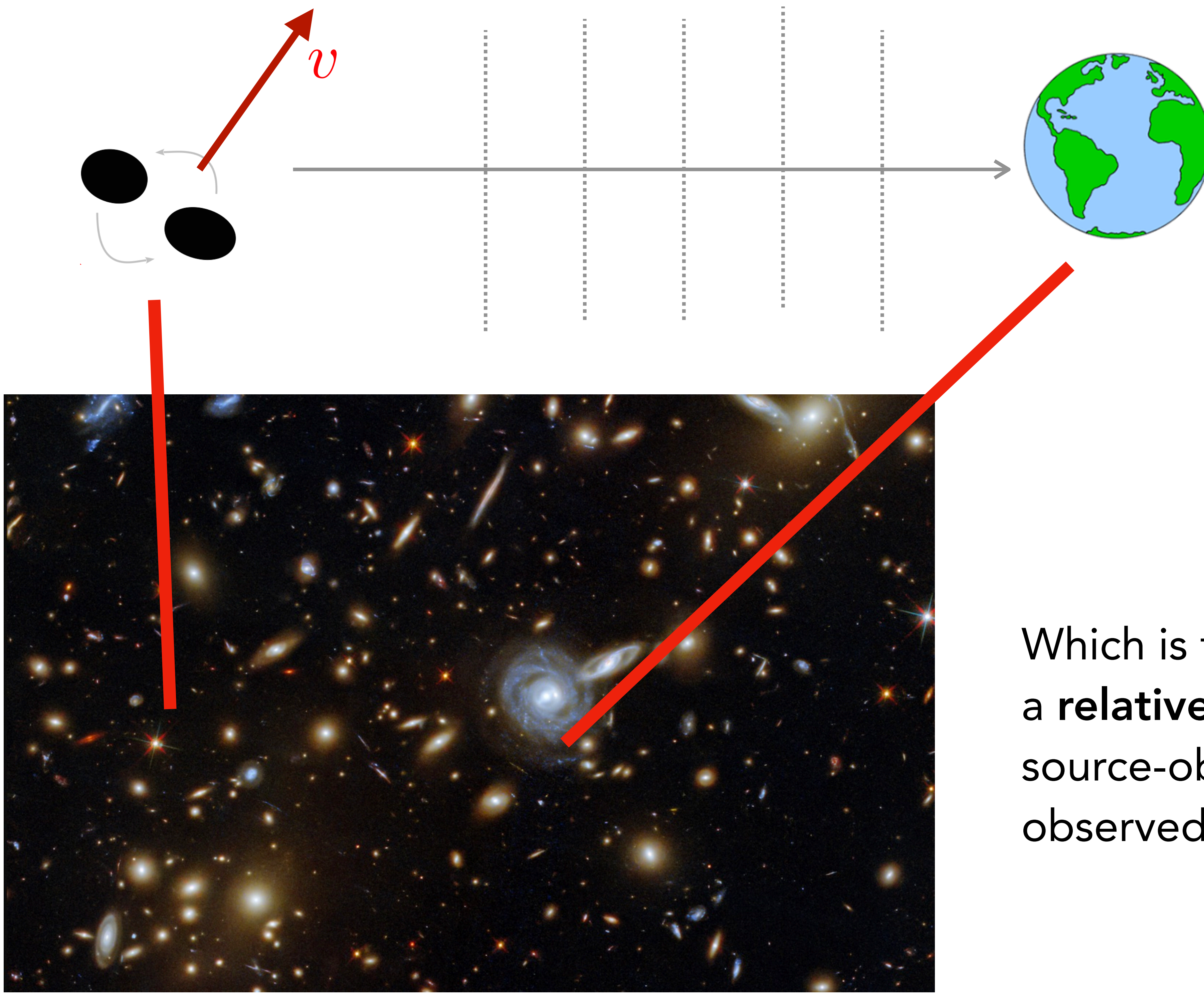
Aberration of gravitational waveforms by peculiar velocities

Giulia Cusin

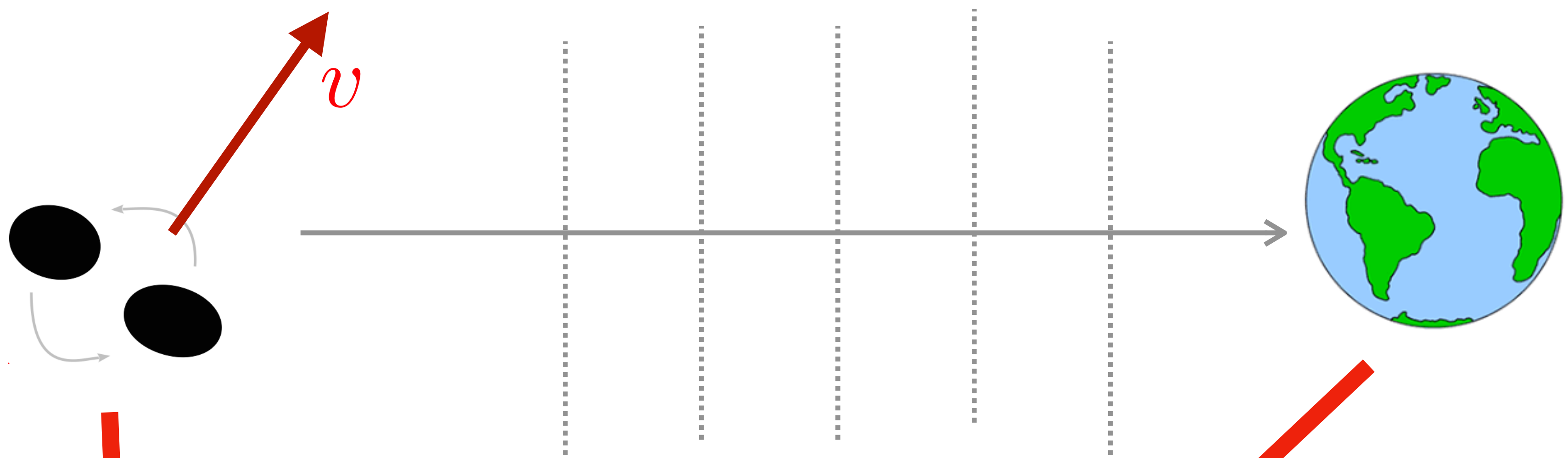
EuCAPT workshop, June 15-17, Rome

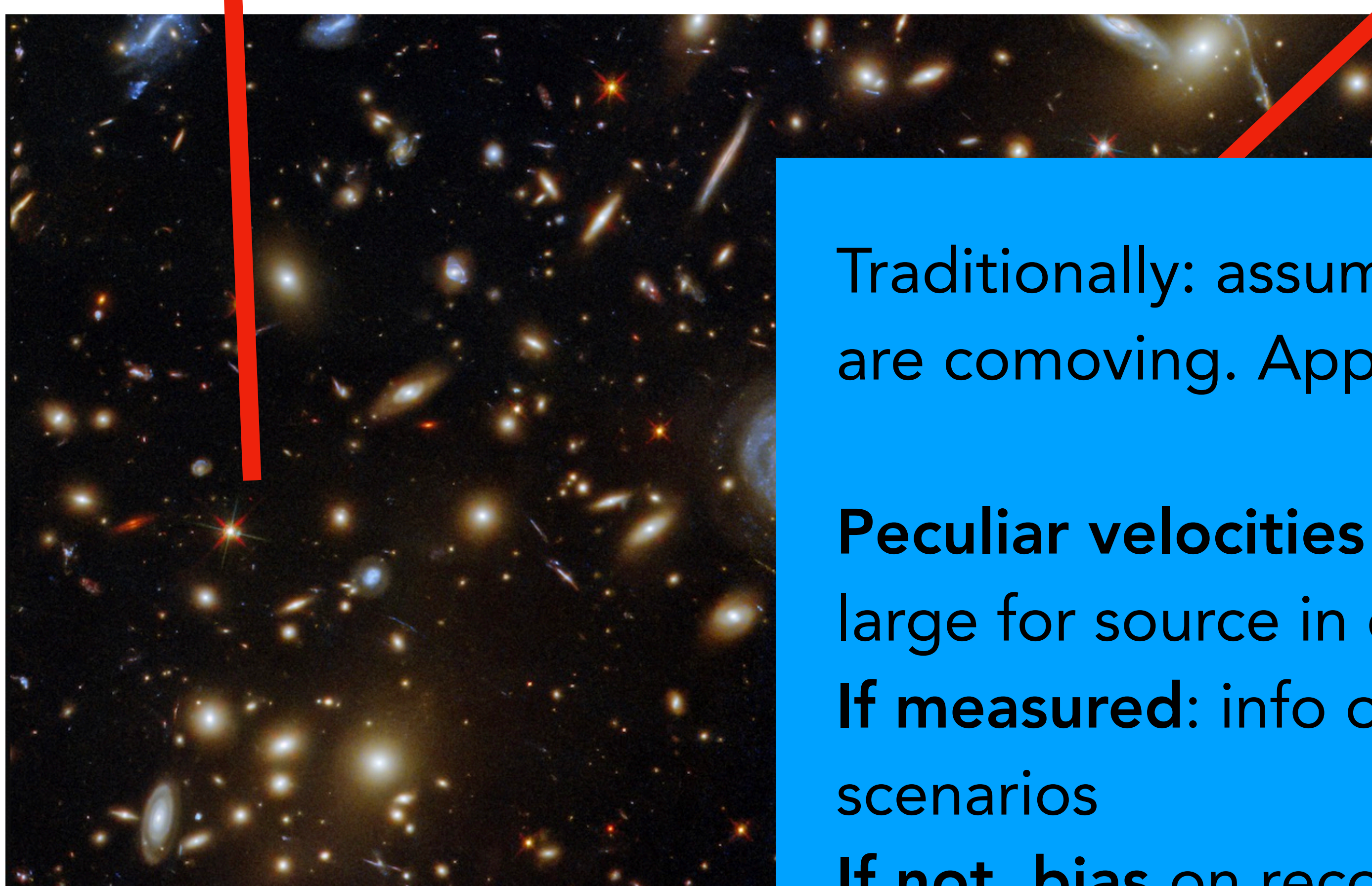
Outline



Which is the effect of a **relative motion** source-observer on an observed waveform?

Outline





Traditionally: assumed that source and observer are comoving. Approximation!

Peculiar velocities and accelerations can be quite large for source in dense environment

If measured: info on environment and formation scenarios

If not, bias on reconstitution orbital parameters, that needs to be accounted for

Outline

We separate effects along line of sight and perpendicular to line of sight

$v_{||}$

Constant velocities along the line of sight: unobservable — shift of chirp mass

Acceleration along the line of sight: modification of the chirp (via time-variation of redshift). When detectable, it can provide us with environmental information (formation scenarios…)

Review

Bonvin et al. 2016, Tamanini et al. 2019

Outline

We separate effects along line of sight and perpendicular to line of sight

$v_\perp$

GW is a spin-2 object, it transforms as a **tensor under boosts**: non-transverse components are generated by the presence of peculiar velocities $v_\perp$

In the observer frame, **spin-1 quantities** are generated

How does this effect manifest itself on observable quantities? Which are the **observational implications**?

Bonvin, Cusin, Mastrogiovanni et al. in prep
Mastrogiovanni, Foffa, Cusin et al. in prep

New

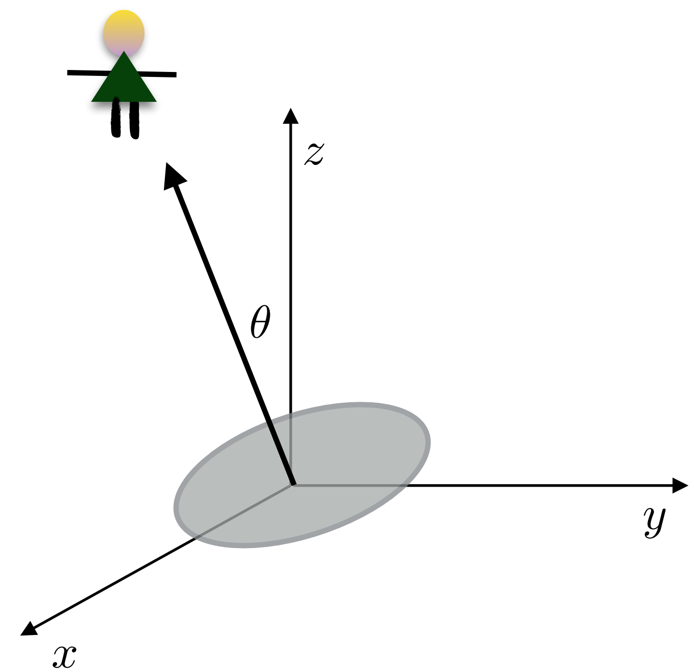
Motion along the line of sight

(waveform modified via time-variation of redshift)

Bonvin et al. 2016, Tamanini et al. 2019

Binary system of compact objects: polarisation modes

$$\boxed{\begin{aligned} h_+(t) &= \frac{4}{r}\left(\frac{GM_c}{c^2}\right)^{5/3}\left(\frac{\pi f_{\rm gw}(t)}{c}\right)^{2/3}\left(\frac{1+\cos^2\theta}{2}\right)\cos[\Phi(t)] \\ h_\times(t) &= \frac{4}{r}\left(\frac{GM_c}{c^2}\right)^{5/3}\left(\frac{\pi f_{\rm gw}(t)}{c}\right)^{2/3}\cos\theta\sin[\Phi(t)] \end{aligned}}$$



Both the frequency and the amplitude increase as the coalescence is approached

$$\dot{f}_{\rm gw} = \frac{96}{5}\pi^{8/3}\left(\frac{GM_c}{c^3}\right)^{5/3} f_{\rm gw}^{11/3} \rightarrow f_{\rm gw}(\tau) = \frac{1}{\pi}\left(\frac{5}{256}\frac{1}{\tau}\right)^{3/8}\left(\frac{GM_c}{c^3}\right)^{-5/8}$$

$$\Phi(t) = 2\pi\int_{t_0}^{t} dt' f_{\rm gw}(t') \longrightarrow \Phi(\tau) = -2\left(\frac{5GM_c}{c^3}\right)^{-5/8}\tau^{5/8} + \Phi_0$$

Binary system of compact objects: polarisation modes

$$h_+(t) = \frac{4}{r}\left(\frac{GM_c}{c^2}\right)^{5/3}\left(\frac{\pi f_{\rm gw}(t)}{c}\right)^{2/3}\left(\frac{1+\cos^2\theta}{2}\right)\cos[\Phi(t)]$$

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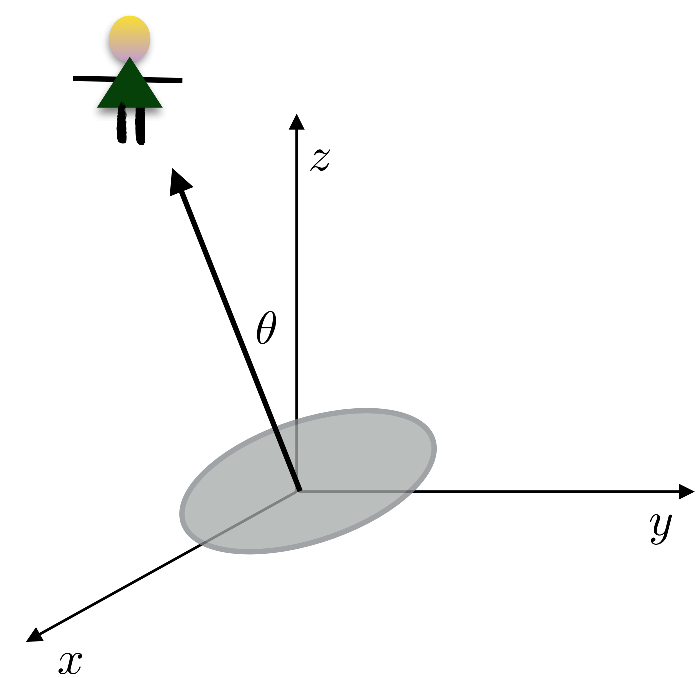
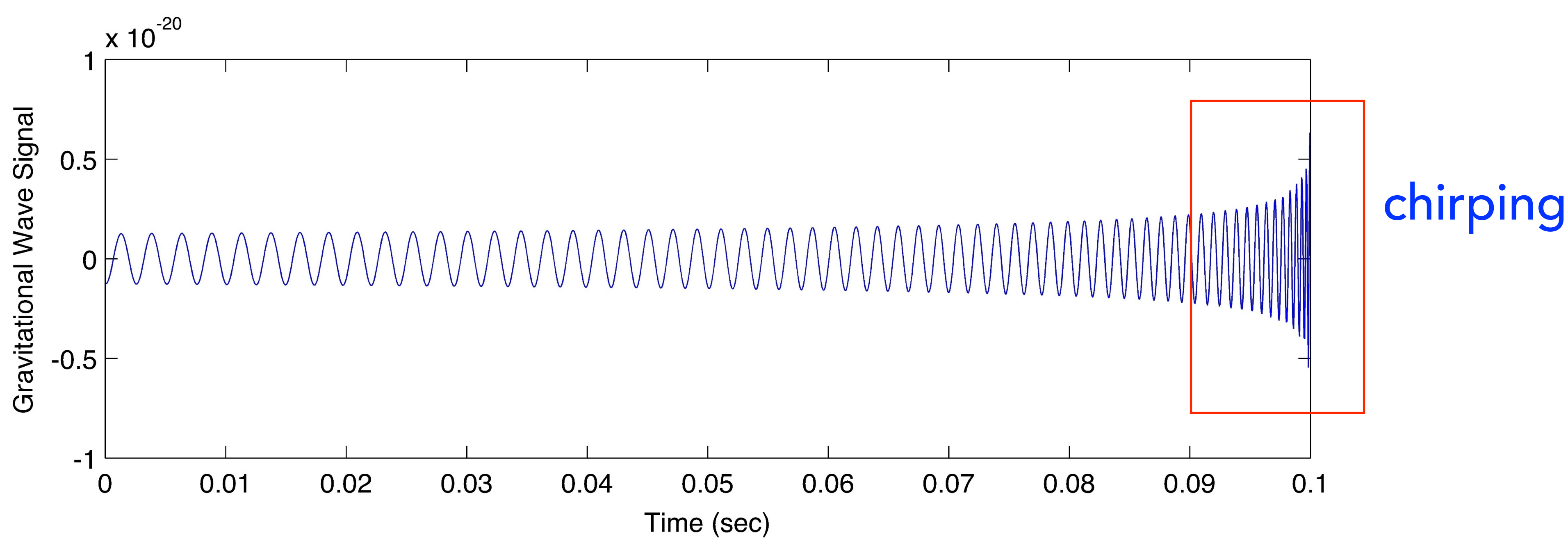


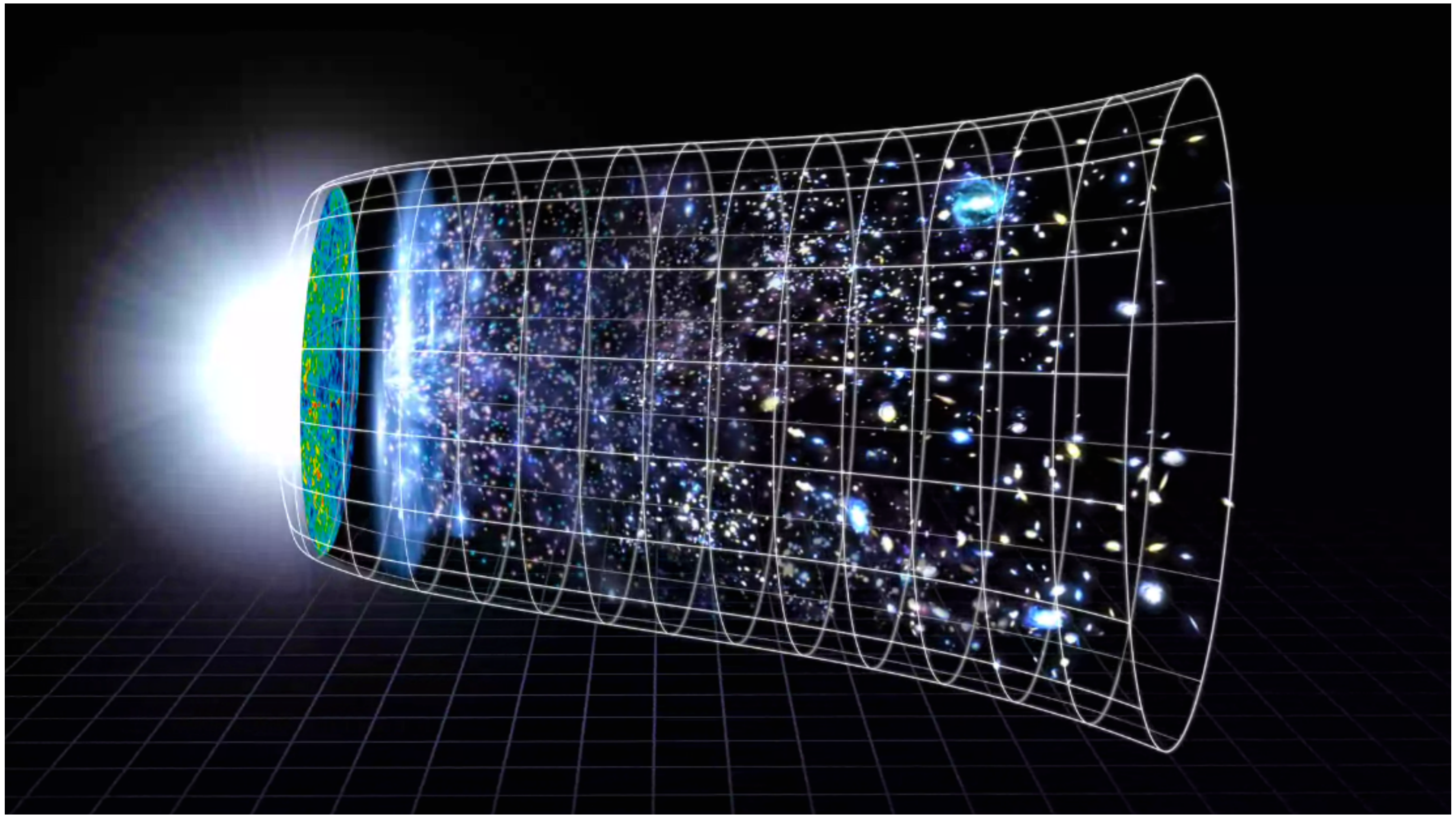
Looking at the chirp I can extract the chirp mass



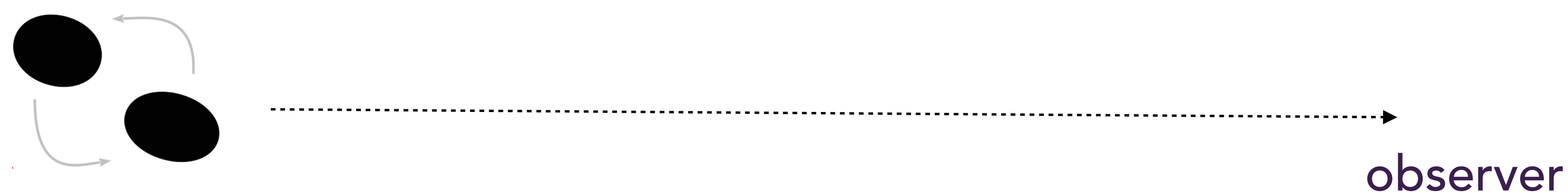
Waveform in a cosmological context

Up to now we considered: flat universe with no expansion and no perturbations

We want to rewrite the waveform at the observer accounting for the fact that Universe is expanding (and that there are cosmological perturbations)



Waveform in the source frame



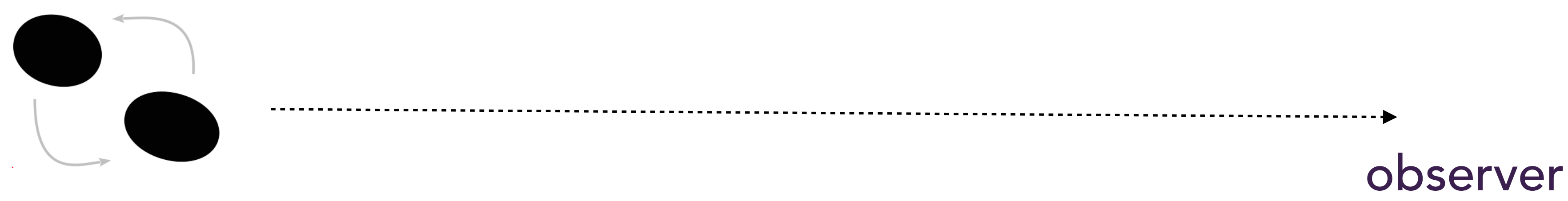
Observed polarisation in terms of quantities defined in the **source frame**

$$h_+(\tau_s) = \frac{4}{a_s r}(GM_c)^{5/3}(\pi f_s(\tau_s))^{2/3}\frac{1+\cos^2\theta}{2}\cos(\phi_s(\tau_s))$$

$$h_\times(\tau_s) = \frac{4}{a_s r}(GM_c)^{5/3}(\pi f_s(\tau_s))^{2/3}\cos\theta\sin(\phi_s(\tau_s))$$

Scale factor accounts for expansion of the Universe

From source frame to observer frame



I want to rewrite this in the **observer frame**

$$h_+(\tau_s) = \frac{4}{a_s r}(GM_c)^{5/3}(\pi f_s(\tau_s))^{2/3}\frac{1+\cos^2\theta}{2}\cos(\phi_s(\tau_s))$$

$$h_\times(\tau_s) = \frac{4}{a_s r}(GM_c)^{5/3}(\pi f_s(\tau_s))^{2/3}\cos\theta\sin(\phi_s(\tau_s))$$

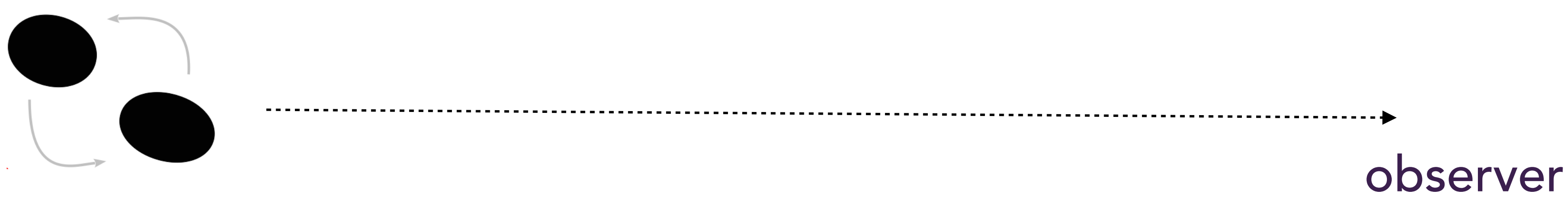
$$\phi_s(\tau_s) = \phi_o(\tau_o(\tau_s))$$

phase is constant along null geodesics

$$f_s = (1+z_s)f_o$$

$$d_p = a_s r = d_L/(1+z_s)$$

Observer frame



Final result in terms of quantities **in the observer frame**

$$M_c(1+z) \qquad f_o = f_s/(1+z)$$

$$h_+(\tau_o) = \frac{4}{d_L}(G\mathcal{M}_c(z))^{5/3}(\pi f_o(\tau_o))^{2/3}\frac{1+\cos^2\theta}{2}\cos(\phi_o(\tau_o))$$

$$h_\times(\tau_o) = \frac{4}{d_L}(G\mathcal{M}_c(z))^{5/3}(\pi f_o(\tau_o))^{2/3}\cos\theta\,\sin(\phi_o(\tau_o))$$

Evolution of frequency in observer frame

Source frame

$$\frac{df_s}{dt_s} = CM_c^{5/3} f_s^{11/3}$$

Observer frame

$$\boxed{(1+z_s)\frac{d((1+z_s)f_o)}{dt_o} = CM_c^{5/3} f_o^{11/3}(1+z_s)^{11/3}}$$

$$1 + z_s = \frac{a_o}{a_s}\left[1 + \mathbf{n}\cdot\mathbf{v_s} + \ldots\right]$$

background contribution

perturbations

Evolution of frequency in observer frame

Source frame

$$\frac{df_s}{dt_s} = C M_c^{5/3} f_s^{11/3}$$

Observer frame

$$\boxed{(1+z_s)\frac{d((1+z_s)f_o)}{dt_o} = C M_c^{5/3} f_o^{11/3} (1+z_s)^{11/3}}$$

$$1 + z_s = \frac{a_o}{a_s}[1 + \mathbf{n}\cdot\mathbf{v_s} + \ldots]$$

background contribution

perturbations

If we **neglect redshift evolution**: effects is a (unobservable) shift in chirp mass

$$\frac{df_o}{dt_o} = C(\overbrace{M_c(1+z_s)}^{\mathcal{M}_c(z_s)})^{5/3} f_o^{11/3}$$

Evolution of redshift: modification of the chirp

If we solve the equation taking into account redshift evolution

$$f(\tau_o) \propto (G\mathcal{M}_c(z_s))^{-5/8}\,[1 + Y(z_s)\tau_o]$$

$$Y(z_s) = X(z_s) + \delta X(z_s)$$

$$X(z_s) = \frac{1}{2}\left(H_0 - \frac{H_s}{1+\bar{z}_s}\right)$$

$$\delta X(z_s) = \frac{1}{2}\left(\frac{\dot{\mathbf{v}}_\mathbf{s}\cdot\mathbf{n}}{1+\bar{z}_s} + \dots\right)$$

background contribution: no significant imprint

Peculiar acceleration along the line of sight: detectable imprint!

Bonvin et al. 2016

Peculiar motion along the line of sight: summary

—Variation of velocities **along line of sight** induces **time-variation in the redshif**t, which in turn modifies chirp

—When going to Fourier space using stationary phase approximation, effect on both **phase** and **amplitude** of the wave

—To be relevant, **we need time for the effects to "accumulate"** : target for LISA

—Long-lived stellar mass black hole: peculiar acceleration measurable by LISA with 10 years mission, giving info on formation scenario (e.g. in AGN disks)

—For large fraction of LISA events: bias in the reconstruction of source parameters (e.g. luminosity distance)

Tamanini et al. 2019

Motion perpendicular the line of sight (polarisation mixing)

Bonvin, **Cusin**, Mastrogiovanni, Gair, Congedo, Tamanini (in prep)

Motion perpendicular to the line of sight: effect on polarisation

Up to now we have treated the two polarisation separately (as if they where scalar objects) and considered effects of motion along the line of sight on each one

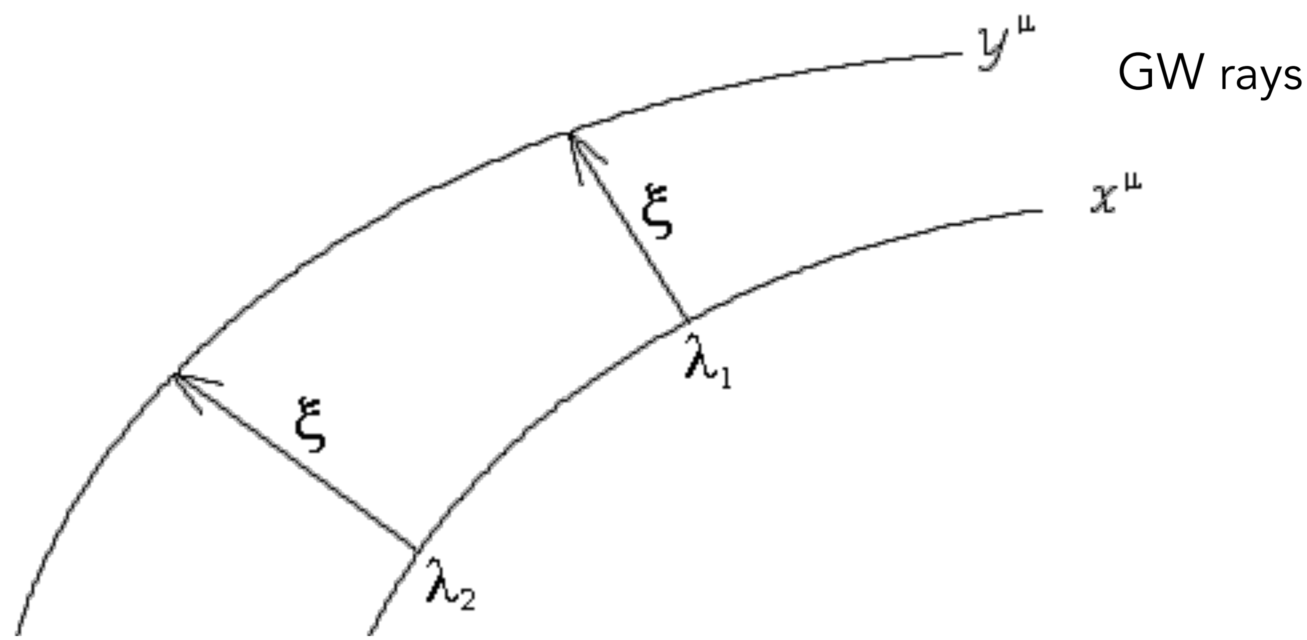
But…GW is a spin-2 object, it transforms as a **tensor under boosts**: full polarisation structure needs to be considered to fully study the effect of peculiar motion

Polarizations of a GW: general concepts

Effect of GW on test particles can be described by

$$\frac{d^2\xi^i}{dt^2} = \boxed{\mathcal{R}_{0i0j}}\,\xi^j$$ **geodesic deviation equation** (ξ^i vector between two nearby rays)

$$P_{ij}(t) \equiv \mathcal{R}_{0i0j}$$ **driving force matrix** (proportional to the GW in TT gauge)



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In a generic theory of gravity: 6 polarisations. For a wave propagating along z

$$P_{ij}(t) = \begin{pmatrix} -\mathrm{Re}\Psi_4 - \Phi_{22} & \mathrm{Im}\Psi_4 & -2\sqrt{2}\mathrm{Re}\Psi_3 \\ \mathrm{Im}\Psi_4 & \mathrm{Re}\Psi_4 - \Phi_{22} & 2\sqrt{2}\mathrm{Im}\Psi_3 \\ -2\sqrt{2}\mathrm{Re}\Psi_3 & 2\sqrt{2}\mathrm{Im}\Psi_3 & -6\Psi_2 \end{pmatrix}$$

general relativity: plus and cross polarisations

Polarisations transverse to the polarisation plane (modified gravity)

Ψ_4

2 dofs

Ψ_3

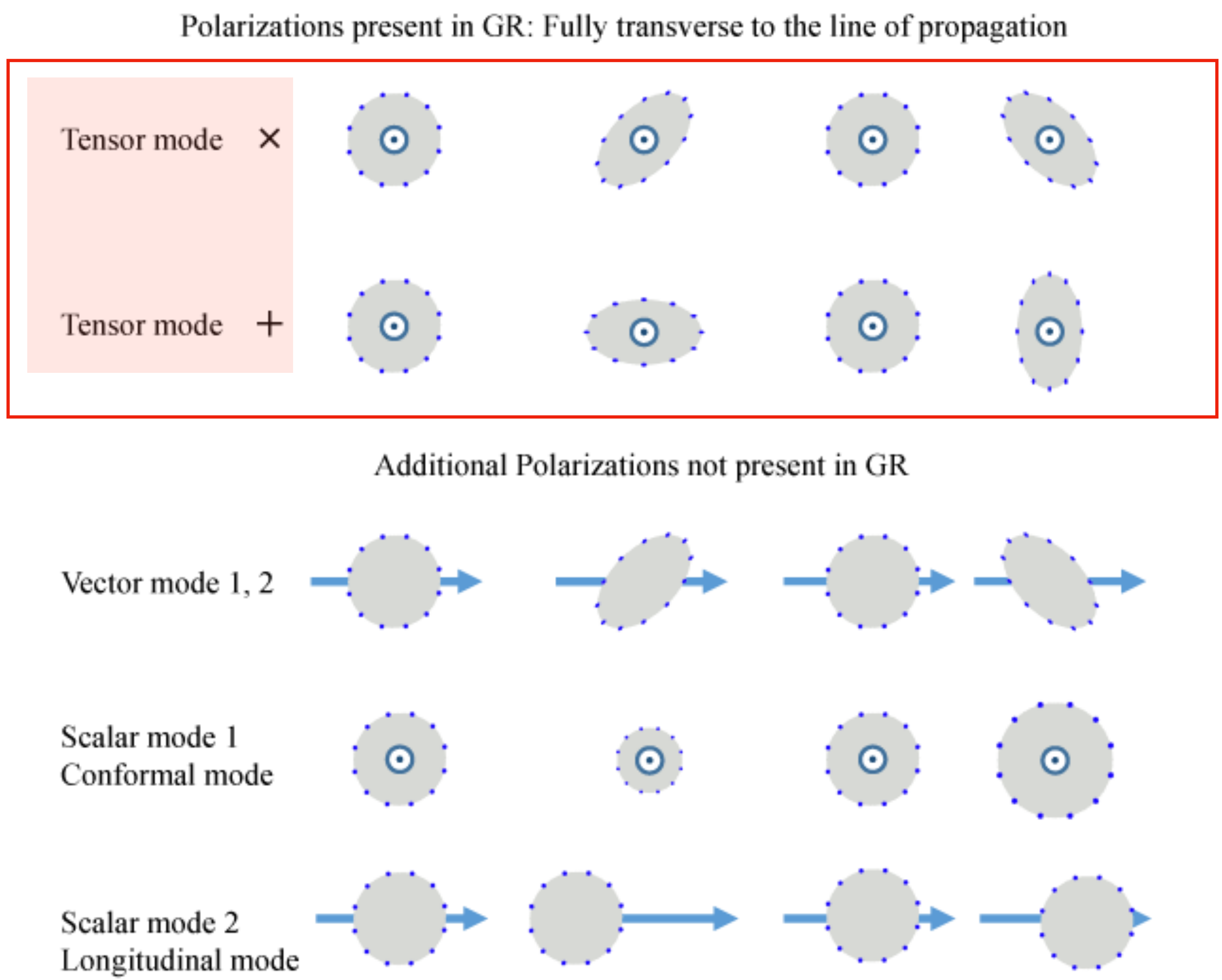
2 dofs

Φ_{22}

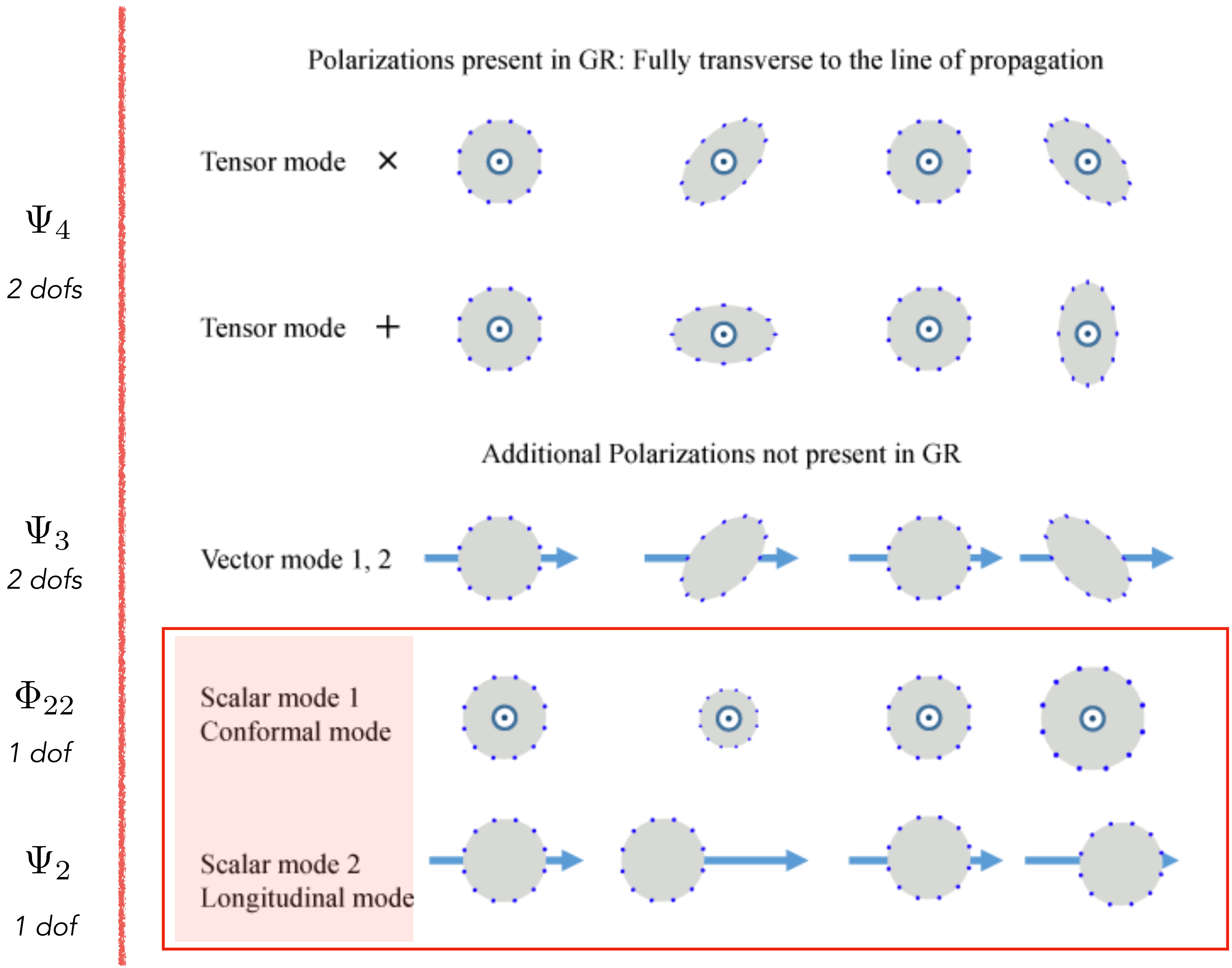
1 dof

Ψ_2

1 dof



C de Rham "Massive gravity"



C de Rham "Massive gravity"

Ψ_4

2 dofs

Ψ_3

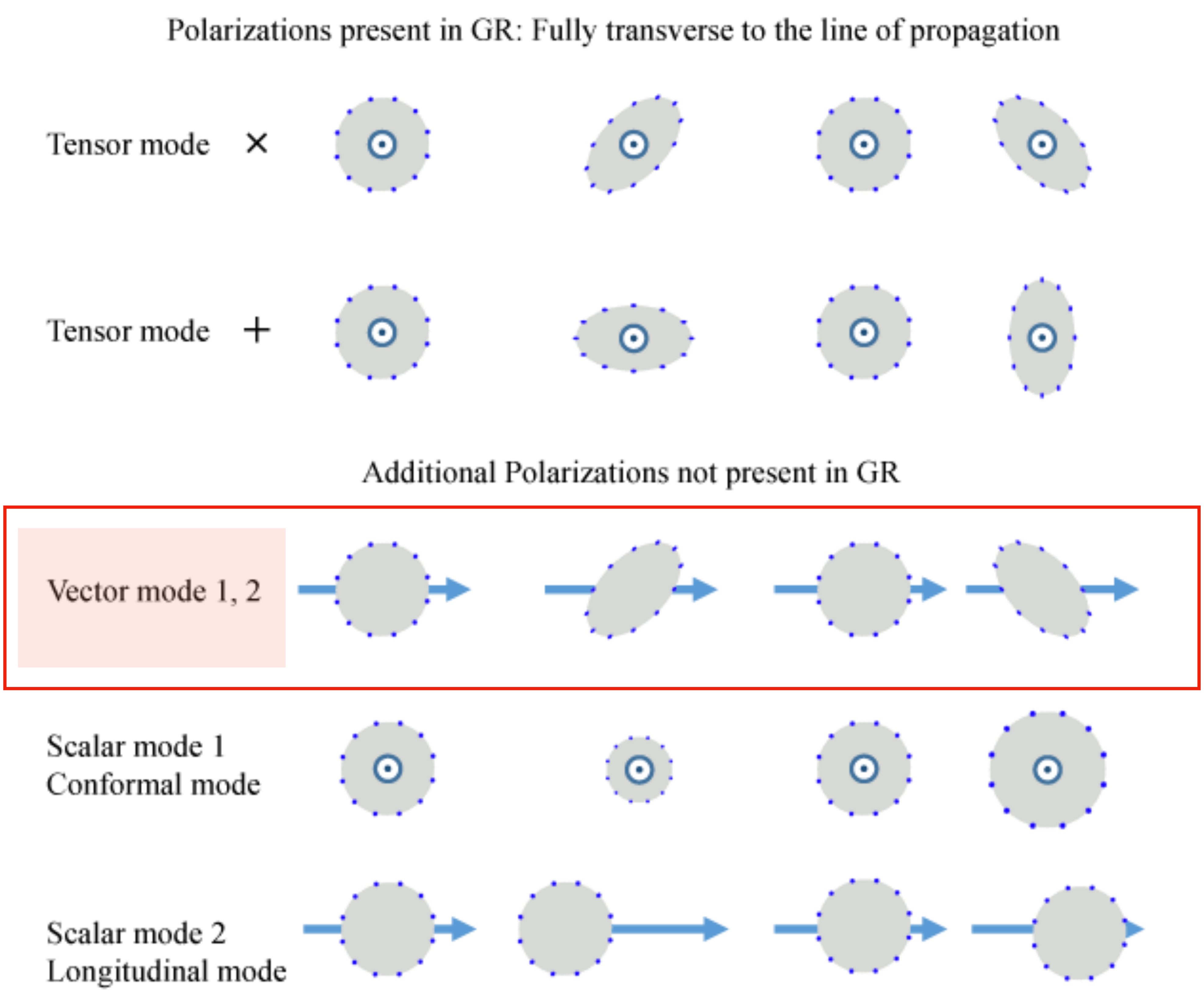
2 dofs

Φ_{22}

1 dof

Ψ_2

1 dof



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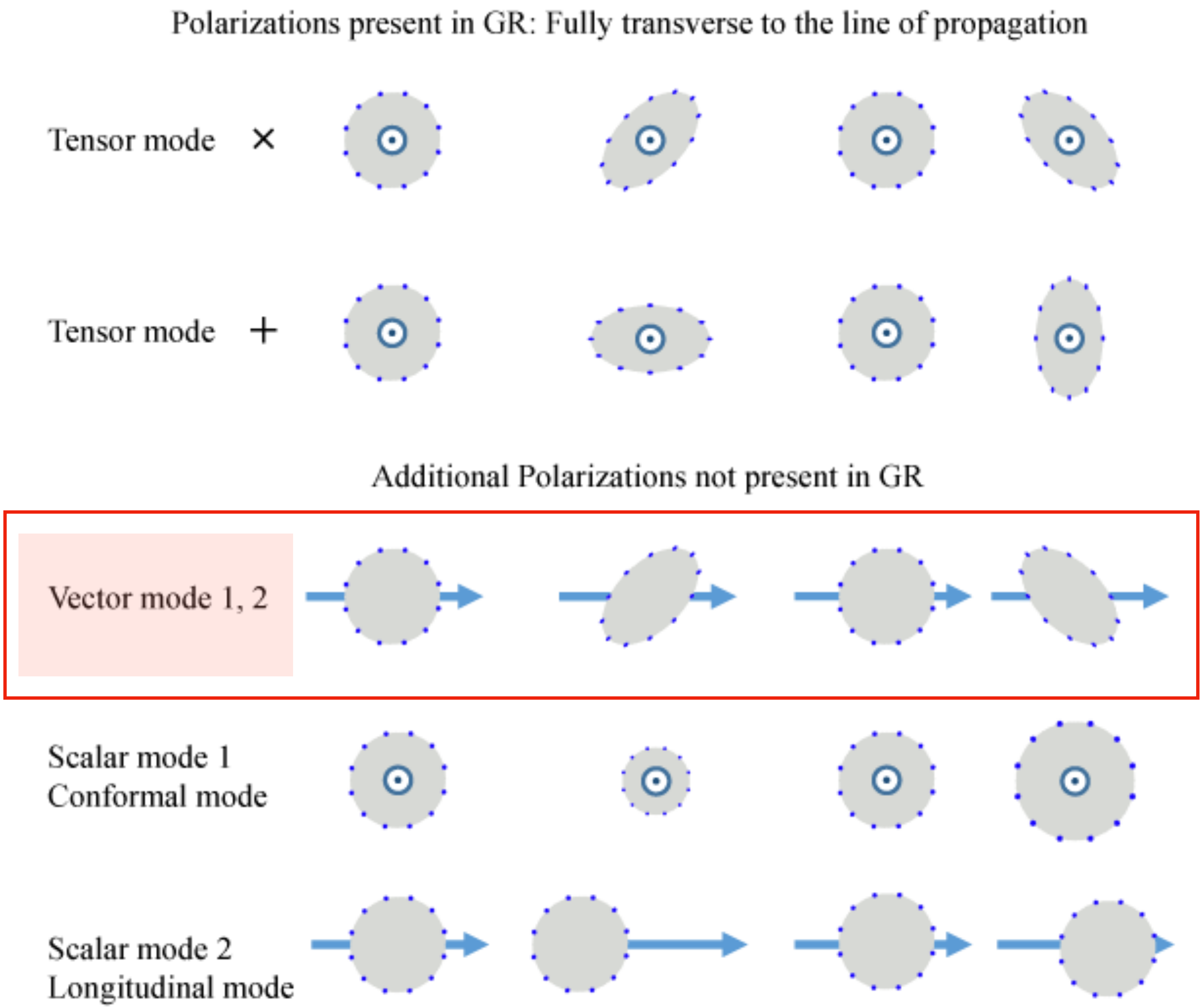

Ψ_4

2 dofs

Ψ_3

2 dofs

Φ_{22}

1 dof

Ψ_2

1 dof

Vector mode excited in GR by peculiar velocities (transverse)

C de Rham "Massive gravity"

Polarizations of a GW: general concepts

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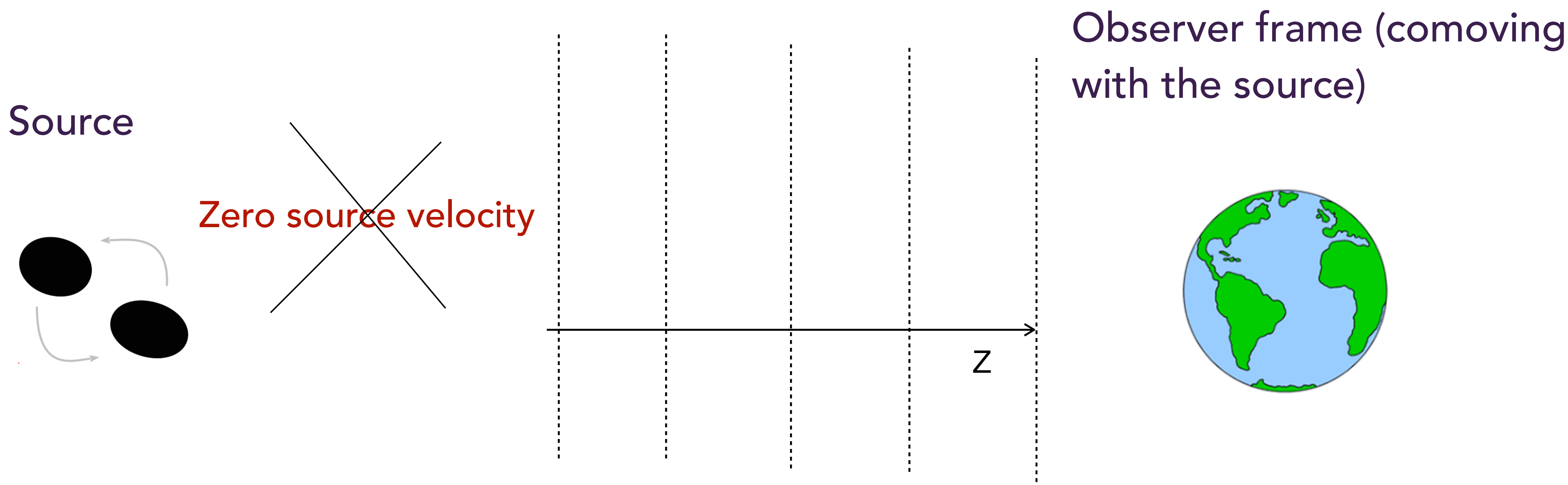
$$P_{ij}(t) \equiv \mathcal{R}_{0i0j}$$ **driving force matrix** (proportional to the GW in TT gauge)

For a wave propagating along the z direction

$$P_{ij}(t) = \begin{pmatrix} -\mathrm{Re}\Psi_4 - \Phi_{22} & \mathrm{Im}\Psi_4 & -2\sqrt{2}\mathrm{Re}\Psi_3 \\ \mathrm{Im}\Psi_4 & \mathrm{Re}\Psi_4 - \Phi_{22} & 2\sqrt{2}\mathrm{Im}\Psi_3 \\ -2\sqrt{2}\mathrm{Re}\Psi_3 & 2\sqrt{2}\mathrm{Im}\Psi_3 & -6\Psi_2 \end{pmatrix}$$

Apparent transverse polarisations appear in general relativity in the presence of a relative motion source-observer

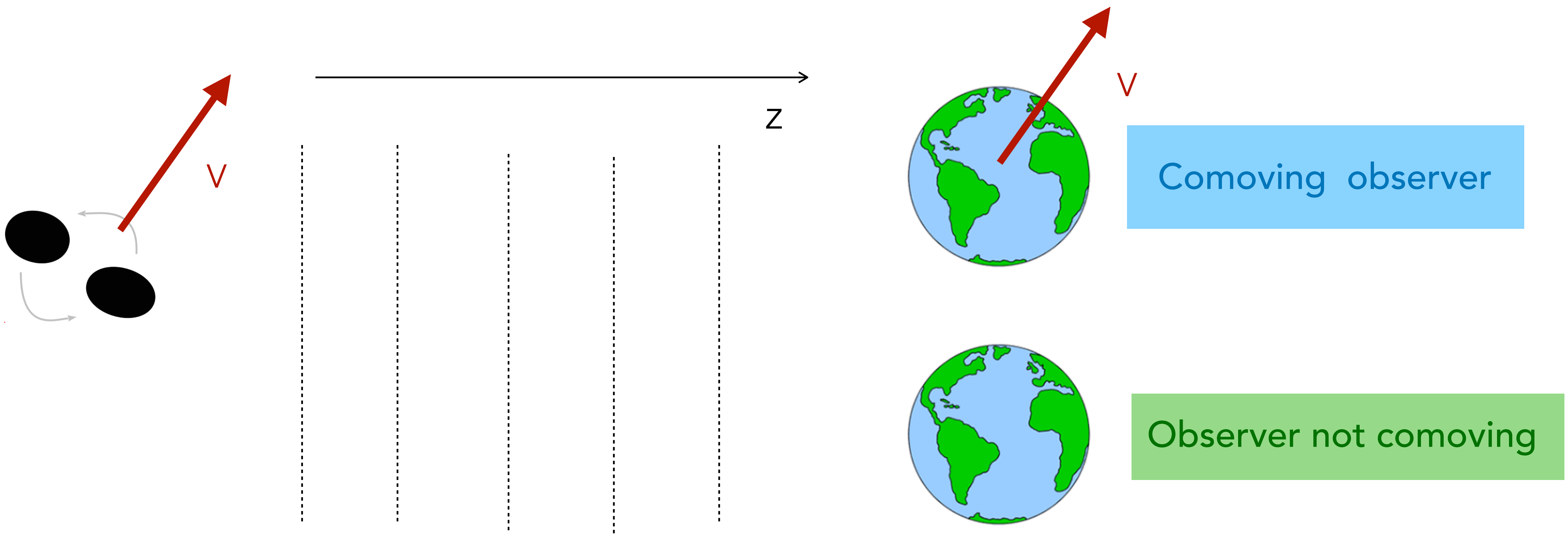
Observed GW signal: no relative velocity observer-source



Observed GW signal for wave propagating along z

$$P_{ij} = \frac{1}{2} \begin{pmatrix} h_+ & h_\times & 0 \\ h_\times & -h_+ & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Observed strain: relative velocity source-observer



GW for observer non-comoving

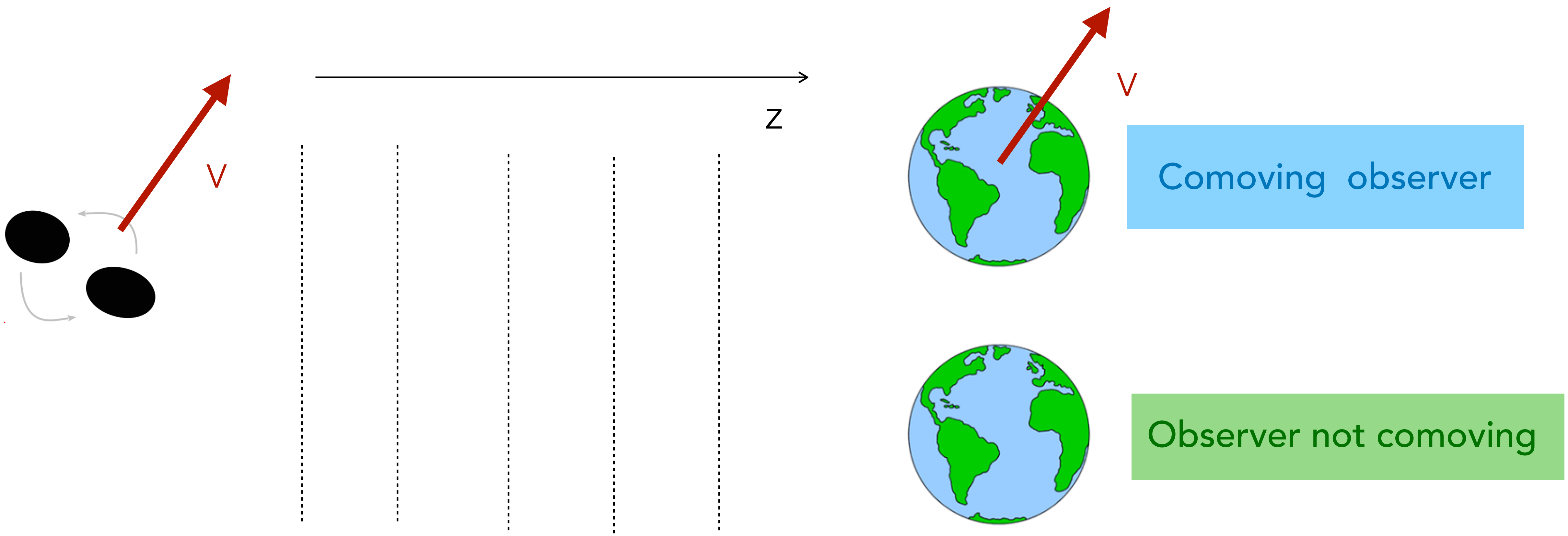
GW for observer comoving

Boost transformation

$$\boxed{h_{\mu\nu} = \Lambda_{\mu}{}^{\alpha}\Lambda_{\nu}{}^{\beta}\tilde{h}_{\alpha\beta}}$$

Lorentz Matrix

Observed strain: relative velocity source-observer



Observed gravitational wave propagating along z (in TT gauge)

$$P_{ij} = \begin{pmatrix} \tilde{h}_+ & \tilde{h}_\times & 0 \\ \tilde{h}_\times & -\tilde{h}_+ & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad P_{ij} = \begin{pmatrix} \tilde{h}_+ & \tilde{h}_\times & v_x\tilde{h}_+ + v_y\tilde{h}_\times \\ \tilde{h}_\times & -\tilde{h}_+ & v_x\tilde{h}_\times - v_y\tilde{h}_+ \\ v_x\tilde{h}_+ + v_y\tilde{h}_\times & v_x\tilde{h}_\times - v_y\tilde{h}_+ & 0 \end{pmatrix},$$

Comoving observer

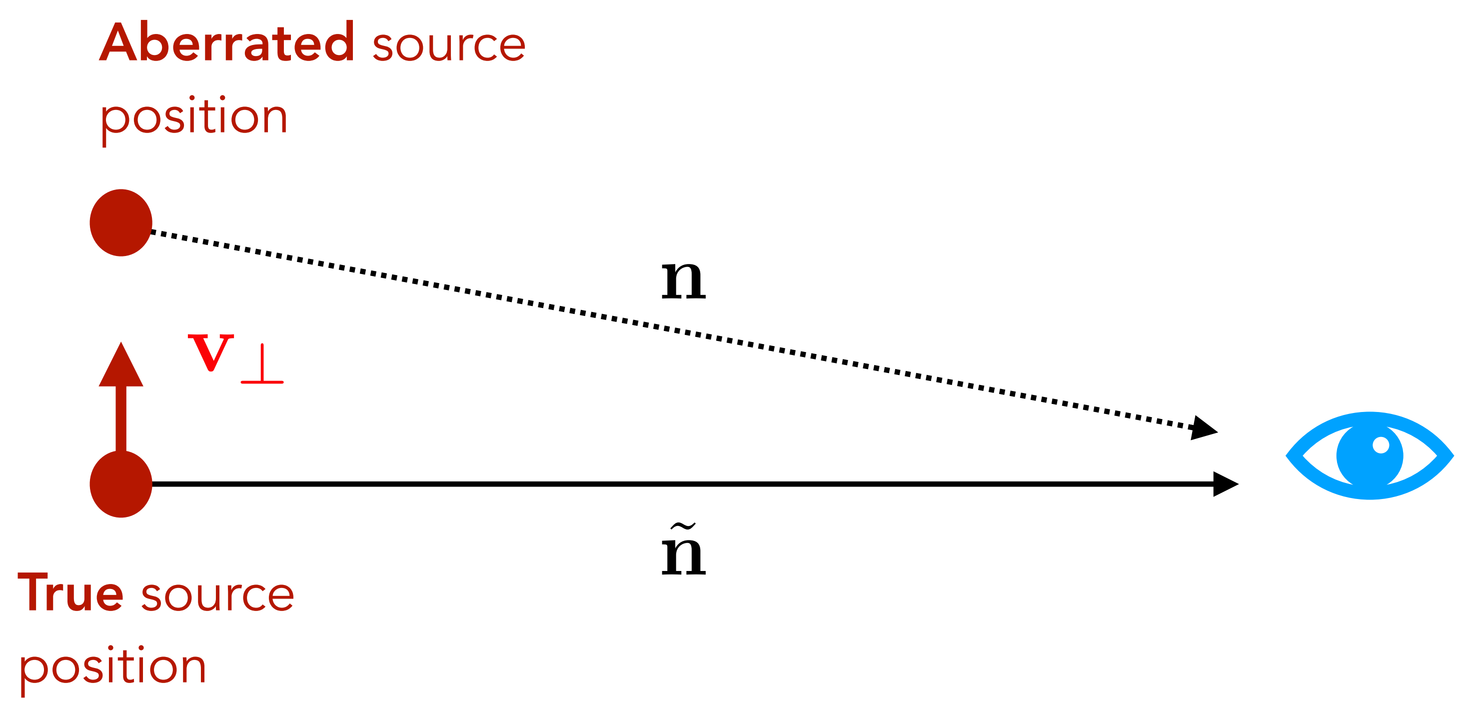
Relative velocity source-observer: spin-1 modes excited as an effect of aberration

Observationally, what do we actually see?

In the presence of a peculiar motion, the **direction of propagation is aberrated**

$$\mathbf{n} = \frac{\mathbf{k}}{|\mathbf{k}|} = \tilde{\mathbf{n}} - \boxed{\mathbf{v}_\perp}$$

Transverse velocity



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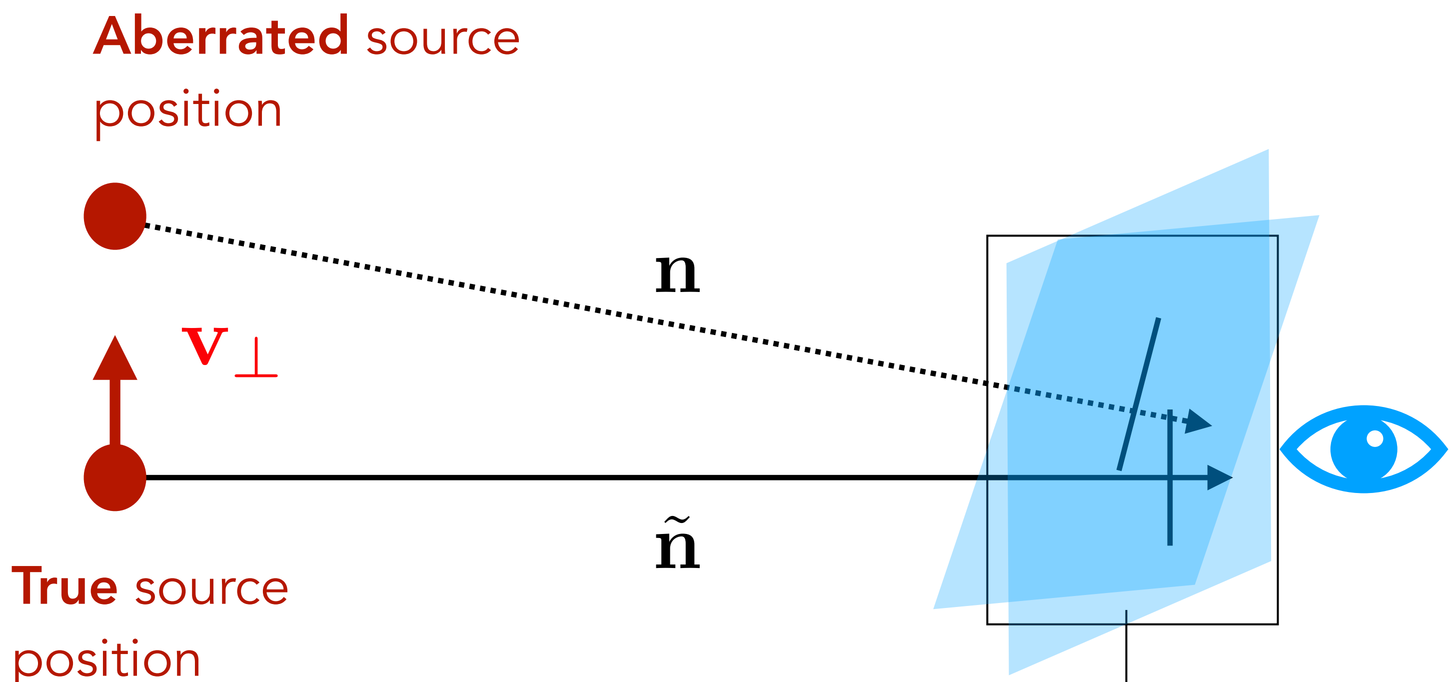
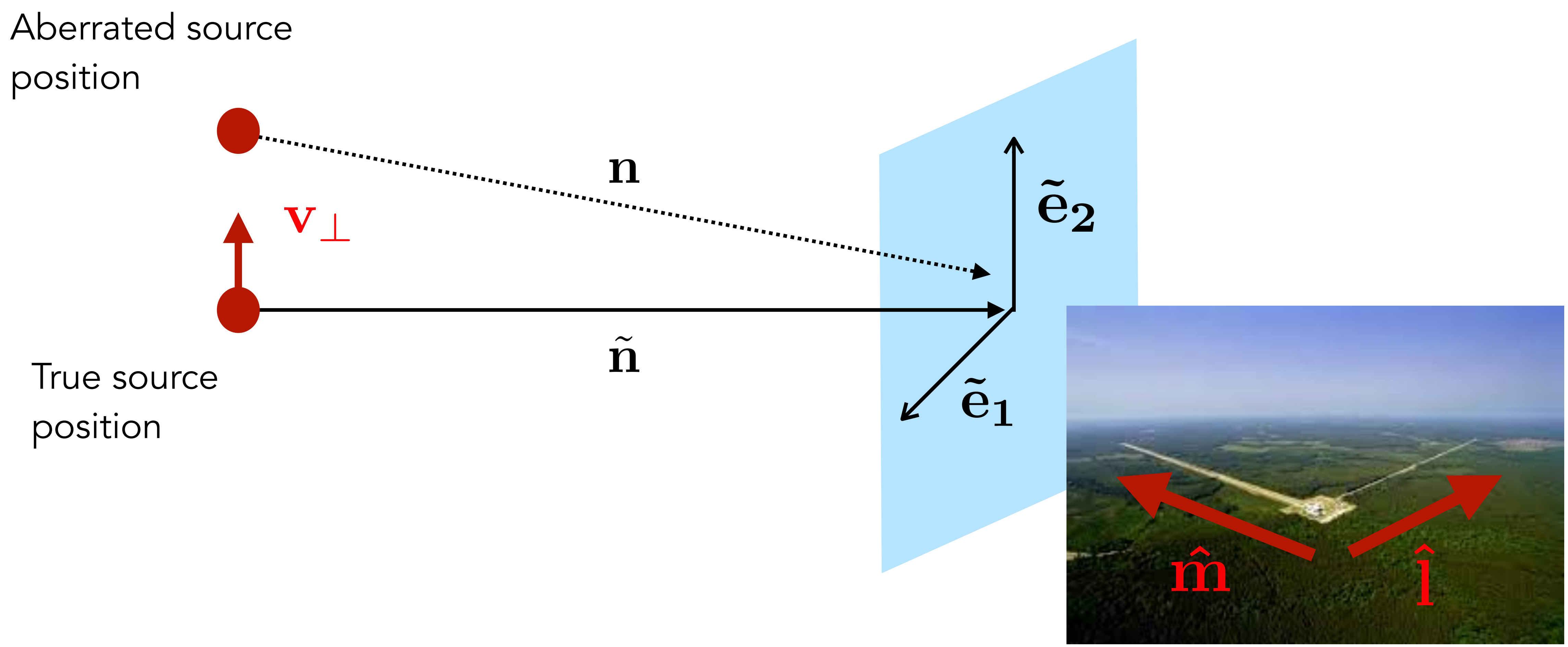


Polarisation plane is orthogonal to the observed direction, hence is aberrated as well

Observed strain with respect to non-aberrated polarisation basis

If we could access the non-aberrated (true) source position $\tilde{\mathbf{n}}$



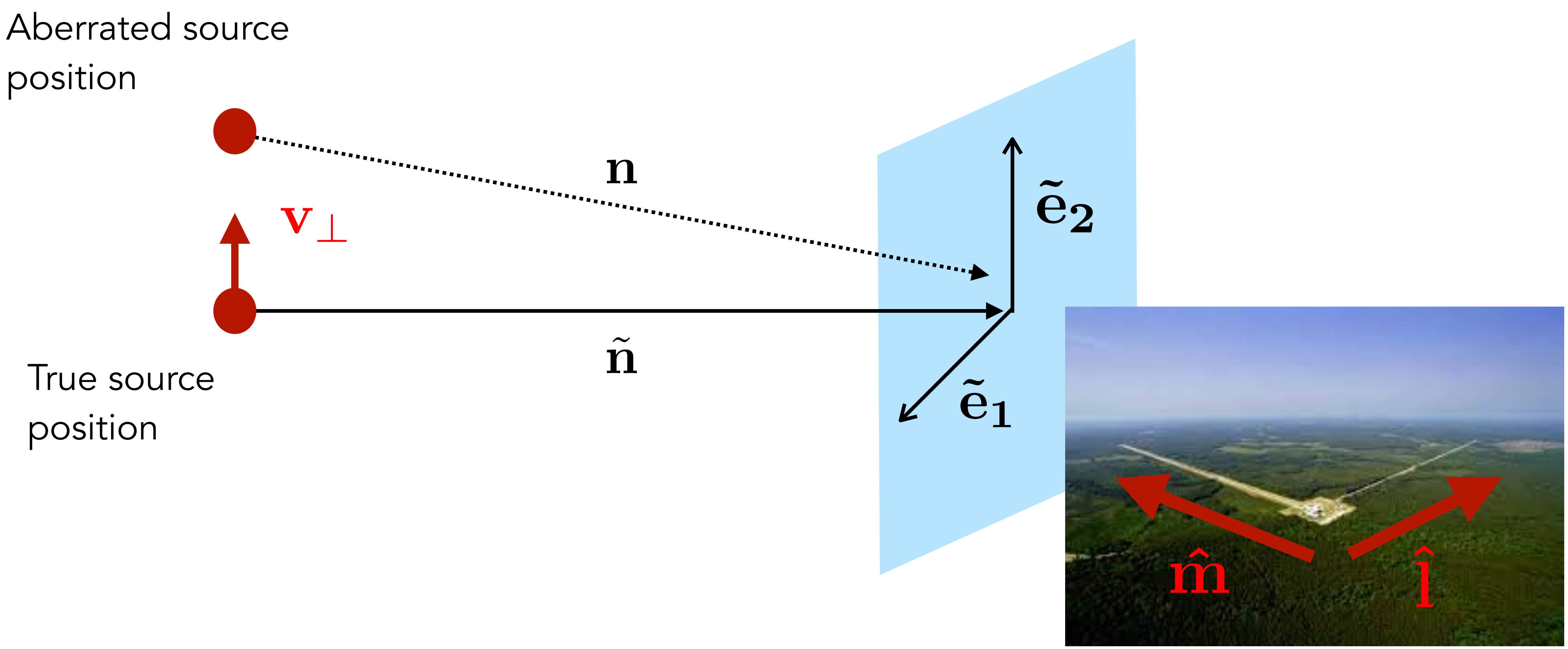
Observed strain with respect to non-aberrated polarisation basis

If we could access the non-aberrated (true) source position $\tilde{\mathbf{n}}$

$$P_{ij}(\hat{l}_i\hat{l}_j - \hat{m}_i\hat{m}_j) = F_+(\tilde{\mathbf{n}})h_+ + F_\times(\tilde{\mathbf{n}})h_\times + F_1(\tilde{\mathbf{n}})h_1 + F_2(\tilde{\mathbf{n}})h_2$$

Detector tensor

Spin-1 modes (longitudinal to polarisation plane)



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Detector tensor

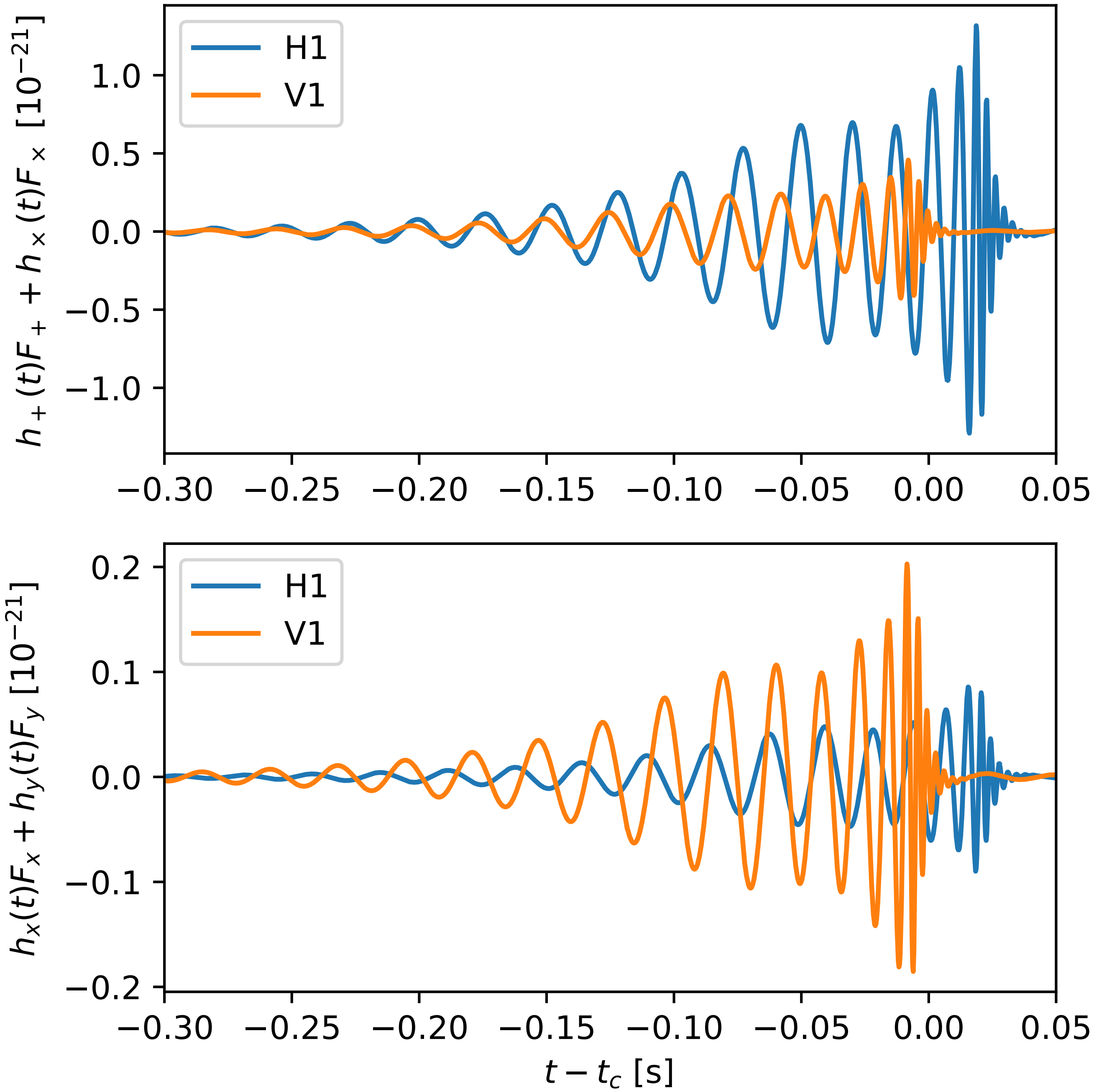
Spin-1 modes (longitudinal to polarisation plane)

$$F_+(\tilde{\mathbf{n}}) = \frac{1}{2}\left(\hat{l}_i\hat{l}_j - \hat{m}_i\hat{m}_j\right)\left(\tilde{e}_{1i}\tilde{e}_{1j} - \tilde{e}_{2i}\tilde{e}_{2j}\right),$$

$$F_\times(\tilde{\mathbf{n}}) = \frac{1}{2}\left(\hat{l}_i\hat{l}_j - \hat{m}_i\hat{m}_j\right)\left(\tilde{e}_{1i}\tilde{e}_{2j} + \tilde{e}_{2i}\tilde{e}_{1j}\right),$$

$$F_1(\tilde{\mathbf{n}}) = \frac{1}{2}\left(\hat{l}_i\hat{l}_j - \hat{m}_i\hat{m}_j\right)\left(\tilde{n}_i\tilde{e}_{1j} + \tilde{e}_{1i}\tilde{n}_j\right),$$

$$F_2(\tilde{\mathbf{n}}) = \frac{1}{2}\left(\hat{l}_i\hat{l}_j - \hat{m}_i\hat{m}_j\right)\left(\tilde{n}_i\tilde{e}_{2j} + \tilde{e}_{2i}\tilde{n}_j\right).$$



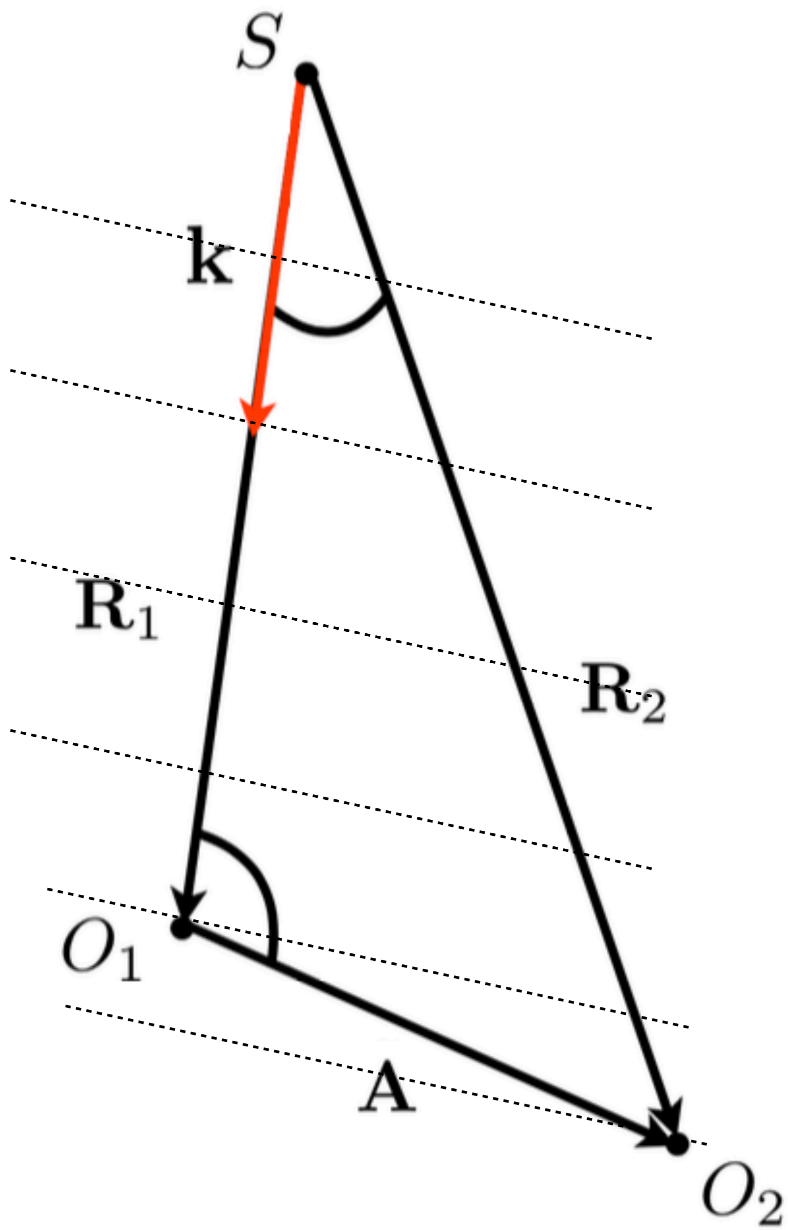
Tensor modes

Vector modes

Binary with $30M_\odot - 30M_\odot$

At 500 Mpc and transverse velocity **0.1 c** (sky position of GW170817)

Is there a way to reconstruct the true position of the source



What about time-delay information

$$\Phi(t, \mathbf{R}_1) = -k^\mu x_{\mu 1} = E\,(t - \mathbf{R}_1 \cdot \mathbf{n})$$
$$\Phi(t, \mathbf{R}_2) = -k^\mu x_{\mu 2} = E\,(t - \mathbf{R}_2 \cdot \mathbf{n})$$

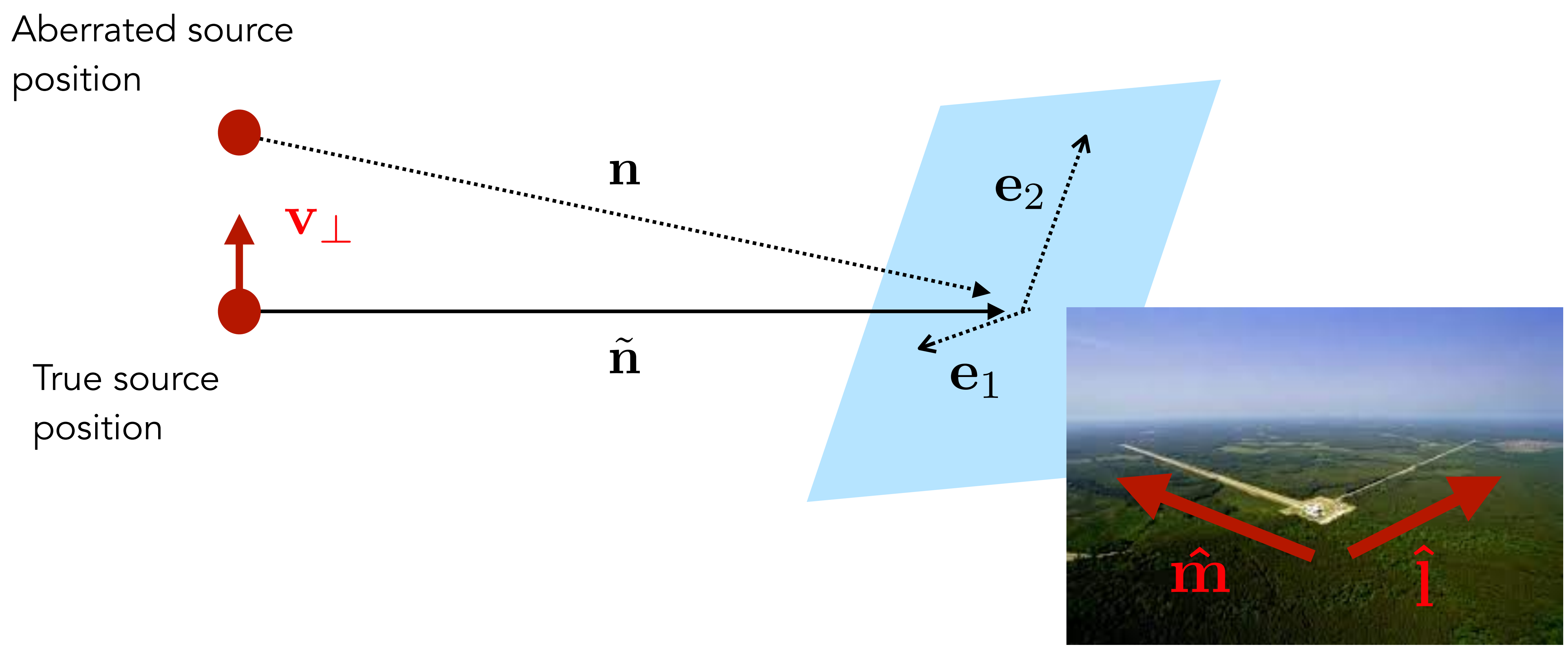
Phase shift $\boxed{\Delta\Phi = -E\,\mathbf{A} \cdot \mathbf{n}}$ Aberrated direction

$$\mathbf{n} = \frac{\mathbf{k}}{|\mathbf{k}|} = \tilde{\mathbf{n}} - \mathbf{v}_\perp$$

If I have multiple interferometers in a network, from **phase shift** I can only reconstruct **aberrated direction**

Observed strain with respect to aberrated polarisation basis

However, we can only access the **aberrated direction**: distorted spin-2



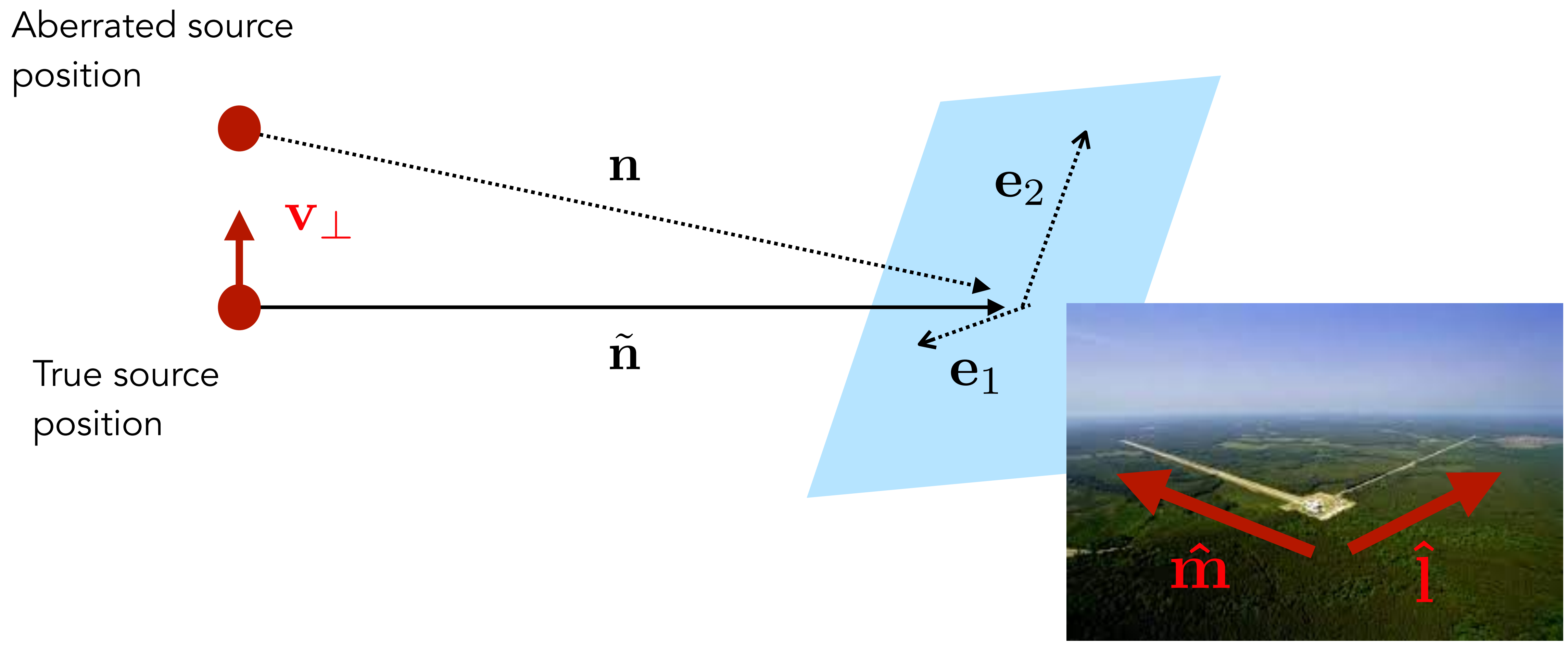
Observed strain with respect to aberrated polarisation basis

However, we can only access the **aberrated direction**: distorted spin-2

$$P_{ij}(\hat{l}_i\hat{l}_j - \hat{m}_i\hat{m}_j) = \hat{h}_+ F_+(\mathbf{n}) + \hat{h}_\times F_\times(\mathbf{n})$$

Detector tensor

$$\hat{h}_\times \sim \tilde{h}_\times - \tilde{h}_+ v_\perp$$
$$\hat{h}_+ \sim \tilde{h}_+ + \tilde{h}_\times v_\perp$$



Observationally: distorted spin-2 polarizations

Kinematic mixing

$$\hat{h}_\times \sim \tilde{h}_\times - \tilde{h}_+ v_\perp$$
$$\hat{h}_+ \sim \tilde{h}_+ + \tilde{h}_\times v_\perp$$

$$P_{ij}(\hat{l}_i\hat{l}_j - \hat{m}_i\hat{m}_j) = \hat{h}_+ F_+(\mathbf{n}) + \hat{h}_\times F_\times(\mathbf{n})$$

From an **observational point of view**, I will see only spin-2 fields but from aberrated direction and with mixed polarisations (with respect to the emitted ones)

Bias when fitting with standard templates

Transverse velocities induce a bias in the reconstruction of orbital parameters
How important is this bias for cosmology (e.g. luminosity distance and sky localisation)?

How important is this kinematic induced bias

We simulate **three different populations** in detector frame masses

1) neutron star binaries with $1.4 M_\odot - 1.4 M_\odot$

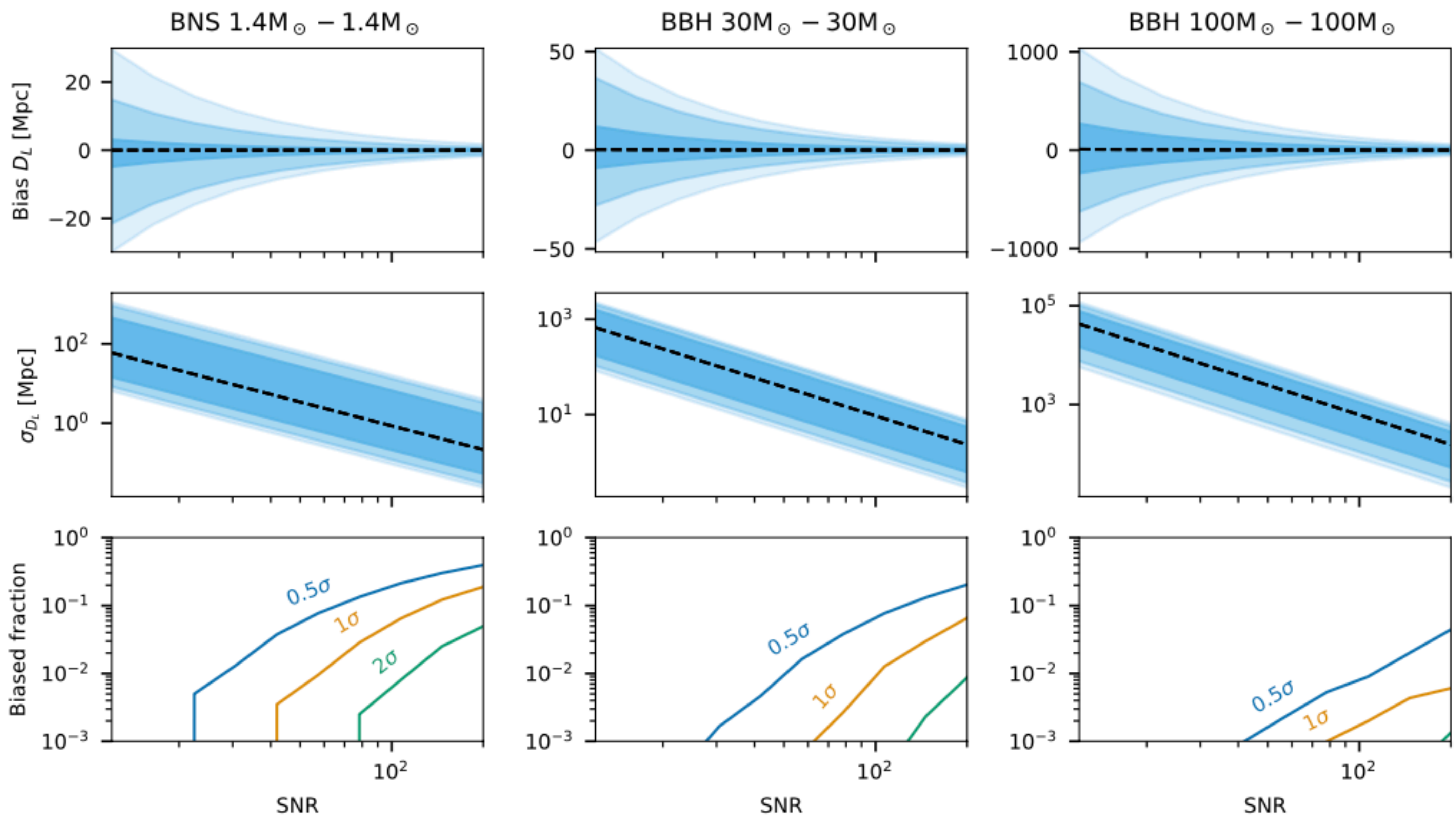
2) black hole-black hole binaries with $30 M_\odot - 30 M_\odot$

3) black hole-black hole binaries with $100 M_\odot - 100 M_\odot$

Assumptions: isotropic sky distribution and orbital orientation, aligned spins, time of arrival uniform in one year

Peculiar motion: isotropic with modulus from Maxwellian distribution with mean 500 km/s. (in agreement with galaxy observations)

Kinematic induced bias on luminosity distance (2LIGO+Virgo)



Bias

Statistical error

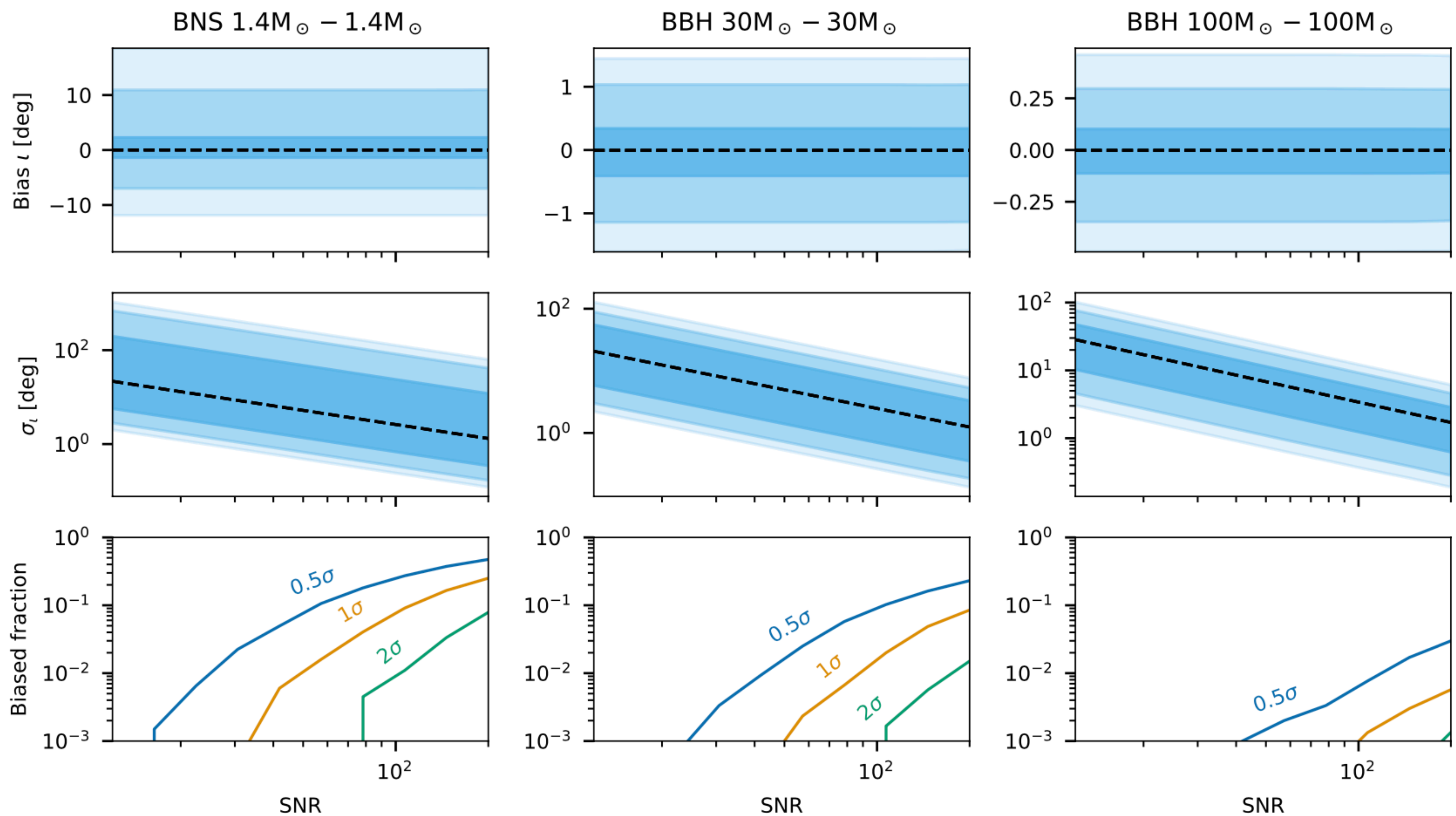
Fraction of biased events

Typical bias scales as $|v|d_L$ hence are larger for low SNR (large distance)

However for these events, the statistical error is also larger: bias important at high SNR

(reconstruction of dL more precise for neutron stars)

Kinematic induced bias on inclination angle (2LIGO+Virgo)



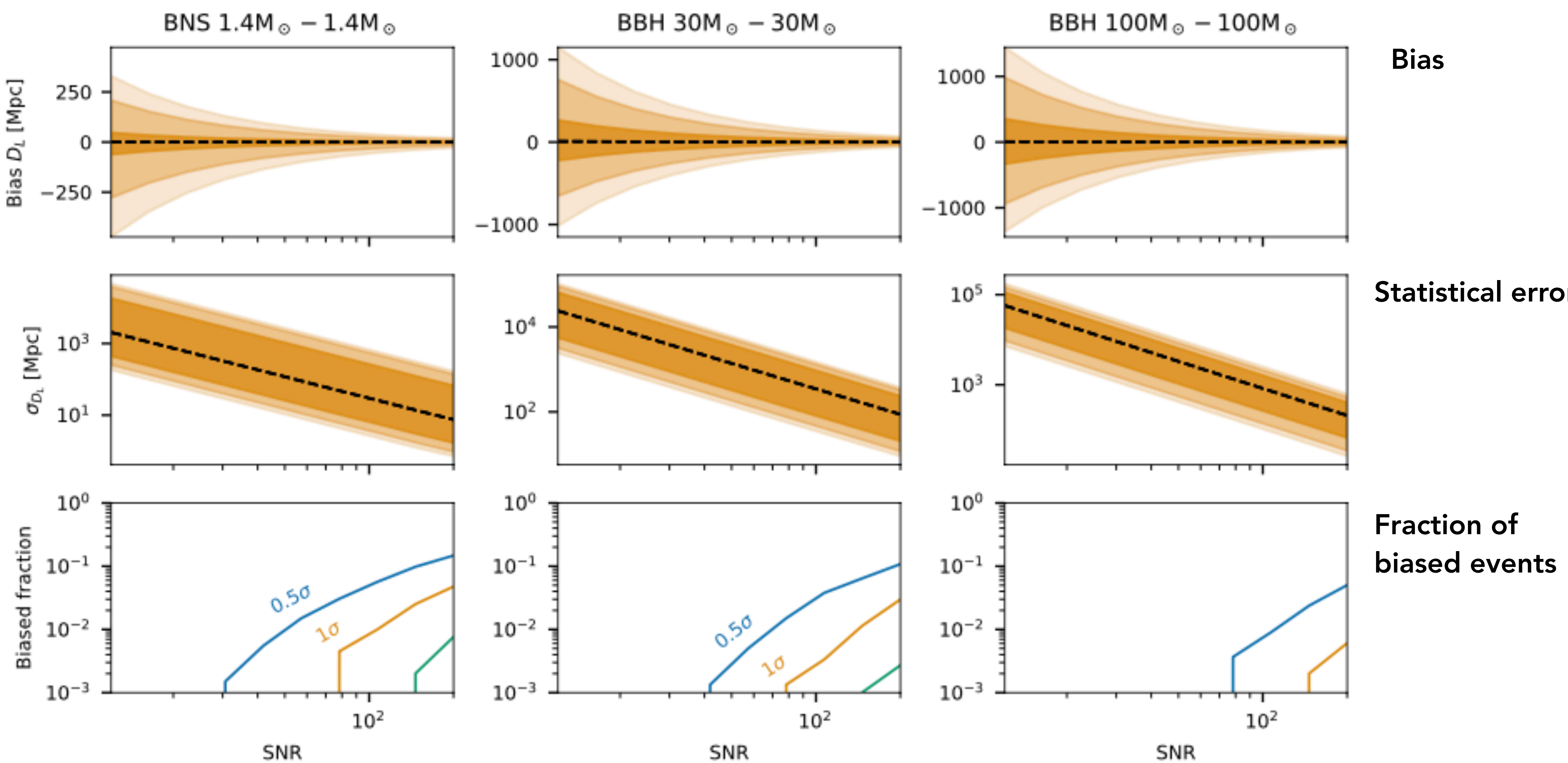
Bias

Statistical error

Fraction of biased events

Situation similar to DL reconstruction due to DL-\iota degeneracy
Lower biased fraction of massive system due to the fact that for short events polarisations are hardly measured

Kinematic induced bias on luminosity distance (ET+CE)

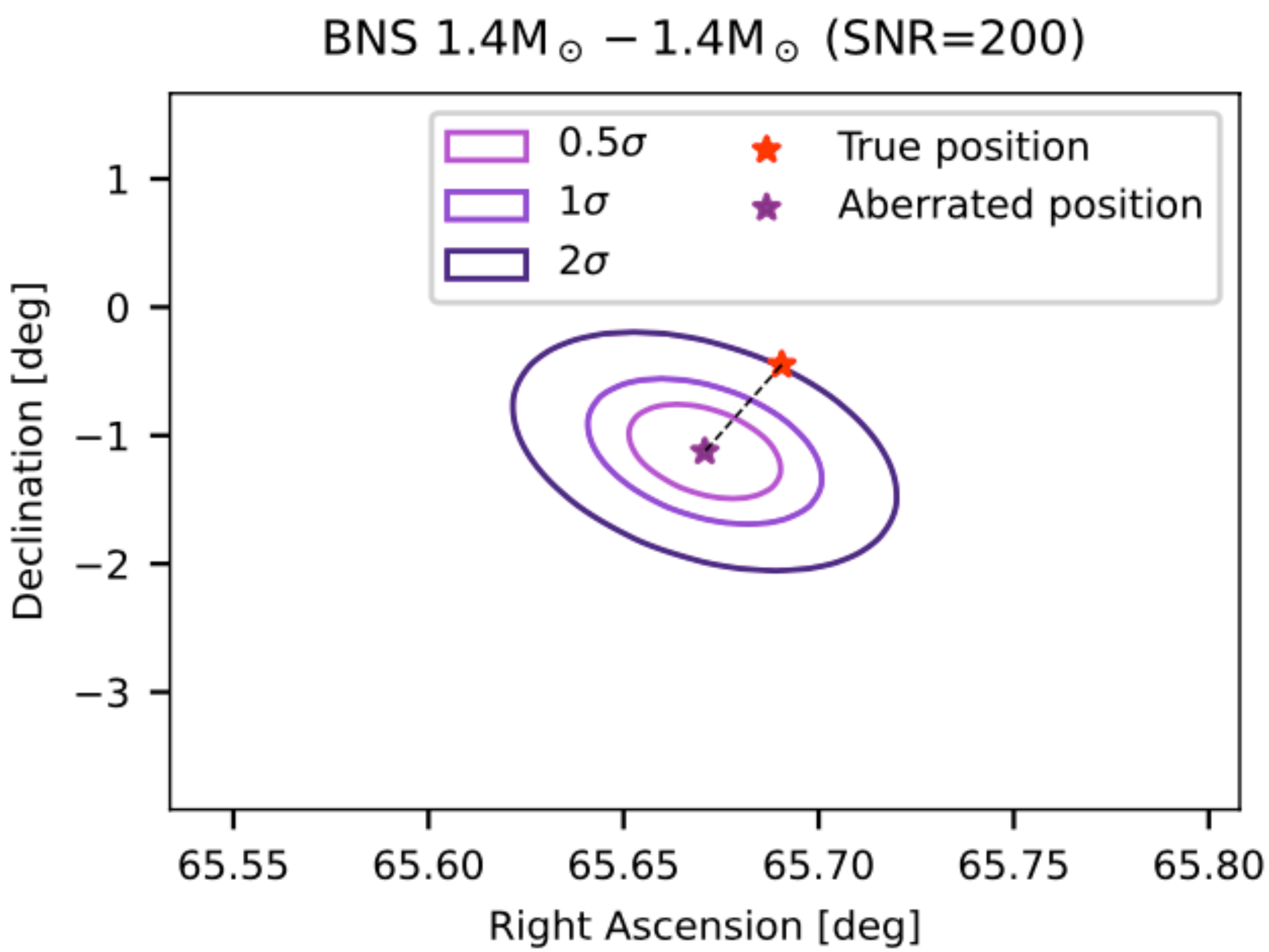


High number of observable events: O(1) binary neutron star merger and O(10) binary black holes with bias on reconstruction dL larger than 1-sigma

(reconstruction of dL more precise for neutron stars)

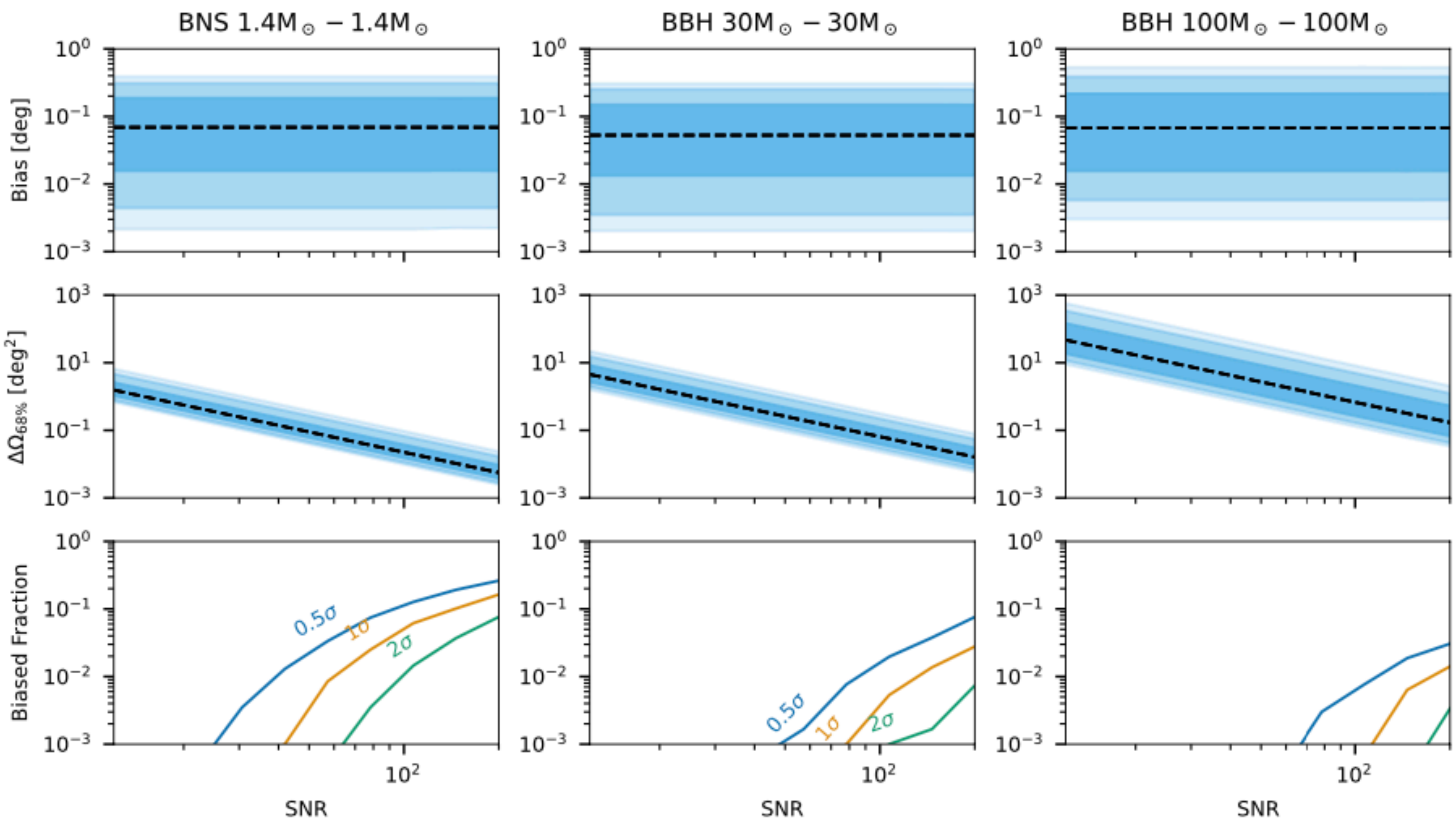
Localisation bias

Impact of sky localisation bias: we compare the sky area associated to statistical uncertainty, with the displacement in sky position induced by the bias.
A GW source is classified as biased if the true **sky position is outside the area** identified by a given confidence interval.



Confidence intervals on sky localisation from Grover et al. 2014

Kinematic induced bias on sky localisation (sky areas) - LVC case



Bias in sky localisation

Sky area associate to statistical error

Fraction of biased events

BNSs detected with a SNR>100 have a **10% probability of having a significant bias**, thus preventing the localization of the host galaxy.

Bias problem less severe for ET+CE (worse sky localisation)

Effect of transverse motion: take home message

—Source velocity transverse to the line of sight: **spin-1** appearing in the observer frame because of aberration

—These are **not new degrees of freedom** (they are proportional to spin-2 polarisations)

—**Observationally** we only have access to aberrated direction. We reconstruct spin-2 modes aberrated, with a **kinematic mixing**

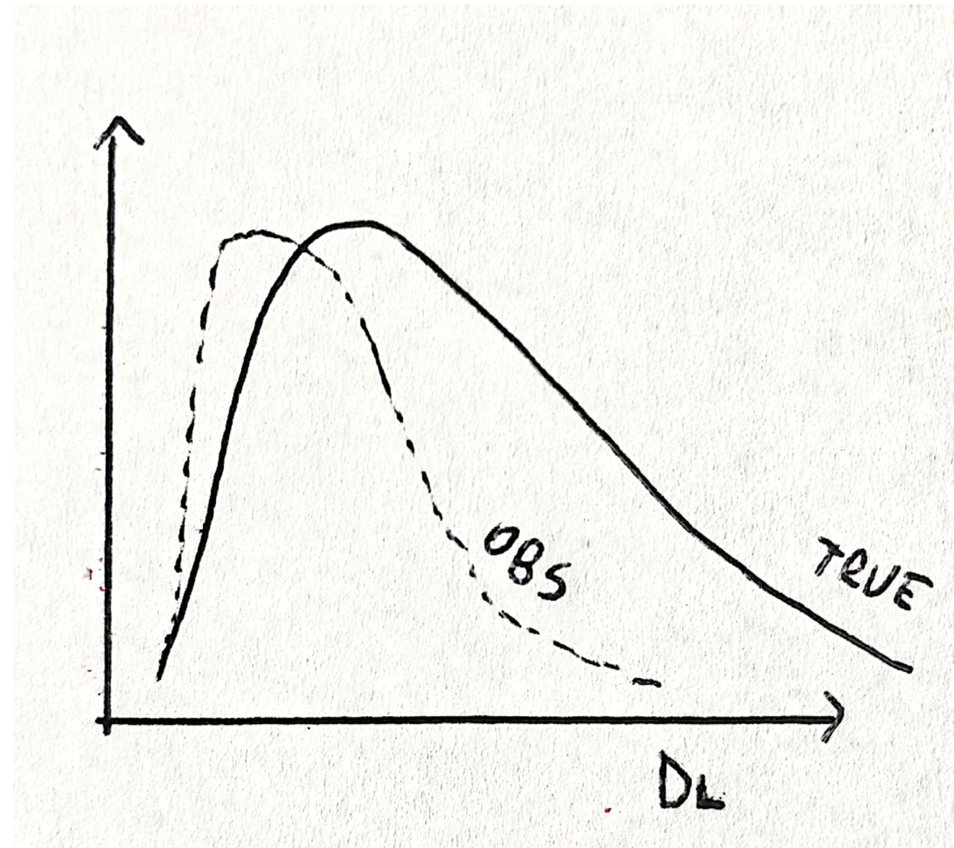
—This gives an **irreducible bias** in the reconstruction of orbital parameters

—Significant fraction of events with **bias larger than 1-sigma** in the reconstruction of **luminosity distance** (important for ET, in standard sirens studies)

—**Localisation bias** relevant for **neutron stars at high SNR** (prevents identification host galaxy)

Effects on the population parameters (in progress)

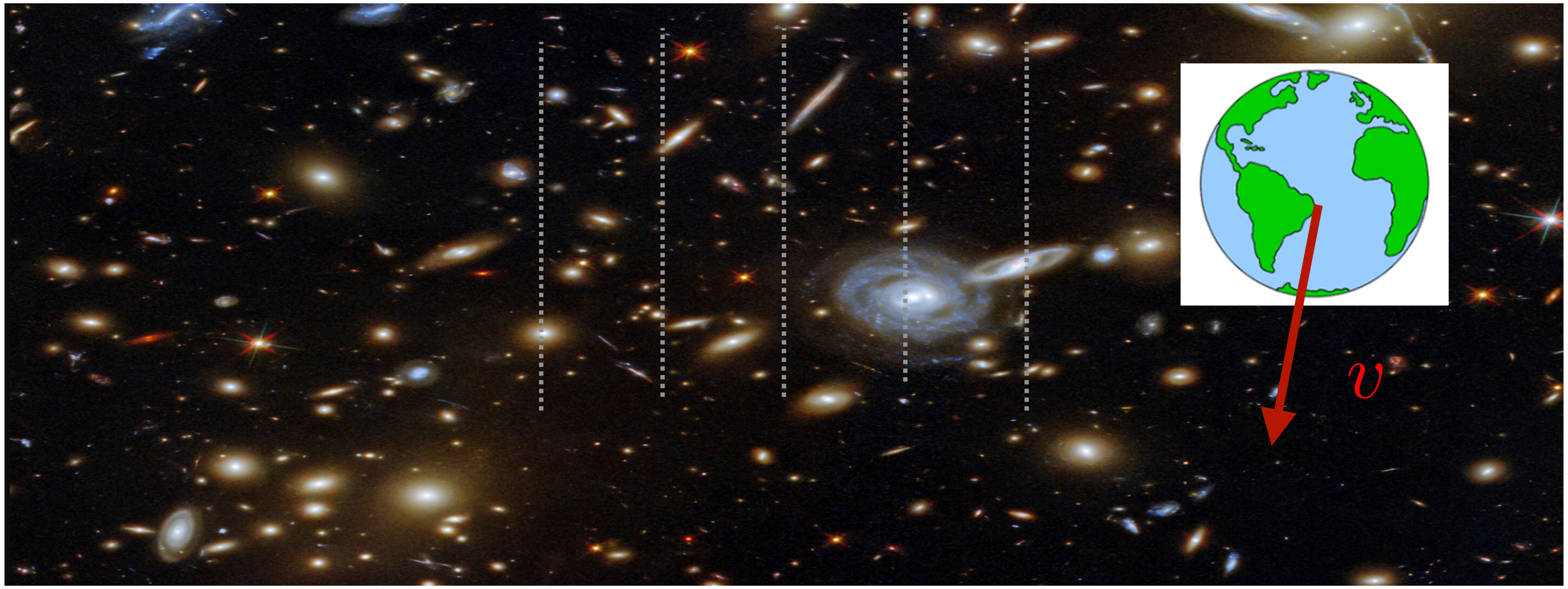
Question: is this kinematic bias **population preserving**?
Primary effect of bias on population will be due to selection, are there other effects? How to fully characterise bias on population?



Example

—**distance estimate** comes from the observed amplitude, which accounts for the expected sensitivity to the observed sky location
—peculiar velocities: true **sky location gets smeared out** and sky location response is some average of the true response within the range of sky locations that a source could be located at
—(stealth) bias on the reconstruction luminosity distance. What happens to distribution?

What about our motion wrt CMB rest frame?

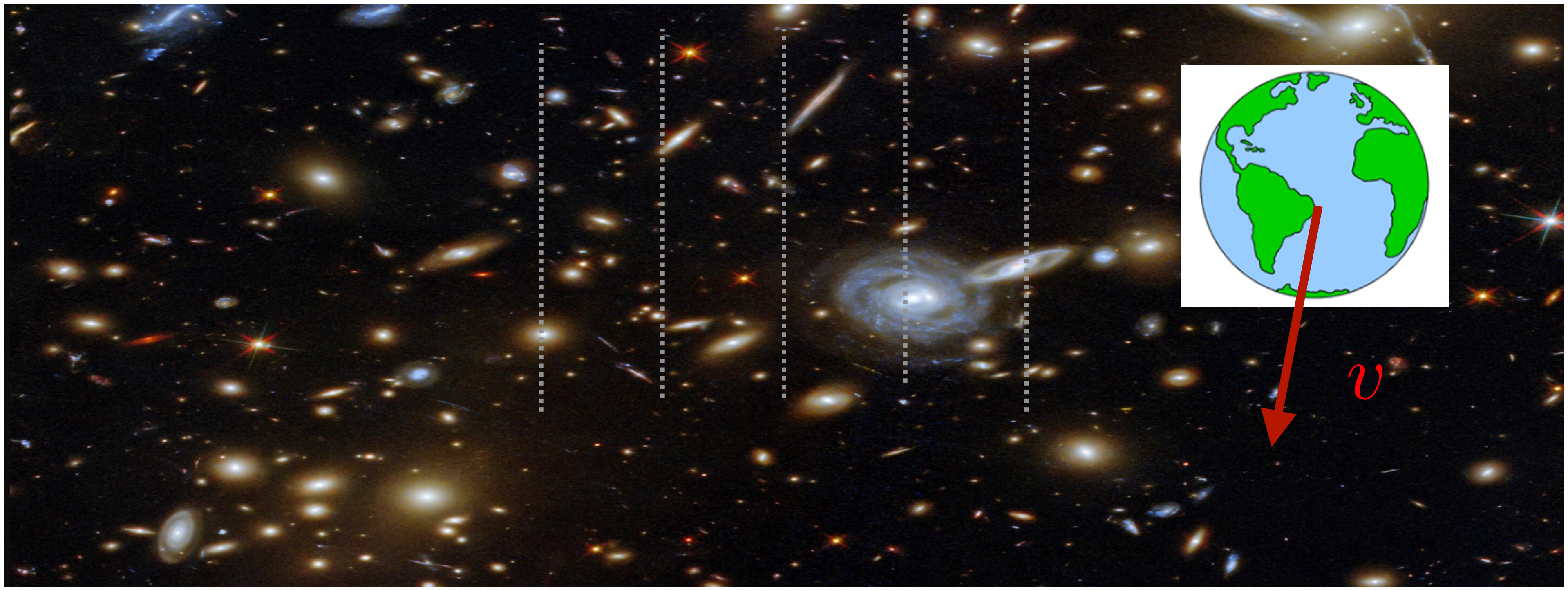


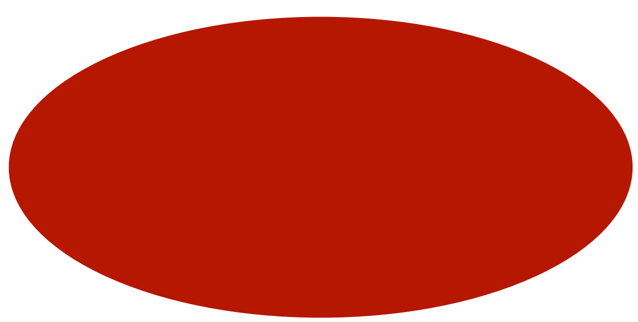
Our reference frame moves wrt to a *universal* reference frame (usually identified with CMB rest frame, or rest frame of galaxies)

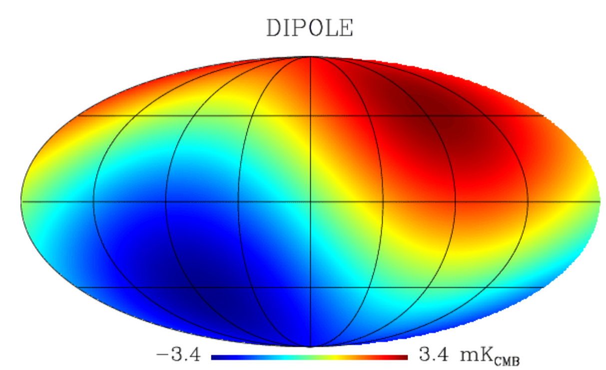
From CMB dipole and galaxy number counts: two estimates for our velocity, which are our of 5 sigma

Idea: can we use GW to get an independent estimate of this velocity?

What about our motion wrt CMB rest frame?







(instrinsic) isotropic distribution of events

We see more events in the direction of the peculiar motion

Kinematic dipole detectable by ET+CE: new constrain on our peculiar velocity!
(interesting, seen discrepancy CMB vs galaxy counts)

Foffa, Mastrogiovanni et al.

Thank you

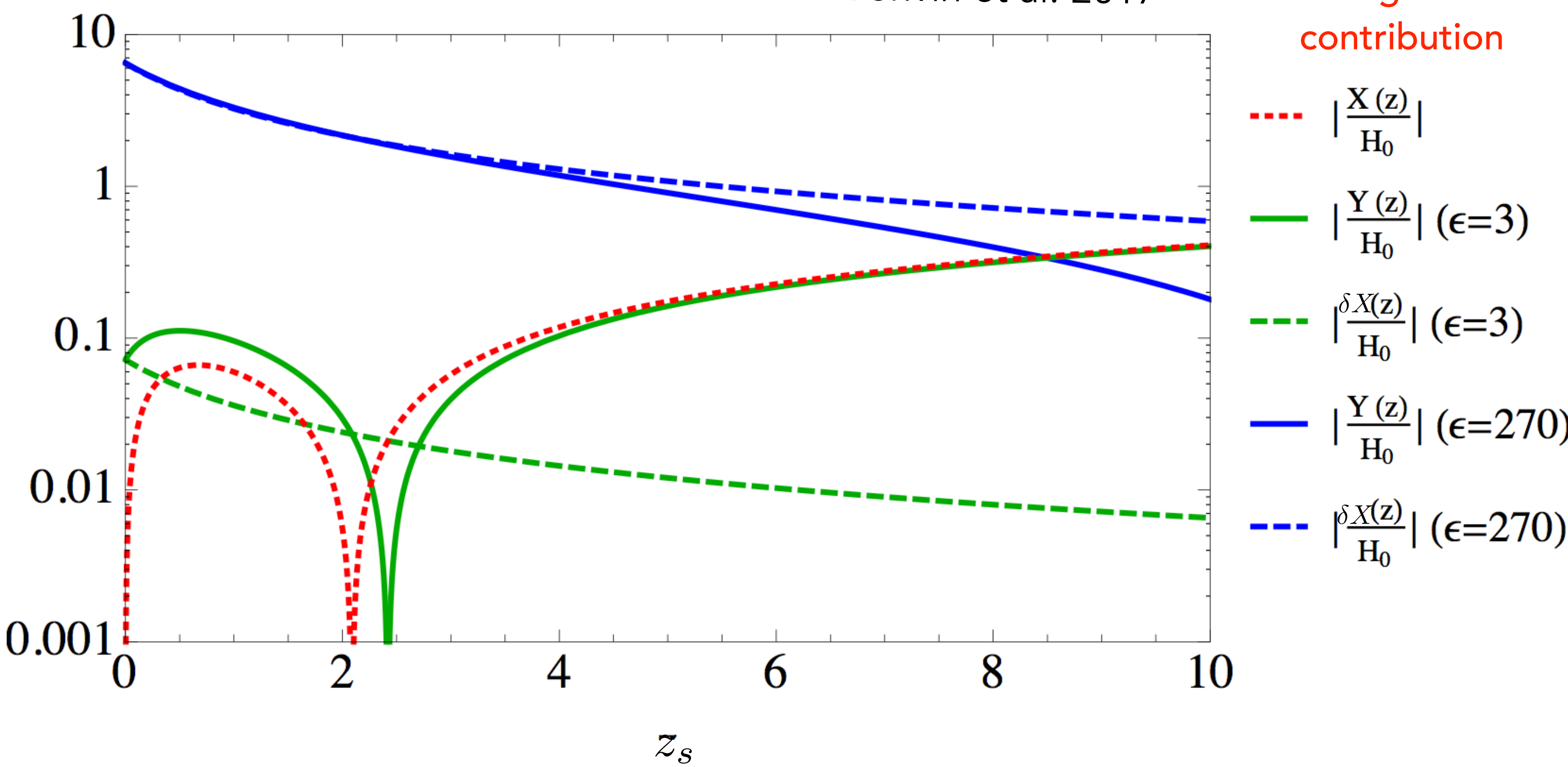
Some numerical estimates

$$\frac{Y(z_s)}{H_0} = \frac{X(z_s)}{H_0} + \frac{\delta X(z_s)}{H_0} \qquad \frac{\delta X}{H_0}(z_s) = \frac{\alpha}{2} \boxed{\frac{v_s^2}{r} \frac{\mathbf{e}\cdot\mathbf{n}}{H_0} \frac{1}{(1+\bar{z})}} \epsilon$$

We compare the two dotted lines (two different value of velocity)

Bonvin et al. 2017

background contribution



binary at the edge of the cluster $r = 100\mathrm{kpc}$ with average velocity $v_2 = 1000\mathrm{km/s}$

binary with very high peculiar velocity $v_s = 3000\mathrm{km/s}$ very close to the center of the cluster $r = 10\mathrm{kpc}$

$$
\begin{aligned}
F_{+}(\tilde{\mathbf{n}}) &= \frac{1}{2}\left(\hat{l}_i\hat{l}_j - \hat{m}_i\hat{m}_j\right)\left(\tilde{e}_{1i}\tilde{e}_{1j} - \tilde{e}_{2i}\tilde{e}_{2j}\right), \\
F_{\times}(\tilde{\mathbf{n}}) &= \frac{1}{2}\left(\hat{l}_i\hat{l}_j - \hat{m}_i\hat{m}_j\right)\left(\tilde{e}_{1i}\tilde{e}_{2j} + \tilde{e}_{2i}\tilde{e}_{1j}\right), \\
F_{1}(\tilde{\mathbf{n}}) &= \frac{1}{2}\left(\hat{l}_i\hat{l}_j - \hat{m}_i\hat{m}_j\right)\left(\tilde{n}_i\tilde{e}_{1j} + \tilde{e}_{1i}\tilde{n}_j\right), \\
F_{2}(\tilde{\mathbf{n}}) &= \frac{1}{2}\left(\hat{l}_i\hat{l}_j - \hat{m}_i\hat{m}_j\right)\left(\tilde{n}_i\tilde{e}_{2j} + \tilde{e}_{2i}\tilde{n}_j\right).
\end{aligned}
$$

Kinematic induced bias on inclination angle (ET+CE)