1 Flux JPA Equations

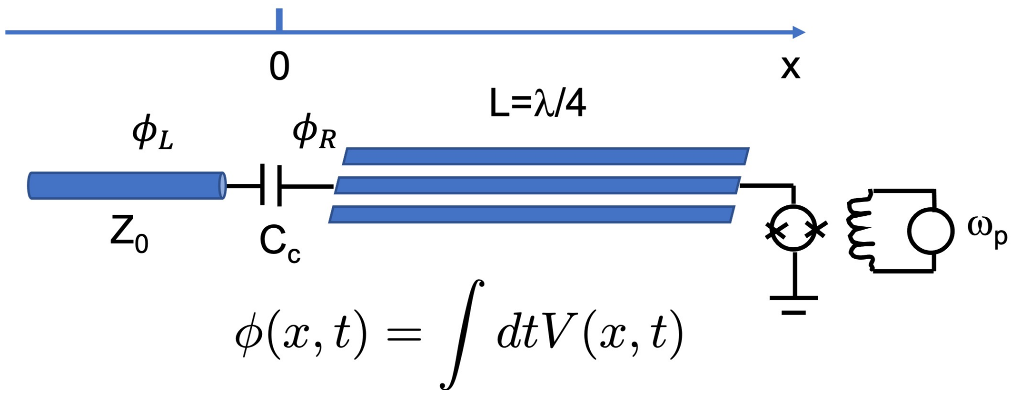

Figure 1: Equivalent circuit

The Lagrangian density of the circuit in figure 1 is obtained by the sum of the capacitive energies minus the inductive energies:

$$\begin{aligned}\mathcal{L} &= \left(\frac{1}{2}C_L\dot{\phi}_L^{\,2} - \frac{1}{2L_L}\phi_L'^2\right)\sigma(-x) + \left(\frac{1}{2}C_R\dot{\phi}_R^{\,2} - \frac{1}{2L_R}\phi_R'^2\right)[\sigma(x)-\sigma(x-L)] \quad (1)\\ &+ \left(\frac{1}{2}C_c(\dot{\phi}_L - \dot{\phi}_R)\right)\delta(x) + \left[\frac{C_J}{2}\dot{\phi}_J^2 + E_J(\phi_{ext})\cos\left(2\pi\frac{\phi_J}{\phi_0}\right)\right]\delta(x-L)\end{aligned}$$

where $\phi_{L,R}(x)$ is the flux variable defined as $\phi(x) = \int^t dt' V(t')$ in the left (right) transmission line and $C_{L,R}$ and $L_{L,R}$ are their capacitances and inductances per unit length, and $\phi_J = \phi_R(x = L)$. $\sigma(x)$ is the Heaviside (step) function. The (symmetric) SQUID Josephson energy is:

$$E_J(\phi_{ext}) = 2E_{0J}\cos\left(\pi\frac{\phi_{ext}}{\phi_0}\right) \quad (2)$$

where E_{0J} is the Josephson energy of the single junction.

We derive the equation of motions of the fields $\phi_{R,L}$ throught the Euler Lagrange equation for the lagrangian density:

$$\frac{d}{dt}\frac{\partial\mathcal{L}}{\partial\dot{\phi}} + \frac{d}{dx}\frac{\partial\mathcal{L}}{\partial\phi'} = \frac{\partial\mathcal{L}}{\partial\phi} \quad (3)$$

We then have:

$$\begin{aligned}\sigma(-x)\left[C_L\ddot{\phi}_L - \frac{1}{L_L}\phi_L''\right] + \delta(x)\left[C_c(\ddot{\phi}_L - \ddot{\phi}_R) + \frac{1}{L_L}\phi_L'\right] &= 0 \quad (4)\\ [\sigma(x)-\sigma(x-L)]\left[C_R\ddot{\phi}_R - \frac{1}{L_R}\phi_R''\right] + \delta(x)\left[-C_c(\ddot{\phi}_L - \ddot{\phi}_R) - \frac{1}{L_R}\phi_R'\right] + &\\ \delta(x-L)\left[C_J\ddot{\phi}_R + \frac{1}{L_R}\phi_R' + \frac{2\pi}{\phi_0}E_J(\phi_{ext})\sin\left(2\pi\frac{\phi_R}{\phi_0}\right)\right] &= 0\end{aligned}$$

that are the wave equations for ϕ_L and ϕ_R and the equations with solution of the form

$$\phi(x,t) = \phi^{in}(t - x/v_p) + \phi^{out}(t + x/v_p) \tag{5}$$

and $v_p = 1/\sqrt{LC}$. The other equations give the boundary conditions:

$$\begin{aligned} -\frac{1}{L_L}\phi_L' &= C_c(\ddot{\phi}_L - \ddot{\phi}_R) \qquad && (x=0) \\ -\frac{1}{L_R}\phi_R' &= C_c(\ddot{\phi}_L - \ddot{\phi}_R) \qquad && (x=0) \\ C_J\ddot{\phi}_R + \frac{1}{L_R}\phi_R' + \frac{2\pi}{\phi_0}E_J(\phi_{ext})\sin\left(2\pi\frac{\phi_R}{\phi_0}\right) &= 0 \qquad && (x=L) \end{aligned} \tag{6}$$

where we recognize the Kirchhoff current conservation laws (KCL). These must be solved in ϕ_L^{out} with ϕ_L^{in} as a initial condition for the input signal (for instance $\phi_{in} = \phi_{in}^0 \sin(\omega t)$). We now replace space derivatives with time derivatives using the wave equations

$$\begin{aligned} \phi' &= \frac{1}{v_p}\left(\dot{\phi}^{out} - \dot{\phi}^{in}\right) \\ \dot{\phi} &= \dot{\phi}^{in} + \dot{\phi}^{out} \end{aligned} \tag{7}$$

and we express the differential equation in terms of *in* and *out* fields. From the first two KLC equations 6 in $x = 0$ we have

$$\begin{aligned} -\frac{1}{Z_0}\left(\dot{\phi}_L^{out} - \dot{\phi}_L^{in}\right) &= C_c(\ddot{\phi}_L - \ddot{\phi}_R) \\ \dot{\phi}_L^{out} - \dot{\phi}_L^{in} &= \dot{\phi}_R^{out} - \dot{\phi}_R^{in} \end{aligned} \tag{8}$$

where $Z_0 = 1/Lv_p$ (we are assuming in the following that transmission line and resonator have same impedance $Z_0 = 50\Omega$). We want to determine the outgoing signal ϕ_L^{out} as a function of the known incoming signal ϕ_L^{in}. Moreover the wave ϕ_R^{out} incoming from the right to the capacitance is determined from the wave ϕ_R^{in} leaving the capacitance a time interval $2L/v_p$ in the past and then reflecting on the junction. Then we look for an equation for ϕ_R^{in} as a function of the known term ϕ_L^{in} and the causally independent wave ϕ_R^{out}. Then

$$\begin{aligned} -\frac{1}{Z_0}\left(\dot{\phi}_R^{out} - \dot{\phi}_R^{in}\right) &= C_c(\ddot{\phi}_L - \ddot{\phi}_R) \qquad && (x=0) \\ \dot{\phi}_L^{out} &= \dot{\phi}_R^{out} - \dot{\phi}_R^{in} + \dot{\phi}_L^{in} \qquad && (x=0) \end{aligned} \tag{9}$$

replacing ϕ_L^{out} in the first equation

$$2\ddot{\phi}_R^{in} + \frac{1}{CcZ_0}\dot{\phi}_R^{in} = \ddot{\phi}_L^{in} + \frac{1}{CcZ_0}\dot{\phi}_R^{out} \qquad (x=0) \tag{10}$$

Both ϕ_R^{out} and ϕ_L^{in} are considered as initial conditions. In particular $\phi_R^{out} = 0$ for $t < 2L/v_p$ that is until the first wave reflects on the junction. Once we know $\phi_R^{in}(t = t_0, x = 0)$ we can solve the third of the KCLs 6 to determine $\phi_R^{out}(t = t_0 + L/v_p, x = L)$. First we replace the space derivative with

$$\frac{1}{L_R}\phi_R' = \frac{1}{Z_0}\left(\dot{\phi}_R - 2\dot{\phi}_R^{in}\right) \tag{11}$$

and obtained obtain the equation for $\phi_J(t) = \phi_R^{out}(t, x = L) + \phi_R^{out}(t, x = L)$:

$$C_J\ddot{\phi}_J + \frac{1}{Z_0}\dot{\phi}_J + \frac{2\pi}{\phi_0}E_J(\phi_{ext})\sin\left(2\pi\frac{\phi_J}{\phi_0}\right) = \frac{2}{Z_0}\dot{\phi}_R^{in} \qquad (x = L) \tag{12}$$

where $\phi_R^{in}(t, x = L) = \phi_R^{in}(t - L/v_p, x = 0)$ is the value determined from equation 10. Once $\phi_J(t)$ is determined we can calculate the outgoing wave

$$\begin{aligned} \phi_R^{out}(t, x = L) &= \phi_R^{out}(t + L/v_p, x = 0) = \phi_J(t) - \phi_R^{in}(t, x = L) \\ &= \phi_J(t) - \phi_R^{in}(t - L/v_p, x = 0) \end{aligned} \tag{13}$$

that is inserted in equation 10 for the next iteration. For each iteration the output field is calculated as

$$\phi_L^{out} = \phi_R^{out} - \phi_R^{in} + \phi_L^{in} \qquad (x = 0) \tag{14}$$