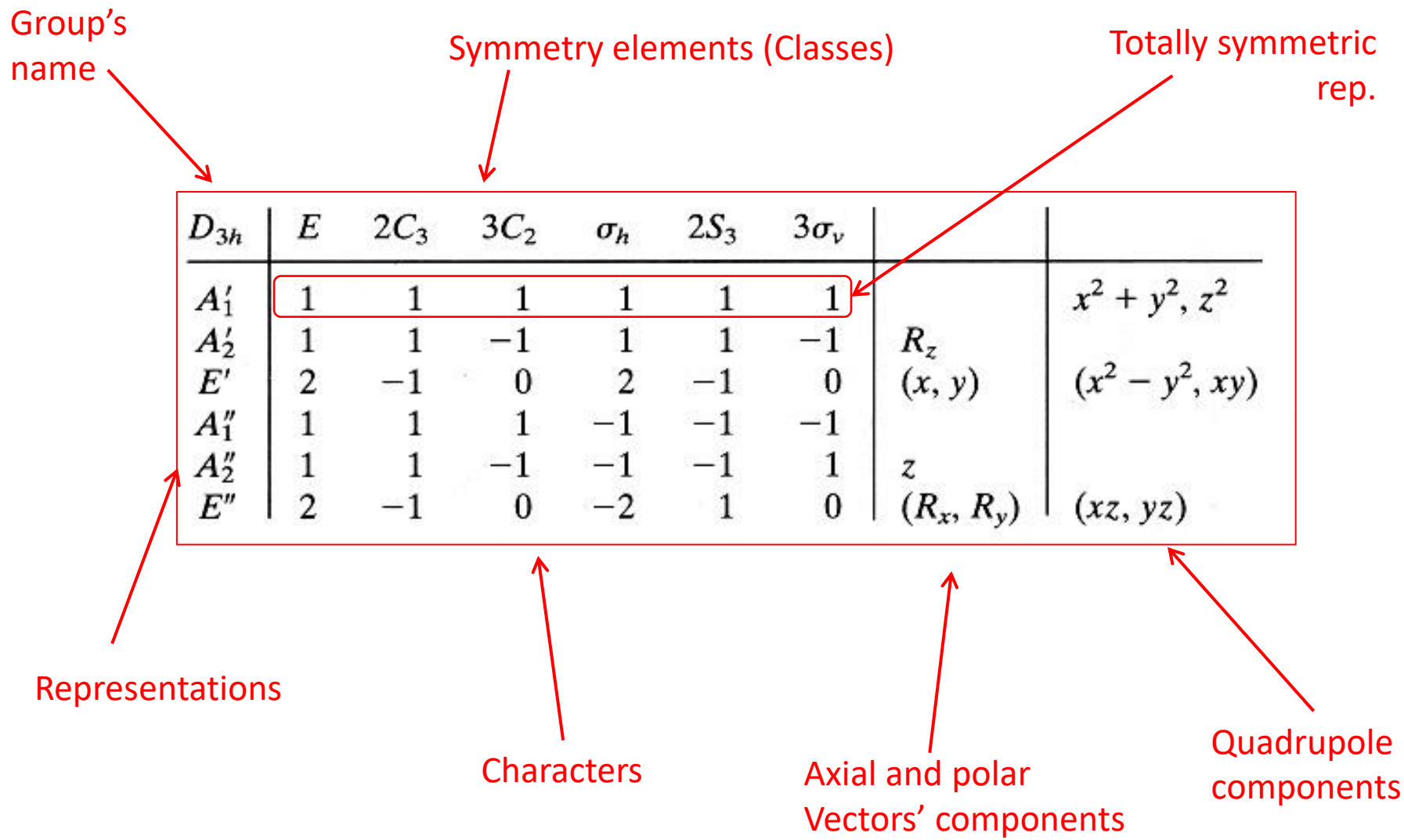

The object itself is NOT a group, it is the full set of operations that leave it invariant (i.e. that give back the same shape after we act upon it with the operation).

In this case the operations that form the group's elements are:

- Rotations of 120° around the perpendicular axis passing through the center $C_3$ , $(C_3)^2$
- Rotations of 180° around the axis passing through the center $3C_2$
- Reflections with respect to the plane containing the triangle $\sigma_h$
- Reflections with respect to the perpendicular plane containing the center and one vertex $3\sigma_v$
- Improper rotations (i.e. rotations followed by reflection in a perp. plane ) $2S_3$

This group takes the name $\mathcal{D}_{3h}$ dihedral prismatic group of order 3

# Character table and representations




| $D_{3h}$ | $E$ | $2C_3$ | $3C_2$ | $\sigma_h$ | $2S_3$ | $3\sigma_v$ | | |
|---|---|---|---|---|---|---|---|---|
| $A_1'$ | 1 | 1 | 1 | 1 | 1 | 1 | | $x^2+y^2, z^2$ |
| $A_2'$ | 1 | 1 | −1 | 1 | 1 | −1 | $R_z$ | |
| $E'$ | 2 | −1 | 0 | 2 | −1 | 0 | $(x, y)$ | $(x^2-y^2, xy)$ |
| $A_1''$ | 1 | 1 | 1 | −1 | −1 | −1 | | |
| $A_2''$ | 1 | 1 | −1 | −1 | −1 | 1 | $z$ | |
| $E''$ | 2 | −1 | 0 | −2 | 1 | 0 | $(R_x, R_y)$ | $(xz, yz)$ |

All the representations are known and have been classified by mathematicians (they might be finite in number or infinite )

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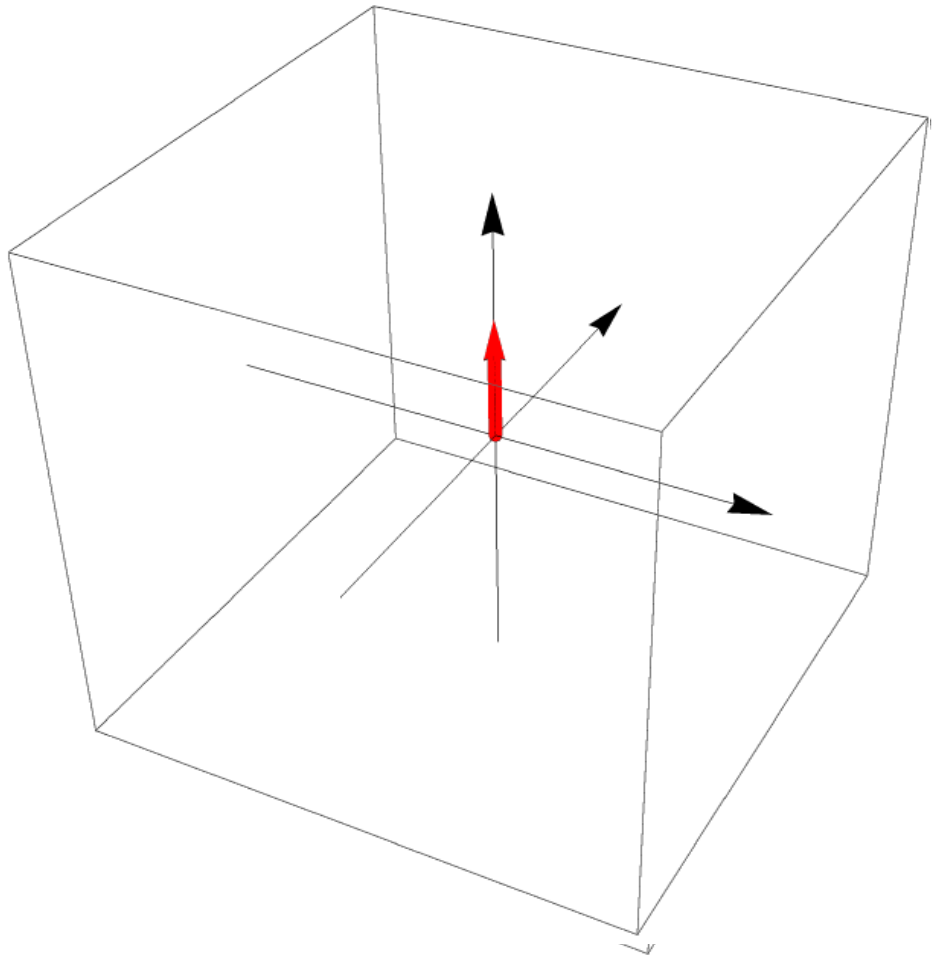
# What are representations and characters ?

The mathematical definitions are somewhat beyond the scope of this lecture, but I will try to get an idea of what a representation is in terms of geometric objects. Take a vector along the z axis and apply the operations of the $\mathcal{D}_{3h}$ group. Assign 1 if the vector remains the same, -1 if it goes to its negative and 0 if it is moved in any other way



| $E$ | $2C_3$ | $3C_2$ | $\sigma_h$ | $2S_3$ | $3\sigma_v$ |
|---|---|---|---|---|---|
| 1 | 1 | -1 | -1 | -1 | 1 |

This classifies all the z-vectors as belonging to the so-called $A_2''$ representation.

| Table II: The Mulliken symbols used to describe the symmetry species of point groups including their meaning with respect to molecular symmetry | |
|---|---|
| **Mulliken Symbols of Symmetry Species (Column 1 In Character Table)** | **Meaning** |
| *A* | Symmetric with respect to principal axis of symmetry |
| *B* | Antisymmetric with respect to principal axis of symmetry |
| *E* | Doubly degenerate, two-dimensional irreducible representation |
| *T* | Triply degenerate, three-dimensional irreducible representation |
| *g* | Symmetric with respect to a center of symmetry |
| *u* | Antisymmetric with respect to a center of symmetry |
| 1 (subscript) | Symmetric with respect to a $C_2$ axis that is perpendicular to the principal axis. Where there is no such axis the subscript indicates that reflection in a $_v$ plane of symmetry is symmetric. |
| 2 (subscript) | Antisymmetric with respect to a $C_2$ axis that is perpendicular to the principal axis. Where there is no such axis the subscript indicates that reflection in a $_v$ plane of symmetry is antisymmetric. |
| ′ (prime) | Symmetric with respect to reflection in a horizontal plane of symmetry |
| ″ (double prime) | Antisymmetric with respect to reflection in a horizontal plane of symmetry |

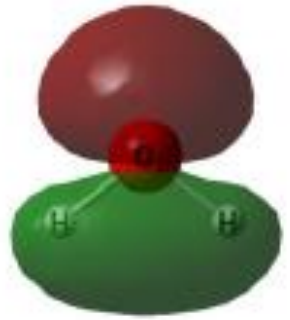
From D. Tuschel – Spectroscopy Molecular Spectroscopy workbench (2014)

# Use of this classification in molecular states

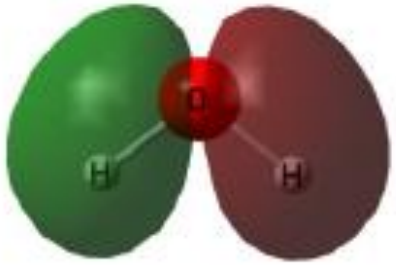
The states of a molecule can be classified according to this terminology

For instance, take a chemical example: water has a $C_{2v}$ point group symmety (isosceles triangle), its electronic wavefunctions are classified according to the representation that they belong to.

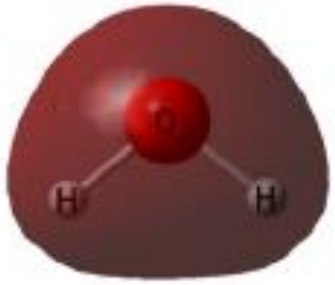
Lowest molecular orbitals of water



$2a_1$



$1b_1$



$1a_1$


BOOKS:
Bright-Wilson, Decius, Cross, *Molecular Vibrations*
Carter, *Introduction to molecular symmetry*
Levine, *Quantum Chemistry*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# Switch to nuclear molecules

I will now show how the mathematics that we have very briefly summarized and that is heavily employed in quantum chemistry, for example, can be used to produce interesting consequences also in nuclear physics.

In particular, I will show how a detailed knowledge of group theoretical structures can be used in alpha cluster models of light nuclei, taking the example of 12C, that was the object of a recent investigation.

# Clusters in nuclei

The existence of clusters, i.e. a smaller lump of particles (protons and neutrons) inside larger nuclei is a proven fact, though many subtleties can be invoked.

For example, it is well-known that many heavy nuclei decay by emitting an alpha-particle, 4He, that is a very well-bound nucleus.

For example, is it well-known that many properties of light-nuclei such as 7Li and 7Be can be well-reproduced in models with two clusters, alpha+t and alpha+3He respectively.

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THE EUROPEAN PHYSICAL JOURNAL A

**Electromagnetic response and breakup of light weakly bound nuclei in a dicluster model**

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# Definition : alpha-conjugate nuclei

Alpha-conjugate nuclei, or $\alpha$-conjugate, or 4n-nuclei are those nuclei made up by a number of protons Z and a number of neutron N that corresponds exactly to a **multiple of an alpha-particle**. Examples:

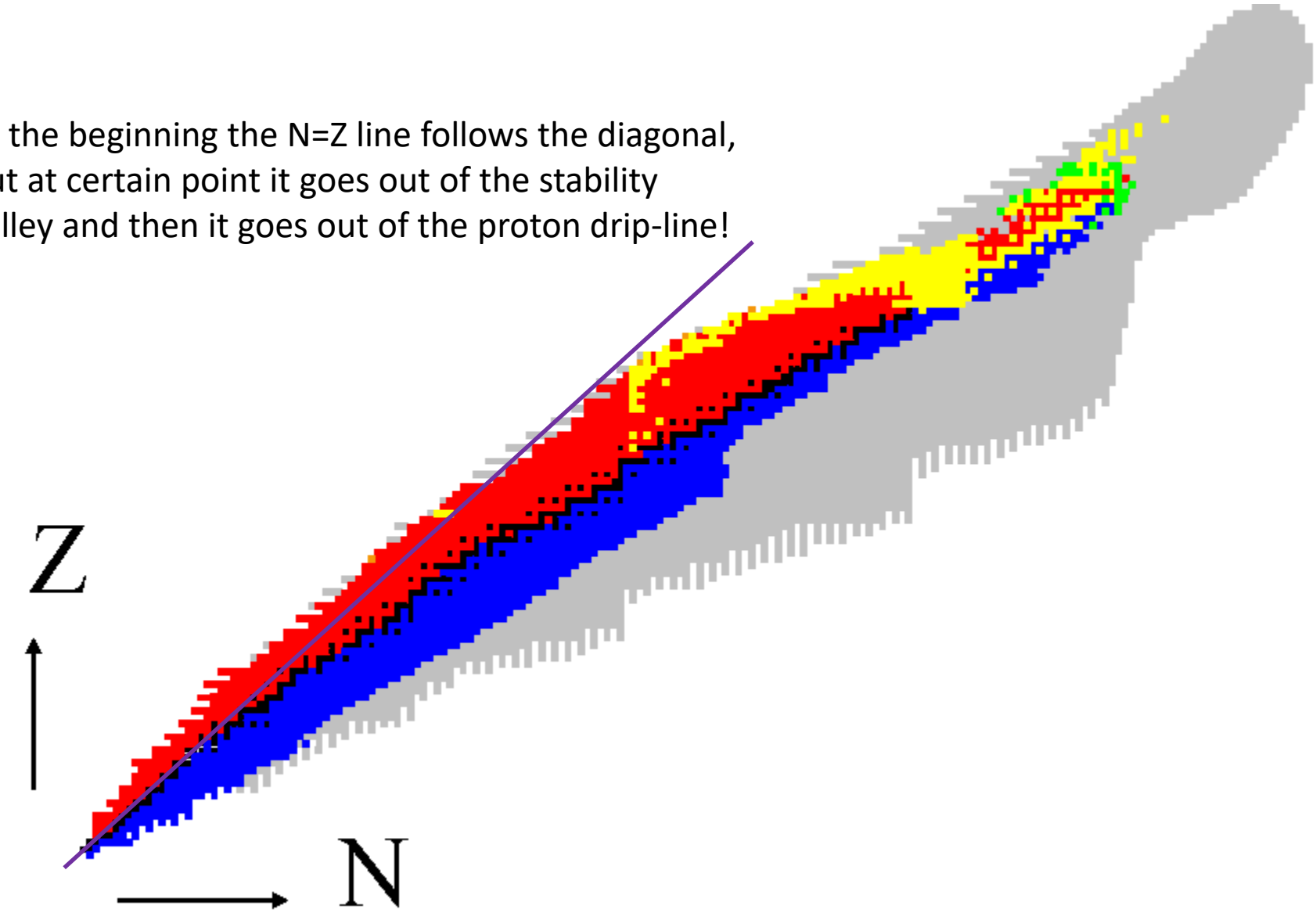
4He : 2p + 2n -> 1 $\alpha$
8Be : 4p + 4n -> 2 $\alpha$
12C : 6p + 6n -> 3 $\alpha$
16O : 8p + 8n -> 4 $\alpha$
20Ne : 10 p + 10 n -> 5 $\alpha$
24Mg : 12 p + 12 n -> 6 $\alpha$
...

- Since N and Z scale linearly , this series of nuclei is represented by the even-even nuclei on the straight line, 2N=2Z , in the nuclear chart and it will eventually get out of the drip-line at about 100Sn : 50p + 50n -> 25 $\alpha$
- *Clearly this DOES NOT MEAN that these nuclei are necessarily 100% made up of a N $\alpha$ particle structure. They MIGHT BE !*

You can read this statement as: The overlap between their true states and the alpha particle cluster states (or the alpha spectroscopic factor) is large.

# N=Z

At the beginning the N=Z line follows the diagonal, but at certain point it goes out of the stability valley and then it goes out of the proton drip-line!

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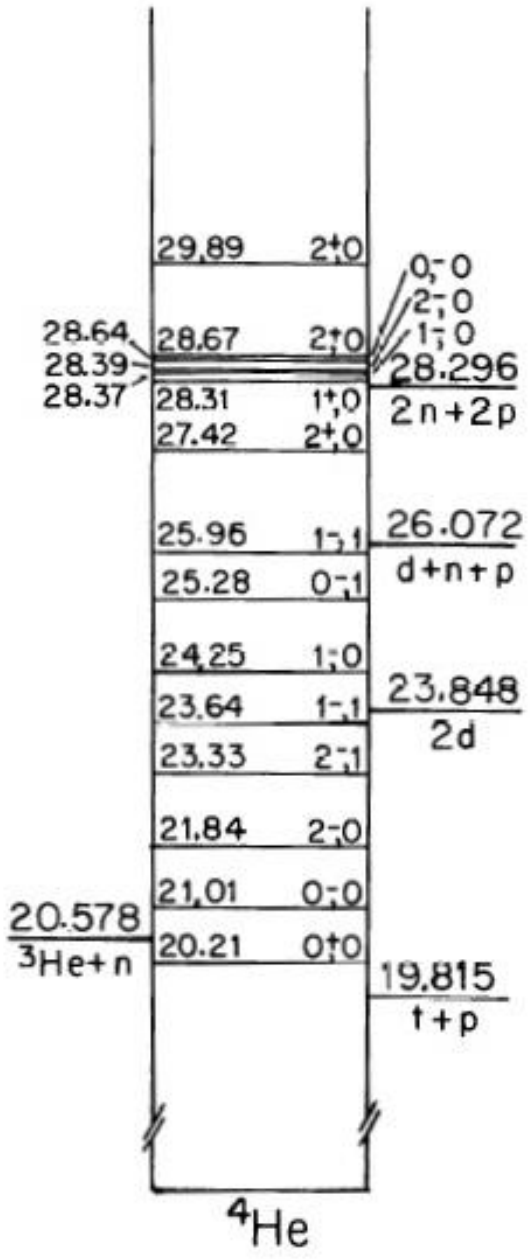
# α - particle




MAIN PROPERTIES:

- The alpha particle or $^{4}_{2}He_{2}$ is a very stable nucleus.
- It's total B.E. = 28.296 MeV
- No excited states up to the first threshold (t+p)
- Threshold for neutron and proton separation at about 20 MeV
- Boson with $J^{\pi} = 0^{+}$ -> Bose-Einstein Condensation (BEC)

Therefore:
It is a very stable system, that does not change state unless there are at least 20 MeV available. One could think it is the ideal building block for clusterized nuclei

OTHER PROPERTIES :

- It has N=Z=2, therefore it is the first nucleus of this series
- It is extremely important in nuclear physics because many nuclei decay by emission of an alpha particle
- It is also very important in reactions because ($\alpha$, $\alpha'$) is a perfect isoscalar probe (for exciting Giant Resonances for example)

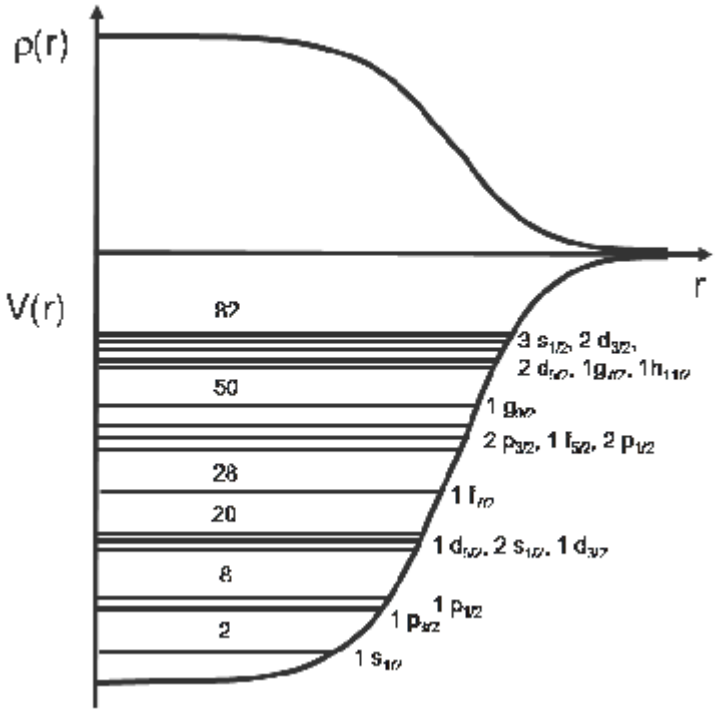
L. Fortunato

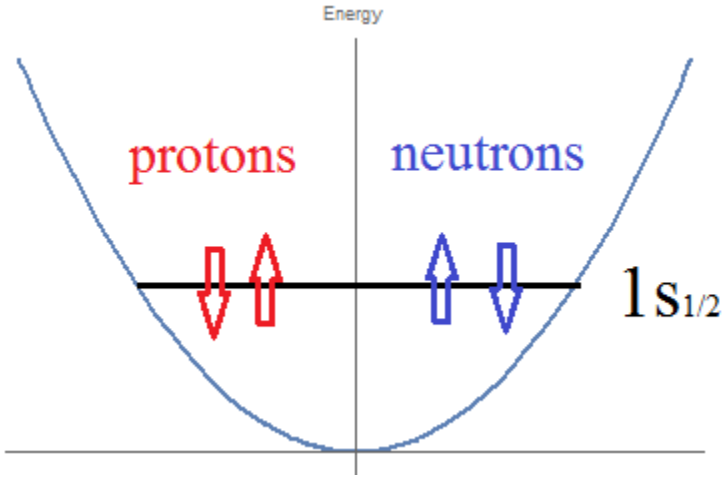
# α - particle in the shell model

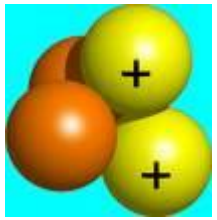
2 protons and 2 neutron fill up the s-states in accordance with Pauli principle.
Each nucleon (proton or neutron) is a fermion with $\ell$=0 and s=1/2

Therefore:

- It is a spherical state L=0 and S=0 , L+S=J=0
- It has a shell gap to p-states (hard to excite s.p. degrees of freedom) and no mixing with p-h configurations



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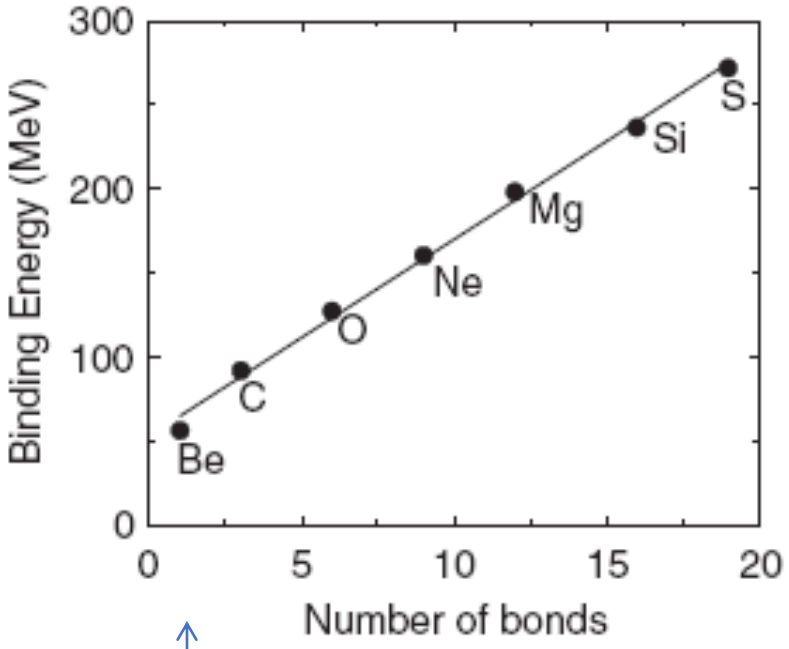
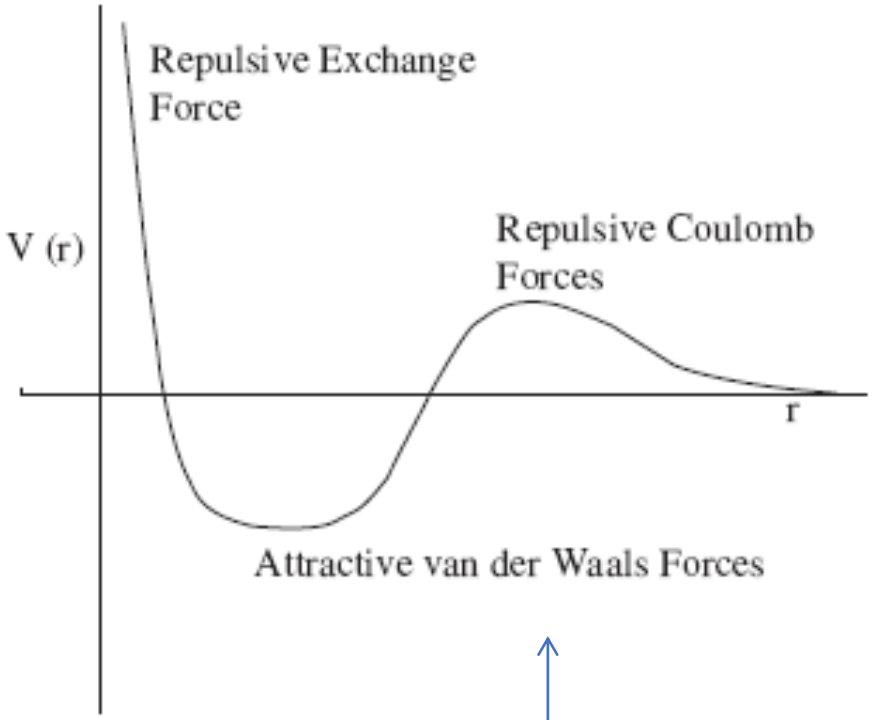
# Saturation of binding energy

*W. von Oertzen et al. / Physics Reports 432 (2006) 43–113*




Heisenberg (1935) noticed that the B.E. is proportional to the number of saturated bonds between alpha constituents.

Hafstad and Teller (1938) developed a model based on a $\alpha$-$\alpha$ potential based on similar well-known phenomena in molecular physics (van der Waals).

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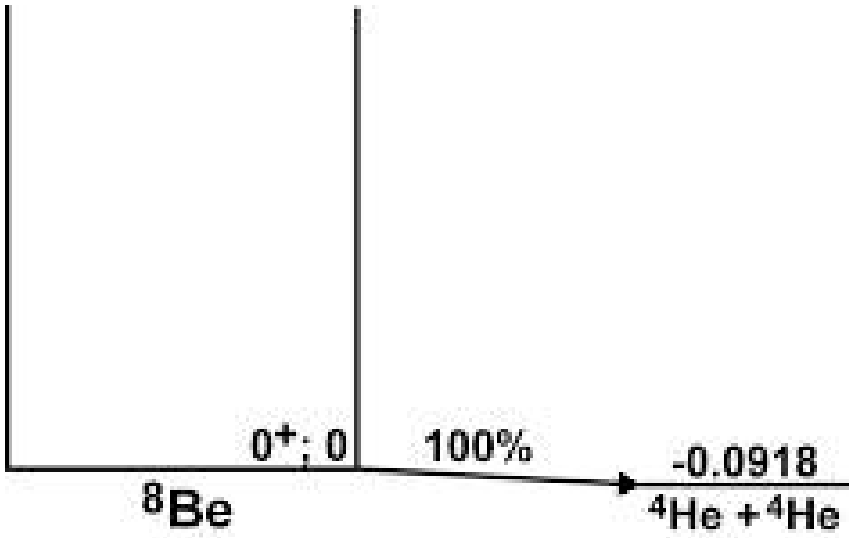
# 8Be is not bound -> low-lying resonance




So Beryllium-8 is not stable, two alpha particles simply DO NOT form a bound state. Nevertheless, they might stay close to each other long enough (on nuclear time scales!) to be observed as a resonance that decay back into two helium-4 nuclei with 100% probability. The energy of the resonance is 92 keV .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# A rotational band is built on top of the g.s.



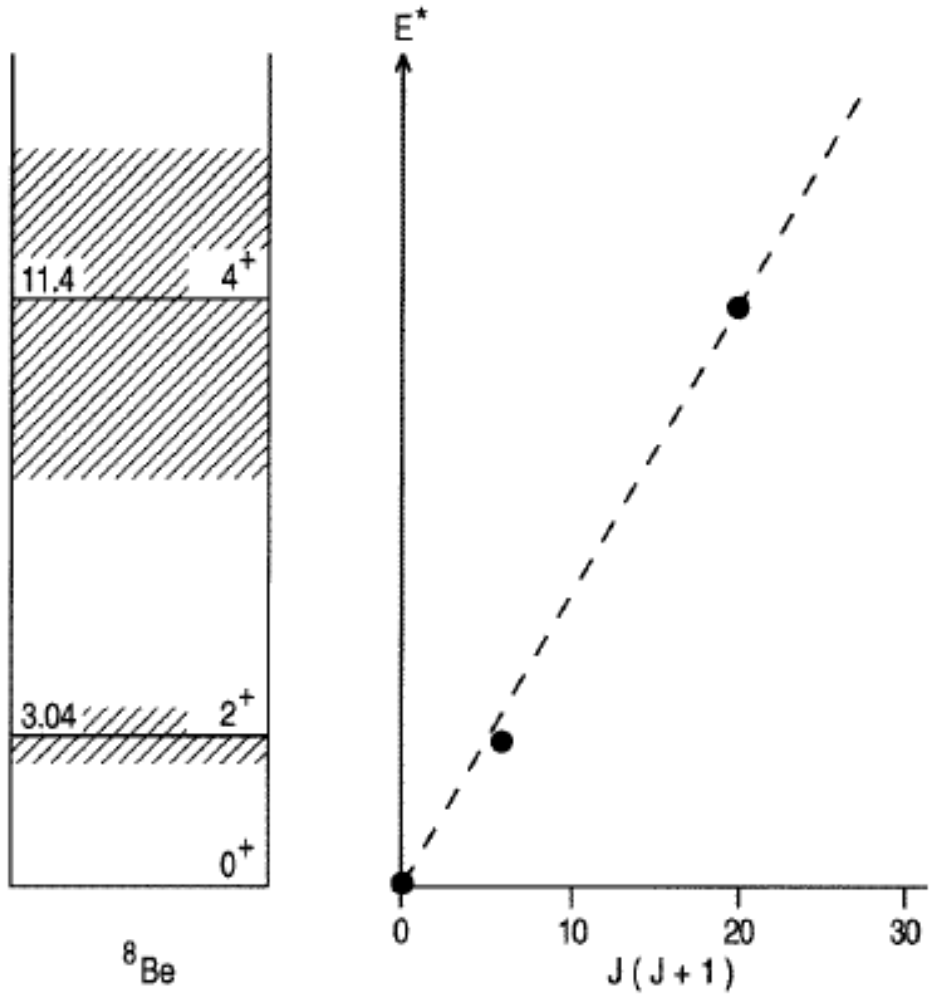


**Figure 2.** Energy level diagram for $^{8}$Be showing the expected rotational sequence for an $\alpha$–$\alpha$ molecule.

From
Betts & Wuosmaa
Rep. Prog. Phys. **60** (1997)
819–861

A rigid rotor in quantum mechanics has the following hamiltonian:

$$\hat{H} = \frac{\hat{J}^2}{2\,I}$$

With eigenenergies:

$$E_J = a\,J\,(J+1)$$

# Analogy with homonuclear diatomic molecuels

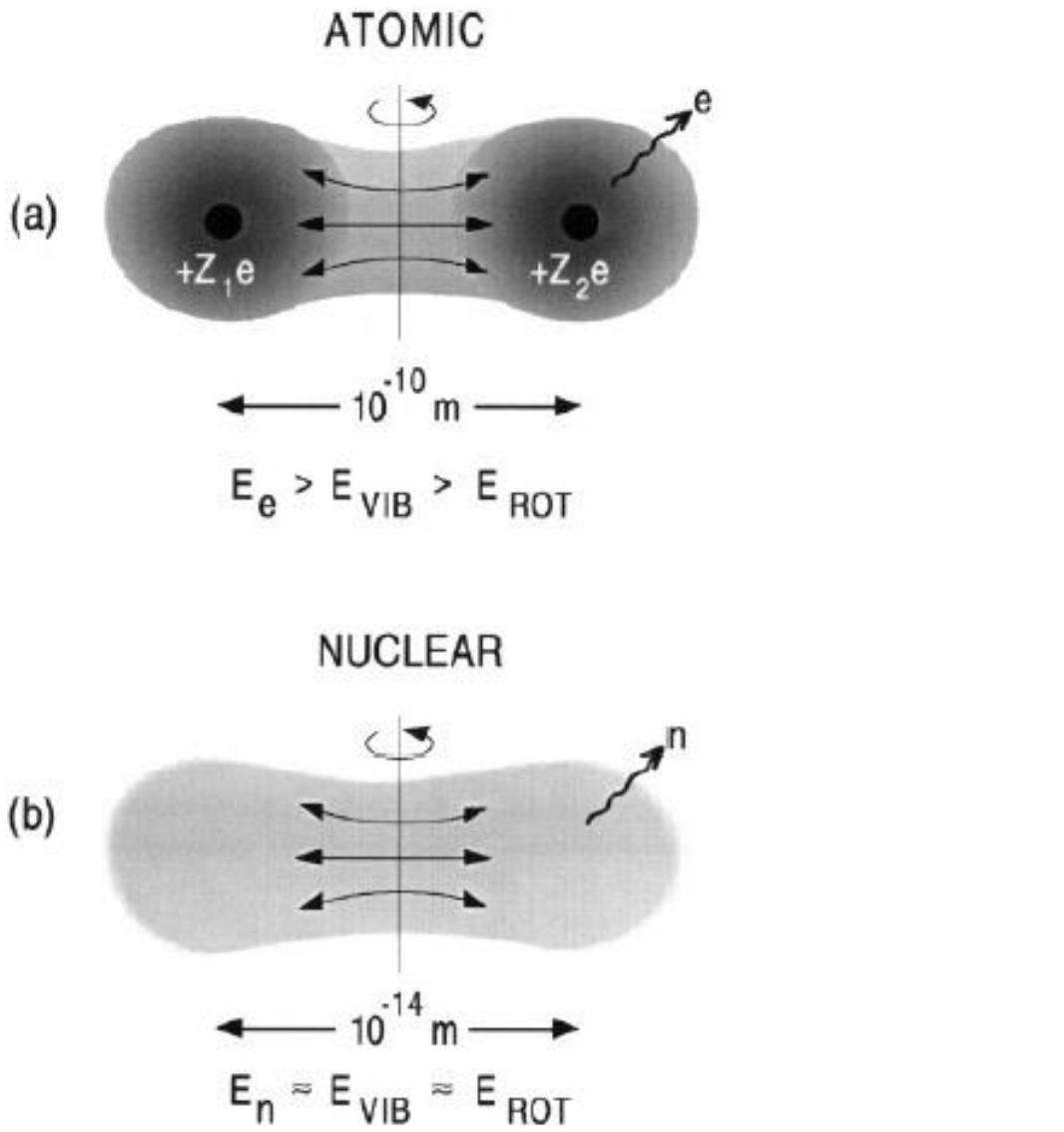


**Figure 1.** Conceptual comparison of (a) atomic and (b) nuclear molecules.

From
Betts & Wuosmaa
Rep. Prog. Phys. **60** (1997)
819–861

The energy regime is very different, but there are many similitudes.
The most important differences are the absence of a field center and the fact that vibrational and rotational degrees of freedom are not separated (i.e. at different energy scales)

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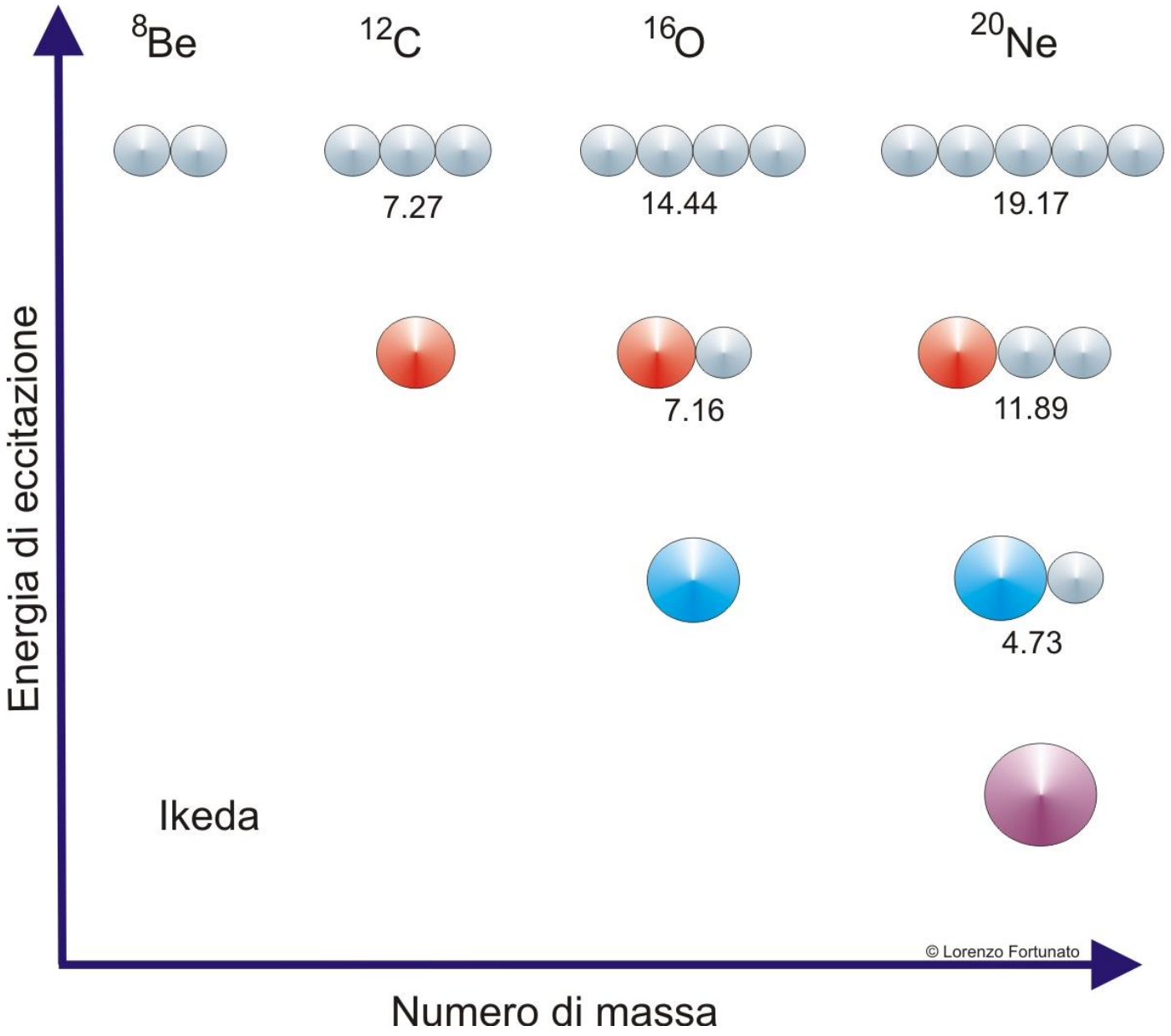
# According to Ikeda it is a threshold phenomenon

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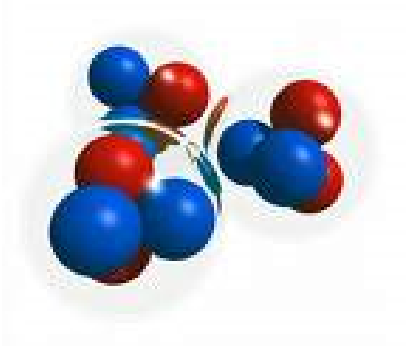
# Carbon-12


Carbon is maybe the most important element on earth together with H and O, because these are the elements of life as we know it.

Carbon-12 is produced in stars via a triple $\alpha$ reaction populating the so-called Hoyle excited state that I will describe briefly in the following.

The question of the precise structure of carbon-12, the most abundant isotope is far from being settled, although much is known about this isotope.



Do three- $\alpha$ cluster states appear only among the excited states as the Ikeda diagram shows ?

Do the g.s. also show three cluster structure?

Have symmetries anything to say about it?

What is the relationship between cluster- and shell-model states?

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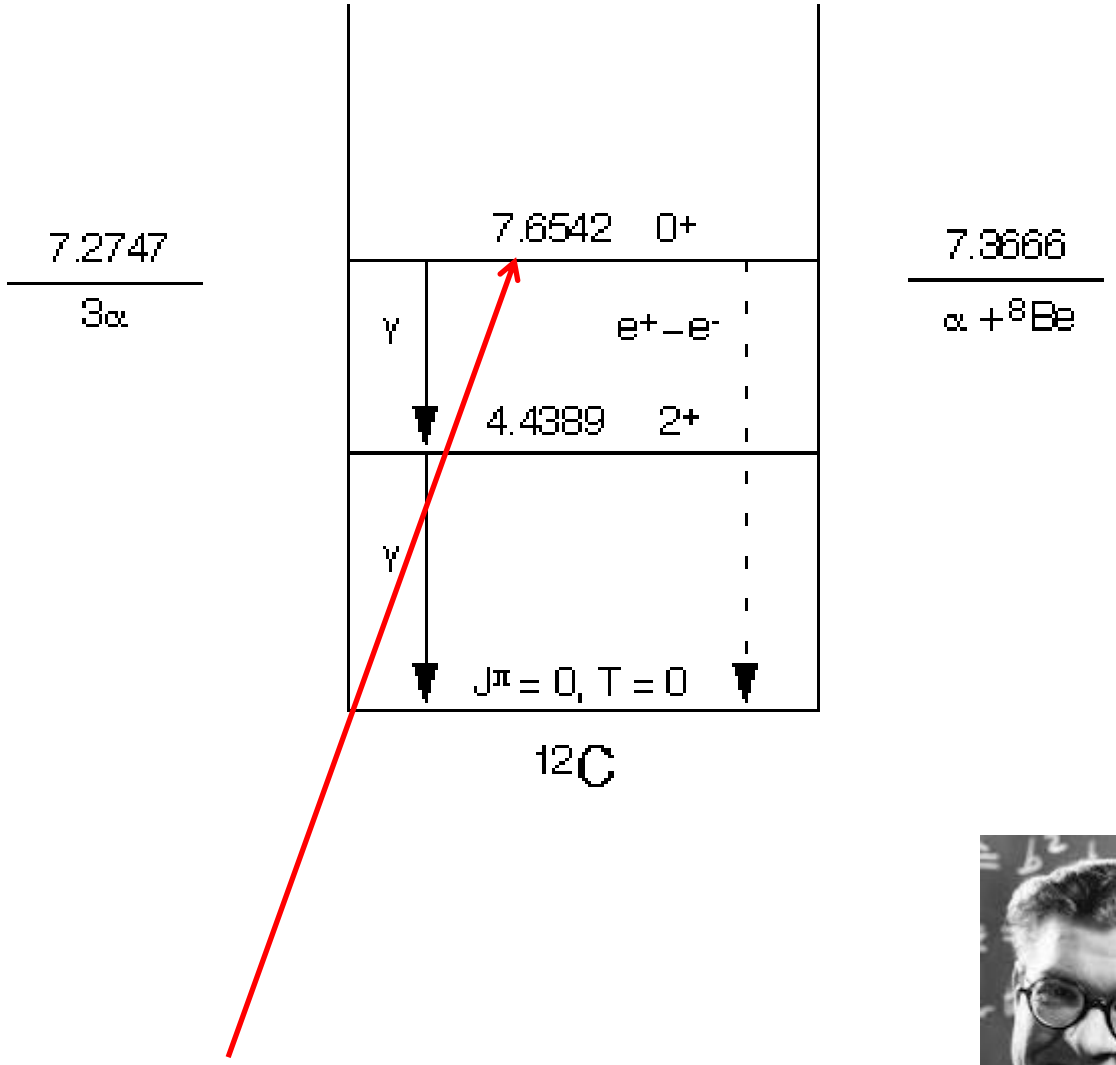
# Excitation spectrum of $^{12}C$




Here you can see the relation of the Hoyle state with the α+8Be threshold and the 3α threshold.

Hoyle state (from the name of Fred Hoyle, an astronomer and scientist from UK).

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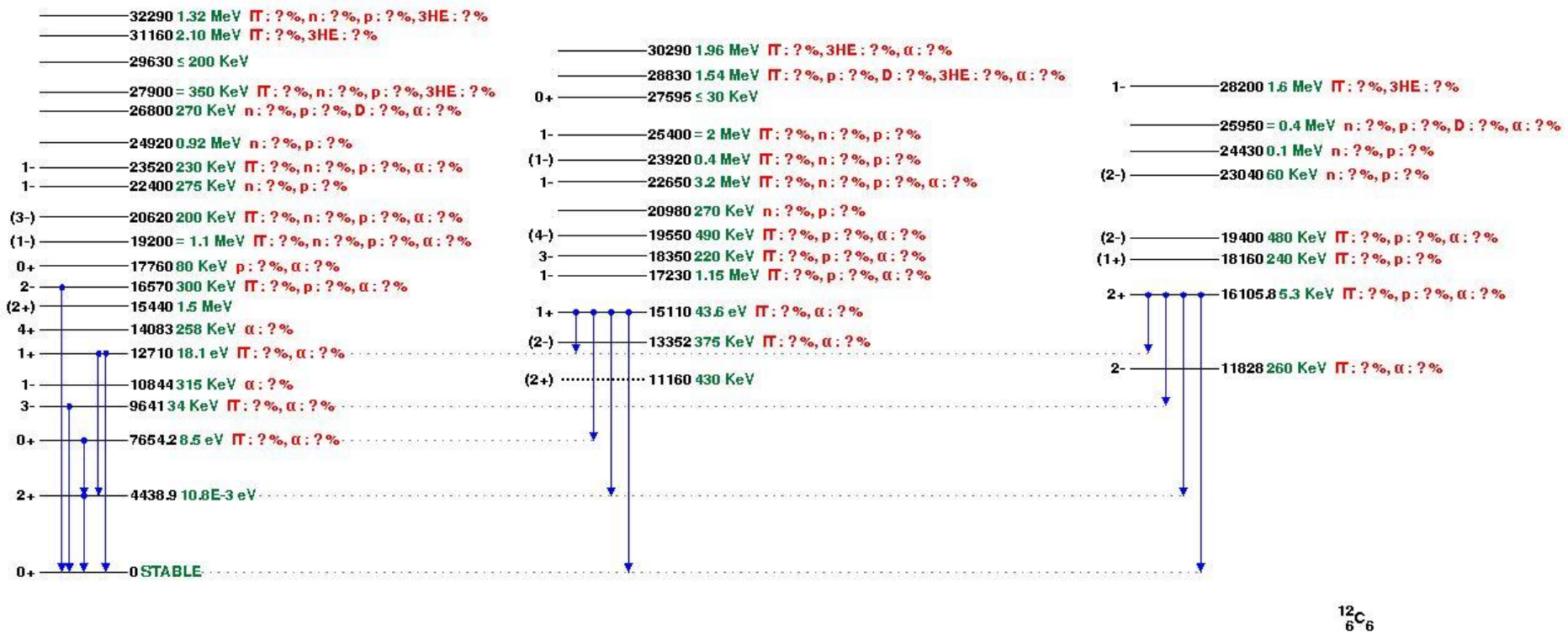
# Excitation spectrum of $^{12}C$

Spectral data from the NNDC database.

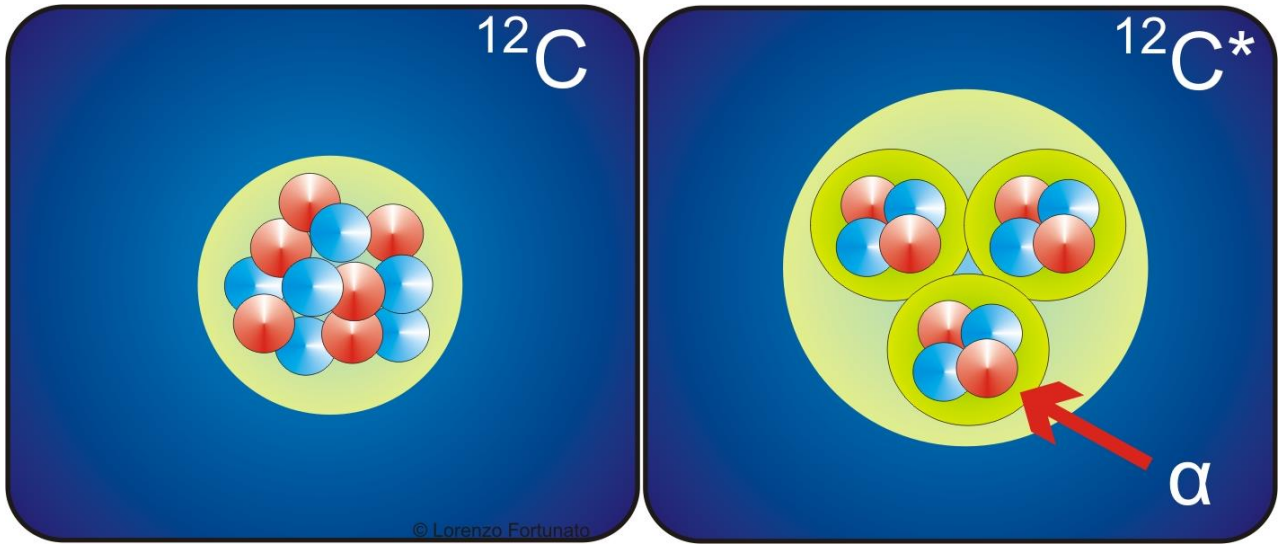



Even if it is almost unreadable, it is enough to understand that the structure of this nucleus is very complex and that there are many experimental uncertainties, expecially about transitions, decay modes, widths/lifetimes, spin parity assignemnts, etc.

There is a lot of important work ahead to be done for experimentalists that want to measure (or re-measure) these data!

L. Fortunato

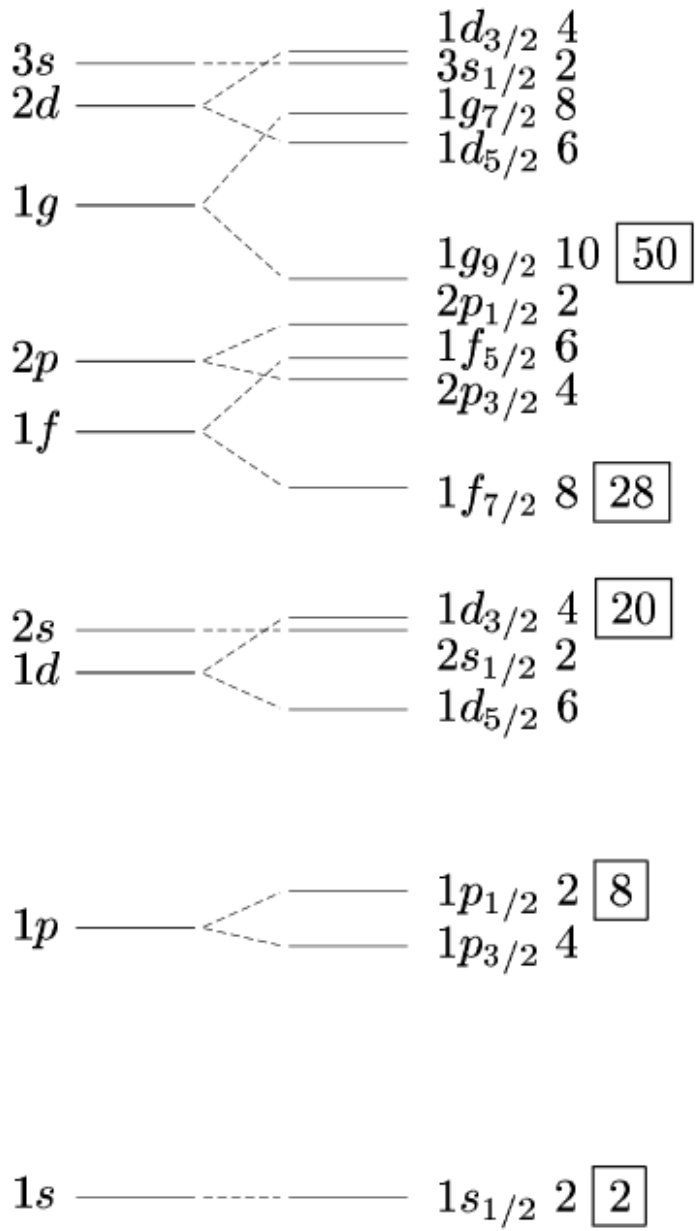
# 12C and shell model




For many years we have treated the g.s. as a shell-model state, made up of 6 protons and 6 neutron filling up the $s_{1/2}$ and $p_{3/2}$ shells. This normally works also for a set of excited states.

Conversely, even *ab initio* shell-model calculations for C-12 have difficulties to reproduce some parts of the excitation spectrum or to obtain density distributions that show any cluster structure at all (it must be forcifully imposed).

# 6 neutrons and 6 protons




This is the standard ordering of shell model states with spin-orbit.
Of course the spin-orbit splitting of p3/2 and p1/2 is not as much as the shell gap, therefore it is possible to excite particle-hole excitations, but it requires quite some energy. Collective rotations and vibrations have more probably a lower energy.

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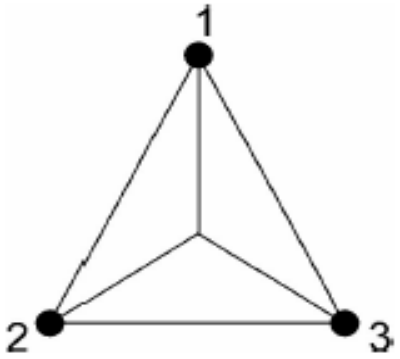
# Role of discrete symmetries in 12C

Recently R.Bijker and F.Iachello have demonstrated the occurence of alpha cluster states in 12C and 16O based on the discrete point-group symmetries and the Algebraic Cluster Model.

PRC 61 (2000) 067305
REV MEX FIS **49 SUPLEMENTO 4**, 7–14 (2003)
R. Bijker, J. Phys. Conf. Ser. **380**, 012003 (2012).
PRL **112,** 152501

対称性
Taishō-sei




Geometry of a three-body system.

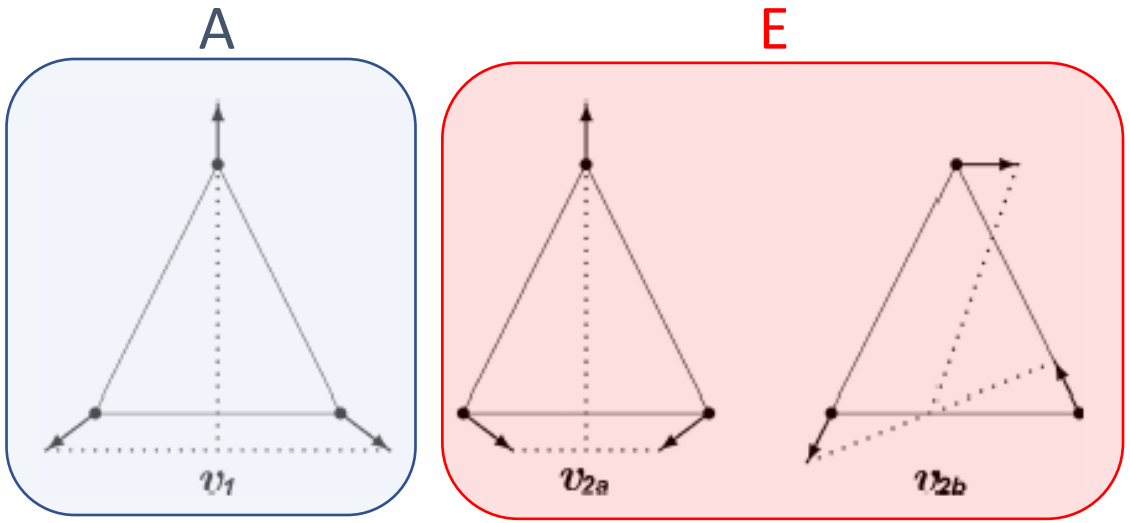



FIGURE 4. Vibrations of an oblate top.

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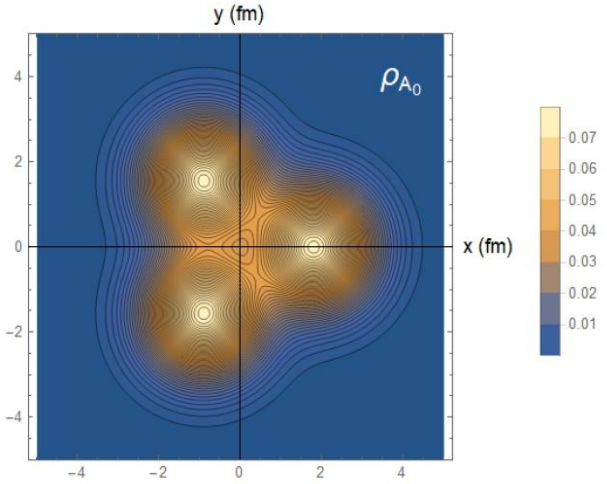
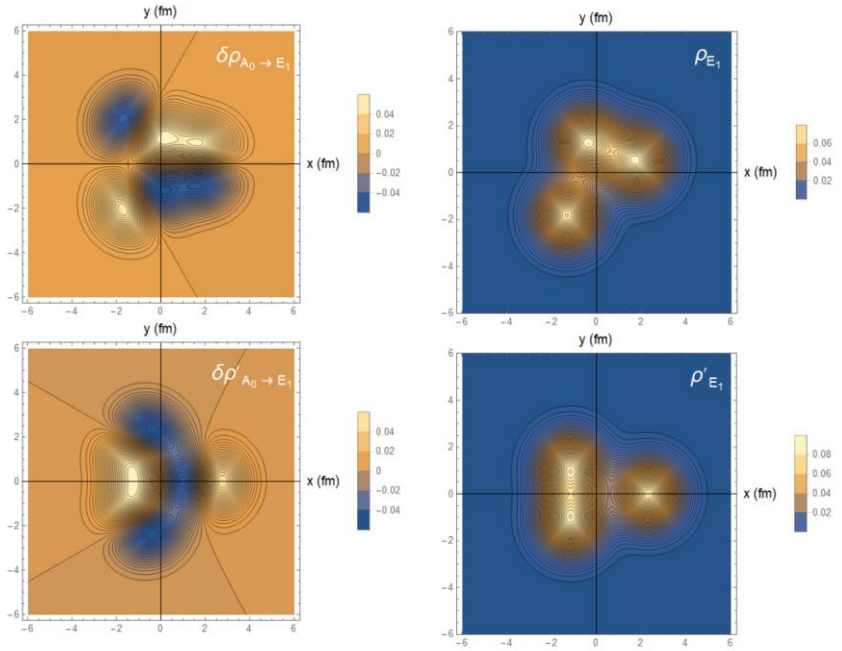
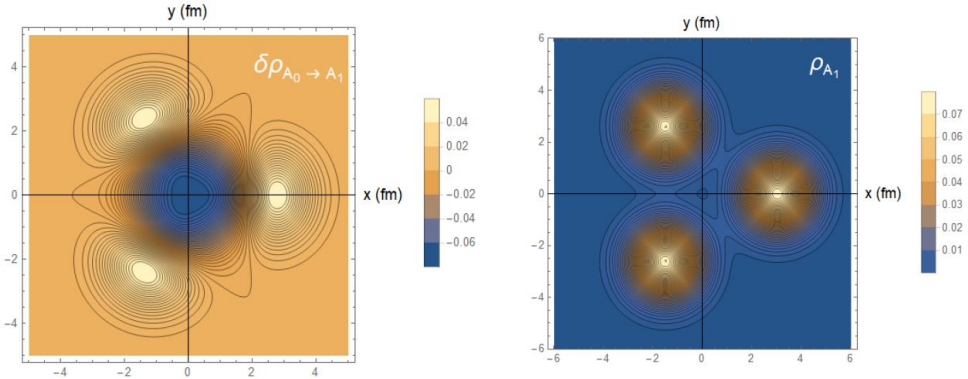
# Transition densities in the cluster model of 12C




Ground state density




E-type vibration




A-type vibration
(Hoyle)

From
Casal, Fortunato,
Vitturi, Lanza,
PRC

L. Fortunato

# Quantum numbers → vibrations and rotations

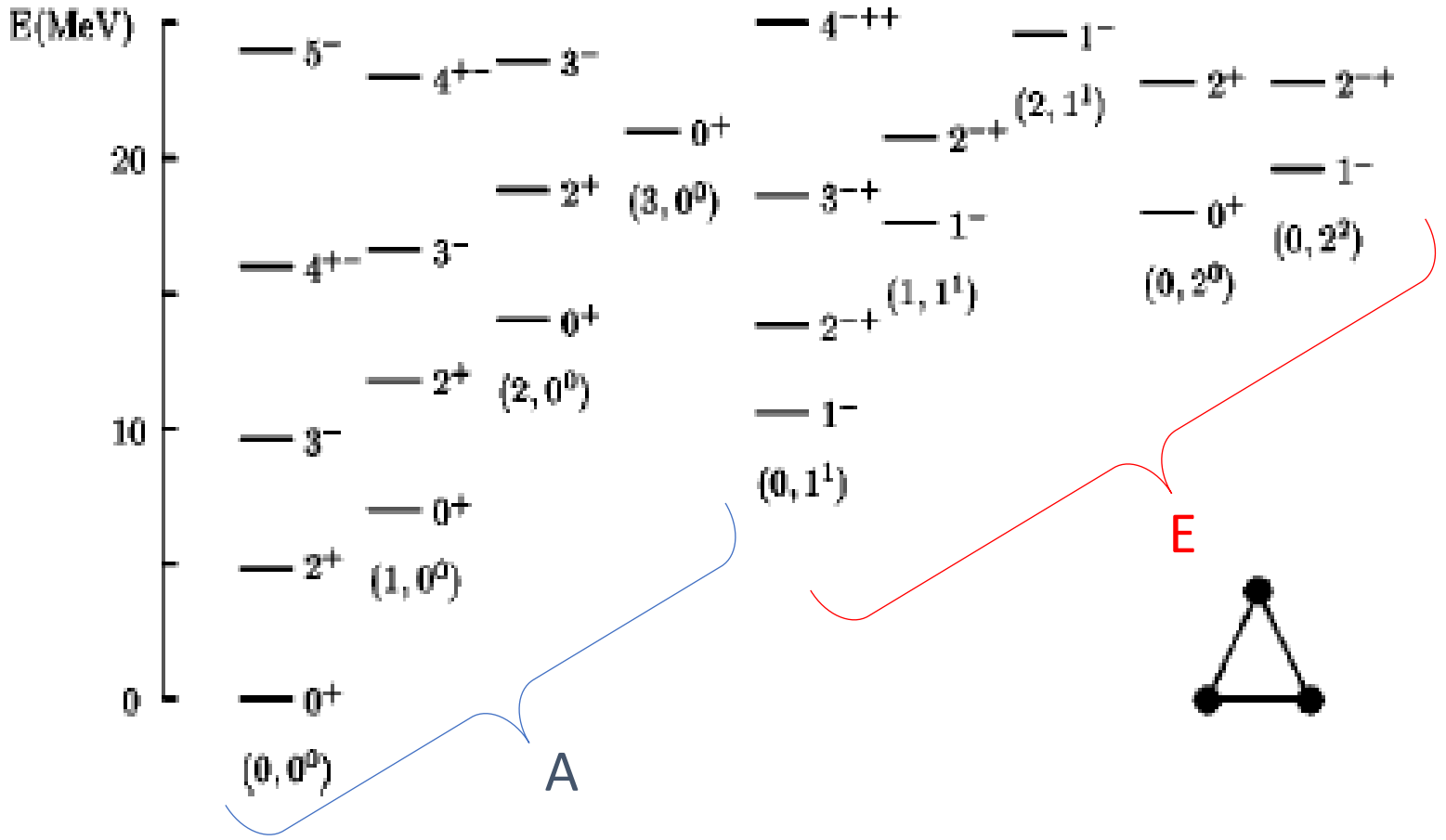



FIG. 1. Spectrum of an equilateral triangle configuration

Notice the 'apparently strange' quantum numbers. They have a perfectly clear interpretation in the theory of point-groups !

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# Algebraic cluster model for 3 alphas

Bijker and Iachello have clearly demonstrated the succesfull application of the ACM, or algebraic cluster model, to the vibrational-rotational spectrum of alpha-alpha conjugate nuclei like 12C and 16O.

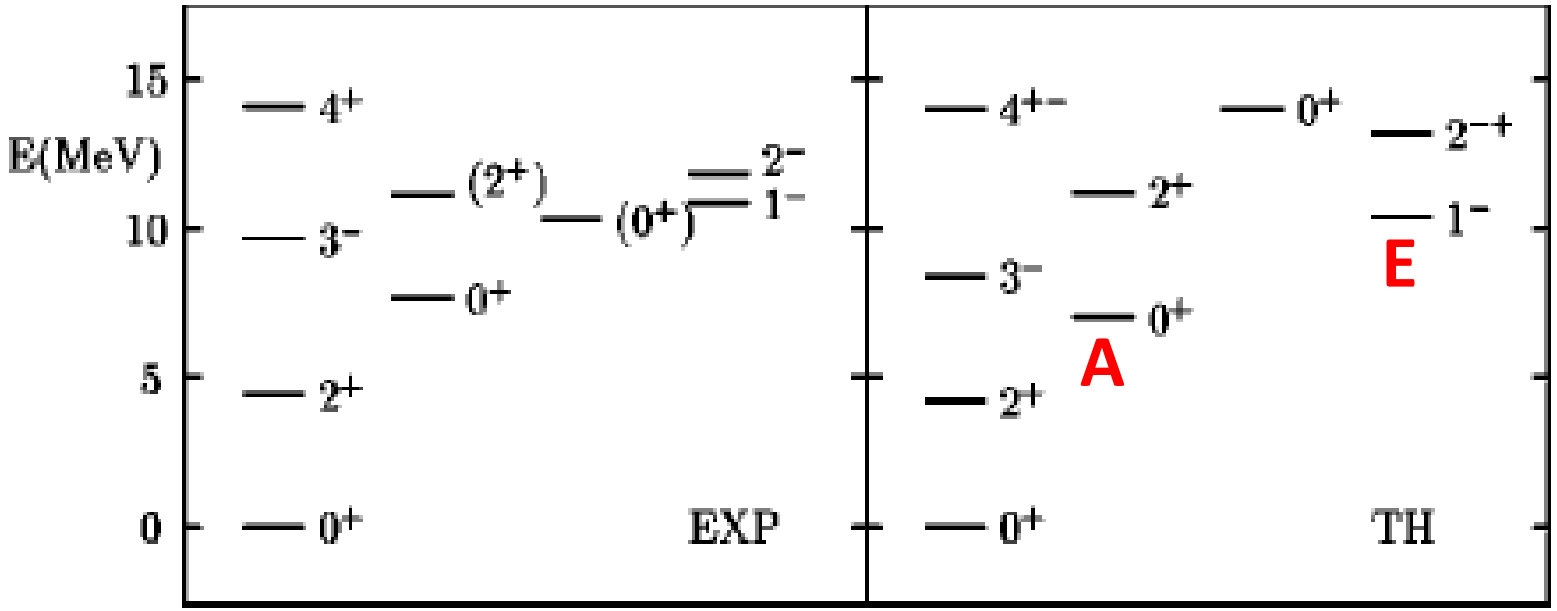



FIG. 2. Comparison between the low-lying experimental spectrum of $^{12}C$ [12] and that calculated using Eq. (6) with $A=7.0$, $B=9.0$, $C=0.7$, and $D=0.0$ MeV. States with uncertain spin-parity assignment are in parentheses.

# Discrete symmetries and polarized gamma-rays in 12C

PHYSICAL REVIEW C **99**, 031302(R) (2019)

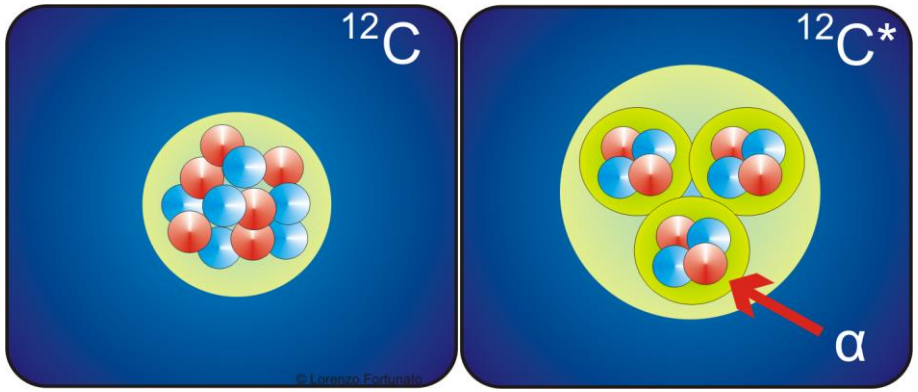
Rapid Communications | Editors' Suggestion

**Establishing the geometry of α clusters in $^{12}$C through patterns of polarized γ rays**

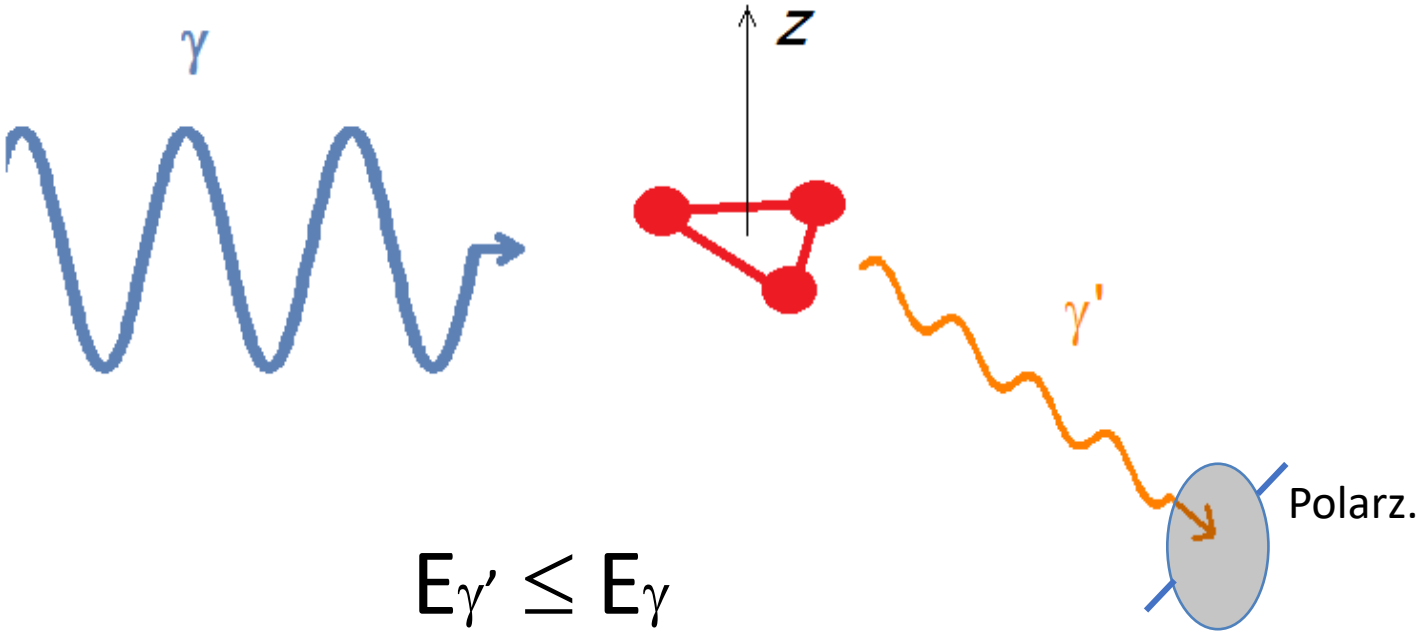
Lorenzo Fortunato

*Dipartimento di Fisica e Astronomia "G.Galilei"-Università di Padova, via Marzolo 8, I-35131 Padova, Italy and INFN-Sez.di Padova, via Marzolo 8, I-35131 Padova, Italy*

(Received 13 December 2018; published 11 March 2019)




One can shoot linearly polarized gamma rays (Electric field oscillating in a given direction constant in time) of appropriate energy (tuned tc match the resonances of interest) and observe the outcoming gammas of the same or different energies with a polarizer/analyzer. If the nucleus has a definite geometrical symmetry (i.e. if there is an underlying discrete group structure), very strict selection rules apply. Experimentally the polarization can be measured with another inverse Compton scattering.




$$E_{\gamma'} \leq E_{\gamma}$$

# Polarized gamma-ray facilites around the globe:

- Mainz Microtron MAMI (Continuous Wave, beam polarization 80%, En. resol. 0.1 MeV, but energy too high 50-800 MeV)
- Triangle University Higs facility (FEL type, quasi CW operation, 2-60 MeV, flux 10^8-10^9 phot./s)
- ELI-NP in Romania (0-20 MeV, high flux, high resolution, 100% polarization)
- LEPS – Japan (very high energy)
- NewSubaru
- …

Extreme Light Infrastructure
Nuclear Physics (ELI-NP)

| Gamma beam parameter | Value |
|---|---|
| Energy [MeV] | 0.2 – 19.5 |
| Spectral density [ph/s/eV] | $0.8 - 4 \cdot 10^4$ |
| Bandwidth rms [%] | $\leq 0.5$ |
| #Photons/shot within FWHM bdw. | $\leq 2.6 \cdot 10^5$ |
| #Photons/s within FWHM bdw. | $\leq 8.3 \cdot 10^8$ |
| Source rms size [μm] | 10 – 30 |
| Source rms divergence [μrad] | 25 – 200 |
| Peak brilliance [$N_{ph}$/s·mm$^2$·mrad$^2$·0.1%bdw] | $10^{20} - 10^{23}$ |
| Pulse length rms [ps] | 0.7 – 1.5 |
| Linear polarization [%] | > 99 |
| Macro repetition rate [Hz] | 100 |
| Number of pulses/macropulse | 32 |
| Pulse–to–pulse separation [ps] | 16 |

With the advent of the new facility in Romania, beams of high brilliance, focused, polarized gamma rays produced with Inverse Compton Scattering will become available with energies ranging from 0.2-20 MeV

# Depolarization ratio

One can measure the so-called depolarization ratio between intensities, by turning the analyzer/polarizer of 90 degrees, i.e.:

$$\rho = \frac{I_{\perp}}{I_{\parallel}}$$

as a tool to determine which modes are totally symmetric modes. In fact from the theory of Raman scattering

$0 \leq \rho \leq \frac{3}{4}$ for polarized bands (symmetric modes)

$\rho = \frac{3}{4}$ for depolarized bands (non-symmetric modes)

even with a randomly oriented sample.

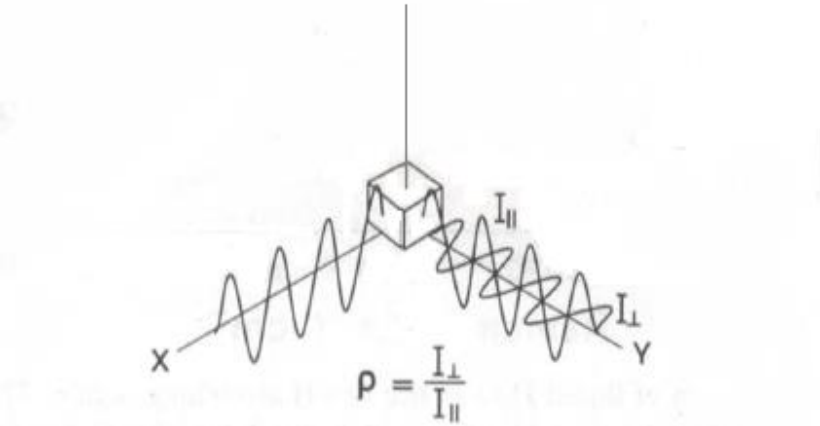



**Figure 8.6.** Parallel and perpendicular Raman scattering.

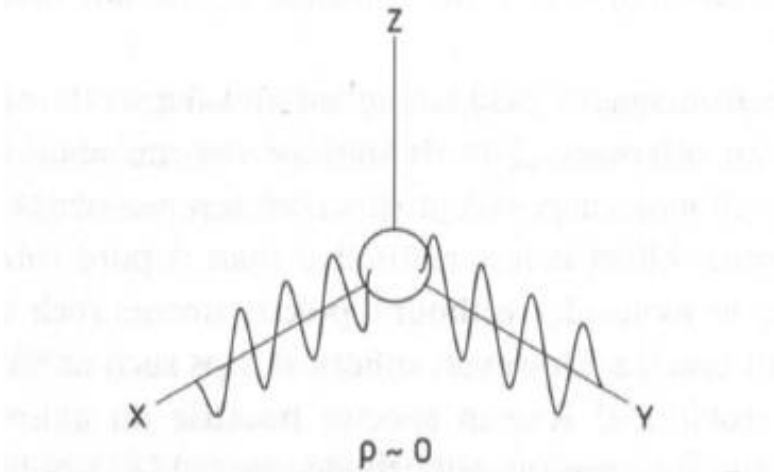



**Figure 8.8.** Polarized light scattering by a sphere.

Figures from book by P.Bernath

# Depolarization ratio : a chemical example $CCl_4$

This kind of measuments of $\rho = \frac{I_\perp}{I_\parallel}$ are absolutely standard in optical spectroscopy (where polarizers and analyzers are easy to do and handle).

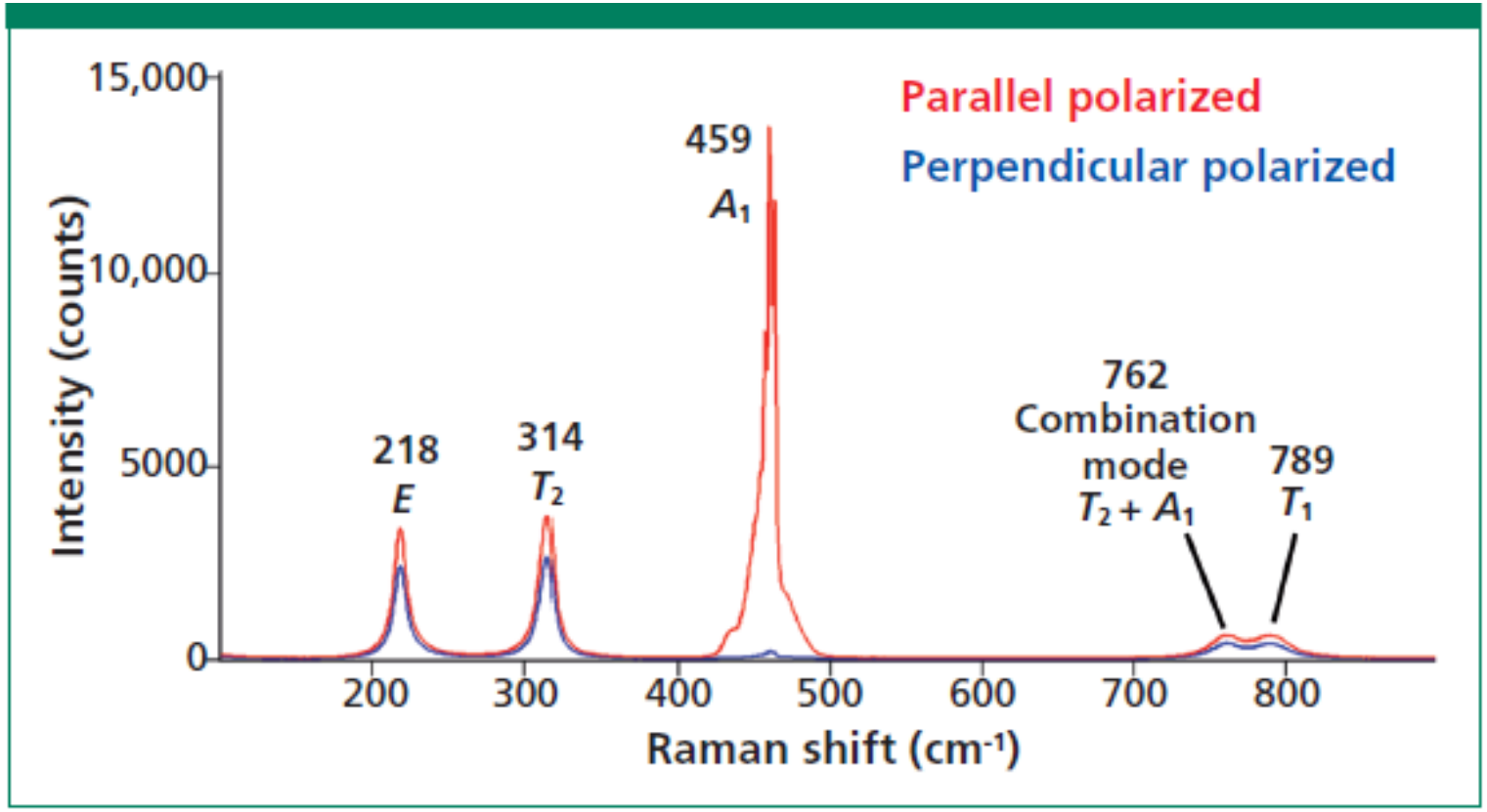
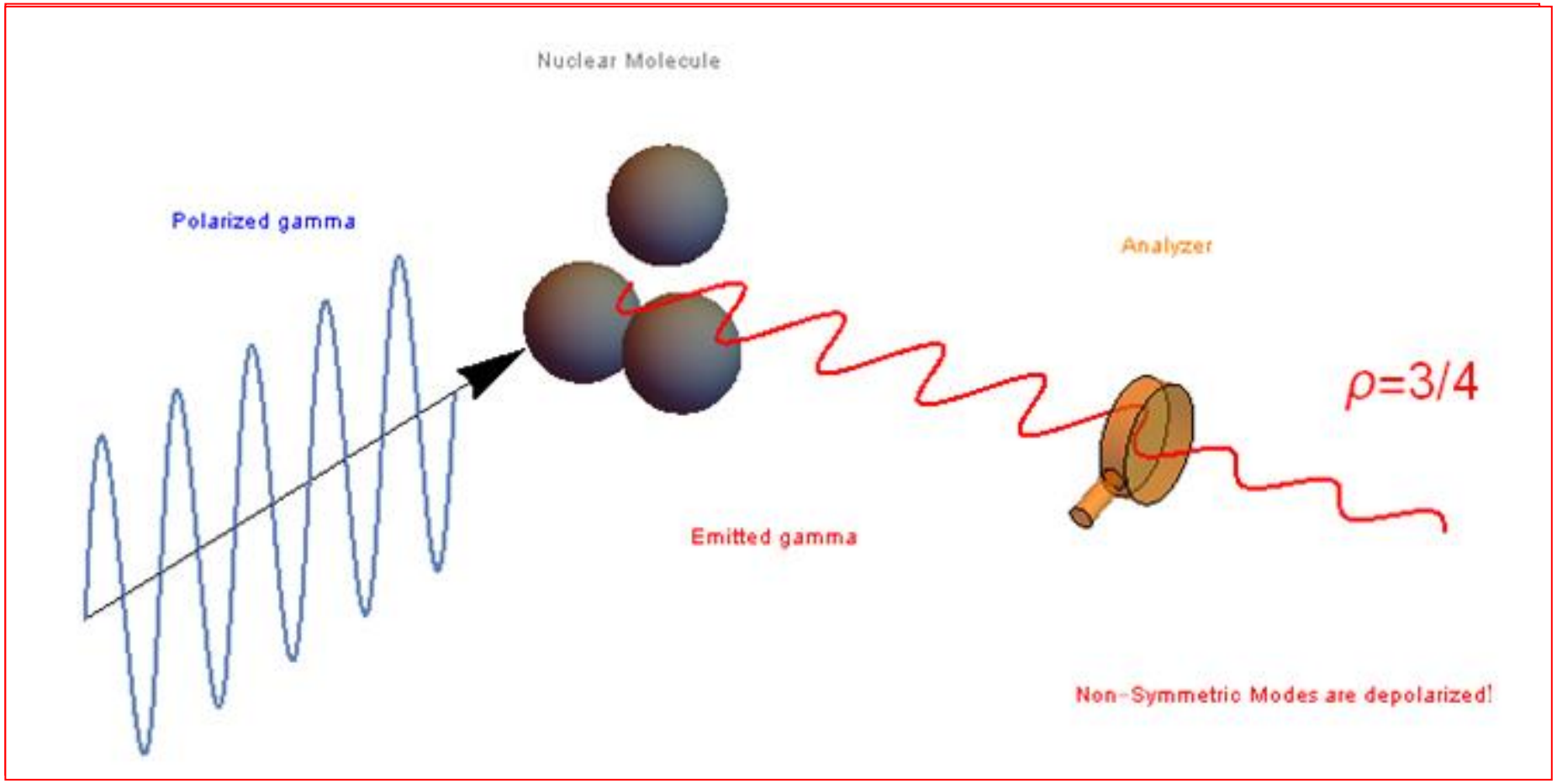



**Figure 2:** Polarized Raman spectra of $CCl_4$.

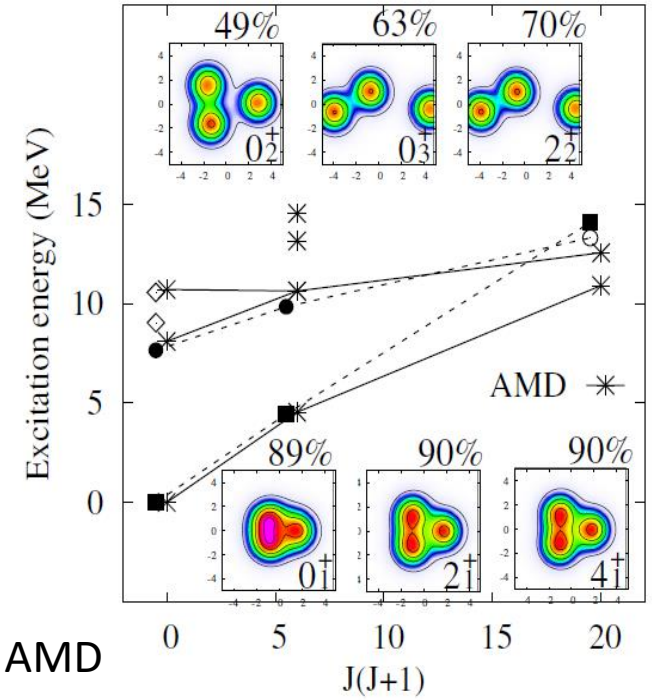
Figure from D.Tuschel – Spectroscopy (2014)

# Depolarization ratio

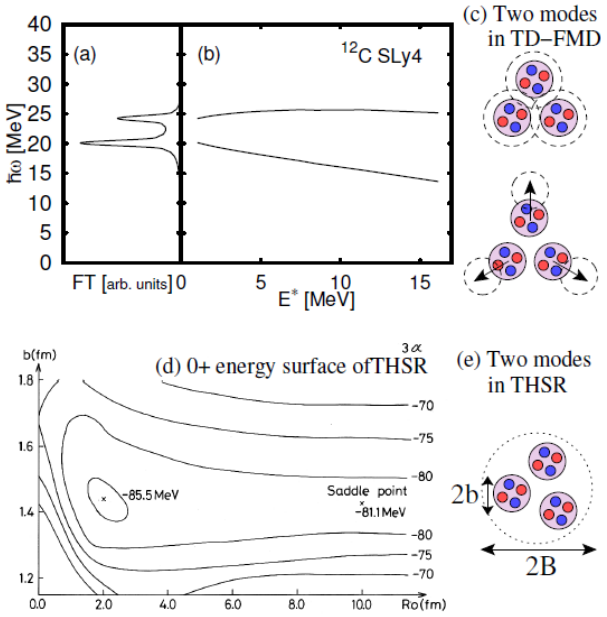


# Panoply of different models … too many… just to name a few !!!

(Too) many models have been proposed for 12C where the triangle is not equilateral, isosceles, scalene or even a linear chain (Morinaga). Therefore. I have set forth to determine all possible outcomes and the patterns that can be predicted are intended as a guidance as to which configuration is right and **the crucial method is clearly through measurements of the depolarization ratio in Raman-like experiments of nuclear fluorescence** that will be feasible at ELI-NP or in other labs where gamma-rays are available.




AMD




FMD

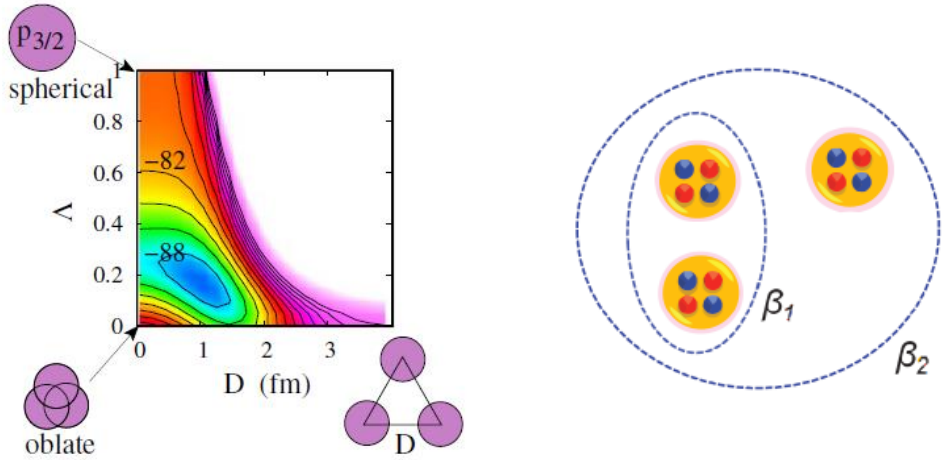



FIG. 10 (Color online) $0^+$-projected energy surface on the $\Lambda$-$D$ plane for $^{12}$C calculated by the AQCM. The interaction and width parameters are same as those in Ref. (Suhara *et al.*, 2013).

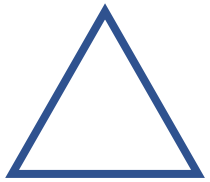
container model

antisymmetrized quasi-cluster model

Figures from: Freer et al. (2018) Microscopic clustering in Light Nuclei

# Algebraic cluster model for 3 alphas

Bijker and Iachello (Ann.Phys. 298, 2002) have clearly demonstrated the succesfull application of the ACM, or algebraic cluster model, to the vibrational-rotational spectrum of alpha-conjugate nuclei like 12C and 16O.



Note that rotational bands DO NOT conform tothe usual quadrupole rotational bands we are used, they have a different symmetry!
Rather, these are the ways in which you can spin a triangle.

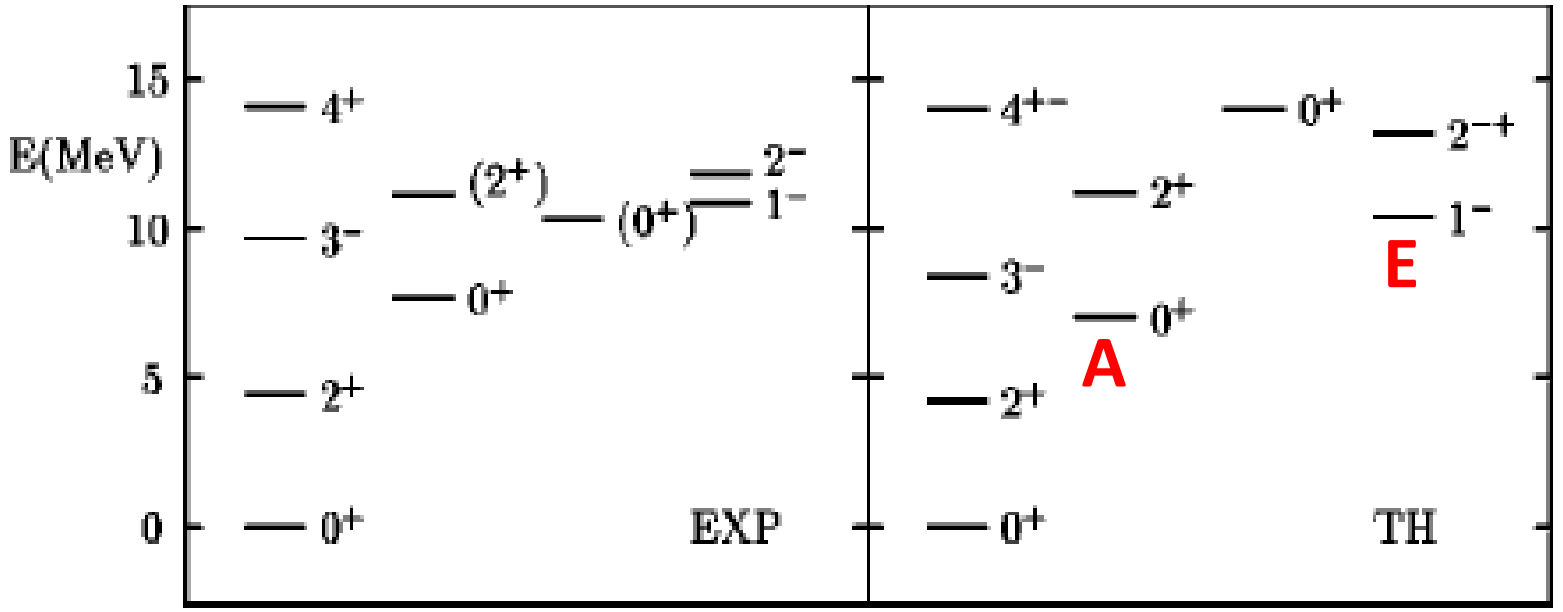
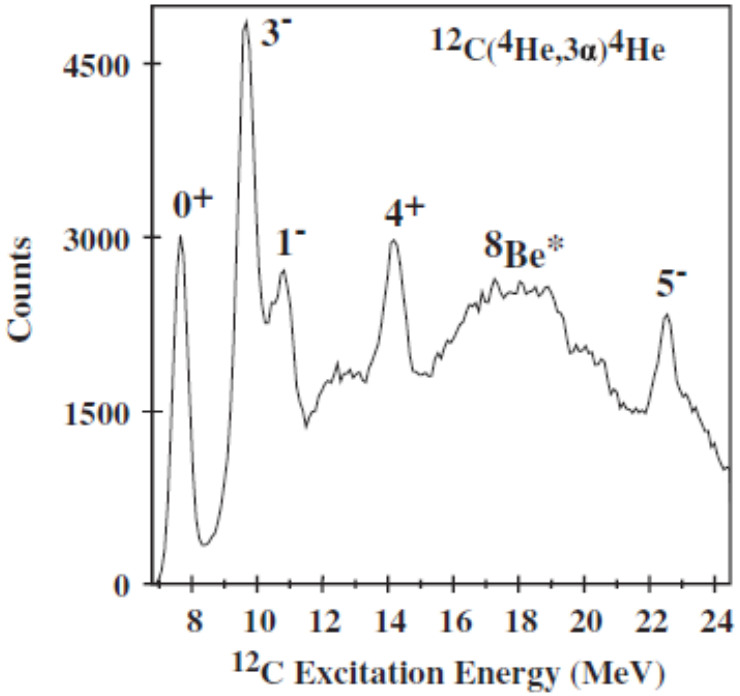


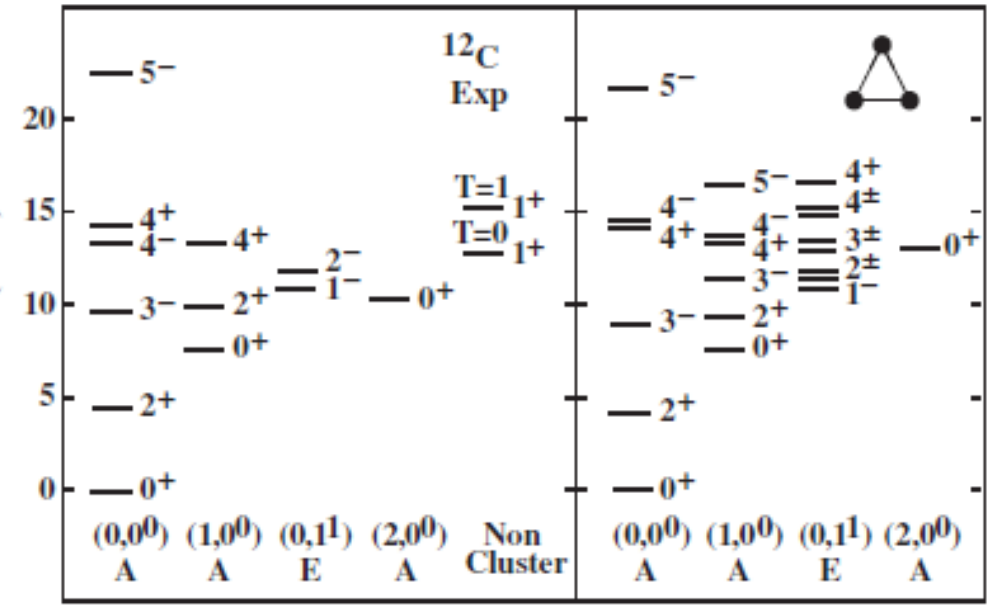

FIG. 2. Comparison between the low-lying experimental spectrum of $^{12}C$ [12] and that calculated using Eq. (6) with $A=7.0$, $B=9.0$, $C=0.7$, and $D=0.0$ MeV. States with uncertain spin-parity assignment are in parentheses.

PRL 113, 012502 (2014) PHYSICAL REVIEW LETTERS week ending 4 JULY 2014

# Evidence for Triangular $\mathcal{D}_{3h}$ Symmetry in $^{12}C$

D. J. Marín-Lámbarri,[1] R. Bijker,[2] M. Freer,[1] M. Gai,[3,4] Tz. Kokalova,[1] D. J. Parker,[1] and C. Wheldon[1]

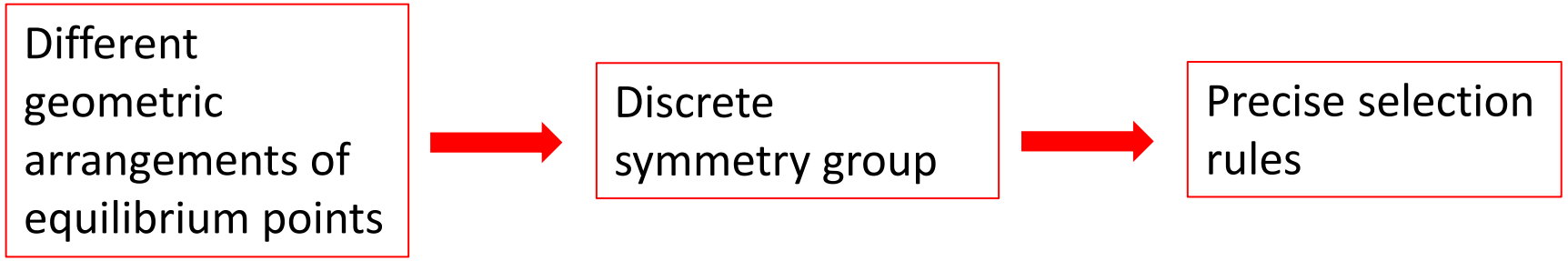



This lovely paper confirms the assignation of the 5- state at 22.4(2) MeV to the g.s. band of an **equilater triangular structure.**

Note the uncommon spin-parity of bands (the doublet 4+/ 4- has a natural explanation in terms of D3h symmetry!).

# Working plan to implement this idea




**Work plan:**

- Decide arrangement of N particles
- This means 3N-6 d.o.f (or 3N-5 d.o.f. for linear arrangement)
- Identify the underlying discrete group structure
- Find the character under transformations of the group $\Gamma_{3N}$
- Subtract translations and rotations to single out character of vibrational modes $\Gamma_{vib}$
- Identify patterns of totally symmetric modes
- Check models against measures of intensities $\rightarrow$ Eureka !!

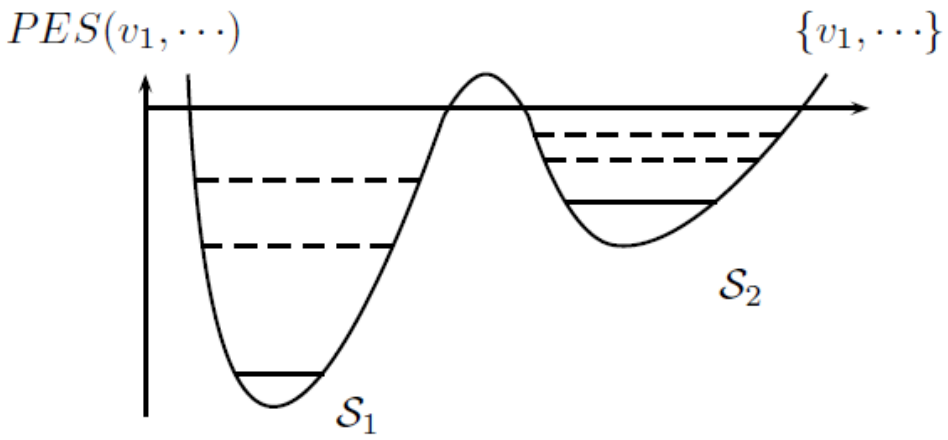
# Tables in PRC 99 (2019) paper: 3 equal clusters

| name | shape | group | $\Gamma_{vib}$ | Patterns |
|---|---|---|---|---|
| linear = |  | $\mathcal{D}_{\infty h}$ | $A_{1g} + A_{1u} + E_{1u}$ |  |
| linear $\neq$ |  | $\mathcal{C}_{\infty \nu}$ | $2A_1 + E_1$ |  |
| equilateral |  | $\mathcal{D}_{3h}$ | $A_1' + E'$ |  |
| isosceles |  | $\mathcal{C}_{2\nu}$ | $2A_1 + B_1$ |  |
| scalene |  | $\mathcal{C}_s$ | $3A'$ |  |

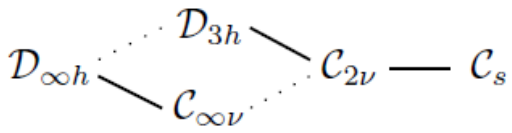
The number of totally symmetric peaks over total is different in each case, therefore one can disentagle the various possibilities

# Tables in PRC 99 (2019) paper

There might be more than just one configuration! The picture complicates a little, but not too much ! One can invoke the concept of descent in symmetry ans still apply some of the rules.




Group chains




| $\mathcal{D}_{\infty h}$ | $A_{1g}+$ | $A_{1u}+$ | $E_{1u}$ |
|---|---|---|---|
| $\mathcal{C}_{\infty\nu}$ | $A_1+$ | $A_1+$ | $E_1$ |
| $\mathcal{C}_{2\nu}$ | $A_1+$ | $A_1+$ | $B_1 + B_2$ |
| $\mathcal{C}_s$ | $A'+$ | $A'+$ | $A'$ |

| $\mathcal{D}_{\infty h}$ | $A_{1g}+$ | $A_{1u}+$ | $E_{1u}$ |
|---|---|---|---|
| $\mathcal{D}_{3h}$ | $A_1'+$ | $A_2''+$ | $E'$ |
| $\mathcal{C}_{2\nu}$ | $A_1+$ | | $A_1 + B_1$ |
| $\mathcal{C}_s$ | $A'+$ | | $A' + A'$ |

FIG. 4. Descent in symmetry restricted to representations of the groups that are relevant to all possible configurations of three identical particles.

# Tetrahedral shape in 16 Oxygen

Bijker, Iachello
PRL 112, 152501 (2014)

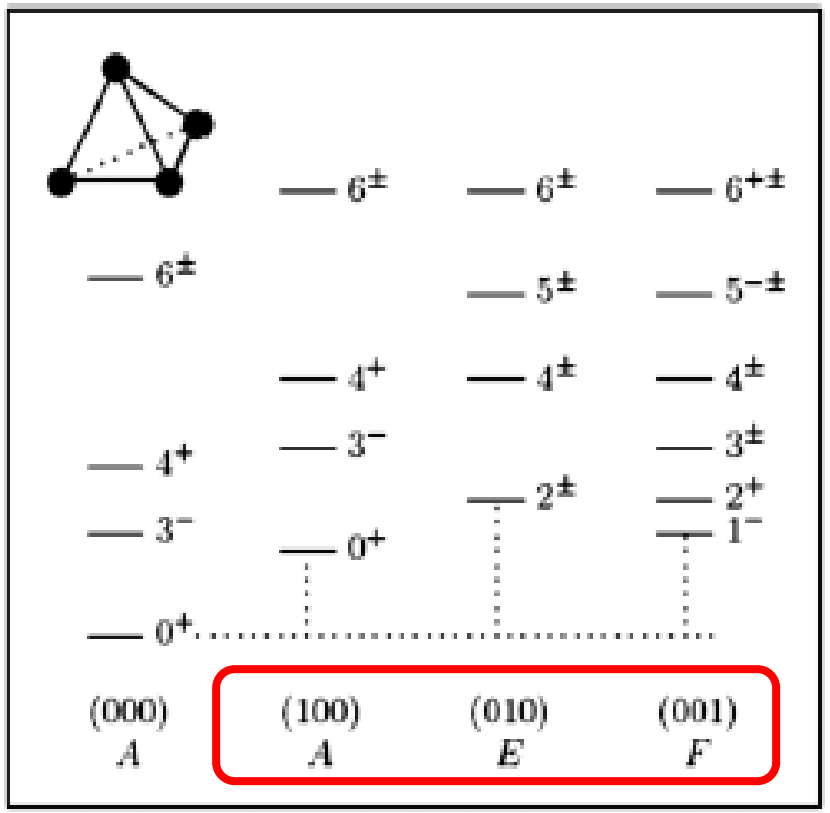



FIG. 1. Schematic spectrum of a spherical top with tetrahedral symmetry and $\omega_1 = \omega_2 = \omega_3$. The rotational bands are labeled by $(v_1, v_2, v_3)$ (bottom). All states are symmetric under $S_4$.

# What are the implication of the triangular structure in reactions?

PHYSICAL REVIEW C **101**, 014315 (2020)

## Transition densities and form factors in the triangular α-cluster model of $^{12}C$ with application to $^{12}C + \alpha$ scattering

A. Vitturi,[1,2] J. Casal,[1,2] L. Fortunato,[1,2] and E. G. Lanza[3,4]
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[2]*I.N.F.N., Sez. di Padova, I-35131 Padova, Italy*
[3]*I.N.F.N., Sez. di Catania, I-95123 Catania, Italy*
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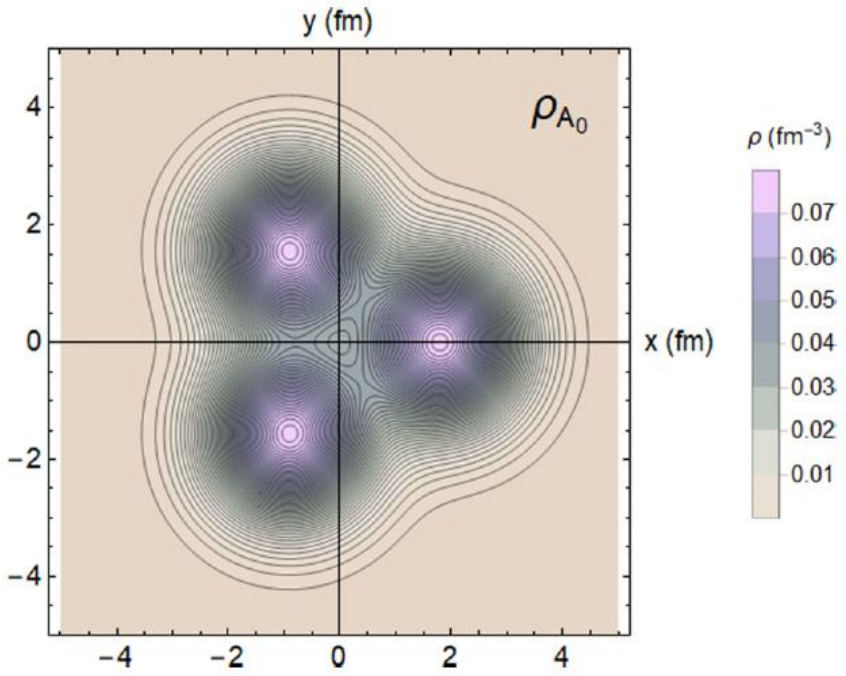
(Received 1 October 2019; published 21 January 2020)

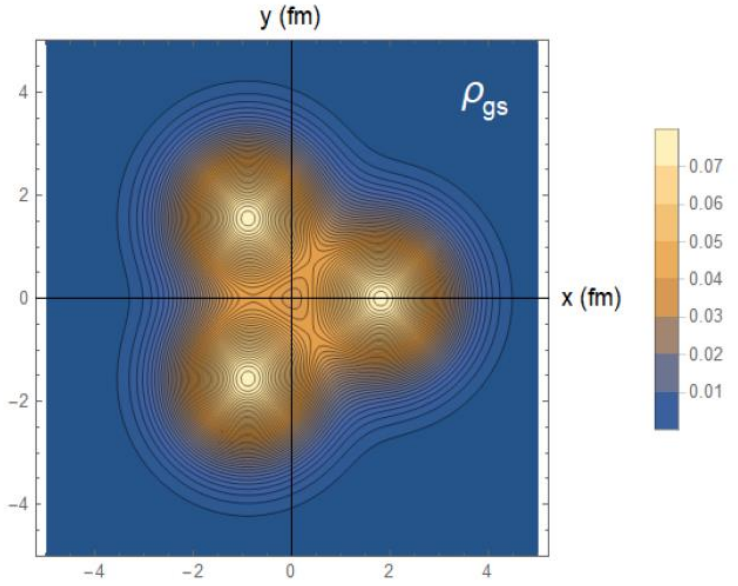
Densities and transition densities are computed in an equilateral triangular α-cluster model for $^{12}C$, in which each α particle is taken as a Gaussian density distribution. The ground state, the symmetric vibration (Hoyle state), and the asymmetric bend vibration are analyzed in a molecular approach and dissected into their components in a series of harmonic functions, revealing their intrinsic structures. The transition densities in the laboratory frame are then used to construct form factors and to compute distorted-wave Born approximation inelastic cross sections for the $^{12}C(\alpha, \alpha')$ reaction. The comparison with experimental data indicates that the simple geometrical model with rotations and vibrations gives a reliable description of reactions where α-cluster degrees of freedom are involved.



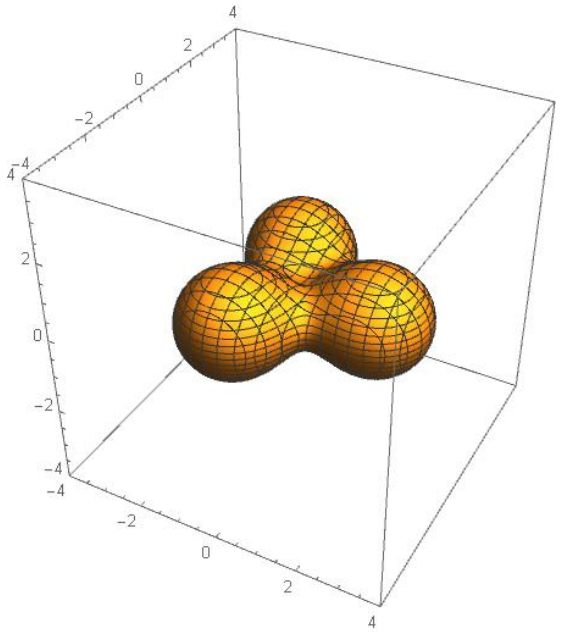

M. Kamimura, Nucl. Phys. A **351**, 456 (1981).
D. C. Cuong, D. T. Khoa, and Y. Kanada En'yo, Phys. Rev. C **88**, 064317 (2013).
M. Ito, Phys. Rev. C **97**, 044608 (2018).
Y. Kanada-En'yo and K. Ogata, Phys. Rev. C **99**, 064601 (2019).

# Transition densities in 12C

In collaboration with A. Vitturi, E. Lanza and J. Casal

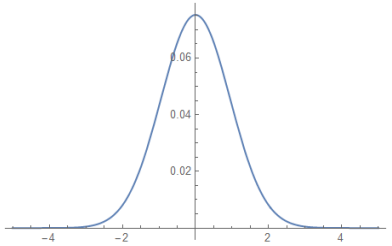



Ground state



This model assume guassian densities for each alpha particle

$$\rho_\alpha(\vec{r}) = \left(\frac{\alpha}{\pi}\right)^{3/2} e^{-\alpha r^2}$$



and a total density that is the sum of three displaced alpha's

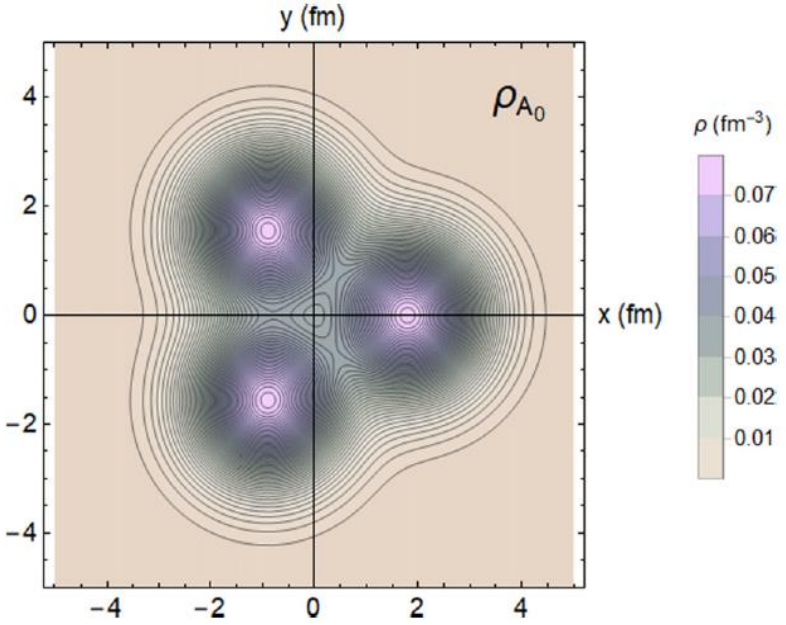
$$\rho_{\rm gs}(\vec{r}, \{\vec{r}_k\}) = \sum_{k=1}^{3} \rho_\alpha(\vec{r} - \vec{r}_k),$$

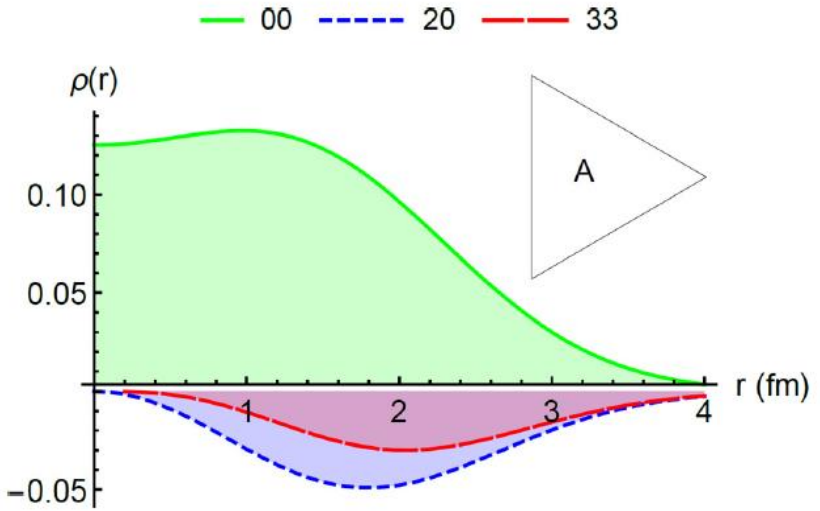
which is then expanded in spherical harmonics

$$\rho_{\rm gs}(\vec{r}) = \sum_{\lambda\mu} \rho_{\rm gs}^{\lambda,\mu}(r) Y_{\lambda,\mu}(\theta, \varphi),$$

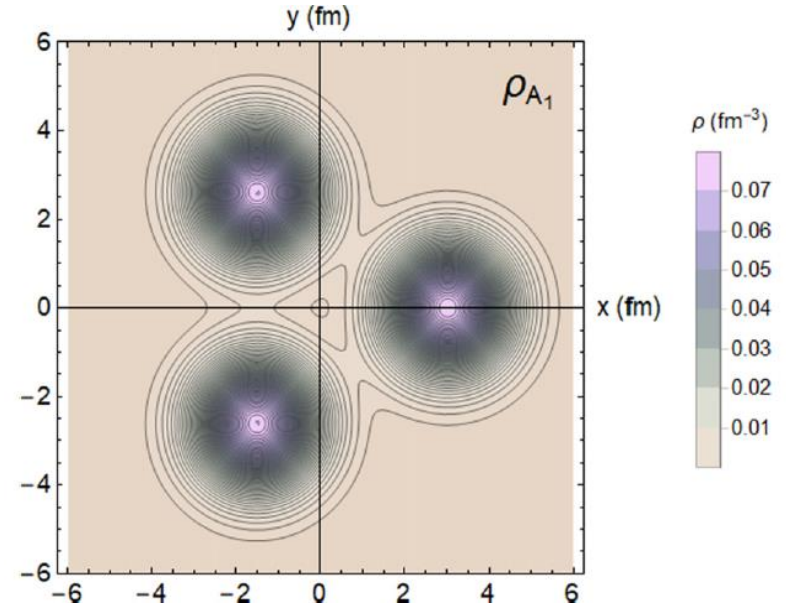
# Ground and Hoyle bands

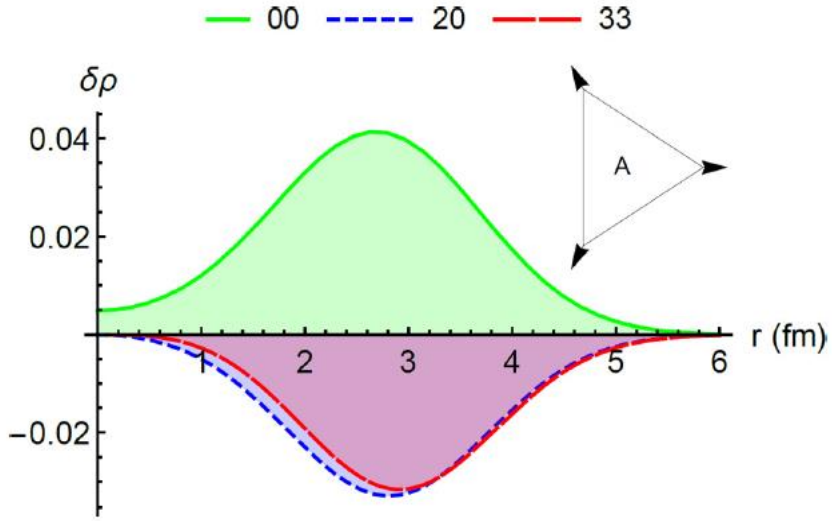
In collaboration with A. Vitturi, E. Lanza and J. Casal



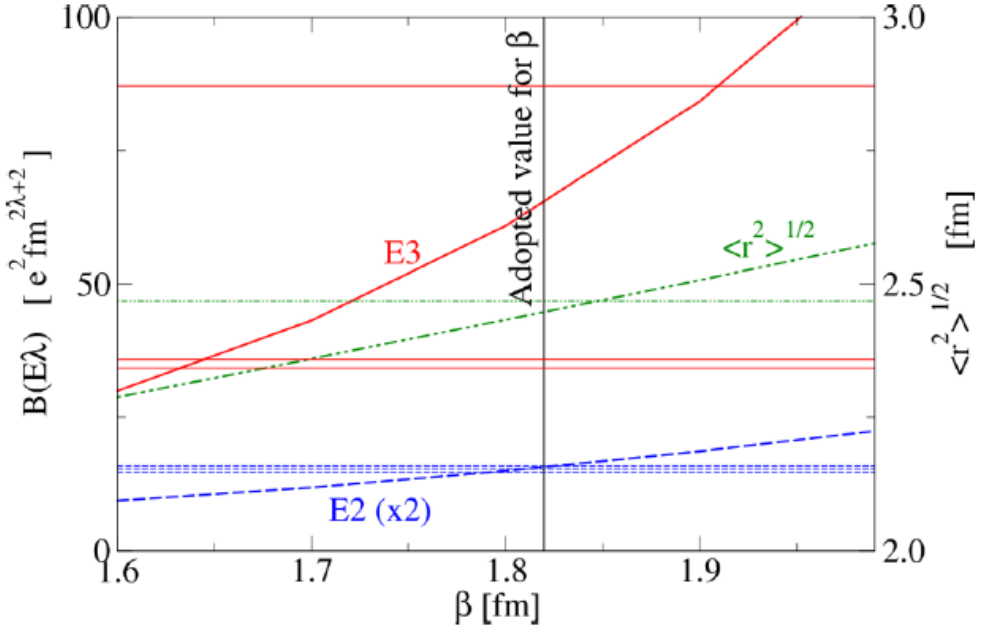

Ground state band




Hoyle state band

# Parameters phenomenologically adjusted



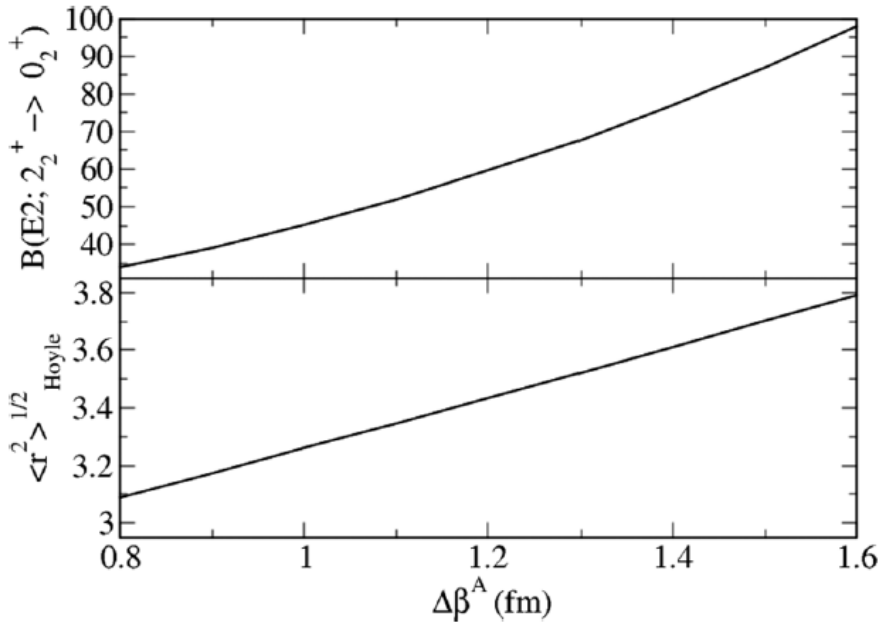




TABLE I. Calculated observables within the ground-state band.

| | |
|---|---|
| $\langle r^2\rangle^{1/2}_{0_1^+}$ | 2.45 (fm) |
| $B(E2; 2_1^+ \rightarrow 0_1^+)$ | 7.86 ($e^2$ fm$^4$) |
| $B(E3; 3_1^- \rightarrow 0_1^+)$ | 65.07 ($e^2$ fm$^6$) |
| $B(E4; 4_1^+ \rightarrow 0_1^+)$ | 96.99 ($e^2$ fm$^8$) |

TABLE II. Quantities calculated in the present work for the Hoyle band using the values of $\beta$ and $\chi_1$ given in the text.

| | |
|---|---|
| $\langle r^2\rangle^{1/2}_{0_2^+}$ | 3.44 (fm) |
| $B(E2; 2_2^+ \rightarrow 0_1^+)$ | 0.58 ($e^2$ fm$^4$) |
| $B(E2; 0_2^+ \rightarrow 2_1^+)$ | 2.90 ($e^2$ fm$^4$) |
| $B(E3; 3_2^- \rightarrow 0_1^+)$ | 70.42 ($e^2$ fm$^6$) |
| $M(E0; 0_2^+ \rightarrow 0_1^+)$ | 5.4 ($e$ fm$^2$) |

# Transition densities in 12C

A-band

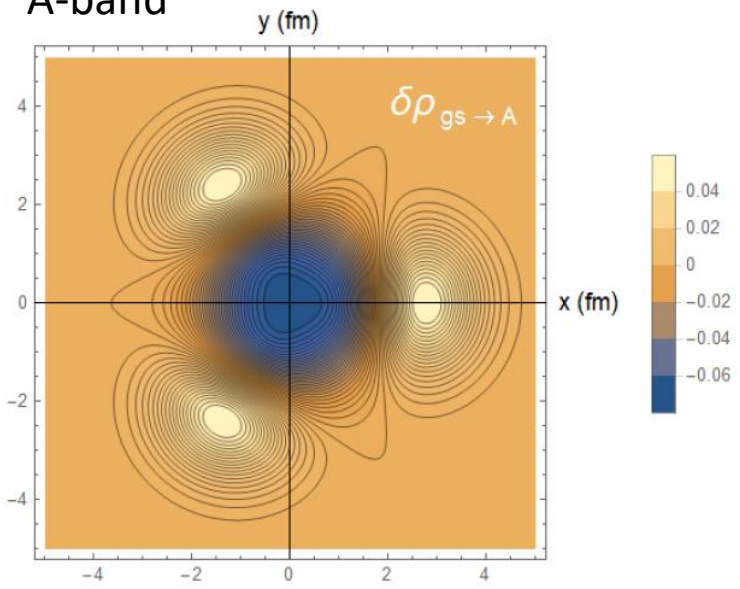
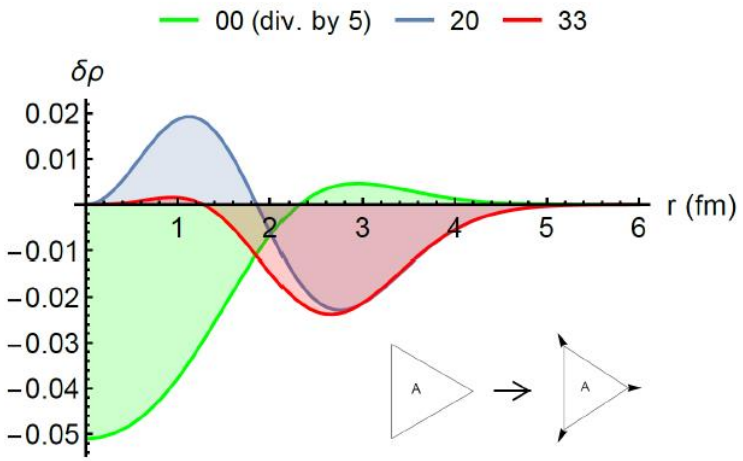
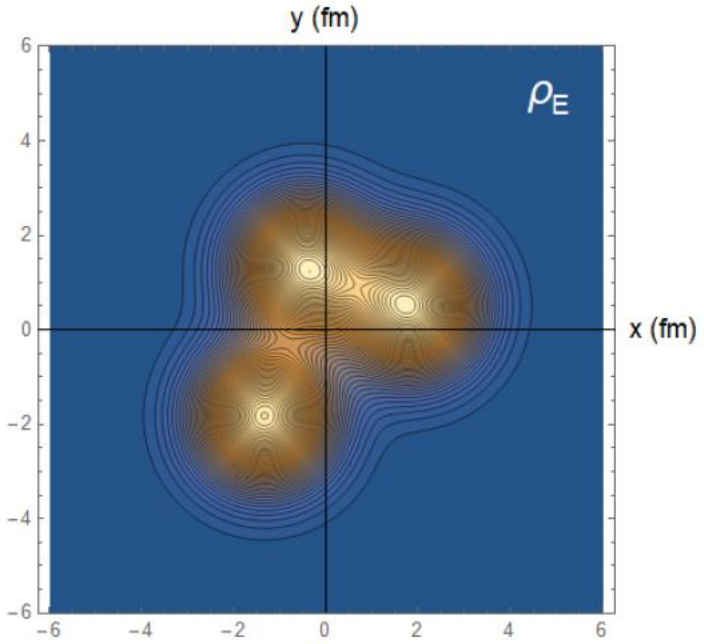
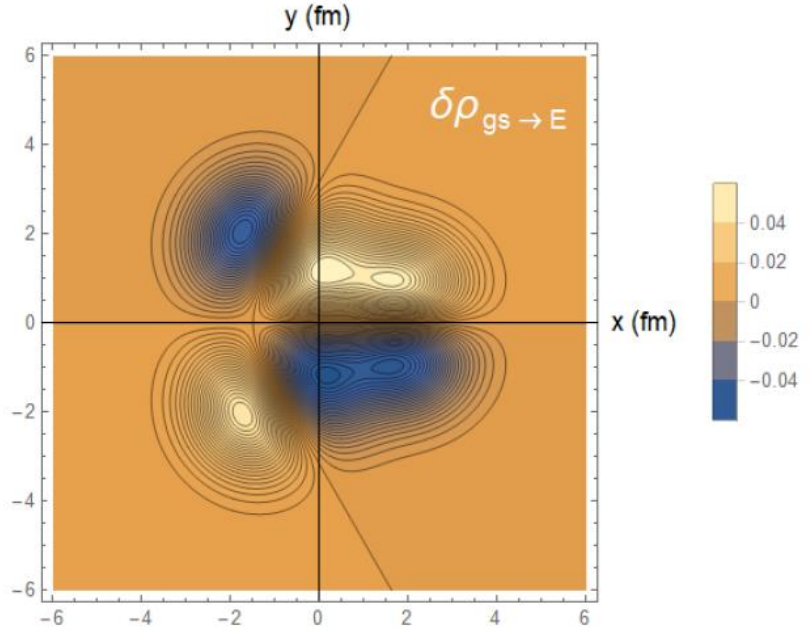
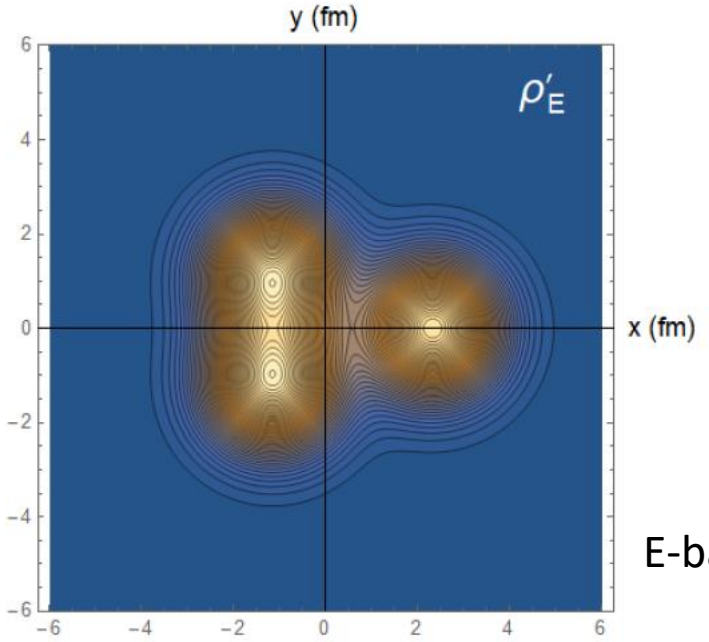


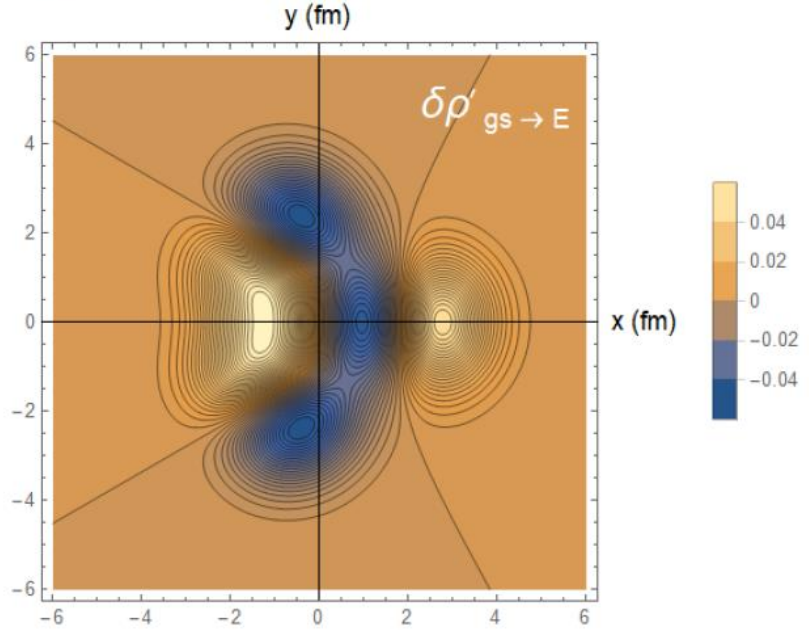

FIG. 7: Transition density for the first A-type vibration.




E-band

# Transition densities -> Form Factors -> Coupled Channels -> Cross-sections

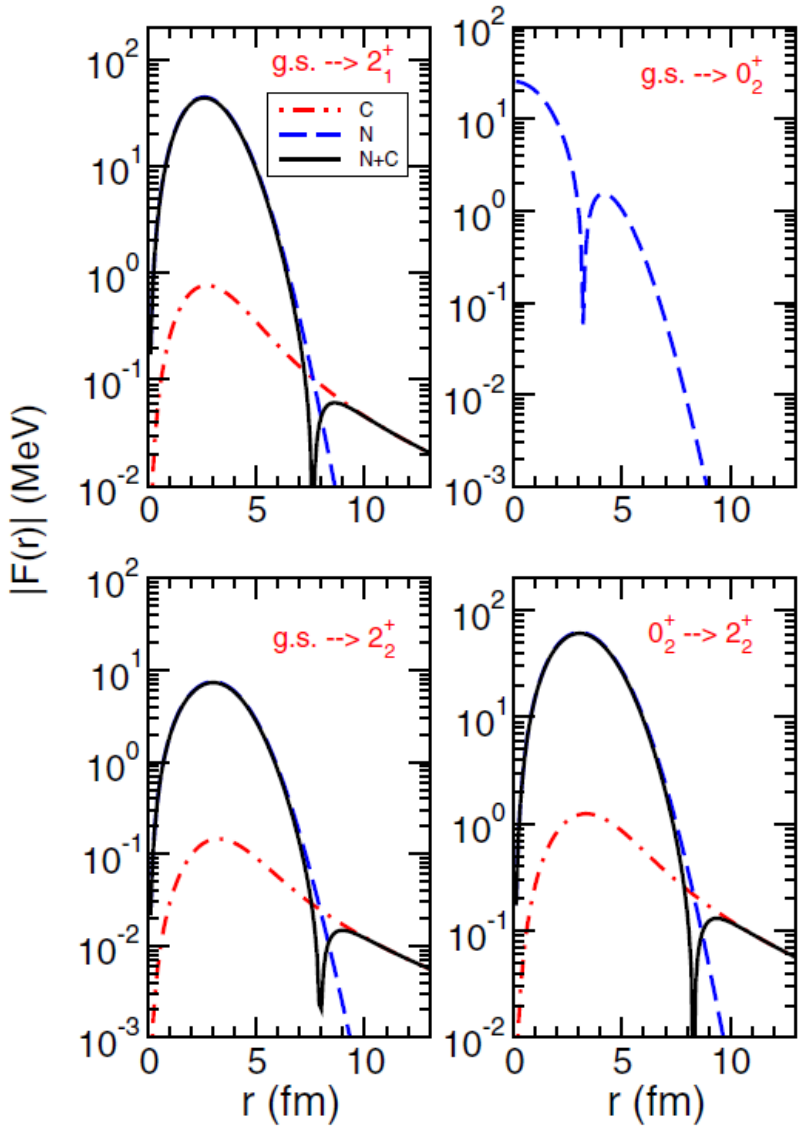



FIG. 15. Form factors in logarithmic scale for a few inelastic excitation processes of interest. We show the nuclear, Coulomb, and total form factors.

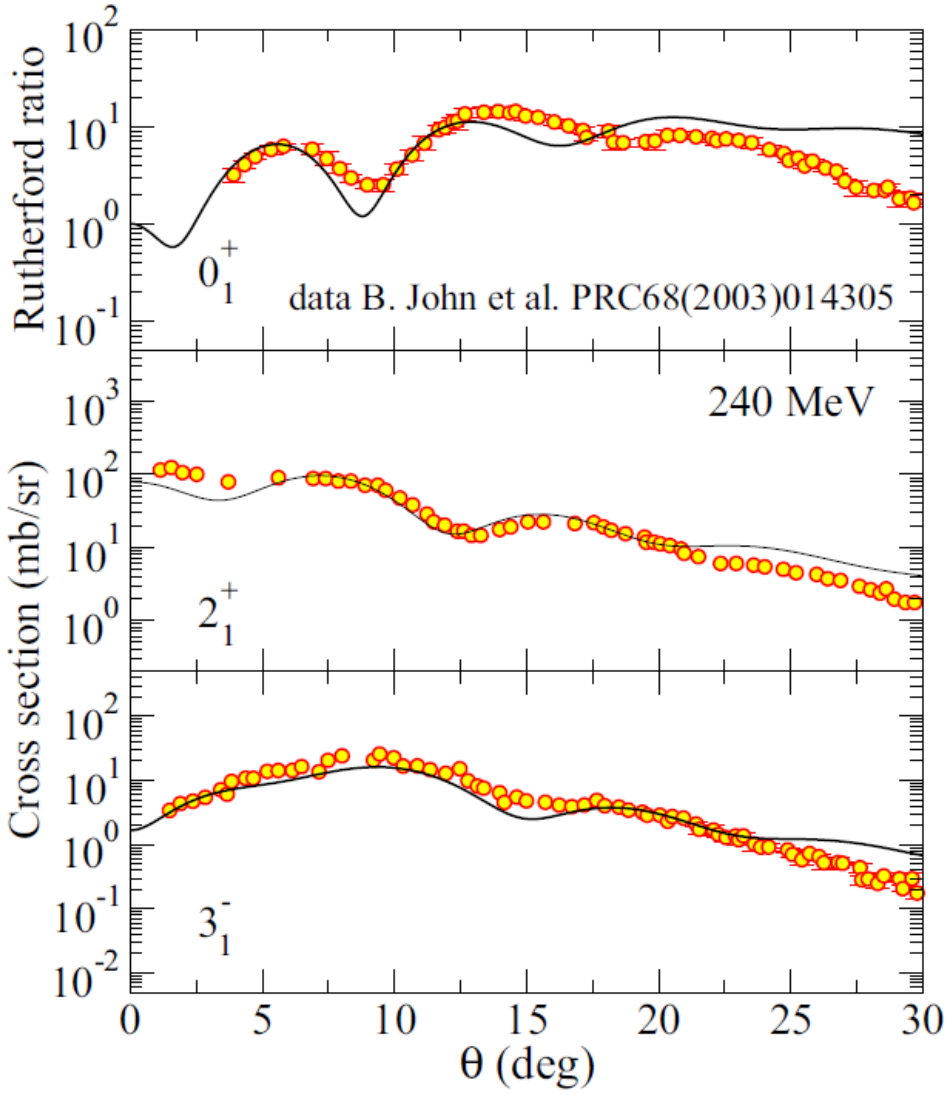



FIG. 16. Differential cross section for the elastic scattering and the transitions $0^+_1 \rightarrow 2^+_1$ and $0^+_1 \rightarrow 3^+_1$ at 240-MeV bombarding energy. Data are from Ref. [41] (retrieved through EXFOR).

Lots of results that I dont'have time to discuss in details.
They confirm that with just a simple triangular model one catches all the gross features, not only of the nuclear structure, but also of reaction dynamics of 12C.

# Importance of the imaginary part of the ion-ion potential

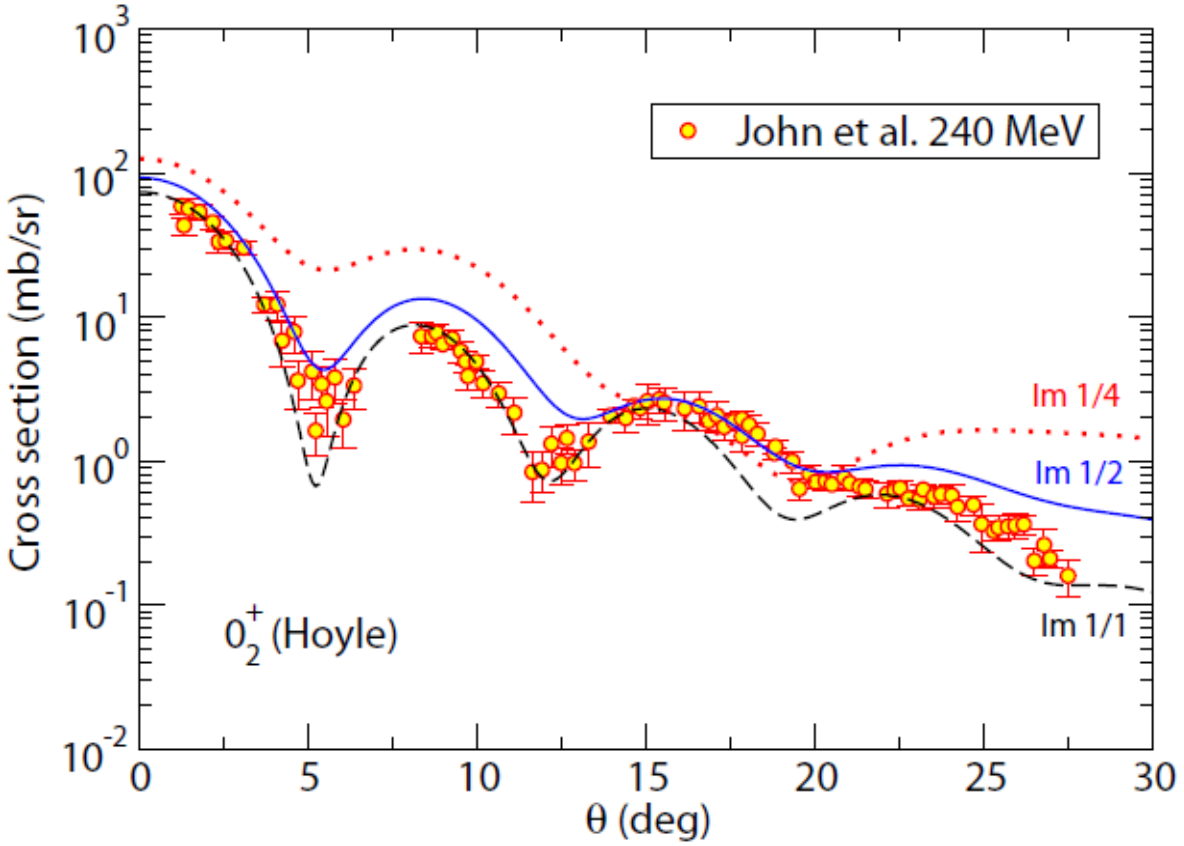
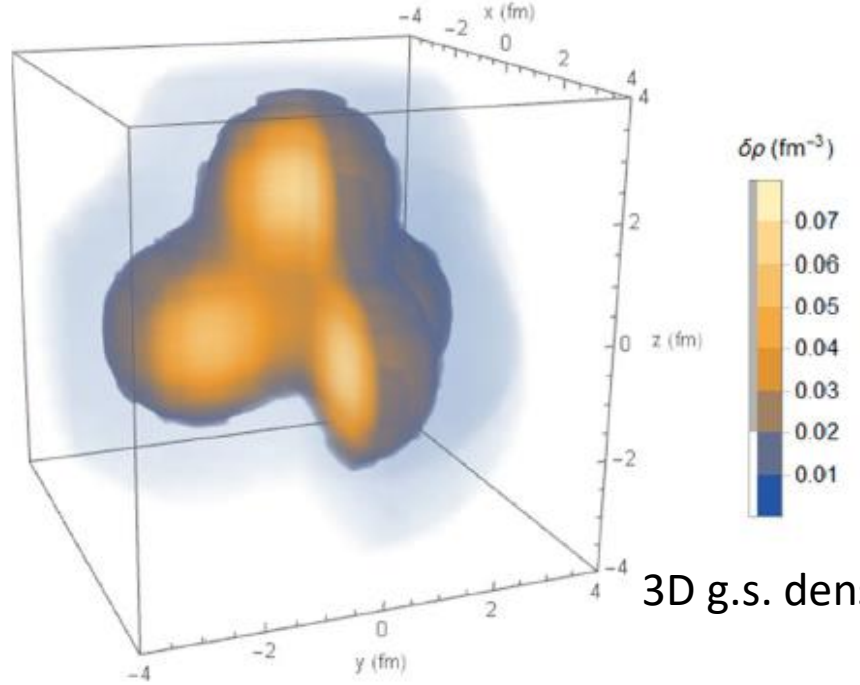



FIG. 17. Differential cross section for the transition $0_1^+ \rightarrow 0_2^+$ at 240-MeV bombarding energy. Data are from Ref. [41] (retrieved through EXFOR) and the three curves have different factors for the depth of the imaginary part as indicated in the figure.

V(r) + iW(r)

# Extended to 16O in a tetrahedral arrangement

Eur. Phys. J. A (2021) 57:33




3D g.s. density plot

Eur. Phys. J. A (2021) 57:33
https://doi.org/10.1140/epja/s10050-021-00347-5

THE EUROPEAN PHYSICAL JOURNAL A

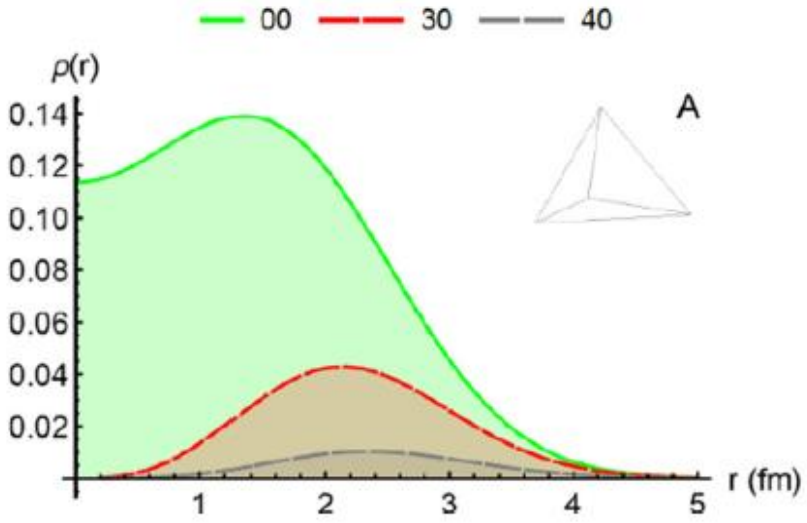
Regular Article - Theoretical Physics

## Alpha-induced inelastic scattering and alpha-transfer reactions in $^{12}$C and $^{16}$O within the Algebraic Cluster Model

Jesus Casal[1,2], Lorenzo Fortunato[1,2], Edoardo G. Lanza[3,4], Andrea Vitturi[1,2,a]

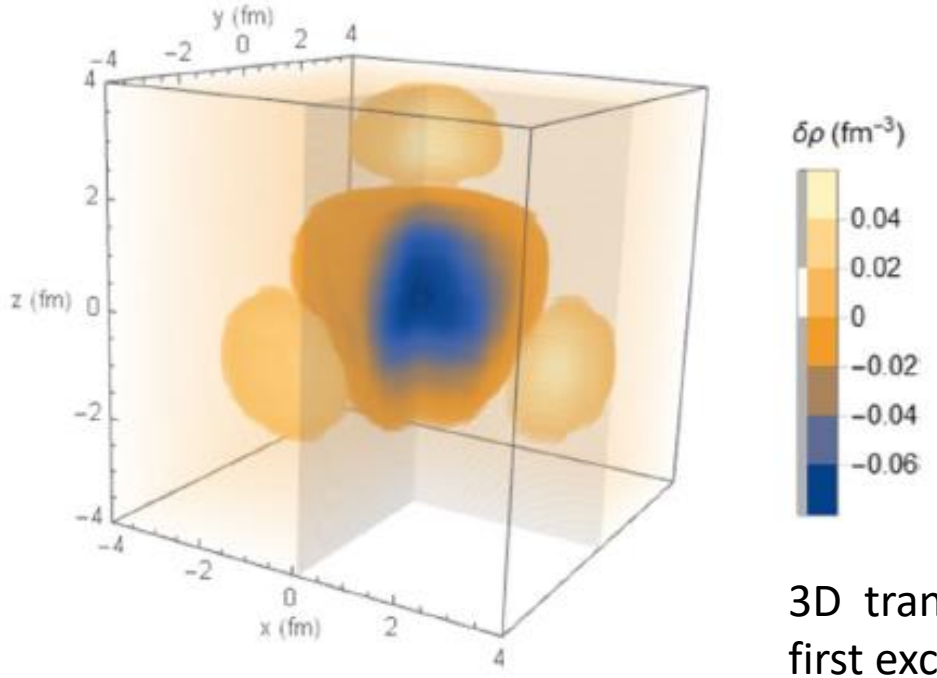
The tetrahedral group Td allows for singly- , doubly- and triply-degenerate representations

→ one can see all of these excitation modes in the spectrum of 16O !

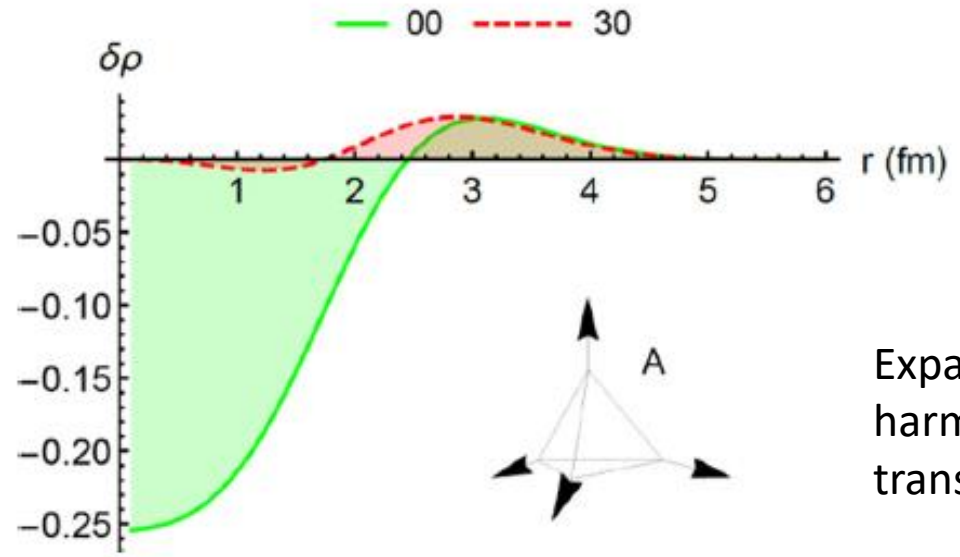



Expansion in spherical harmonics

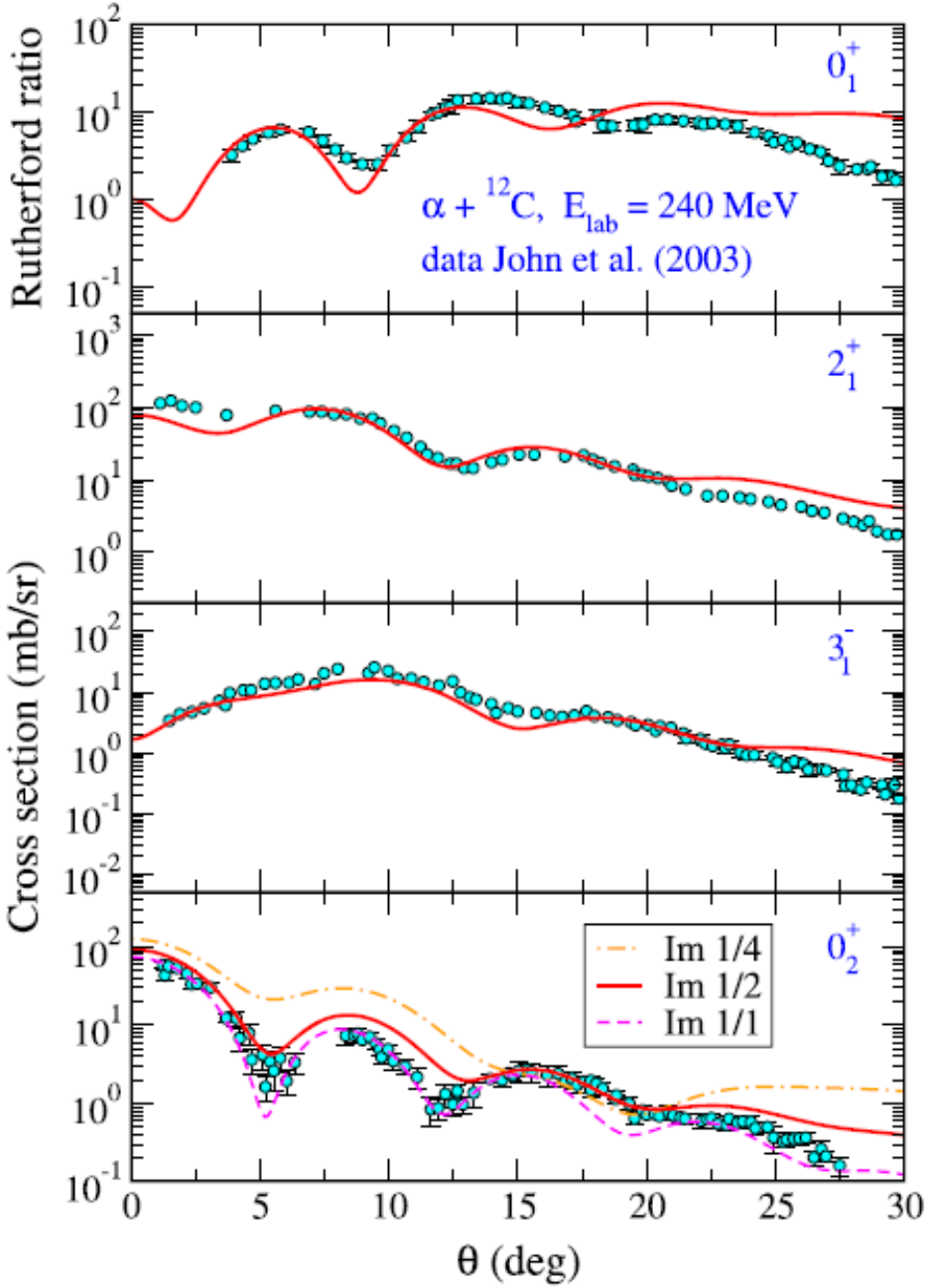
# Extended to 16O in a tetrahedral arrangement -> 12C(α,γ)16O




3D transition density to first excited A band (0+)




Expansion in spherical harmonics of the transition densities



# Selection rule for alpha transfer

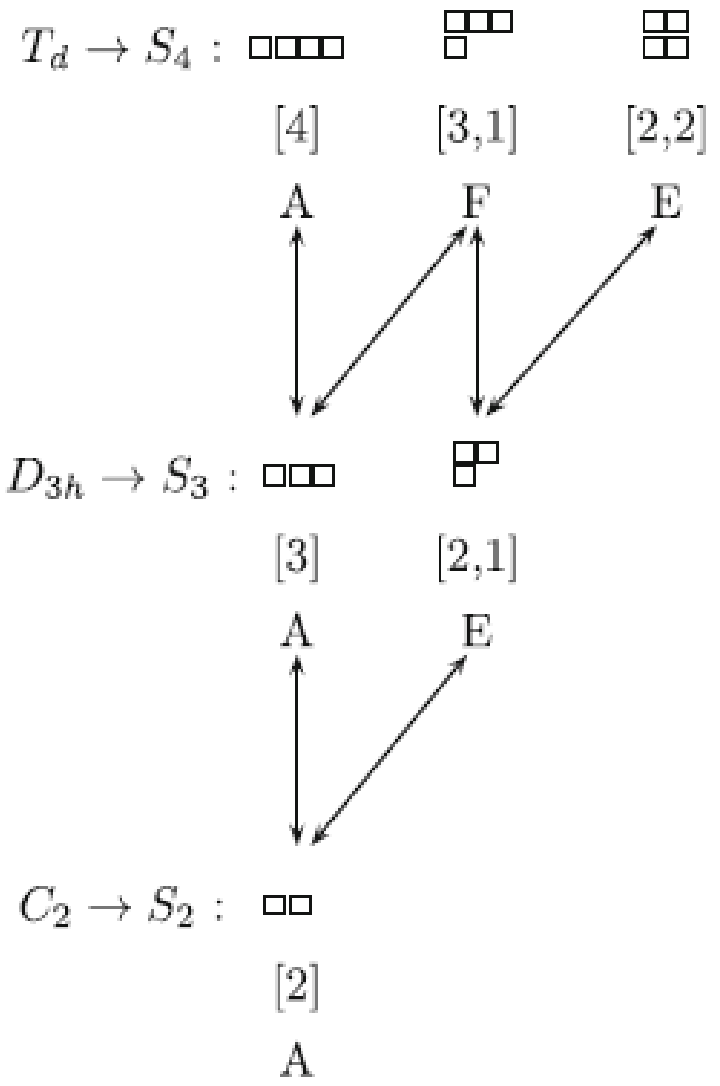



**Fig. 8** The representations of the systems with 4,3 and 2 $\alpha$ particles connected by arrows corresponding to processes of induction/restriction that amounts to the addition/removal of one box from the corresponding Young diagrams

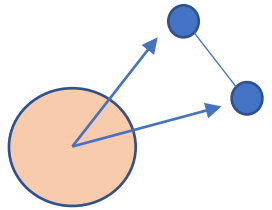
Taking into account the symmetric groups of 2-3-4 identical objects, one can up with a scheme for alpha-transfer (I mean the addition or removal of 1 alpha particle) based on Young tableaux, i.e. on the representations of those groups.

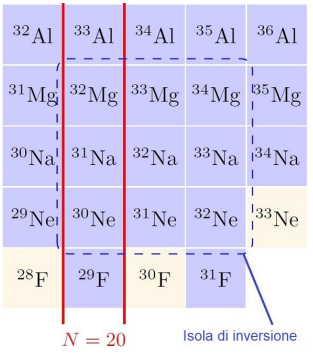
It turns out that not all of them can be connected, for instance one cannot go from the A states of 12C to the E states of oxygen, the alpha transfer should be identically zero !

But,… violated in exp. data ?

# Fluorine-29 stands on the coast of the island of inversion


COMMUNICATIONS PHYSICS






PERSPECTIVE

https://doi.org/10.1038/s42005-020-00402-5 OPEN

# The $^{29}$F nucleus as a lighthouse on the coast of the island of inversion

L. Fortunato [1,2]✉, J. Casal [1,2], W. Horiuchi [3], Jagjit Singh [4] & A. Vitturi[1,2]

# Fluorine-29 stands on the coast of the island of inversion

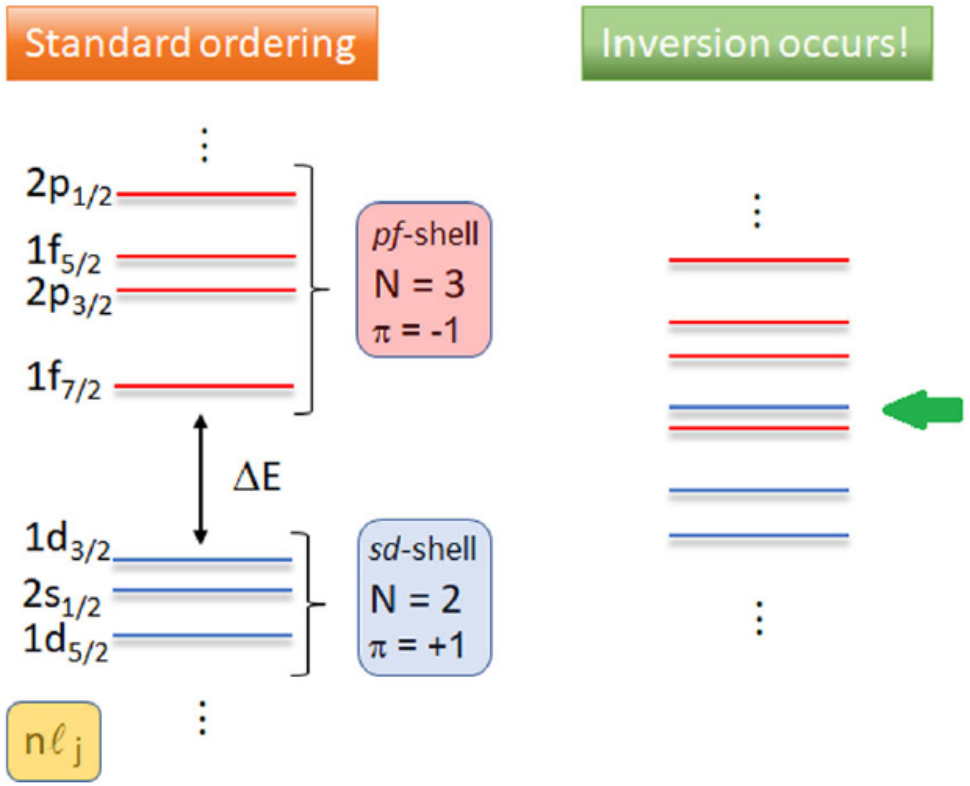



**Fig. 1 Standard ordering of shell-model energy levels and typical inversion mechanism.** The $N=2$ *sd*-shell and the $N=3$ *pf*-shell with positive and negative parity $\pi$, respectively, are shown on the left in the standard ordering (states are labeled by the standard set of quantum number $n\ell_j$). Inversion occur (right) when the shell gap, $\Delta E$, associated with the filling of 20 neutrons, disappears and one level (or more) of the $N=3$ *pf*-shell gets lower than one (or more) of the levels of the $N=2$ *sd*-shell.

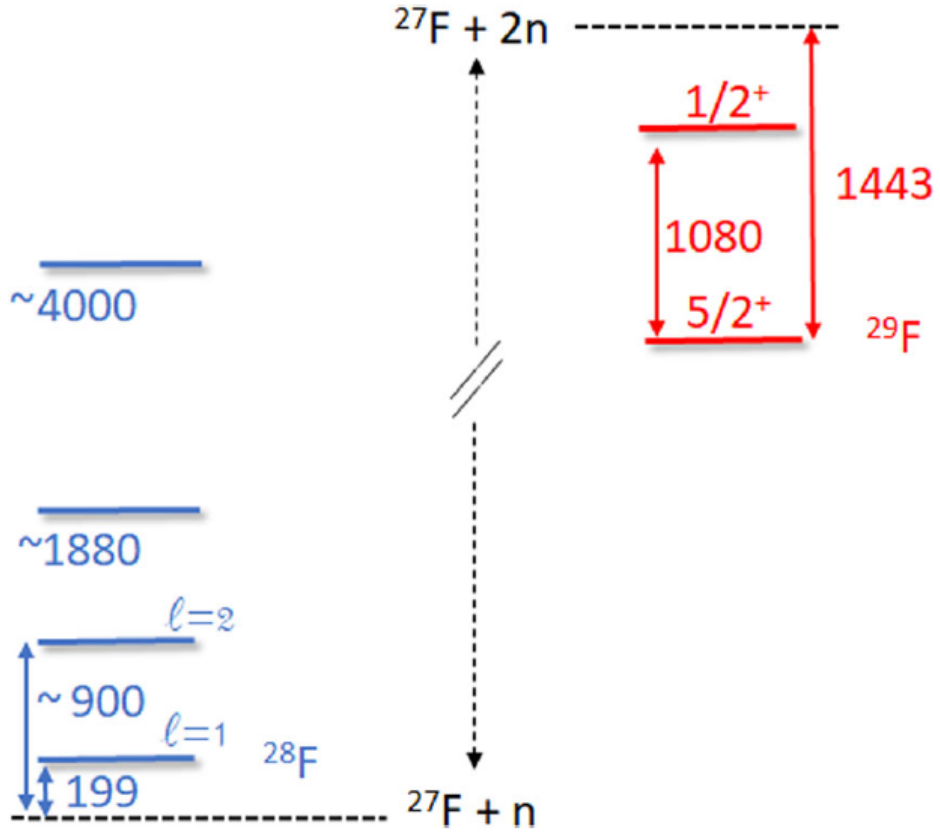



**Fig. 2 Synopsis of known experimental data on $^{28,29}$F.** All energies are in keV (not to scale) from refs. [4 --6,11]. States in red are labelled by the $J^\pi$ quantum numbers and energies are referred to the $^{27}$F + 2n threshold. States in blue are inferred from the $^{29}$F(−1n) column of Fig. 2 of ref. [6], and correspond only to the states decaying to the ground state of $^{27}$F. They are labelled by the orbital angular momentum quantum number, $\ell$, when available. Energies are referred to the $^{27}$F + n threshold.

# Fluorine-29 stands on the coast of the island of inversion

We had previously PRC **101**, 024310 (2020) proposed 4 scenarios for the structure of the very exotic nucleus 29F, called A,B,C,D , based on the three-body hyperspherical formalism by J.Casal

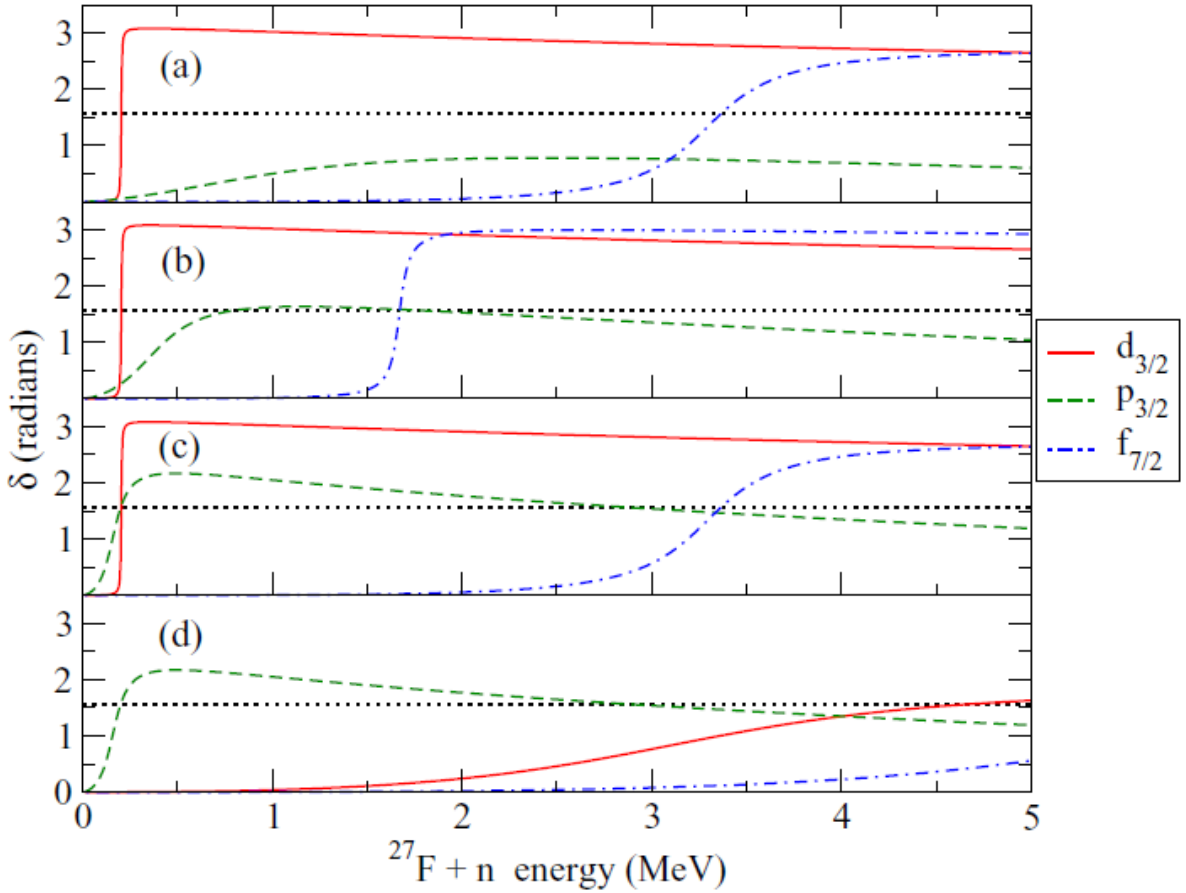


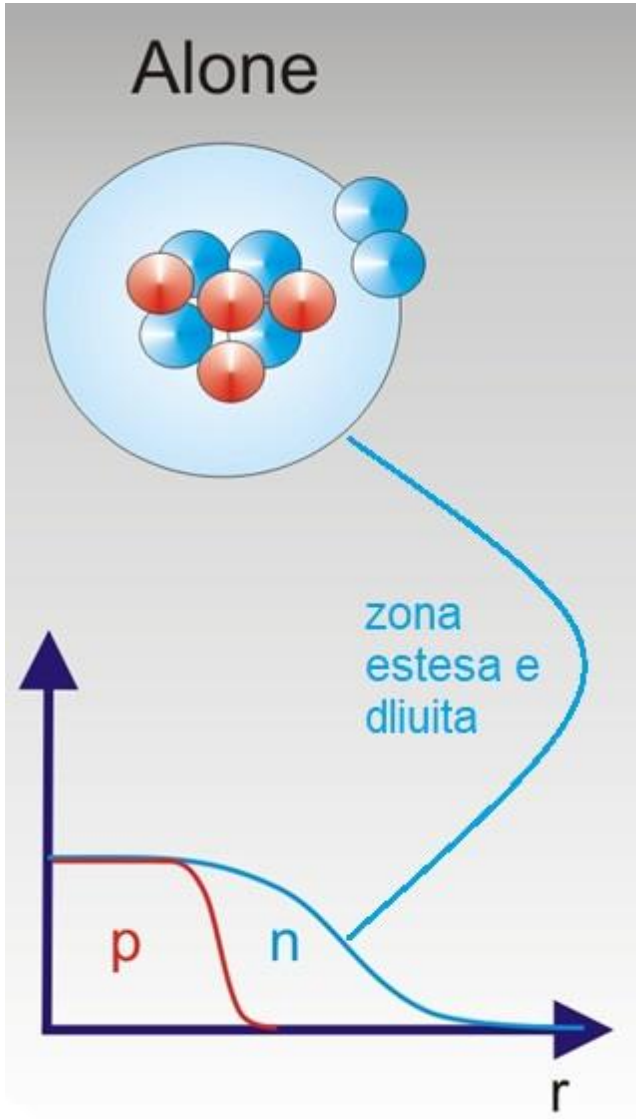

FIG. 1. $^{27}$F $+n$ phase shifts for $d_{3/2}$, $p_{3/2}$, and $f_{7/2}$ states, corresponding to different sets (A–D). The dotted black line corresponds to $\pi/2$.

Standard

Intruder

Degenerated

Inverted



# Fluorine-29 stands on the coast of the island of inversion

New experiments by
1) Revel, A. et al. "Extending the southern shore of the island of inversion to 28F" PRL 124, 152502 (2020)
and then
2) Bagchi, S. et al. "Two-neutron halo is unveiled in 29F" PRL 124, 222504 (2020)

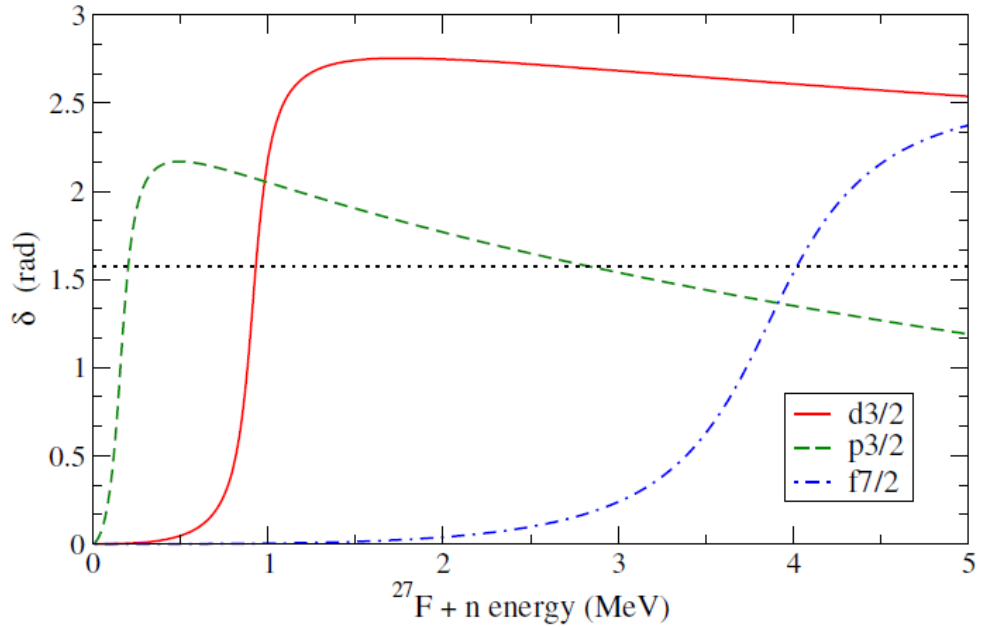
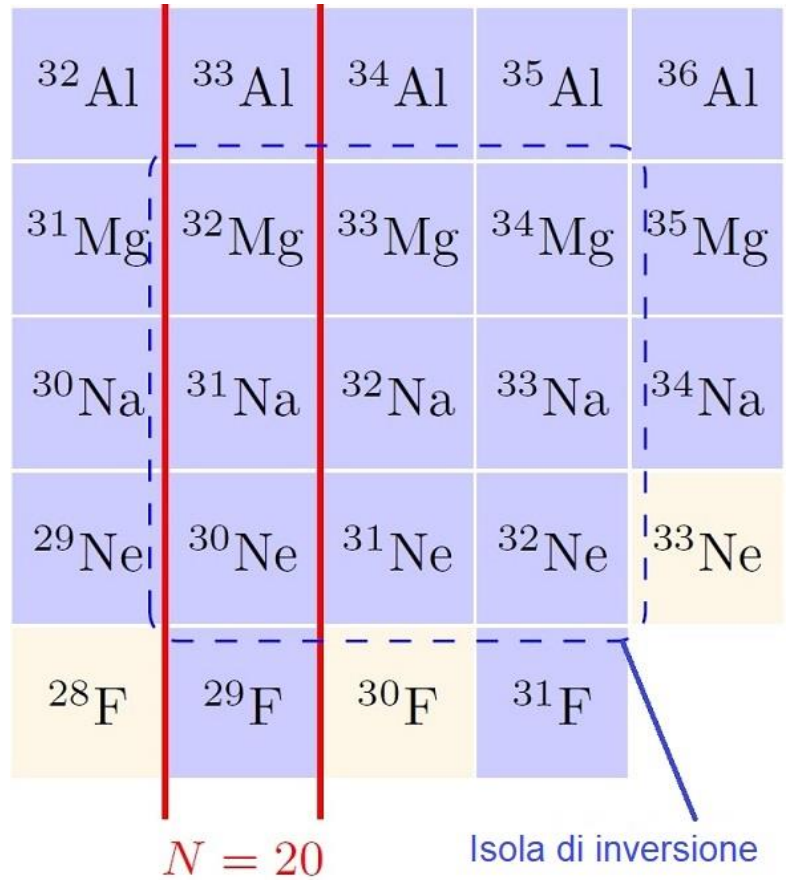
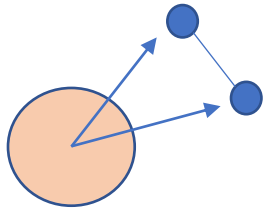



**Fig. 3 Phase-shifts, $\delta$, for the $^{27}$F + n system in the D$^{\flat}$ scenario as a function of the neutron-core relative energy.** Adjusting the red curve to reproduce the *d*-resonance at about 0.9 MeV, also gives the *f*-wave state (blue curve) at about 4 MeV.

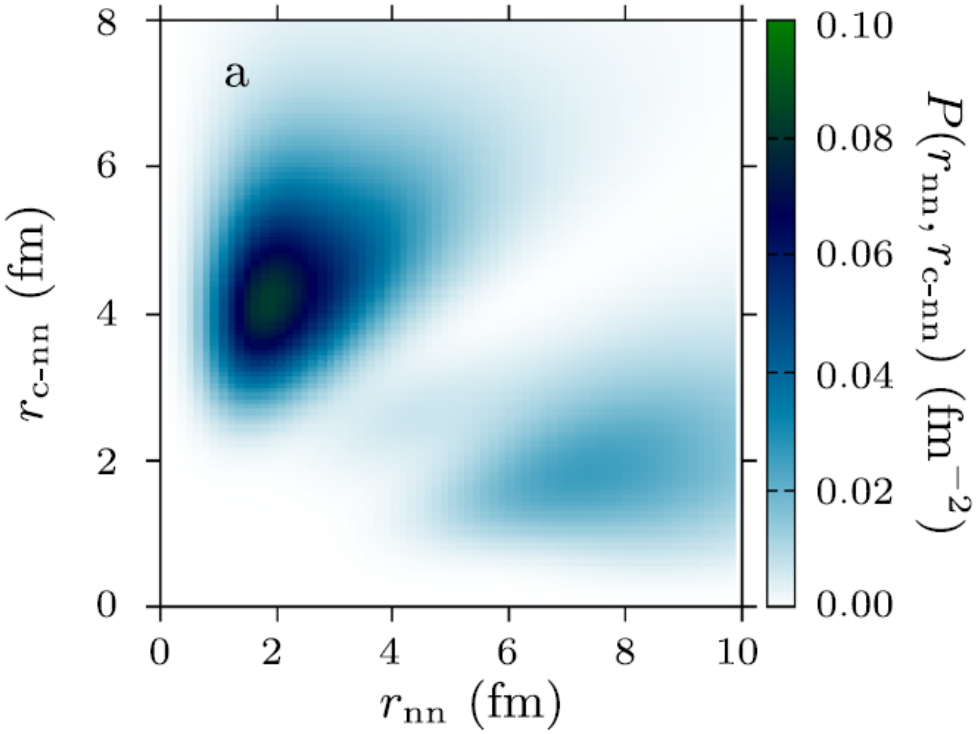



Final picture … Scenario D ♭

# Fluorine-29 stands on the coast of the island of inversion



$$\frac{d\sigma}{d\varepsilon} = (\alpha Z_2)^2 \sum_{\pi\lambda\mu} \left(\frac{\Delta E}{\hbar c}\right)^{2\lambda-2} \frac{dB}{d\varepsilon}(\pi\lambda; \mathrm{gs} \rightarrow \mathrm{Cont.}) |G_{\pi\lambda\mu}(\beta^{-1})|^2 g_\mu(\xi)$$



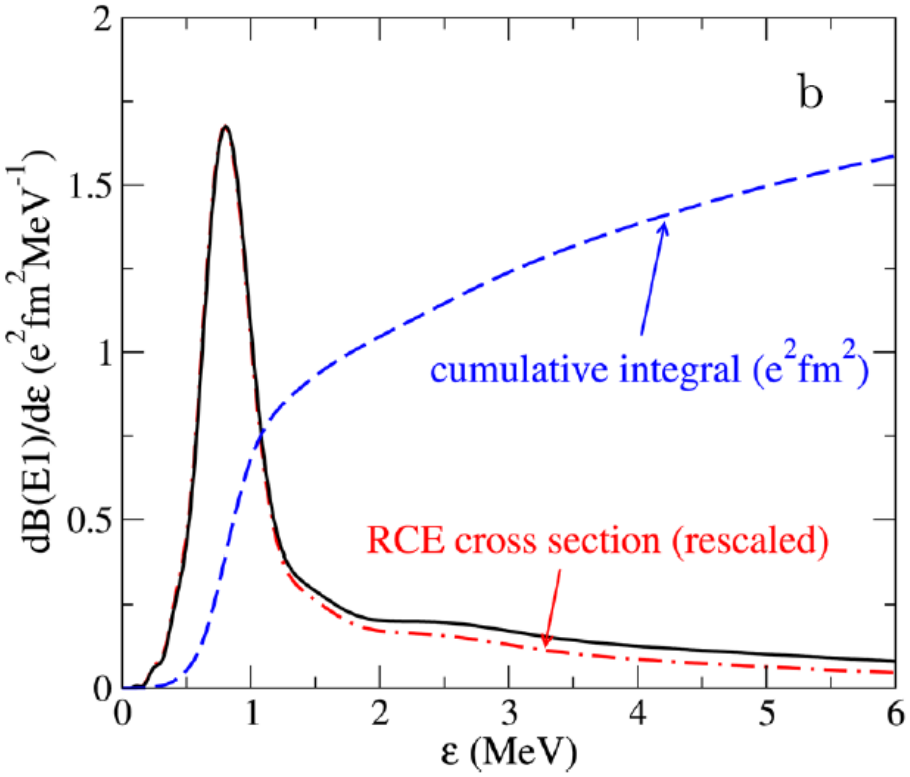




**Fig. 4 Results on $^{29}$F within the new $D^b$ scenario. a** Ground-state probability density of $^{29}$F as a function of the distance between the two valence neutrons ($r_{nn}$) and that between their center of mass and the core ($r_{c-nn}$). The maximum probability density corresponds to the dineutron configuration. **b** Electric dipole ($E1$) strength function from the ground state to continuum as a function of the $^{27}$F + n + n energy. The dashed line indicates the cumulative integral. The dash-dotted line is the corresponding Relativistic Coulomb Excitation (RCE) cross-section, scaled to the same maximum to illustrate the decreasing proportionality with the energy.

# Summary

- ✓ **PRC 99 (2019)** I have suggested to use the highly polarized monochromatic gamma rays that will be available at ELI-NP as a tool to study the molecular vibrations of clusterized nuclei, taking as a definite example the 12C nucleus as composed of 3 $\alpha$ particles. A measure of depolarization ratio could be done in a sort of Raman nuclear fluorescence experiment. This would yield precise patterns of vibrational spectra, that will correlate directly with a given geometric configuration possessing a discrete point-group symmetry.

- ✓ **PRC 101, 014315 (2021)** We have calculated transition densities and form factors for 12C in a triangular molecular model for the g.s., A and E bands. We applied this model to 12C+ $\alpha$ scattering showing a very good agreement.
- ✓ **EPJA 57 (2021)** We have extended these results to 16O in a tetrahedral arrangement and we have calculated alpha-induced reaction observables. I have found a new selection rule for alpha-transfer based on group theory.

- ✓ **Comm.Physics 3 (2020)** and also **PRC 101, 024310 (2020)** We have succesfully interpreted new experimental results on the structure of 29F indicating it lies at the border of the island of inversion.

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# A few points for discussion

- if the g.s. rotational band contains the same multipolarity that one is trying to excite in the vibrational bands, this is also to be included in the above patterns.

- in principle the degree of polarization might be close to 3/4 also for polarized (A) bands, therefore it might become hard to distinguish between them

- non-cluster degrees of freedom might come into play at a certain energy, thus blurring the picture

- in nuclei with a cluster structure including *t* or *h* clusters, the interplay with single-particle orbits around a molecular center might also be very relevant

- I guess a BEC gas would show no geometric arrangments (no equilibirum points) and would behave as an L=0 state (a sphere), thus offering only 1 such bands of A type (polarized).

Nomenclature

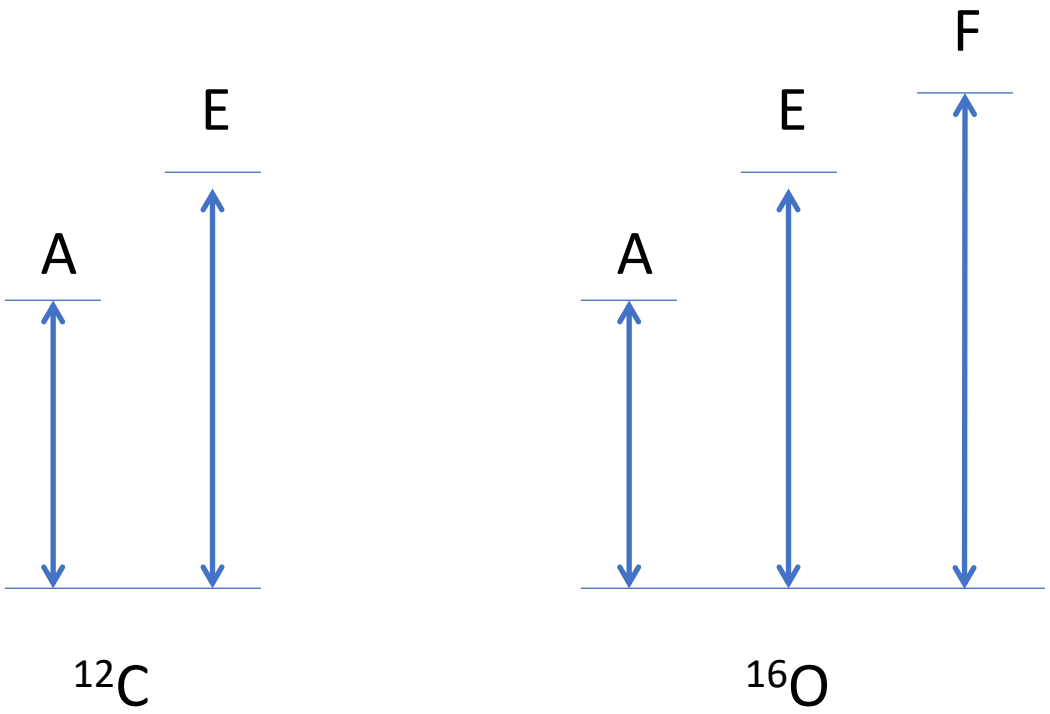
**Table II: The Mulliken symbols used to describe the symmetry species of point groups including their meaning with respect to molecular symmetry**

| Mulliken Symbols of Symmetry Species (Column 1 In Character Table) | Meaning |
|---|---|
| *A* | Symmetric with respect to principal axis of symmetry |
| *B* | Antisymmetric with respect to principal axis of symmetry |
| *E* | Doubly degenerate, two-dimensional irreducible representation |
| *T* | Triply degenerate, three-dimensional irreducible representation |
| *g* | Symmetric with respect to a center of symmetry |
| *u* | Antisymmetric with respect to a center of symmetry |
| 1 (subscript) | Symmetric with respect to a $C_2$ axis that is perpendicular to the principal axis. Where there is no such axis the subscript indicates that reflection in a $_v$ plane of symmetry is symmetric. |
| 2 (subscript) | Antisymmetric with respect to a $C_2$ axis that is perpendicular to the principal axis. Where there is no such axis the subscript indicates that reflection in a $_v$ plane of symmetry is antisymmetric. |
| ′ (prime) | Symmetric with respect to reflection in a horizontal plane of symmetry |
| ″ (double prime) | Antisymmetric with respect to reflection in a horizontal plane of symmetry |

From D. Tuschel – Spectroscopy : Molecular Spectroscopy workbench (2014)

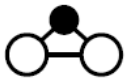
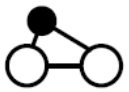
# Tetrahedral shape in 16 Oxygen

They use a somewhat simplified notation based on the permutation (sub)groups $S_3$ and $S_4$ of the full discrete groups $D_{3h}$ and $T_d$ respectively , but the essence is the same.



# Tables for 2 clusters and for 3 clusters of type AAB

| name | shape | group | $\Gamma_{vib}$ | Patterns |
|---|---|---|---|---|
| linear AA |  | $\mathcal{D}_{\infty h}$ | $A_{1g}$ |  |
| linear AB |  | $\mathcal{C}_{\infty\nu}$ | $A_1$ |  |

| name | shape | group | $\Gamma_{vib}$ | Patterns |
|---|---|---|---|---|
| linear ABA |  | $\mathcal{D}_{\infty h}$ | $A_{1g} + A_{1u} + E_{1u}$ |  |
| linear AAB |  | $\mathcal{C}_{\infty\nu}$ | $2A_1 + E_1$ |  |
| isosceles AAB |  | $\mathcal{C}_{2\nu}$ | $2A_1 + B_1$ |  |
| scalene AAB |  | $\mathcal{C}_s$ | $3A'$ |  |

# Tables for 4clusters, only some have been worked out

| name | shape | group | $\Gamma_{vib}$ | Patterns |
|---|---|---|---|---|
| linear aaa |  | $\mathcal{D}_{\infty h}$ | $2A_{1g} + E_{1g} + E_{4g} + A_{1u} + E_{1u}$ | 2/6 |
| linear aba |  | $\mathcal{D}_{\infty h}$ | $2A_{1g} + E_{1g} + A_{1u} + E_{1u}$ | 2/5 |
| square $a^4b^2$ |  | $\mathcal{D}_{4h}$ | $A_{1g} + B_{1g} + B_{2g} + B_{2u} + E_u$ | 1/5 |
| kite $a^4bc$ |  | $\mathcal{D}_{2h}$ | $2A_g + B_{1g} + B_{1u} + B_{2u} + B_{3u}$ | 2/6 |
| centered eq. triangle $a^3b^3$ |  | $\mathcal{D}_{3h}$ | $A_1' + 2E' + A_2''$ | 1/4 |
| rectangle $a^2b^2c^2$ |  | $\mathcal{D}_{2h}$ | $2A_g + B_{1g} + A_u + B_{2u} + B_{3u}$ | 2/6 |
| tetrahedron $a^6$ |  | $\mathcal{T}_d$ | $A_1 + E + T_2$ | 1/3 |
| uneq. tetrah. $a^3b^3$ |  | $\mathcal{C}_{3\nu}$ | $2A_1 + 2E$ | 2/4 |
| wedge $a^4b^2$ |  | $\mathcal{D}_{2d}$ | $2A_1 + B_1 + B_2 + E$ | 2/5 |
| 2 triangles at $90^o$ $a^5b$ |  | $\mathcal{C}_{2\nu}$ | $3A_1 + A_2 + B_1 + B_2$ | 3/6 |

One might gather information on polarization due to alpha particles' substructures