PRIMORIDAL BLACK HOLES AND CONTEMPORARY GRAVITATIONAL WAVES

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Stochastic backgound of GW from the early universe is expected to be the final proof of inflation.
However, there could be other (natural?) mechanisms of generation of primordial GW with similar features.
Based on:
A.D. Dolgov, P.D. Naselsky, I.D. Novikov, e-Print: astro-ph/0009407;
A.D. Dolgov, D. Ejlli, work in progress.

Basic features:

1. Efficient PBH creation in the early universe (not unusual).

2. PBH dominance in the cosmological energy density (MD stage).

3. PBH evaporation and 2nd RD stage with forgotten past.

Modified cosmological thermal history: matter dominance (by nonrelativisitc PBH) in the early universe, all heavy relics are noticeably diluted.
Early structure formation at very small scales creating dense PBH clusters.
Possibly observable GW from PBH scattering and binaries in clusters.
Universe heating by PBH evaporation, 2nd RD stage, "return to normality".

PBH life-time with initial mass M is

$$\tau_{BH} \approx N_3 \, M^3 / m_{Pl}^4 \, ,$$

where $N_3 = N_{eff}/3 \cdot 10^3$ is the number of particle species with $m < T_{BH}$,

$$T_{BH} = m_{Pl}^2 / (8\pi M).$$

To avoid a conflict with BBN we need $\tau_{BH} < 10^{-2}$ s and hence

$$M < 1.2 \cdot 10^9 N_3^{1/3} \; g.$$

PBH production. **Density contrast at horizon scale should be $\delta\rho/\rho \sim 1$.** **Hence PBHs formed at cosmological time t_p, would have masses:**

$$M = t_p m_{Pl}^2 \simeq 4 \cdot 10^{38} \left(\frac{t}{\text{sec}}\right) \text{g}\,.$$

If $M < 10^9 g$, the production time $t_p < 10^{-29}$ s and $T_p > 10^{11}$ GeV. **Non-zero $\delta\rho$ was either generated at inflation or e.g. by inhomogeneous baryogenensis (AD and J. Silk).**

Relative cosmological energy density of BHs at production is

$$\Omega_{BH}(t_p) \equiv \Omega_p,$$

model dependent parameter.
At production $\Omega_p \ll \Omega_{tot} \approx 1$ and the universe was radiation dominated.
Inflationary $\delta\rho$ with flat spectrum lead to a low $\Omega_p \leq 10^{-6}$ but inhomogeneous baryogenesis could create much larger $\delta\rho$ at small scales.

Prior to BH decay $n_{BH}(t)a^3 = const$, hence at RD stage:

$$n_{BH}(t) = n_p \left(\frac{a_p}{a(t)}\right)^3 = n_p \left(\frac{t_p}{t}\right)^{3/2}.$$

Mass fraction of BH rose with time as

$$\Omega_{BH}(t) = \frac{n_{BH}(t)M}{\rho_c} \sim t^{1/2}.$$

When Ω_{BH} approached 1, MD stage started which lasted till BHs evaporated, $\Omega_{BH} \to 0$, and RD came back.

RD stage would turn into MD stage if PBH decayed after the onset of the MD stage.

From $\Omega_{BH} = \Omega_p \left(t/t_p\right)^{1/2}$ follows that $\Omega_{BH} \sim 1$ at $t_{eq} = t_p/\Omega_p^2$.

The necessary condition:

$t_{eq} < \tau_{BH} \sim M^3/m_{Pl}^4$ leads to:

$$M > m_{Pl}/\Omega_p \, .$$

Rising perturbations and GW.

At MD stage primordial density perturbations rise as $\Delta \equiv \delta\rho/\rho \sim a(t)$. For sufficiently long MD stage, Δ would reach unity and after that quickly rises to $\Delta \gg 1$. Density of PBH grows and GW emission is strongly amplified.

The regions with high n_{BH} would emit GW much more efficiently than in the homogeneous case. The emission of GW is proportional to vn_{BH}^2 and, both the BH velocity in dense regions and n_{BH} would be by several orders of magnitude larger than those in the homogeneous universe.

The BH life-time, τ_{BH}, must be large enough, so that the density fluctuations in BH matter would rise up to the values of the order unity:

$$\left(\frac{\delta\rho}{\rho}\right)_{in}\left(\frac{\tau_{BH}}{t_{eq}}\right)^{2/3} \sim 1\,,$$

where $(\delta\rho/\rho)_{in} \sim 10^{-5} - 10^{-4}$ is the magnitude of primordial density perturbations, not necessarily true at small scales.

Perturbations would have time to reach unity if $M > M_1$:

$$M_1 = 4 \cdot 10^3\,\mathrm{g}\,\frac{10^{-6}}{\Omega_p}\left(\frac{10^{-4}}{\Delta_{in}}\right)^{\frac{3}{4}}\left(\frac{N_{eff}}{100}\right)^{\frac{1}{2}}.$$

After $\Delta \approx 1$, rapid structure formation would take place, violent relaxation with non-dissipating dark matter.

After $\Delta = 1$ the cluster would stopped to expand it would begin to shrink since gravity took over the free streaming of PBHs. The density contrast would quickly rise. It looks reasonable that the density contrast of the evolved bunch could rise up to $\Delta_b = 10^6$, as in contemporary galaxies.

Let us assume that a perturbation with some wave length λ crossed horizon at moment t_{in}. The mass inside horizon at this moment was:

$$M_b(t_{in}) = m_{Pl}^2 t_{in}.$$

It is the mass of the would-be high density cluster of PBHs.

After horizon crossing the perturbations would continue to grow up as the scale factor, $\Delta(t) = \Delta_{in}(t/t_{in})^{2/3}$. Such rise would continue till moment $t_1(t_{in})$ such that:

$$\Delta[t_1(t_{in})] = \Delta_{in}[t_1(t_{in})/t_{in}]^{2/3} = 1\,,$$

or $t_1(t_{in}) = t_{in}\Delta_{in}^{-3/2}$.

After the size of the cluster stabilized, the number and mass densities of PBH would be constant too. But the density contrast, Δ_b would continue to rise as $(t/t_1)^2$ because ρ_c drops down as $1/t^2$. From time $t = t_1$ to $t = \tau_{BH}$ the density contrast would additionally rise by the factor:

$$\Delta(\tau_{BH}) \sim \left(\frac{\tau_{BH}}{t_1}\right)^2 = \left(\frac{M}{M_1}\right)^2 .$$

The size of the cluster at $t = \tau_{BH}$:

$$R_b = \Delta_b^{-1/3} t_1^{2/3} t_{in}^{1/3}\,.$$

and the average distance between the PBHs can be estimated as:

$$d_b = (M/M_b)^{1/3}\, R_b = \Delta_b^{-1/3} t_1^{2/3} r_g^{1/3}\,.$$

It does not depend upon t_{in}.

The virial velocity inside the cluster would be

$$v = \sqrt{\frac{2M_b}{m_{Pl}^2 R_b}} \approx 0.14 \left(\frac{\Delta_b}{10^6}\right)^{\frac{1}{6}} \left(\frac{\Delta_{in}}{10^{-4}}\right)^{\frac{1}{2}} .$$

So PBHs in the cluster can be moderately relativistic.

The frequency f_* of GW produced at time t_* during PBH evaporation, is redshifted down to the present day value, f, according to:

$$f = f_* \left[\frac{a(t_*)}{a_0}\right] = 0.34\, f_* \frac{T_0}{T_*} \left[\frac{100}{g_S(T_*)}\right]^{1/3},$$

where $T_0 = 2.725$ K, $T_* \equiv T(t_*)$ is the plasma temperature at the moment of radiation of the gravitational waves, and $g_S(T_*)$ is the number of species contributing to the entropy of primeval plasma at temperature T_*.

Taking $t_* = \tau_{BH}$ we find

$$\frac{T_*(\tau_{BH})}{m_{Pl}} \approx 0.04 \left[\frac{100}{g_*(T_*)}\right]^{\frac{1}{4}} \left(\frac{m_{Pl}}{M}\right)^{\frac{3}{2}} .$$

For comparison at the PBH production moment the temperature of the primeval plasma was:

$$T_p \approx 0.2 m_{Pl} \left(\frac{m_{Pl}}{M}\right)^{1/2} .$$

The frequency of such GW today would be equal to:

$$f \sim 5.2 \cdot 10^{11}\,\mathrm{Hz}\left(\frac{f_*}{m_{Pl}}\right)\left(\frac{M}{m_{Pl}}\right)^{3/2}.$$

For $f_* \leq r_g$ we find:

$$f \leq 2.6 \cdot 10^{11}\mathrm{Hz}\left(\frac{M}{m_{Pl}}\right)^{1/2}.$$

Bremsstrahlung of gravitons. Bakker, Gupta, Kaskas, Phys. Rev. 182 (1969), 1391:

$$d\sigma = \frac{64M^2m^2}{15m_{Pl}^6}\frac{d\xi}{\xi}\left[5\sqrt{1-\xi}+ \frac{3}{2}(2-\xi)ln\frac{1+\sqrt{1-\xi}}{1-\sqrt{1-\xi}}\right],$$

where ξ is the ratio of the emitted graviton frequency $\omega = 2\pi f_*$ to the kinetic energy of the incident black hole, i.e. $\xi = 2M\omega/p^2$.

The energy density of the gravitational waves radiated in the fequency interval $\delta\omega = \omega_2 - \omega_1$ at the time of emission would be:

$$\Omega_{GW}(\omega_1, \omega_2) \sim 10^6 v \frac{\Delta}{10^5} \frac{N_{eff}}{100} \frac{Q}{10} \frac{\delta\omega}{M}.$$

$Q \sim 10$ reflects the uncertainty in the cross-section due to the unaccounted for Sommerfeld enhancement.

With $v \approx 0.1$, and the density contrast as large as $\Delta \sim 10^5$ the fraction of the cosmological energy density of the GW emitted by the bremsstrahlung of gravitons from the PBH collisions when the universe age was equal to the life-time of the PBH could reach:

$$\Omega_{GW}(\omega_{max};\tau_{BH}) \sim 10^5 \left(\frac{m_{Pl}}{M}\right)^2$$

For very light PBH, $M < 300 m_{Pl}$, the fraction of GW might exceed unity, which is evidently a senseless result. However, one should remember the lower bound on the PBH mass and that $m_{Pl}/M < \Omega_p/20$ and $m_{Pl}/M < 10^{-7}(\Omega_p/10^{-6})$.

BH binaries.

GW luminosity GW radiation from a single binary in stationary approximation:

$$L = \frac{32 M_1^2 M_2^2 (M_1 + M_2)}{5 R^5} \approx \frac{64}{5} \frac{M^5}{R^5 m_{Pl}^8}.$$

The rotation frequency is expressed through R as

$$\omega_{rot}^2 = \frac{M}{m_{Pl}^2 R^3}.$$

Validity of stationary approximation:

$$R > R_{min} = 5 \left(\frac{100}{N_{eff}} \right)^{1/4} \frac{M^{3/2}}{m_{Pl}^{5/2}}.$$

The contribution of GW radiation into cosmological energy density from the binaries in stationary orbits can be estimated as

$$\Omega_{GW}^{(stat)} = \frac{\epsilon L \tau_{BH}}{M} \sim 0.1\epsilon \left(\frac{m_{Pl}}{M}\right)^{1/2} F(R)\,.$$

where $F(R)$ is the fraction of PBH binaries with $R \geq R_{min}$.

Inspiriling phase. The cosmological density parameter of GW at $t = \tau_{BH}$:

$$\frac{d\Omega_{GW}(f_s;\tau_{BH})}{d\log f_*} \sim \epsilon \left(\frac{M}{m_{Pl}^2}\right)^{5/3} \frac{f_s^{2/3}}{M},$$

where ϵ is the fraction of binaries.

The density parameter of gravitational waves today in terms of the present day frequency is:

$$\Omega_{GW}(f;t_0) \approx \frac{\epsilon}{10^5}\left(\frac{f}{10^{12}\mathrm{Hz}}\right)^{\frac{2}{3}}\left(\frac{m_{Pl}}{M}\right)^{\frac{1}{3}} .$$

As we saw above, $10^{-5}v\Delta \sim 1$.
However, this product may be larger due to possible Sommerfeld enhancement and central cusps in BH bunches.
The existing and near-future detectors are not sensitive to such GW but Ultimate DECIGO (2035), which will be sensitive to $\Omega = 10^{-20}$ at $f = 1$ Hz may put the limit:

$$M > 10^{3.6} m_{Pl}$$

or discover them.

High frequency detectors, not of existing projects, but of next generations may have better chance.

VII. Gravitons from BH evaporation.

Average graviton energy:

$$\omega_{av} = 3T_{BH} = \frac{3m_{Pl}^2}{8\pi M}.$$

Gravitons carry about 1% of the total evaporated energy and thus their contribution into cosmological energy density would be about 10^{-6}.

For $\omega < \omega_{av}$ the graviton density fraction drops down to $10^{-6}(\omega/\omega_{av})^4$.

IX. Cosmological evolution with BH dominance at an early stage.

To create BH with mass M the temperature before the production moment should be:

$$T_{heat} > 0.2 m_{Pl}^{3/2} / M^{1/2} \, .$$

Demanding $T_{heat} < T_{GUT} = 10^{15}$ GeV gives: $M > 10^7 M_{Pl}$.

If $M > 10^7 m_{Pl} \approx 10^2$ g, the BH temperature

$$T_{BH} = \frac{m_{Pl}^2}{8\pi M} < 5 \times 10^{10}\ GeV\,.$$

Reheating temperature after BH evaporation:

$$T_{reh} \approx 0.1 m_{Pl} \left(\frac{m_{Pl}}{M}\right)^{\frac{3}{2}} < 2 \times 10^7\ GeV.$$

However, lighter BHs are also possible due to matter accretion.

The bounds presented in the previous page may not be valid in the case of the mechanism based on inhomogeneous Affleck-Dine baryogenesis.

Baryogenesis might proceed according to AD or through asymmetric BH evaporation.

Heating after inflation is almost forgotten.

DM is produced by BH evaporation or by secondary thermalization.

The universe would be clumpy at very small scales, $M_b \sim M/\Omega_p^2$.

Formation of larger BH from bunches of small BH by dynamical friction?

X. Conclusion

Cosmological scenario with dominance of PBH is plausible.

This early MD-stage may be observable through high frequency GW.

GW from inflation could be noticebly diluted at PBH MD stage.

Heavy relics from after-inflationary heating would be forgotten.

Baryogenesis might successfully proceed in the course of BH evaporation.

If DM and baryon asymmetry are produced in BH evaporation, it is natural to expect that $\Omega_{DM} \sim \Omega_b$.

Cosmological energy density of dark matter particles is not related to their annihilation cross-section.

BBN is safe, though some distortion is possible.

Impact on CMB is weak or high frequency GW could distort it (?).

THE END

Mass density of BH: $\rho_{BH} = n_{BH} M$.
Define the average distance between BHs as $d = n_{BH}^{-1/3}$.
Express the average distance at production d_p through t_p and Ω_p:

$$\Omega_p = \frac{\rho_p}{\rho_c} = \frac{32\pi t_p^2 M n_p}{3m_{Pl}^2} = \frac{32\pi}{3}\left(\frac{t_p}{d_p}\right)^3$$

and $d_p = (4\pi/3)^{1/3} r_g \Omega_p^{-1/3}$.

while at MD stage

$$n_{BH}(t) = n_p \left(\frac{t_p}{t_{eq}}\right)^{3/2} \left(\frac{t_{eq}}{t}\right)^2,$$

where t_{eq} is the onset of BH dominance, if $t_{eq} > \tau_{BH}$.

II. Initial GW emission.

BHs start to interact and thus to radiate GW when they enter inside horizon of each other. The corresponding time moment is t_h is determined by the condition $2t_h = d(t_h)$ so:

$$d_p(t_h/t_p)^{1/2} = t_h,$$

Correspondingly:

$$t_h = \frac{1}{2}\left(\frac{4\pi}{3}\right)^{2/3} r_g \Omega_p^{-2/3}.$$

For $t > t_h$, the curvature effects can be neglected and the BH motion is determined by the Newtonian gravity:

$$\ddot{r} = -\frac{M}{m_{Pl}^2 r^2}$$

with the initial condition $r_{in} = d(t_{in})$, where $d(t)$ is the average distance between BHs. E.g. at RD stage:
$d(t) = d_p(a_p/a(t)) = d_p(t_p/t)^{1/2}$,
and initial velocity equal to the Hubble one, $\dot{r}_{in} = H(t_{in})r_{in}$.

For $t_{in} = t_h$: $\dot{r}_{in} = c = 1$ and the non-relativistic approximation is invalid. To avoid that and we should choose $t_{in} > t_h$ such that $v_{in} \ll 1$.

The solution of the equation of motion demonstrates that the effects of mutual attraction at this stage and production of GW are weak.

The acceleration of BH towards each other would be noticeable when their Hubble velocity becomes smaller than the capture velocity. The corresponding time moment, t_c, when it started is determined from the condition:

$$\frac{1}{2}v_H^2 \equiv \frac{1}{2}(H_c d_c)^2 = \frac{M}{m_{Pl}^2 d_c},$$

where sub-index "c" means that the corresponding quantity is taken at t_c.

At RD-stage we obtain:

$$\frac{d_p^3}{4t_c^2}\left(\frac{t_c}{t_p}\right)^{3/2} = r_g$$

and thus:

$$t_c = \frac{8\pi^2}{9}\frac{r_g}{\Omega_p^2}\,.$$

However, the mass fraction of BHs at $t = t_c$ is:

$$\Omega_c = \Omega_p \left(\frac{t_c}{t_p}\right)^{1/2} = \frac{4\pi}{3} > 1\,.$$

Thus at $t = t_c$ the universe is already matter dominated and we need to use another law for the scale factor, namely $a \sim t^{2/3}$.

Let us find time t_{eq} when the energy density of relativistic matter and non-relativistic BHs became equal. This can be found from the condition $\Omega_{BH}(t_{eq}) = \Omega_p \left(t_{eq} t_p\right)^{1/2} = 1$ and thus:

$$t_{eq} = \frac{t_p}{\Omega_p^2} = \frac{r_g}{2\Omega_p^2}.$$

We take $\Omega_{BH}(t_{eq}) = 1/2$, or even better, solve analytically the Friedman equations...

Recalculate now t_c with an account of the change of expansion regime from RD to MD. The average distance between BHs when $t > t_{eq}$, behaves as:

$$d(t) = d_p \left(\frac{t_{eq}}{t_p}\right)^{1/2} \left(\frac{t}{t_{eq}}\right)^{2/3} .$$

Now we find that the condition that the Hubble velocity is smaller than the virial one, for average values, reads:

$$\frac{4d_p^3}{9r_g t_p^{3/2} t_{eq}^{1/2}} < 1\,.$$

Here we took $H(t_c) = 2/(3t_c)$. But this condition is never fulfilled.

However, this negative result does not mean that the acceleration of BHs and GW emission are suppressed.

The life-time of BH w.r.t. evaporation is

$$\tau_{BH} \approx \frac{3 \times 10^3}{N_{eff}} \frac{M^3}{m_{Pl}^4},$$

where $N_{eff} \sim 100$ is the number of species with $m < T_{BH} = m_{Pl}^2/(8\pi M)$.

$$\frac{M}{m_{Pl}} > 2 \cdot 10^{7} \left(\frac{10^{-6}}{\Omega_p} \right) \sqrt{\frac{10^{-4}}{\Delta_{in}} \frac{N_{eff}}{100}} .$$

Such high density bunch of PBH would have mass:

$$M_b = \frac{16}{9} m_{Pl}^2 t_{eq} = \frac{M}{\Omega_p^2},$$

i.e. the mass inside horizon at $t = t_{eq}$.

The virial velocity inside this bunch is

$$v = \sqrt{\frac{2M_b}{m_{Pl}^2 R_b}} = \frac{4}{3}\Delta^{\frac{1}{6}} \left(\frac{m_{Pl}}{\Omega_p M}\right)^{\frac{2}{3}} \left(\frac{N_{eff}}{3000}\right)^{\frac{1}{3}}$$

Maximum velocity in the bunch is limited by the condition of sufficiently large M i.e. that $\Delta \equiv \delta\rho/\rho \geq 1$ **(p. 24)** **and reads:**

$$v_{max} \approx 0.06\Delta^{1/6} \left(\frac{\Delta_{in}}{10^{-4}}\right)^{-1/3}$$

and with Δ as large as 10^6 BHs can be moderately relativistic.

Taking $t_e = \tau_{BH}$, we find:

$$\Omega_{GW}(\tau_{BH}) = \frac{QN_{eff}}{18\pi \cdot 10^3} \frac{\omega_{max}}{M} v\Delta \, .$$

The first numerical factor is of order unity, ω_{max} can be about $r_g^{-1} \sim m_{Pl}^2/M$, $v \leq 0.1$, and $\Delta \sim 10^6$. So

$$\Omega_{max}(\tau_{BH}) \sim 10^5 \left(\frac{m_{Pl}}{M} \right)^2$$

NB: recall the bounds $m_{Pl}/M < \Omega_p/20$ and $m_{Pl}/M < 10^{-7}(\Omega_p/10^{-6})$.

Maximum value of ω is determined by the sensitivity of GW detectors. The red-shifted ω today:

$$\omega_0 = \omega \frac{T_0}{T(\tau_{BH})}.$$

$T(\tau_{BH})$ is found approximately from:

$$\rho = \frac{m_{Pl}^2}{6\pi\tau_{BH}^2} = \frac{\pi^2 g_* T^4}{30},$$

where $g_* = g_*(T) \approx 10^2$ is the number of particle species at this T.

$$T(\tau_{BH}) = \left(\frac{30}{6\pi g_*}\right)^{\frac{1}{4}} \left(\frac{N_{eff}}{3 \cdot 10^3}\right)^{\frac{1}{2}} \frac{m_{Pl}^{\frac{5}{2}}}{M^{\frac{3}{2}}}.$$

Substituting numbers:

$$T(\tau_{BH}) \approx 0.06 m_{pl} \left(\frac{m_{Pl}}{M}\right)^{3/2} .$$

For comparison at production moment:

$$T_p \approx 0.2 m_{Pl} \left(\frac{m_{Pl}}{M}\right)^{1/2} .$$

The frequency of such GW today:

$$\omega_0 = 2.7^o \left(\frac{M}{m_{Pl}}\right)^{3/2} \frac{\omega}{0.06 m_{Pl}}.$$

If we take maximum $\omega \sim m_{Pl}^2/M$, the GW frequency today would be:
$\omega_0 \sim (6 \cdot 10^{12}/s)\,(M/m_{Pl})^{1/2}$,
i.e. for $M = 10^5 g$, $\omega_0 \sim 1$ keV.
Usually results are expressed in terms of $f = \omega/(2\pi)$.

Now we can calculate Ω_{GW} today as a function of measured $f = \omega_0/(2\pi)$:

$$\Omega_{GW}^{(0)} = 10^{-5} v \Delta \left(\frac{f}{10^{11} Hz} \right) \left(\frac{m_{Pl}}{M} \right)^{5/2} .$$

In this expression the factor 10^{-4} is included which comes from dilution of relativistic gravitons at the late MD stage and it is assumed that $QN_{eff}/(18 \cdot 10^3 \pi) \approx 1$.

V. Classical (in contrast to quantum) emission of gravitational waves:

$$\frac{dE}{dt} = \frac{1}{45m_{Pl}^2}\frac{d^3D}{dt^3},$$

where $D \sim Mr^2$ is the quadrupole moment of two colliding BHs.

We calculate D using equation of motion:

$$\ddot{r} = -\frac{M}{m_{Pl}^2 r^2}.$$

Convenient variables:

$$z = r/R, \;\; t = m_{Pl} R \sqrt{r/M}\, t$$

where R is the initial value of the distance between black holes taken to be equal to the average distance.
In this variables the equation of motion takes a very simple form:

$$z'' = -1/z^2\,,$$

where prime means differentiation w.r.t. dimensionless time η.

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The estimated energy loss in a collision of two black holes:

$$\delta E = \frac{M}{315}\left(\frac{r_g}{r_{min}}\right)^{7/2},$$

where r_{min} is the minimal distance between the black holes during the collision.

A reasonable estimate $r_{min} = Sr_g$,, where S is a numerical factor larger than 1 but probably smaller than 10.

For $r \sim r_g$ our flat-space estimates are invalid.

The number of collisions per unit time and volume is:

$$\dot{n}_{coll} = \sigma n_{BH}^2 v \,,$$

where σ is the cross-section of the collision, which we take as $\sigma = S^2 r_g^2$; probably it is an underestimate.

Now we can evaluate the cosmological fraction of energy of the produced gravitational waves:

$$\Omega_{GW} = \frac{4S^{-3/2}}{1890\pi K}\left(\frac{m_{Pl}}{M}\right)^2 \Omega_{BH}^2 v\Delta$$

Taking into account that $\Omega_{BH} = 1$, $v\Delta \sim 10^{-5}$, $K = N_{eff}/3000 \approx 0.03$, and that relativistic matter is diluted by 10^4 at the contemporary matter dominated stage, we obtain:

$$\Omega_{GW}^{(0)} \approx 0.3 S^{-3/2} (m_{Pl}/M)^2 .$$

The expected frequency of such GW today can be estimated as follows. The characteristic frequency at production is $\omega_{prod} \sim 1/Sr_g$.

It is redshifted by the ratio of the temperature at production, T_{prod} to the present day $T_0 = 2.7^o$. Thus the detected frequency today should be:

$$\omega_{obs} \sim 7 \cdot 10^9 S^{-1} (M/m_{Pl})^{1/2} Hz$$

VI. GW from PBH binaries.

Gravitationally bound systems of PBH pairs captured by dynamical friction.

Luminosity of GW radiation from a single binary:

$$L = \frac{32 M_1^2 M_2^2 (M_1 + M_2)}{5 r^5} \approx \frac{64}{5} \frac{M^5}{r^5 m_{Pl}^8}.$$

If we take $r = \kappa r_g$, $\kappa \gg 1$, then

$$L = \frac{2 m_{Pl}^2}{5 \kappa^5}.$$

Average distance between BHs in the high density bunch:

$$d_b = 0.1 r_g \Omega_p^{\frac{2}{3}} \left(\frac{M}{m_{Pl}}\right)^{\frac{4}{3}} \left(\frac{100}{N}\right)^{\frac{2}{3}} \left(\frac{10^6}{\Delta}\right)^{\frac{1}{3}},$$

so $\kappa < 0.1 \Omega_p^{2/3} (M/m_{Pl})^{4/3}$.

Total rate of GW radiation from the binaries, if the binaries make fraction ϵ in the high density bunch:

$$\dot{\rho}_{GW} = \epsilon L n_{BH}^{b} = 10^6 \epsilon \rho_c(\tau_{BH}) \frac{\Delta}{10^6} \frac{L}{M}.$$

Total emitted energy $\rho_{GW} = \dot{\rho}_{GW} t_{eff}$, $t_{eff} \sim \kappa r_g$ is the coalescence time. Total cosmological fraction of GW energy now:

$$\Omega_{GW}^{(0)} = 10^{-4+6} \epsilon t_{eff} \frac{\Delta}{10^6} \frac{L}{M}.$$

Substituting proper numbers and expressions:

$$\Omega_{GW}^{(0)} = 10^2 \epsilon \kappa^{-4} > \frac{10^6 \epsilon}{\Omega_p^{8/3}} \left(\frac{m_{Pl}}{M} \right)^{16/3} .$$

This is a lower bound because we took maximum κ.

Since $M/m_{Pl} > 20/\Omega_p$:

$$\Omega_{GW}^{(0)} \sim 10^6 \epsilon \left(\frac{\Omega_p}{400} \right)^{8/3}$$

VIII. BH formation mechanism.

With flat spectrum of perturbations the probability of BH formation is low, $\Omega_p \ll 1$. Large density perturbations at small scales after inflation could be created by a massive scalar field with general renormalizable coupling to the inflaton (AD, J. Silk):

$$\lambda(\Phi - \Phi_1)^2|\chi|^2 .$$

Log-normal mass spectrum of BH.

Assume that χ lives in CW potential.
If $m_\chi < H_{infl}$ but $m_\chi > H_{heat}$,
χ would acquire large value during inflation but with low probability.
MD bubbles of χ in small fraction of volume could be formed with large density contrast.
The bubbles are matter dominated and cold, hence the bounds presented above may be not applicable.