Modeling of JPA

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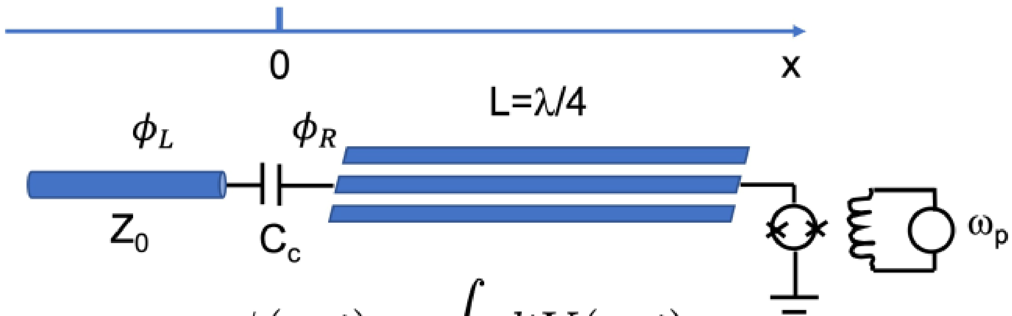
Aim: modelling of a JPA without simplifications of the nonlinear components

Model by Claudio Gatti



$$\phi(x,t) = \int dt V(x,t)$$

The Lagrangian density of the circuit in figure 1 is obtained by the sum of the capacitive energies minus the inductive energies:

$$\begin{aligned}\mathcal{L} \;=\; & \left(\frac{1}{2}C_L\dot{\phi_L}^2 - \frac{1}{2L_L}\phi_L'^2\right)\sigma(-x) + \left(\frac{1}{2}C_R\dot{\phi_R}^2 - \frac{1}{2L_R}\phi_R'^2\right)[\sigma(x)-\sigma(x-L)] \quad (1)\\ & + \left(\frac{1}{2}C_c(\dot{\phi}_L - \dot{\phi}_R)\right)\delta(x) + \left[\frac{C_J}{2}\dot{\phi}_J^2 + E_J(\phi_{ext})\cos\left(2\pi\frac{\phi_J}{\phi_0}\right)\right]\delta(x-L)\end{aligned}$$

where $\phi_{L,R}(x)$ is the flux variable defined as $\phi(x) = \int^t dt'V(t')$ in the left (right) transmission line and $C_{L,R}$ and $L_{L,R}$ are their capacitances and inductances per unit length, and $\phi_J = \phi_R(x = L)$. $\sigma(x)$ is the Heaviside (step) function. The (symmetric) SQUID Josephson energy is:

$$E_J(\phi_{ext}) = 2E_{0J}\cos\left(\pi\frac{\phi_{ext}}{\phi_0}\right) \qquad (2)$$

where E_{0J} is the Josephson energy of the single junction.

Wave equations for ϕ_L and ϕ_R

$$\sigma(-x)\left[C_L\ddot{\phi}_L - \frac{1}{L_L}\phi_L''\right] + \delta(x)\left[C_c(\ddot{\phi}_L - \ddot{\phi}_R) + \frac{1}{L_L}\phi_L'\right] = 0 \quad (4)$$

$$[\sigma(x) - \sigma(x-L)]\left[C_R\ddot{\phi}_R - \frac{1}{L_R}\phi_R''\right] + \delta(x)\left[-C_c(\ddot{\phi}_L - \ddot{\phi}_R) - \frac{1}{L_R}\phi_R'\right] +$$
$$\delta(x-L)\left[C_J\ddot{\phi}_R + \frac{1}{L_R}\phi_R' + \frac{2\pi}{\phi_0}E_J(\phi_{ext})\sin\left(2\pi\frac{\phi_R}{\phi_0}\right)\right] = 0$$

The solution is the sum of an outgoing and an incoming wave

$\phi(x,t) = \phi^{in}(t - x/v_p) + \phi^{out}(t + x/v_p)$ and $v_p = 1/\sqrt{LC}$. The other equations give the boundary conditions:

$$-\frac{1}{L_L}\phi_L' = C_c(\ddot{\phi}_L - \ddot{\phi}_R) \qquad (x=0)$$

$$-\frac{1}{L_R}\phi_R' = C_c(\ddot{\phi}_L - \ddot{\phi}_R) \qquad (x=0)$$

$$C_J\ddot{\phi}_R + \frac{1}{L_R}\phi_R' + \frac{2\pi}{\phi_0}E_J(\phi_{ext})\sin\left(2\pi\frac{\phi_R}{\phi_0}\right) = 0 \qquad (x=L)$$

$$\begin{aligned}\phi' &= \frac{1}{v_p}\left(\dot{\phi}^{out} - \dot{\phi}^{in}\right)\\ \dot{\phi} &= \dot{\phi}^{in} + \dot{\phi}^{out}\end{aligned}$$

in and *out* fields.

$$2\ddot{\phi}_R^{in} + \frac{1}{CcZ_0}\dot{\phi}_R^{in} = \ddot{\phi}_L^{in} + \frac{1}{CcZ_0}\dot{\phi}_R^{out} \qquad (x=0)$$

$$C_J\ddot{\phi}_J + \frac{1}{Z_0}\dot{\phi}_J + \frac{2\pi}{\phi_0}E_J(\phi_{ext})\sin\left(2\pi\frac{\phi_J}{\phi_0}\right) = \frac{2}{Z_0}\dot{\phi}_R^{in} \qquad (x=L) \qquad \phi_R^{in}(t, x=L) = \phi_R^{in}(t - L/v_p, x=0)$$

$$\begin{aligned}\phi_R^{out}(t, x=L) &= \phi_R^{out}(t + L/v_p, x=0) = \phi_J(t) - \phi_R^{in}(t, x=L)\\ &= \phi_J(t) - \phi_R^{in}(t - L/v_p, x=0)\end{aligned}$$

$$\phi_L^{out} = \phi_R^{out} - \phi_R^{in} + \phi_L^{in} \qquad (x=0)$$

Transient Analysis of Lossless Transmission Lines

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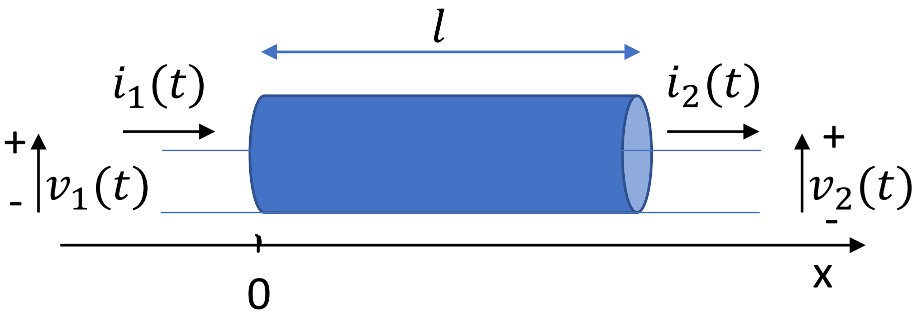

$\tau = \sqrt{LC}\, l$ line delay

$Z_0 = \sqrt{\frac{L}{C}}$ line impedance

$\Delta e = +Z_0 \Delta i$ $\qquad$ $\Delta e = -Z_0 \Delta i$

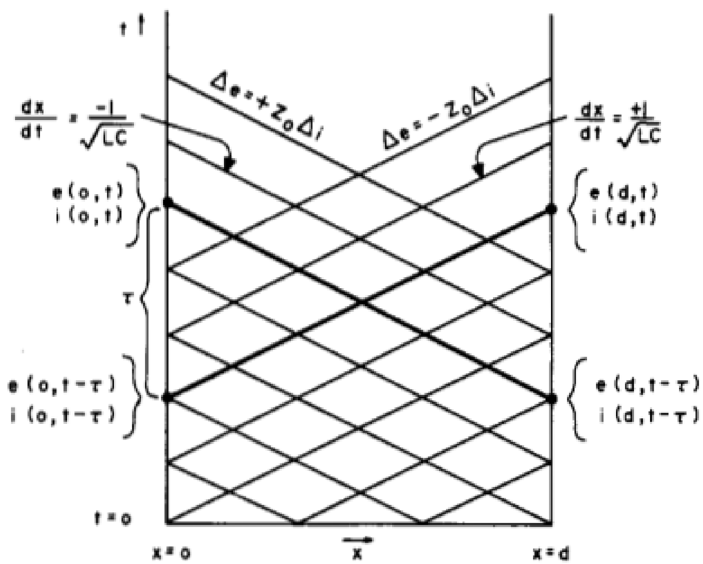

Fig. 1. Characteristic curves for a uniform transmission line.

$$v = v(x,t) \quad i = i(x,t)$$

$$L\,\frac{\partial i}{\partial t} + \frac{\partial v}{\partial x} = 0 \qquad\qquad i_{tt} - \frac{1}{L\,C}\, i_{xx} = 0$$

$$C\,\frac{\partial v}{\partial t} + \frac{\partial i}{\partial x} = 0 \qquad\qquad v_{tt} - \frac{1}{L\,C}\, v_{xx} = 0$$

The method of characteristics for solving these equations is based on a transformation in the $x-t$ plane which accomplishes the conversion of (1) and (2) into a pair of ordinary differential equations. Each of these two ordinary differential equations holds true along a different family of *characteristic curves* in the $x-t$ plane, one family corresponding to the forward or incident wave and the other to the backward or reflected wave. These two families of characteristic curves are, in effect, the basis for a new coordinate system instead of the canonical coordinates $x=$constant and $t=$constant.

$$dx/dt = 1/\sqrt{LC} \quad \text{and} \quad dx/dt = -1/\sqrt{LC}$$

$$\sqrt{\frac{L}{C}}\,di + \left(Ri + \sqrt{\frac{L}{C}}\,Ge\right)dx + de = 0 \tag{3}$$

which holds true only along the *forward* characteristics, defined as $dx/dt = 1/\sqrt{L/C}$, and

$$-\sqrt{\frac{L}{C}}\,di + \left(Ri - \sqrt{\frac{L}{C}}\,Ge\right)dx + de = 0 \tag{4}$$

which holds true only along the *backward* characteristics, defined as $dx/dt = -1/\sqrt{LC}$.

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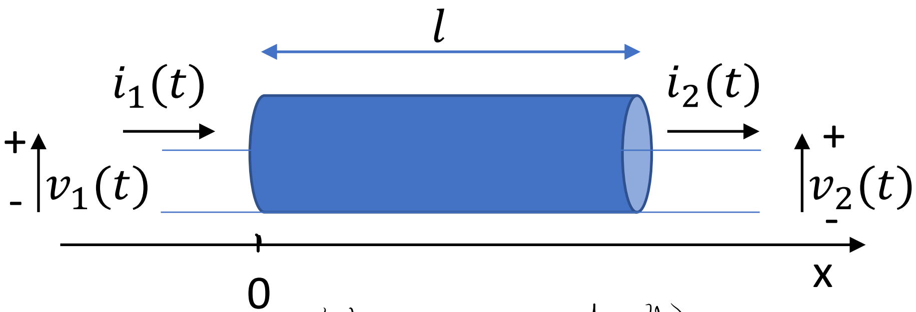
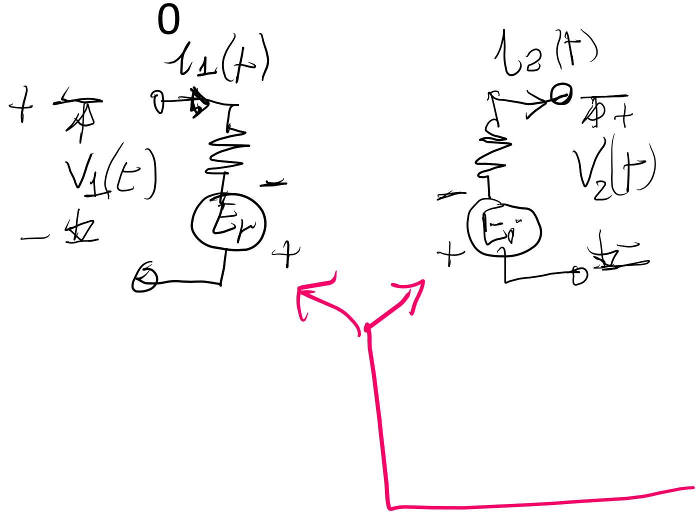




$$v = v(x,t) \quad i = i(x,t)$$

$\tau = \sqrt{LC}\, l$ line delay

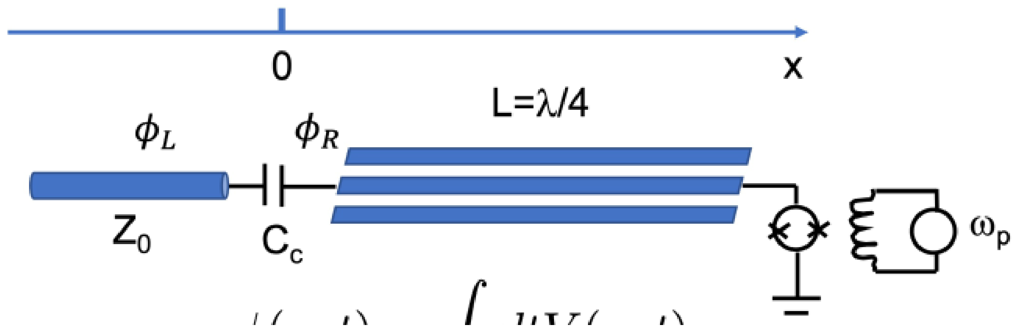
$Z_0 = \sqrt{\frac{L}{C}}$ line impedance

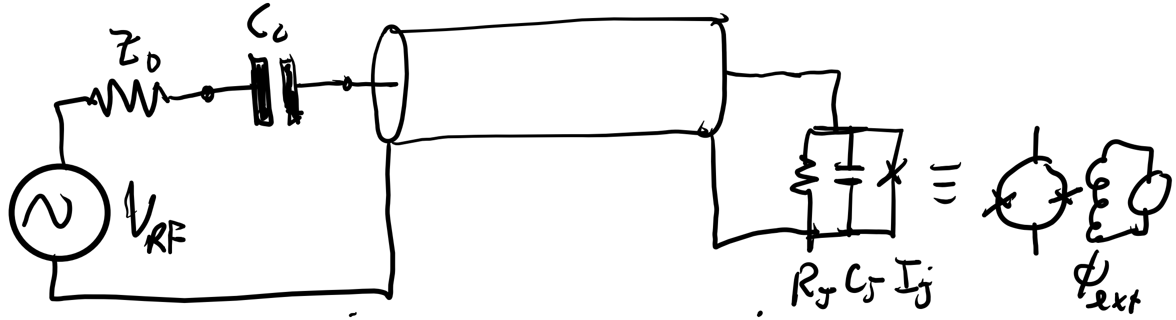
$$v(l,t) = -Z_0 i(l,t) + [v(0,t-\tau) + Z_0 i(0,t-\tau)]$$
$$v(0,t) = +Z_0 i(0,t) + [v(l,t-\tau) - Z_0 i(l,t-\tau)]$$

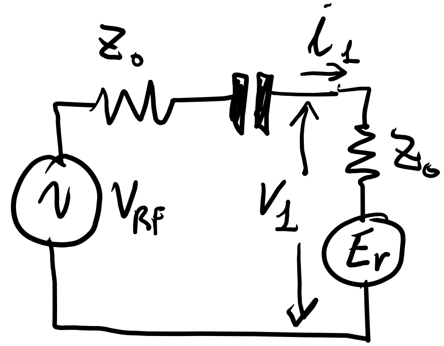
$$v_2(t) = -Z_0 i_2(t) + [v_1(t-\tau) + Z_0 i_1(t-\tau)]$$
$$v_1(t) = +Z_0 i_1(t) + [v_2(t-\tau) - Z_0 i_2(t-\tau)]$$

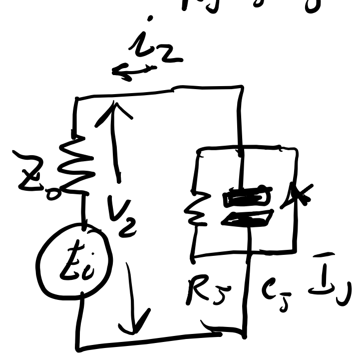
$$v_2(t) = -Z_0 i_2(t) - E_i(t-\tau)$$
$$v_1(t) = +Z_0 i_1(t) - E_r(t-\tau)$$

$$E_i(t) = -[2v_1(t) + E_r(t-\tau)]$$
$$E_r(t) = -[2v_2(t) + E_i(t-\tau)]$$


0
x
L=λ/4
ϕ_L
ϕ_R
Z_0
C_c
ω_p


Z_0
C_c
V_{RF}
R_J C_J I_J
ϕ_{ext}


Z_0
i_1
V_{RF}
V_1
Z_0
E_r


i_2
Z_0
V_2
E_i
R_J C_J I_J

In our case we have:

$$\dot{v}_1 = -\frac{1}{2\,C_c\,Z_0}\,v_1 + \frac{1}{2}\dot{v}_{rf}(t) - \frac{1}{2}\dot{E}_r(t-\tau) - \frac{1}{2\,C_c\,Z_0}\,E_r(t-\tau)$$

$$\dot{v}_2 = -\left(\frac{1}{C_j\,Z_0} + \frac{1}{C_j\,R_j}\right) v_2 - \frac{2\,I_j}{C_j}\cos\left(2\pi\frac{\phi_{ext}}{\phi_0}\right)\sin\varphi - \frac{1}{C_j\,Z_0}\,E_i(t-\tau)$$

$$\dot{\varphi} = \frac{2\,e}{\hbar}v_2$$

where

$$\dot{v}_{rf}(t) = v_{rf}\,\omega\,\cos\omega t$$

$$\phi_{ext}(t) = \phi_{ext}\cos\omega_p\,t$$

$$Z_0 = 50\;\Omega$$

$$\tau = \frac{T}{4} = \frac{\pi}{2\,\omega}$$

We start with Ei and Er set to zero

The values of Ei and Er are updated during the integration according to:

$$E_i(t) = -\left[2v_1(t) + E_r(t-\tau)\right]$$

$$E_r(t) = -\left[2v_2(t) + E_i(t-\tau)\right]$$

Next step is to integrate the system and do some checks with published data