Leading hadronic contribution to the muon $g-2$ from lattice QCD

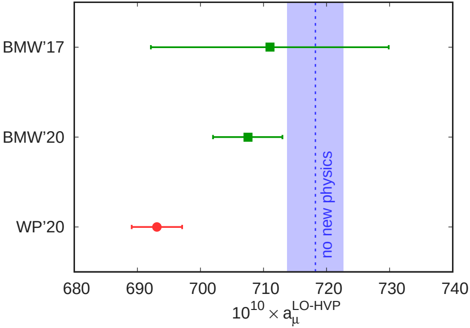
Bálint C. Tóth

University of Wuppertal

Budapest–Marseille–Wuppertal collaboration

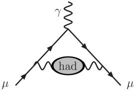
Tensions in $a_\mu^{\text{LO–HVP}}$

- 4.2σ between WP'20 and experiment
- 1.5σ between BMW'20 and experiment
- 2.1σ between WP'20 and BMW'20

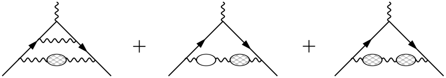


Hadronic contributions

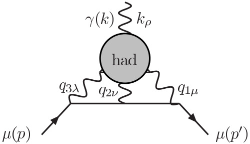
- LO hadron vacuum polarization (LO-HVP, $(\frac{\alpha}{\pi})^2$)



- NLO hadron vacuum polarization (NLO-HVP, $(\frac{\alpha}{\pi})^3$)



- Hadronic light-by-light (HLbL, $(\frac{\alpha}{\pi})^3$)



- pheno $a_\mu^{\text{HLbL}} = 9.2(1.9)$

  [Colangelo, Hoferichter, Kubis, Stoffer et al '15–'20]

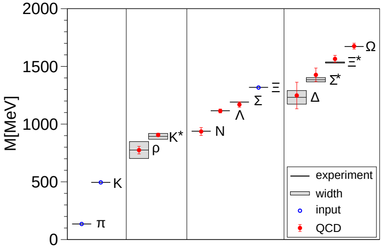
- lattice a_μ^{HLbL}=7.9(3.1)(1.8) or 10.7(1.5)

  [RBC/UKQCD '19 and Mainz '21]

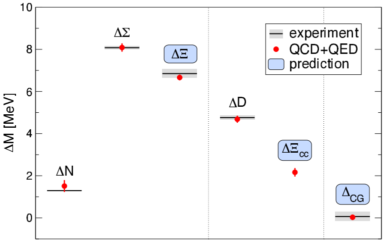
HVP from lattice QCD

Lattice QCD: examples

- Hadron spectrum [BMWc '08]

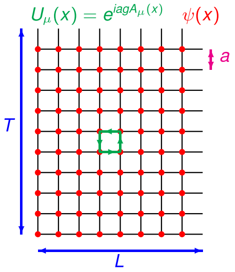


- Neutron–proton difference [BMWc '14]



Lattice QCD

- Lattice gauge theory:
  - 1st principles method
  - non-perturbative $\longrightarrow$ sum over all Feynman-diagrams at once (and beyond)
  - only imaginary (Euclidean) time is accessible (no problem for a_μ)
- Discretize space-time with lattice spacing: a



- quarks on sites, gluons on links
- discretize action + operators

$$\int d^4x \longrightarrow a^4 \sum_x$$

$$\partial_\mu \longrightarrow \text{finite differences}$$

- Different fermion discretizations: staggered, Wilson, twisted mass, domain wall, overlap, ...

- To get physical results, need to perform:
  1. Infinite volume limit $(V \to \infty)$ $\longrightarrow$ numerically or analytically
  2. Continuum limit $(a \to 0)$ $\longrightarrow$ min. 3 different a

Lattice QCD

- Integrate over all classical field configurations

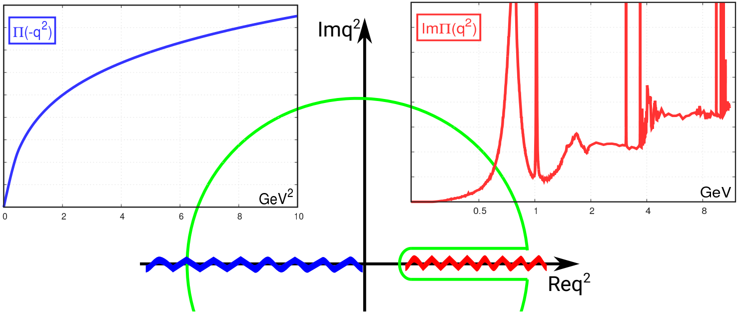
$$\int [\mathrm{d}U]\,[\mathrm{d}\overline{\psi}]\,[\mathrm{d}\psi]\;\mathcal{O}\;e^{-S_{\mathrm{g}}(U)-\overline{\psi}\,M(U)\,\psi}$$

- E.g. $96^3 \times 144$ lattice $\longrightarrow$ $\approx 4 \cdot 10^9$ dimensional integral
- Stochastic integration

- 100000 years for a laptop $\longrightarrow$ 1 year for supercomputer

Hadronic vacuum polarization

- $\Pi_{\mu\nu}(q) = \left(q_\mu q_\nu - g_{\mu\nu}q^2\right)\Pi(q^2)$ analytic + branch-cut



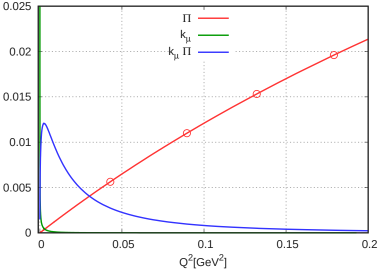
- Minkowski from R-ratio experiments
- Euclidean from lattice QCD or exp. like MUonE
- Minkowski → Euclidean via dispersion relation ($Q^2 = -q^2$)

$$\Pi(Q^2) = \int_{s_{th}}^{\infty} ds\, \frac{Q^2}{s(s+Q^2)} \frac{1}{\pi} \mathrm{Im}\Pi(s)$$

$a_\mu^{\text{LO-HVP}}$ from lattice QCD

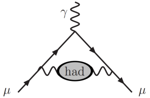
- get Π from Euclidean current-current correlator [Blum '02]

$$\Pi_{\mu\nu} = \int dx\, e^{iQx} \langle J_\mu(x) J_\nu(0) \rangle = \left(Q_\mu Q_\nu - \delta_{\mu\nu} Q^2 \right) \Pi(Q^2)$$



$$a_\mu^{\text{HVP}} = \frac{\alpha^2}{\pi^2} \int dQ^2\, k_\mu(Q^2)\, \Pi(Q^2)$$

$k_\mu(Q^2)$ describes the leptonic part of diagram

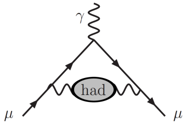


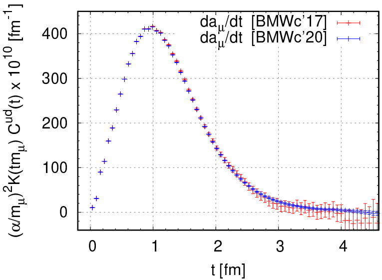
- Q is available at discrete momenta only
- need $\Pi(Q^2) - \Pi(0)$, but $\Pi(0)$ is not directly accessible
- smooth interpolation in Q and prescription for $\Pi(0)$

[Bernecker,Meyer '11], [HPQCD'14], ...

$a_\mu^{\text{LO-HVP}}$ from lattice QCD

- $a_\mu^{\text{LO-HVP}} = \alpha^2 \int_0^\infty dt\, K(t)\, C(t)$





- $C(t)$: Current-current correlator

$$C(t) = \frac{1}{3}\sum_{i=1}^{3}\langle J_i(t) J_i(0)\rangle$$

- $K(t)$ describes the leptonic part of diagram [Bernecker,Meyer '11], [HPQCD'14], ...

$$K(t) = \int_0^{Q^2_{\max}} \frac{dQ^2}{m_\mu^2}\, \omega\left(\frac{Q^2}{m_\mu^2}\right)\left[t^2 - \frac{4}{Q^2}\sin^2\left(\frac{Qt}{2}\right)\right]$$

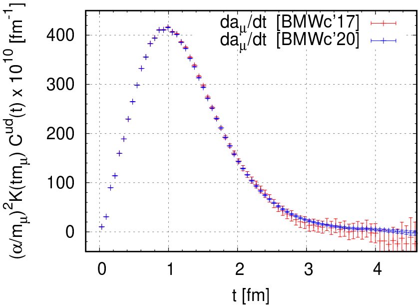
$$\omega(r) = [r + 2 - \sqrt{r(r+4)}]^2 / \sqrt{r(r+4)}$$

- only integrate up to $Q^2_{\max} = 3\,\text{GeV}^2$
- $Q^2 > Q^2_{\max}$: perturbation theory

Challenges for lattice

Noise reduction

- noise/signal in $C(t) = \langle J(t)J(0)\rangle$ grows for large distances



- Low Mode Averaging: use exact (all2all) quark propagator in IR and stochastic in UV
- decrease noise by replacing $C(t)$ by upper/lower bounds above t_c

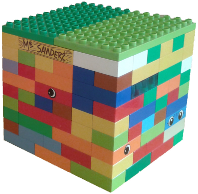
$$0 \leq C(t) \leq C(t_c)e^{-E_{2\pi}(t-t_c)}$$

⟶ few permil level accuracy on each ensemble

Finite-size effects

- Typical lattice runs use $L < 6\,\mathrm{fm}$, earlier model estimates gave $O(2)\%$ FV effect.

$L_{\mathrm{ref}} = 6.272\,\mathrm{fm}$



$L_{\mathrm{big}} = 10.752\,\mathrm{fm}$

1. $a_\mu(\mathrm{big}) - a_\mu(\mathrm{ref})$
   - perform numerical simulations in $L_{\mathrm{big}} = 10.752\,\mathrm{fm}$
   - perform analytical computations to check models

lattice	NLO XPT	NNLO XPT	MLLGS	HP	RHO
$18.1(2.0)_{\mathrm{stat}}(1.4)_{\mathrm{cont}}$	11.6	15.7	17.8	16.7	15.2

2. $a_\mu(\infty) - a_\mu(\mathrm{big})$
   - use models for remnant finite-size effect of “big” $\sim 0.1\%$

Scale determination

Lattice spacing a enters into a_μ determination:

- physical values of m_μ, m_π, m_K

$\longrightarrow$ $\Delta_{\text{scale}} a_\mu \sim 1.8 \cdot \Delta(\text{scale})$ [Della Morte *et.al.* '17]

1. For final results: M_{Ω^-} scale setting $\longrightarrow$ $a = (aM_{\Omega^-})^{\text{lat}} / M_{\Omega^-}^{\text{exp}}$
   - Experimentally well known: 1672.45(29) MeV [PDG 2018]
   - Moderate m_q dependence
   - Can be precisely determined on the lattice

2. For separation of isospin breaking effects: w_0 scale setting
   No experimental value [Lüscher 2010] [BMWc 2012]
   $\longrightarrow$ Determine value of w_0 from $M_\Omega \cdot w_0$

$$w_0 = 0.17236(29)(63)[70]\ \text{fm}$$

QCD+QED

- Reach sub-percent level: include isospin breaking effects for
  - $\langle jj \rangle$
  - masses
  - scale
- Rewrite dynamical QED as quenched QED expectation values

$$\langle O \rangle_{\text{QCD+unquenched QED}} = \frac{\left\langle \left\langle O(U,A) \frac{\det M(U,A)}{\det M(U,0)} \right\rangle_{\text{quenched QED}} \right\rangle_{\text{QCD}}}{\left\langle \left\langle \frac{\det M(U,A)}{\det M(U,0)} \right\rangle_{\text{quenched QED}} \right\rangle_{\text{QCD}}}$$

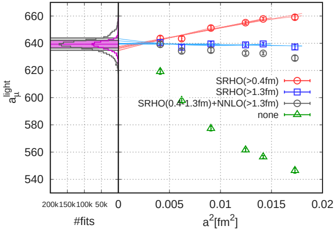
- Take isospin symmetric gluon configurations: U
- Compute derivatives

$$m_l \frac{\partial X}{\partial \delta m} \qquad \frac{\partial X}{\partial e} \qquad \frac{1}{2}\frac{\partial^2 X}{\partial e^2}$$

- Hybrid approach:
  - sea effects: derivatives [De Divitiis *et.al.* 2013]
  - valence effects: finite differences [Eichten *et.al.* 1997]

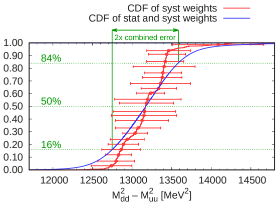
Continuum limit

Controlled $a \to 0$ extrapolation



- 6 lattice spacings: 0.132 fm $\longrightarrow$ 0.064 fm
- Leading cutoff effects at large *t* are taste breaking effects $\longrightarrow$ mass effects
- Distortion in spectrum: cured by taste improvement rho-pion-gamma model (SRHO)

  [Sakurai '60][Bijnens *et.al.* '99][Jegerlehner *et.al.* '11][Chakraborty *et.al.* '17]

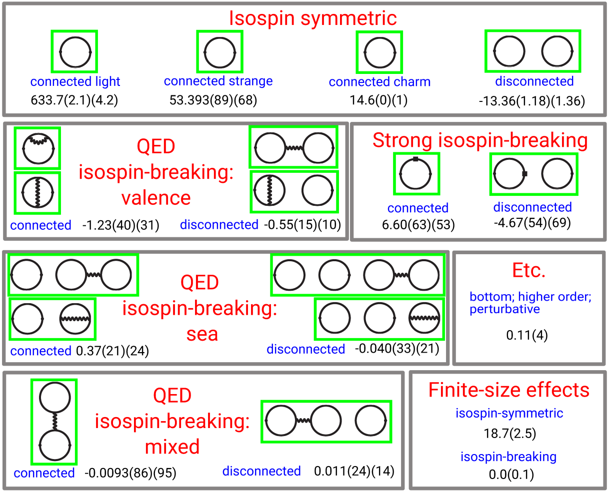


- Several hundreds of thousands of analyses, combined using histogram method

  linear vs. quadratic, a^2 vs $a^2\alpha_s(1/a)^3$ [Husung *et.al* 2020]
  cuts in lattice spacing, hadron mass fit ranges, ...

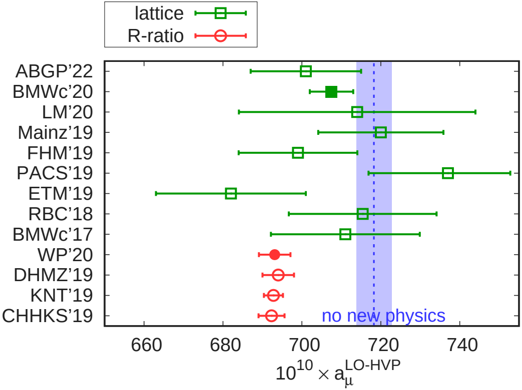
- Uncertainty arising from choice of taste improvement: Added to systematic error in quadrature

Overview of contributions



$$10^{10}\times a_\mu^{\text{LO-HVP}} = 707.5(2.3)_{\text{stat}}(5.0)_{\text{sys}}[5.5]_{\text{tot}}$$

Final result



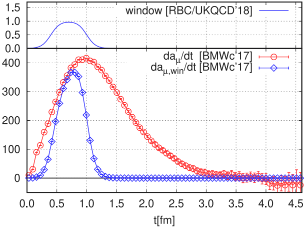
- $10^{10} \times a_\mu^{\text{LO-HVP}} = 707.5(2.3)(5.0)[5.5]$ with 0.8% accuracy
- consistent with new FNAL experiment
- 2.0σ larger than [DHMZ'19], 2.5σ than [KNT'19]

Window observable

Window observable

- Restrict correlator to window between $t_1 = 0.4\,\text{fm}$ and $t_2 = 1.0\,\text{fm}$

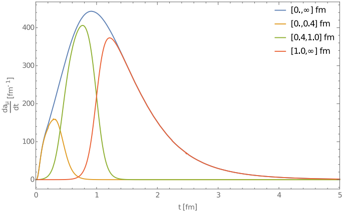
[RBC/UKQCD'18]



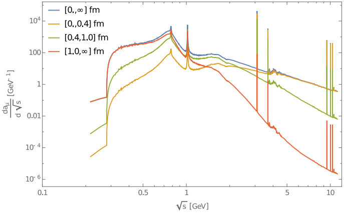
(144 × 96^3, $a \sim 0.064$ fm, $M_\pi \sim 135$ MeV)

- Less challenging than full a_μ
  - signal/noise
  - finite size effects
  - lattice artefacts (short & long)

- Effect in $\sqrt{s}$

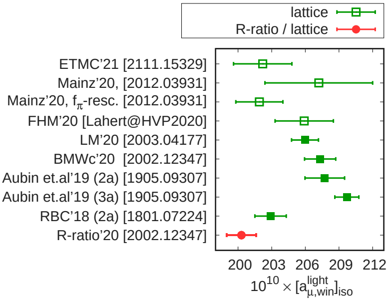


$\longleftrightarrow$

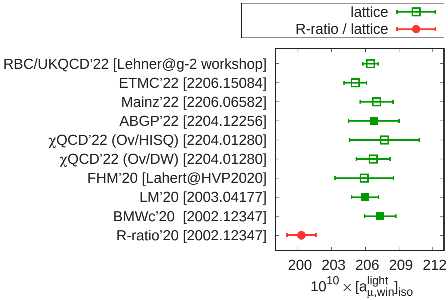


Window observable

- Status in 2021:



- Status now:

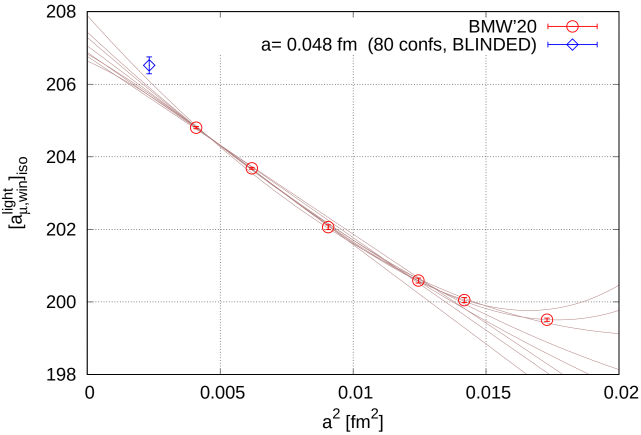


- Latest result from each group $\longrightarrow$ consensus within lattice community
- R-ratio vs lattice discrepancy has to be understood

Ongoing improvements

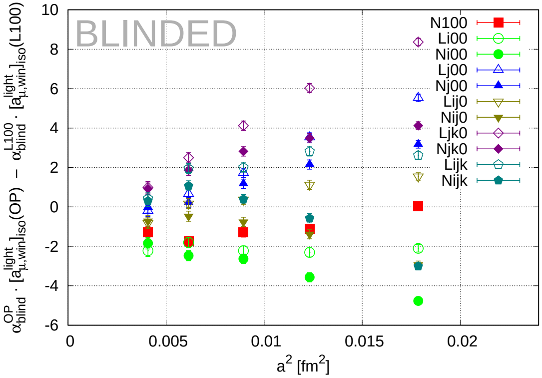
Finer lattice

- $a = 0.048\,\text{fm} \quad 128^3 \times 192 \qquad$ (previously $\quad a = 0.064\,\text{fm} \quad 96^3 \times 144$)



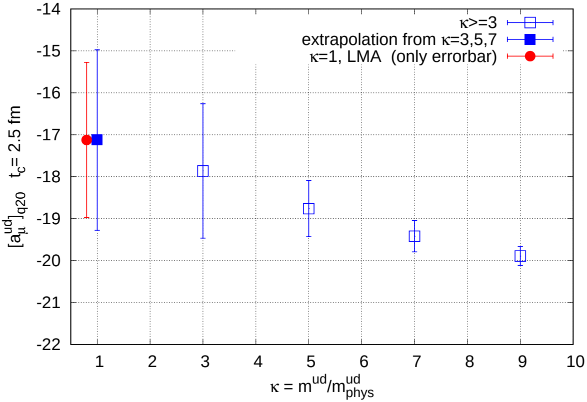
More operators

- Current operator can be discretized in different ways
- Different result at finite lattice spacing, more control over continuum extrapolation



QED contribution

- Eliminating a chiral extrapolation by direct computation at the physical mass



Conclusions & Outlook

Conclusions & Outlook

- Reduce uncertainty on $a_\mu^{\text{LO-HVP}}$
- Understand window discrepancy