Istituto Nazionale di Fisica Nucleare

NOW 2022

Neutrino Oscillation Workshop

Analytic treatment of Neutrino decay and oscillation in Matter

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In collaboration with Dibya S. Chattopadhyay Aomol Dighe, Kaustav Chakraborty, S.M. Lakshmi

Plan

- What is the problem ?
- How do we solve it ?
- Two generation probabilities in matter
- Three generation probabilities in matter
- Some interesting observations

The Problem

- In presence of decay the Hamiltonian is non-hermitian
- $$\mathscr{H} = H - i\Gamma/2$$
- The decay eigenstates and mass eigenstates need not be the same

  → $[H, \Gamma] \neq 0$
- Hence these are not simultaneously diagonalisable by Unitary transformations [Berryman, De Gouvea, Hernandez, Oliveira Phys. Lett. B, 2015]
- Even if they are same in vacuum, matter effects make this mismatch inevitable
- We consider invisible decay

The Hamiltonian in Matter

$$\mathcal{H}_m = H_m - i\Gamma_m/2.$$

- The Hamiltonian in matter can be written in general as

$$\mathcal{H}_m = \begin{pmatrix} a_1 - i b_1 & -\frac{1}{2} i \gamma e^{i\chi} \\ -\frac{1}{2} i \gamma e^{-i\chi} & a_2 - i b_2 \end{pmatrix}$$

- Choose matter basis such that the hermitian part is diagonal
- The flavour evolution is given as $\nu(t) = e^{-i\mathcal{H}_m t} \nu(0).$
- Since $[H_m, \Gamma_m] \neq 0$ → $e^{-i\mathcal{H}_m t} \neq e^{-iH_m t} e^{-\Gamma_m t/2}.$
- One needs to express $e^{-i\mathcal{H}_m t}$ as a chain of commutators using Zassenhaus formula (inverse Baker-Campbell- Hausdroff formula)

Probabilities (two-flavour)

$$\mathcal{H}_m = \begin{pmatrix} a_1 - i b_1 & -\frac{1}{2} i \gamma e^{i\chi} \\ -\frac{1}{2} i \gamma e^{-i\chi} & a_2 - i b_2 \end{pmatrix}$$

$d_i \equiv a_i - ib_i,$ $\Delta_a \equiv a_2 - a_1,$ $\Delta_b \equiv b_2 - b_1, \Delta_d \equiv d_2 - d_1$

$$\bar{\gamma} \equiv \frac{\gamma}{|\Delta_d|}, \quad \bar{\Delta}_a \equiv \frac{\Delta_a}{|\Delta_d|}, \quad \bar{\Delta}_b \equiv \frac{\Delta_b}{|\Delta_d|}.$$

- Survival probability to $\mathcal{O}(\bar{\gamma})$

$$P_{\alpha\alpha} = \frac{e^{-(b_1+b_2)t}}{2}\left[(1+|A|^2)\cosh(\Delta_b t) + (1-|A|^2)\cos(\Delta_a t) - 2\mathrm{Re}(A)\sinh(\Delta_b t) + 2\mathrm{Im}(A)\sin(\Delta_a t)\right]$$

- Conversion Probability $\mathcal{O}(\bar{\gamma})$

$$P_{\beta\alpha} = \frac{e^{-(b_1+b_2)t}}{2} |B(\chi)|^2 [\cosh(\Delta_b t) - \cos(\Delta_a t)]$$

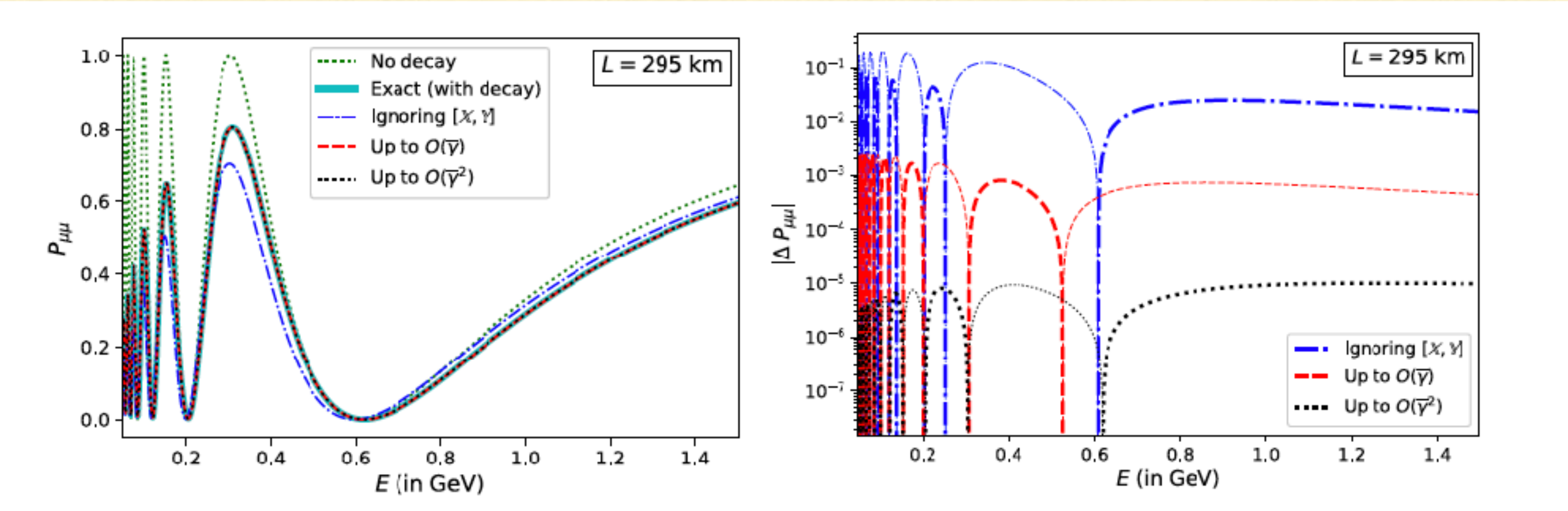
Term	Expression
$\mathrm{Re}(A)$	$-\cos 2\theta_m + \bar{\gamma}\bar{\Delta}_b \sin 2\theta_m \cos\chi$
$\mathrm{Im}(A)$	$-\bar{\gamma}\bar{\Delta}_a \sin 2\theta_m \cos\chi$
$\lvert A\rvert^2$	$\cos^2 2\theta_m - 2\bar{\gamma}\bar{\Delta}_b \sin 2\theta_m \cos 2\theta_m \cos\chi$
$\lvert B\rvert^2$	$\sin^2 2\theta_m + 2\bar{\gamma}\sin 2\theta_m \left(\bar{\Delta}_a \sin\chi + \bar{\Delta}_b \cos 2\theta_m \cos\chi\right)$

- $P_{\alpha\alpha} \neq P_{\beta\beta}\,,\ P_{\alpha\beta} \neq P_{\beta\alpha}$ unlike oscillation

- Probabilities to $\mathcal{O}(\bar{\gamma}^2)$ and expressions expanding $e^{-i\mathcal{H}_m t}$ in terms of the Pauli matrices were also done

Accuracy of the approximations

$$\Delta P_{\mu\mu} \equiv P_{\mu\mu}(\text{analytical}) - P_{\mu\mu}(\text{exact})$$



- $L = 295$ km, $E \sim 1$ GeV, $\Delta_a = 2.56 \times 10^{-3}$ eV$^2/(2E)$, $\theta_m = 45^\circ$, $(b_1, b_2, \gamma) = (3,6,8) \times 10^{-5}$ eV$^2/(2E)$, $\chi = \pi/4$.

Chattopadhyay, Chakroborty, Dighe, SG, Mohan, PRL 2022

More realistic : three generation

Chattopadhyay, Chakroborty, Dighe, SG, arXiv 2111.13128

A specific example

- The Hamiltonian in vacuum with mass and decay states same

$$H_{\rm vac} = \frac{1}{2E}\begin{pmatrix} m_1^2 - i\alpha_1 & 0 \\ 0 & m_2^2 - i\alpha_2 \end{pmatrix} \qquad \alpha_j = m_j/\tau_j$$

- In matter of constant density

$$H_{\rm matter} = \frac{1}{2E}\left[U\begin{pmatrix} m_1^2 & 0 \\ 0 & m_2^2 \end{pmatrix}U^\dagger + \begin{pmatrix} A_{CC} & 0 \\ 0 & 0 \end{pmatrix} + U\begin{pmatrix} -i\alpha_1 & 0 \\ 0 & -i\alpha_2 \end{pmatrix}U^\dagger\right]$$

Usual two flavour mass matrix in matter ← →

Decay part

- $$H_{eff}^{\rm matt} = \frac{1}{4E}\begin{pmatrix} A_{CC} - \Delta m_{21}^2 \cos 2\theta & \Delta m_{21}^2 \sin 2\theta \\ \Delta m_{21}^2 \sin 2\theta & \Delta m_{21}^2 \cos 2\theta \end{pmatrix} \rightarrow H_m = \frac{1}{2E}\begin{pmatrix} M_1^2 & 0 \\ 0 & M_2^2 \end{pmatrix}$$

$$\tan 2\theta_m = \frac{\Delta m_{21}^2 \sin 2\theta}{-A_{CC} + \Delta m_{21}^2 \cos 2\theta}.$$

$$\Delta M_{21}^2 = \sqrt{(-A_{CC} + \Delta m_{21}^2 \cos 2\theta)^2 + (\Delta m_{21}^2 \sin 2\theta)^2}.$$

L. Wolfenstein, PRD 17, 1978 S.P. Mikhyev, A.Yu. Smirnov, SJNP 42, 1985

The decay part

- In the matter mass basis, the decay matrix

$$\Gamma_m = \frac{1}{2E}\begin{pmatrix}\cos\theta_m & -\sin\theta_m \\ \sin\theta_m & \cos\theta_m\end{pmatrix}\left[\begin{pmatrix}\cos\theta & \sin\theta \\ -\sin\theta & \cos\theta\end{pmatrix}\begin{pmatrix}-i\alpha_1 & 0 \\ 0 & -i\alpha_2\end{pmatrix}\begin{pmatrix}\cos\theta & -\sin\theta \\ \sin\theta & \cos\theta\end{pmatrix}\right]\begin{pmatrix}\cos\theta_m & \sin\theta_m \\ -\sin\theta_m & \cos\theta_m\end{pmatrix},$$

$$\mathcal{H}_m = \frac{1}{2E}\begin{pmatrix} M_1^2 - \frac{i}{2}(\alpha_1+\alpha_2-(\alpha_2-\alpha_1)\cos 2\theta_\Delta) & -\frac{i}{2}(\alpha_2-\alpha_1)\sin 2\theta_\Delta \\ -\frac{i}{2}(\alpha_2-\alpha_1)\sin 2\theta_\Delta & M_2^2 - \frac{i}{2}(\alpha_1+\alpha_2+(\alpha_2-\alpha_1)\cos 2\theta_\Delta)\end{pmatrix}.$$

Off diagonal terms generated in matter

$$\theta_\Delta = \theta - \theta_m$$

- Comparing with the general form

$$\mathcal{H}_m = \begin{pmatrix} a_1 - i\,b_1 & -\frac{1}{2}i\,\gamma\,e^{i\chi} \\ -\frac{1}{2}i\,\gamma\,e^{-i\chi} & a_2 - i\,b_2\end{pmatrix}$$

$$a_1 = M_1^2/2E$$
$$a_2 = M_2^2/2E$$

$$b_1 = \frac{1}{4E}[\alpha_1+\alpha_2-(\alpha_2-\alpha_1)\cos 2\theta_\Delta)]$$
$$b_2 = \frac{1}{4E}[\alpha_1+\alpha_2+(\alpha_2-\alpha_1)\cos 2\theta_\Delta]$$
$$\chi = 0$$
$$\gamma = \frac{1}{2E}(\alpha_2-\alpha_1)\sin 2\theta_\Delta$$

- With this mapping probabilities can be calculated using the general formula

Single decaying neutrino

- If only the 2nd state decays => $\alpha_1 = 0$

$$a_{1,2} = \frac{\tilde{m}^2_{1,2}}{2E}, \qquad b_{1,2} = \frac{\alpha_2}{4E}[1 \mp \cos[2(\theta - \theta_m)]],$$

$$\chi = 0, \qquad \gamma = \frac{\alpha_2}{2E}\sin[2(\theta - \theta_m)].$$

- Interestingly even if we started with only one decaying state in vacuum, in matter both states can decay $(b_1, b_2, \gamma \neq 0)$

- Conversion Probability:

$$P_{\alpha\beta} = \frac{1}{2}e^{-\alpha_2 L/2E}[(\sin^2 2\theta_m + (\frac{\alpha_2}{2\Delta M^2_{21}})^2 \sin 4\theta_m \sin 4\theta_\Delta) (\cosh(\frac{1}{2E}\alpha_2 L \cos 2\theta_\Delta) - \cos(\Delta M^2_{21} L/2E))]$$

$$\longrightarrow \quad P_{\alpha\beta} = \sin^2 2\theta_m \sin^2\left(\frac{\Delta M^2_{21} L}{4E}\right)$$

No decay limit

Three Generation: Decay of only ν_3

$$\mathscr{H}_f^{(\gamma_3)} = \frac{1}{2E_\nu} U \left[\Delta m_{31}^2 \begin{pmatrix} 0 & 0 & 0 \\ 0 & \alpha & 0 \\ 0 & 0 & 1 \end{pmatrix} - i\,\Delta m_{31}^2 \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \gamma_3 \end{pmatrix} \right] U^\dagger + \begin{pmatrix} V_{cc} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\alpha \approx 0.03 \simeq O(\lambda^2)\,, \qquad s_{13} \equiv \sin\theta_{13} \simeq 0.14 \simeq O(\lambda)$$

$$\gamma_i \Delta m_{31}^2 = m_i/\tau_i\,.$$

- OMSD approximation ($\Delta m_{21}^2 = 0$)
- Probabilities expanded in α, s_{13}, γ_3
- Probabilities expanded in α, s_{13} and exact in γ_3

Using Cayley-Hamilton Theorem

OMSD probabilities

$$\widetilde{\mathcal{H}}_f^{(\mathrm{OMSD})} = \frac{\Delta m_{31}^2}{2E_\nu}\left[R_{13}\begin{pmatrix}0 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 1-i\gamma_3\end{pmatrix}R_{13}^\dagger + \begin{pmatrix}A & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{pmatrix}\right] .$$

$$= \widetilde{H}_f^{(\mathrm{OMSD})} - \frac{i}{2}\widetilde{\Gamma}_f^{(\mathrm{OMSD})}$$

$\widetilde{H}_f^{(\mathrm{OMSD})}$ can be diagonalised as

$$\widetilde{H}_f^{(\mathrm{OMSD})} = \frac{\Delta m_{31}^2}{2E_\nu} R_{13}^m \begin{pmatrix}\Lambda_1 & 0 & 0\\ 0 & \Lambda_2 & 0\\ 0 & 0 & \Lambda_3\end{pmatrix} R_{13}^{m\,\dagger} .$$

$$\Lambda_{1,3} = \frac{\Delta m_{31}^2}{4E}\left[1 + A \mp C_{13}\right] , \qquad \Lambda_2 = 0 ,$$

$$C_{13} \equiv \sqrt{(\cos 2\theta_{13} - A)^2 + (\sin 2\theta_{13})^2}$$

$$\tan 2\theta_{13}^m = \frac{\sin 2\theta_{13}}{\cos 2\theta_{13} - A} .$$

$$\widetilde{\mathcal{H}}_m^{(\mathrm{OMSD})} = \mathrm{diag}[(\Lambda_1, \Lambda_2, \Lambda_3)] - \frac{i}{2}\widetilde{\Gamma}_m^{(\mathrm{OMSD})} .$$

$$\widetilde{\Gamma}_m^{(\mathrm{OMSD})} = R_{13}^{m\,\dagger}\,\widetilde{\Gamma}_f^{(\mathrm{OMSD})}\,R_{13}^m = \frac{\Delta m_{31}^2}{E_\nu}\begin{pmatrix}\gamma_1^m & 0 & \frac{1}{2}\gamma_{13}^m\\ 0 & 0 & 0\\ \frac{1}{2}\gamma_{13}^m & 0 & \gamma_3^m\end{pmatrix}$$

$$\gamma_1^m \equiv \gamma_3 \sin^2\delta\theta , \qquad \gamma_3^m \equiv \gamma_3\cos^2\delta\theta , \qquad \gamma_{13}^m \equiv -\gamma_3 \sin(2\,\delta\theta) ,$$

$$\delta\theta \equiv \theta_{13}^m - \theta_{13}$$

- In the matter mass basis γ_{13}^m generated

$P_{e\mu}$ in matter

$$A(\nu_\alpha \to \nu_\beta) = [e^{-i\mathcal{H}_f^{(\mathrm{OMSD})}L}]_{\beta\alpha} = [R_{23} R_{13}^m \widetilde{A}_m R_{13}^{m\,\dagger} R_{23}^\dagger]_{\beta\alpha}$$

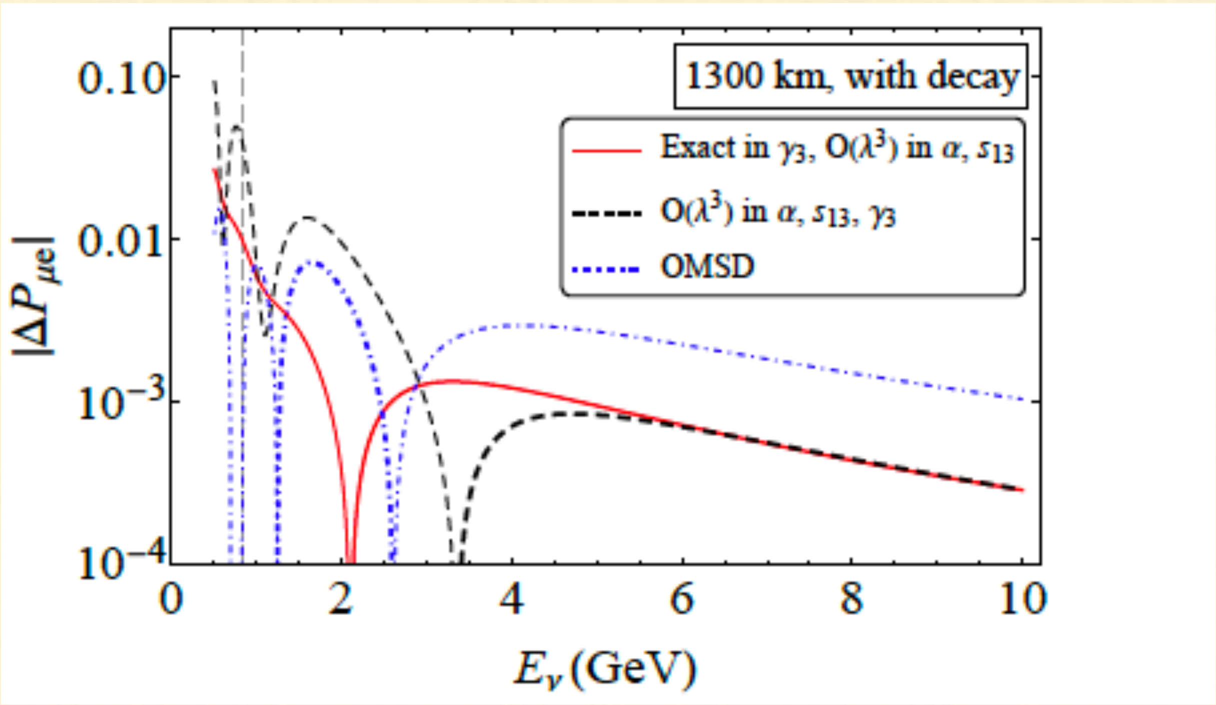
$$\widetilde{A}_m = \exp[-i\widetilde{\mathcal{H}}_m^{(\mathrm{OMSD})} L]$$

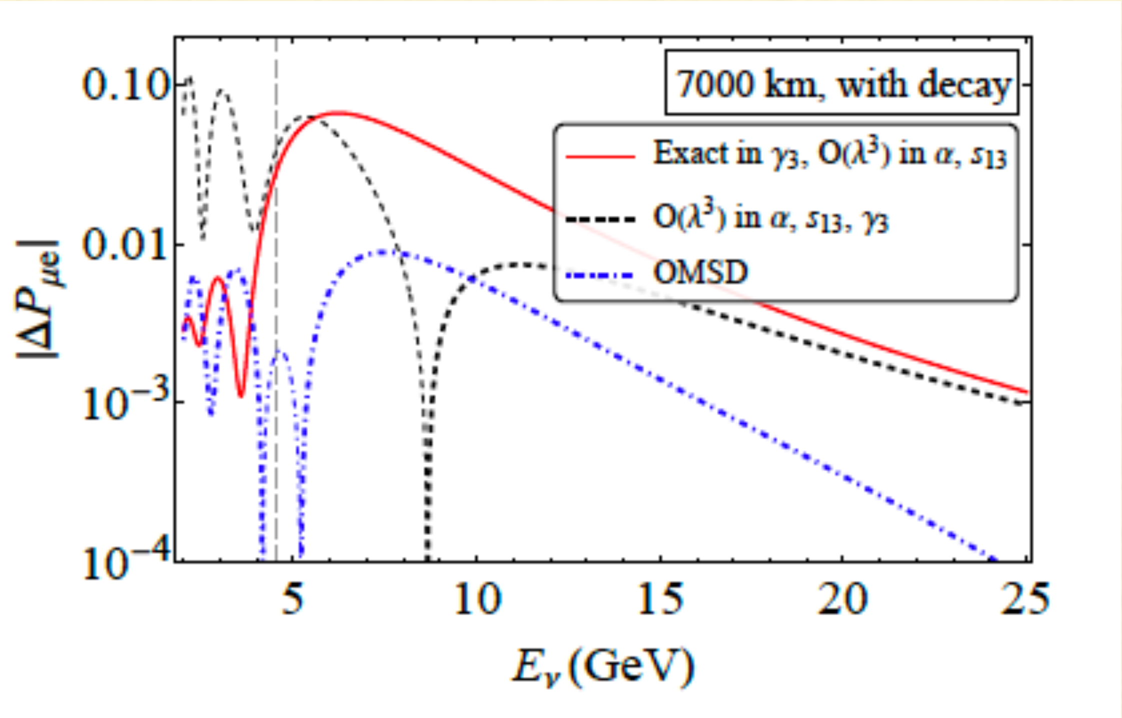
$$P_{e\mu} = s_{23}^2 \sin 2\theta_{13}^m \left[\sin 2\theta_{13}^m - 2\widetilde{\gamma}\gamma_- \cos 2\theta_{13}^m\right] \left[\frac{1}{4}\left(e^{-2\gamma_1^m \Delta} - e^{-2\gamma_3^m \Delta}\right)^2 + e^{-2\gamma_+ \Delta} \sin^2 \Delta_m\right]$$

$$\theta_{12} = 33^\circ\,, \quad \theta_{23} \simeq 45^\circ\,, \quad \theta_{13} \simeq 8.5^\circ\,, \quad \delta_{\mathrm{CP}} = 0^\circ\,,$$
$$\Delta m_{21}^2 = 7.37 \times 10^{-5}\ \mathrm{eV}^2\,, \quad \Delta m_{31}^2 = 2.56 \times 10^{-3}\ \mathrm{eV}^2\,.$$

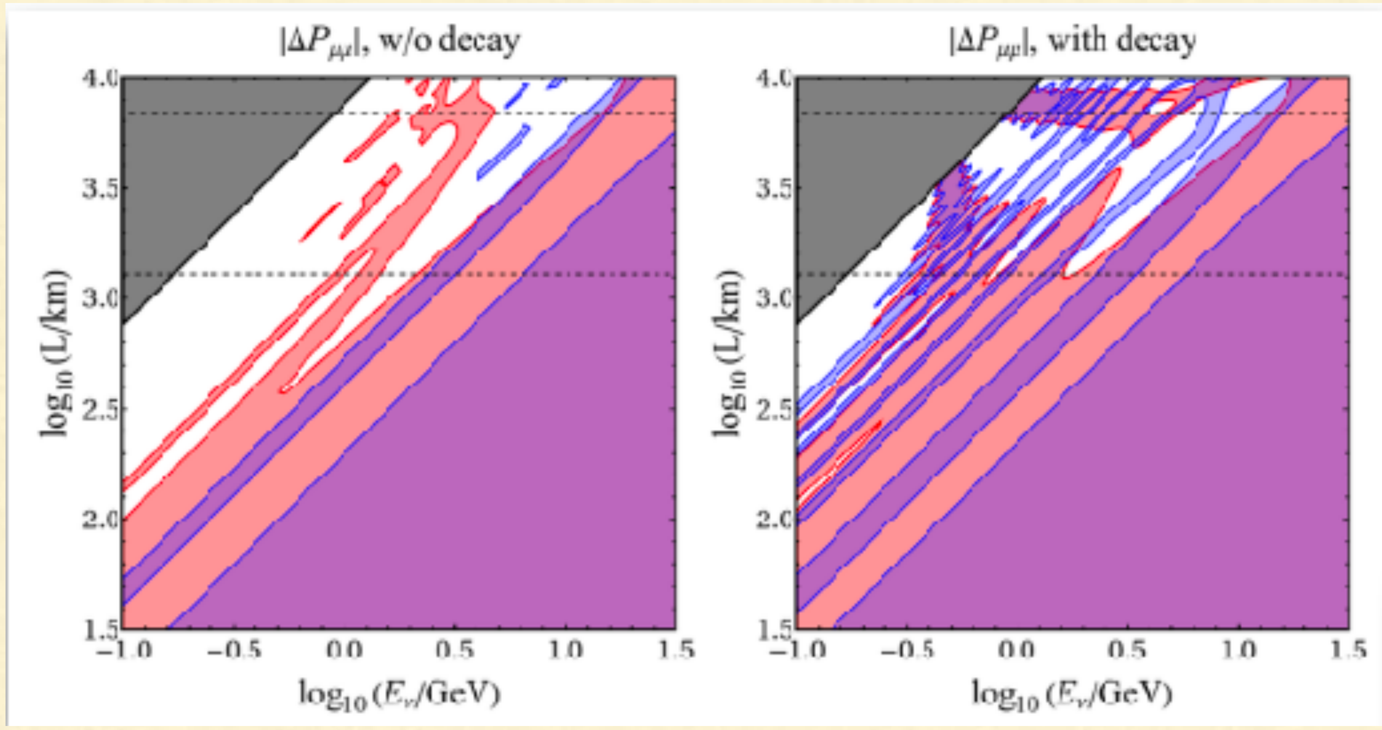
$$\gamma_3 = 0.1$$

$$\gamma_\pm \equiv \gamma_1^m \pm \gamma_3^m\,, \quad \widetilde{\gamma} \equiv \frac{\gamma_{13}^m}{C_{13}^2 + \gamma_-^2}\,.$$



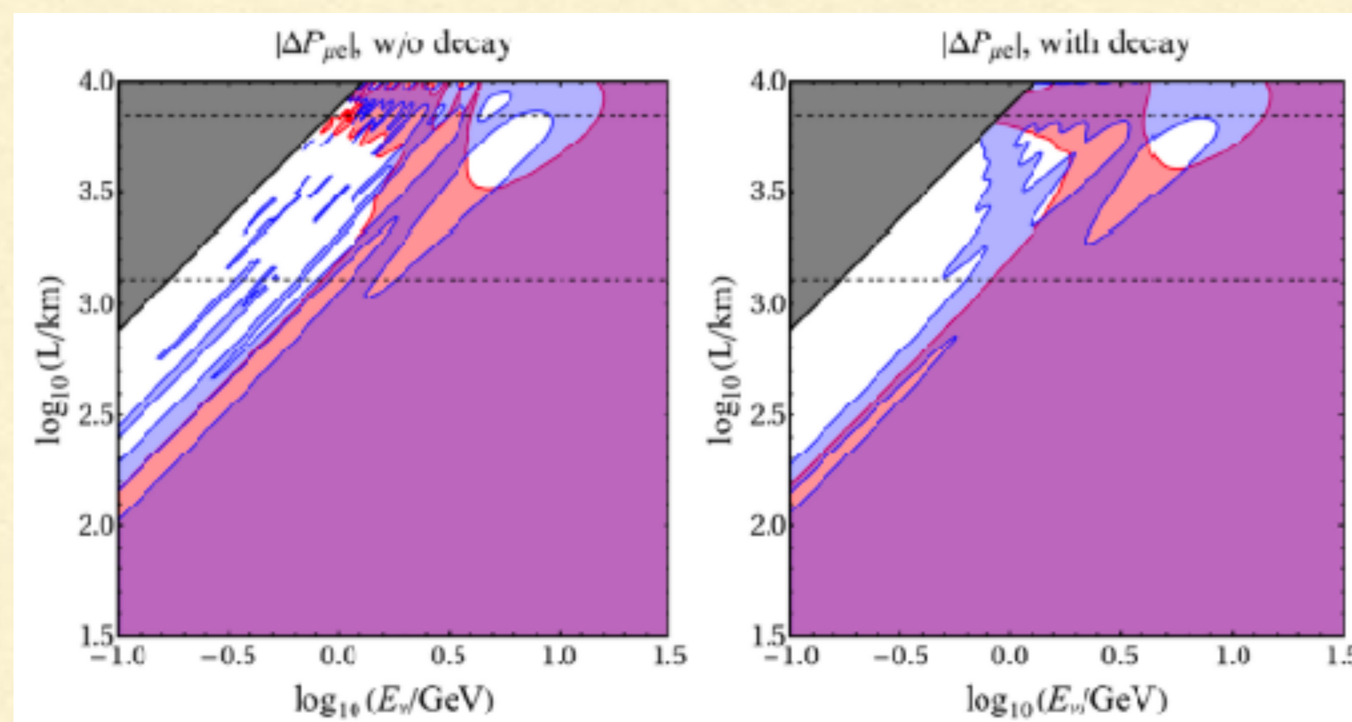


Accuracy of the approximations



- The regions in (L-E) plane where $|\Delta P_{\alpha\beta}| < 1\%$
- Grey region : $\alpha\Delta > 1$ and analytic expressions not valid

- OMSD approximation: Blue
- Expansion: Red
- Both: Purple



The general decay matrix

$$\mathcal{H}_f^{(\Gamma)} = U\left[\frac{1}{2E_\nu}\begin{pmatrix} 0 & 0 & 0 \\ 0 & \Delta m^2_{21} & 0 \\ 0 & 0 & \Delta m^2_{31}\end{pmatrix} - \frac{i}{2}\Gamma\right]U^\dagger + \begin{pmatrix} V_{cc} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0\end{pmatrix},$$

$$\alpha \approx 0.03 \simeq O(\lambda^2)\,,$$

$$s_{13} \equiv \sin\theta_{13} \simeq 0.14 \simeq O(\lambda)$$

$$\Gamma = \frac{\Delta m^2_{31}}{E_\nu}\begin{pmatrix} \gamma_1 & \frac{1}{2}\gamma_{12}e^{i\chi_{12}} & \frac{1}{2}\gamma_{13}e^{i\chi_{13}} \\ \frac{1}{2}\gamma_{12}e^{-i\chi_{12}} & \gamma_2 & \frac{1}{2}\gamma_{23}e^{i\chi_{23}} \\ \frac{1}{2}\gamma_{13}e^{-i\chi_{13}} & \frac{1}{2}\gamma_{23}e^{-i\chi_{23}} & \gamma_3\end{pmatrix}.$$

- Decay is a sub-leading effect => length scale associated with decay < that of oscillation

$$\gamma_1\Delta m^2_{31} \lesssim O(\lambda)\Delta m^2_{21}\,, \qquad \gamma_2\Delta m^2_{31} \lesssim O(\lambda)\Delta m^2_{21}\,, \qquad \gamma_3\Delta m^2_{31} \lesssim O(\lambda)\Delta m^2_{31}\,,$$

$$\gamma_1 \lesssim O(\lambda^3)\,, \qquad \gamma_2 \lesssim O(\lambda^3)\,, \qquad \gamma_3 \lesssim O(\lambda)$$

- The decay matrix is positive definite => $O(\gamma_{ij}^2) \lesssim O(\gamma_i)O(\gamma_j)$

$$\gamma_1\,,\ \gamma_2 \sim O(\lambda^3)\,, \qquad \gamma_3 \sim O(\lambda)\,, \qquad \gamma_{12} \sim O(\lambda^3)\,, \qquad \gamma_{13}\,,\ \gamma_{23} \sim O(\lambda^2)$$

Probabilities expanded in γ_3

$$P_{\alpha\beta} = P^{(0)}_{\alpha\beta} + P^{(\gamma_3)}_{\alpha\beta} + P^{(\Gamma)}_{\alpha\beta}$$

$$\Delta = \frac{\Delta m^2_{31} L}{4E_\nu},$$

$$\begin{aligned}
P^{(0)}_{\mu\mu} =& 1 - \sin^2 2\theta_{23} \sin^2\Delta - \frac{2}{A-1} s^2_{13} \sin^2 2\theta_{23} \\
&\times \left(\sin\Delta \cos A\Delta \frac{\sin[(A-1)\Delta]}{A-1} - \frac{A}{2}\Delta \sin 2\Delta \right) \\
&- 4 s^2_{13} s^2_{23} \frac{\sin^2[(A-1)\Delta]}{(A-1)^2} \\
&+ \alpha c^2_{12} \sin^2 2\theta_{23} \Delta \sin 2\Delta + O(\lambda^3), \\
P^{(\gamma_3)}_{\mu\mu} =& -\gamma_3 \Delta \left(\sin^2 2\theta_{23} \cos 2\Delta + 4 s^4_{23} \right) \\
&+ \gamma_3^2 \Delta^2 \left(\sin^2 2\theta_{23} \cos 2\Delta + 8 s^4_{23} \right) + O(\lambda^3), \\
P^{(\Gamma)}_{\mu\mu} =& \sin 2\theta_{23} \left(\gamma_{13} s_{12} \cos\chi_{13} - \gamma_{23} c_{12} \cos\chi_{23} \right) \\
&\times \sin 2\Delta + O(\lambda^3) .
\end{aligned}$$

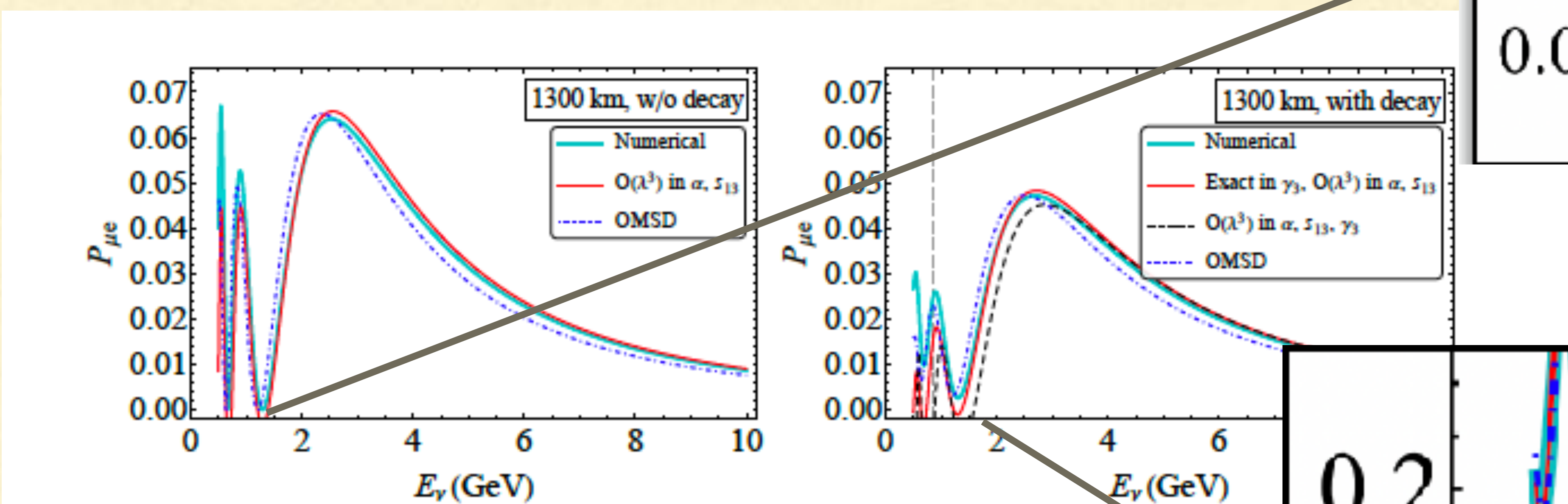
$$\begin{aligned}
P^{(0)}_{e\mu} =& 4 s^2_{13} s^2_{23} \frac{\sin^2[(A-1)\Delta]}{(A-1)^2} \\
&+ 2\alpha s_{13} \sin 2\theta_{12} \sin 2\theta_{23} \cos(\Delta - \delta_{CP}) \\
&\times \frac{\sin[(A-1)\Delta]}{A-1} \frac{\sin A\Delta}{A} + O(\lambda^4), \\
P^{(\gamma_3)}_{e\mu} =& -8\gamma_3 s^2_{13} s^2_{23} \Delta \frac{\sin^2[(A-1)\Delta]}{(A-1)^2} + O(\lambda^4), \\
P^{(\Gamma)}_{e\mu} =& -4 s_{13} s^2_{23} \left(\gamma_{23} s_{12} \sin[\delta_{CP} + \chi_{23}] \right. \\
&\left. + \gamma_{13} c_{12} \sin[\delta_{CP} + \chi_{13}] \right) \frac{\sin^2[(A-1)\Delta]}{(A-1)^2} \\
&+ O(\lambda^4),
\end{aligned}$$

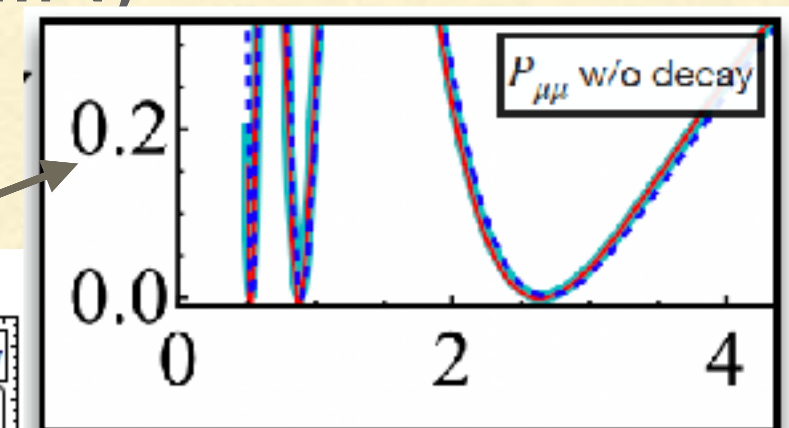
Decay part no matter dependence

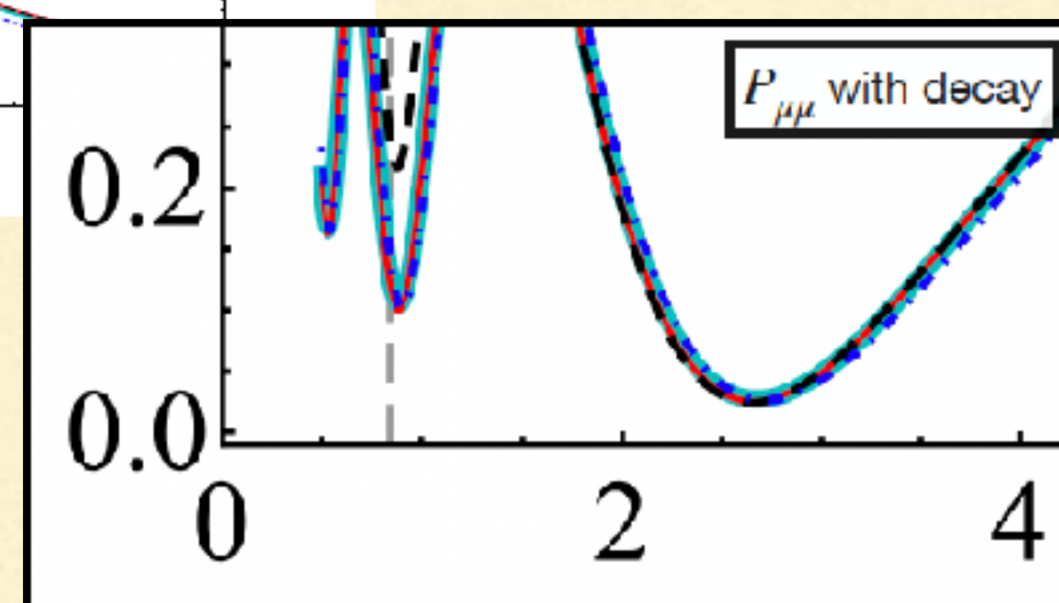
Effect of off-diagonal decay terms as important as γ_3

Increase in probability due to decay ?

$P_{\mu\mu}$ at survival probability Minima (SPMIN)

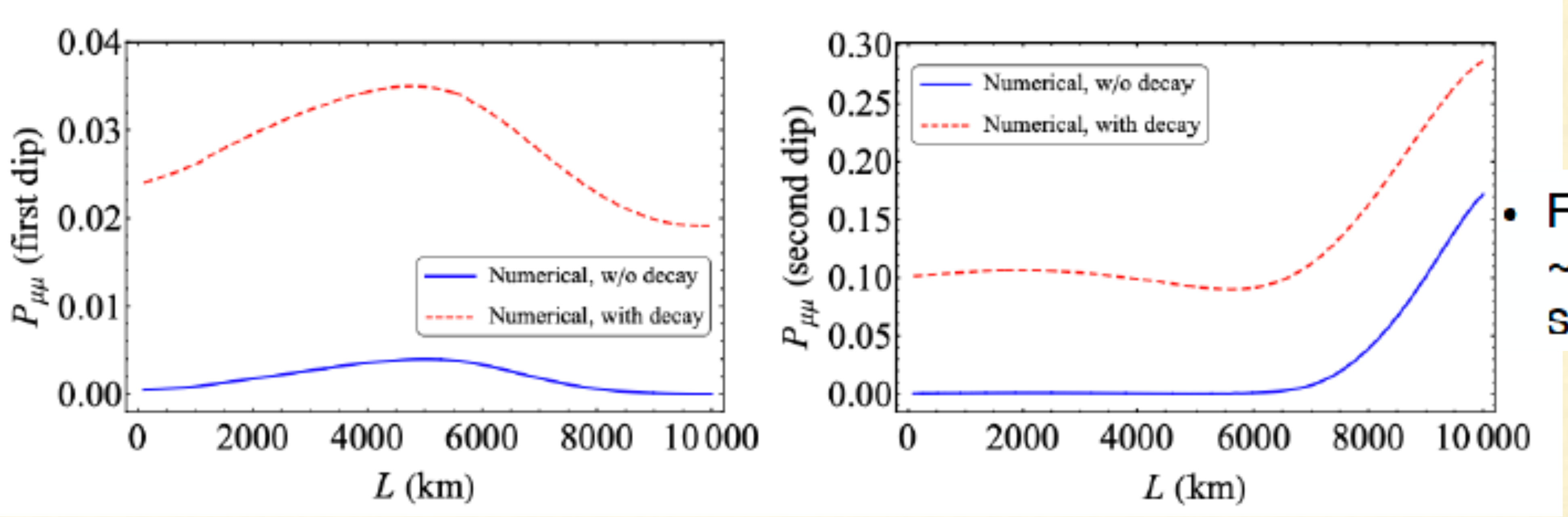






- Increase in the survival probability due to decay

The first two Dips in $P_{\mu\mu}$



- For $\gamma_3 = 0.1$, increase of ~0.02 at first and ~0.1 at second oscillation dip.

$$P_{\mu\mu}^{\text{leading}}(\text{dip}) = 1 - \sin^2 2\theta_{23} - s_{23}^4\left(1 - e^{-4\gamma_3\Delta}\right) + 2s_{23}^2c_{23}^2\left(1 - e^{-2\gamma_3\Delta}\right)$$

$$P_{\mu\mu}(\text{first dip}) \simeq P_{\mu\mu}^{\text{leading}}(\Delta \simeq \pi/2) = \frac{1}{4}\left(1 - e^{-\pi\gamma_3}\right)^2 \geq 0$$

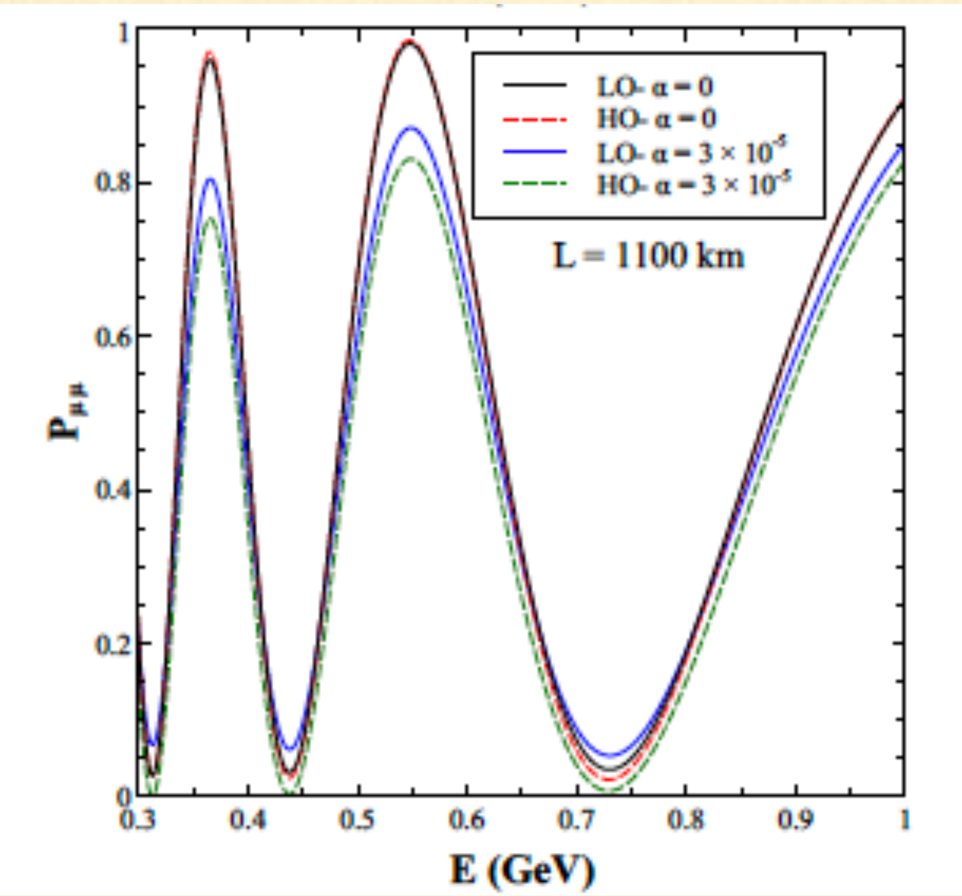
$$P_{\mu\mu}(\text{second dip}) \simeq P_{\mu\mu}^{\text{leading}}(\Delta \simeq 3\pi/2) \simeq \frac{1}{4}\left(1 - e^{-3\pi\gamma_3}\right)^2 \geq 0$$

Second dip $E_\nu \simeq 0.69\left(\frac{L}{1000\text{ km}}\right)$ GeV → Possible to observe in ESSNUSB/T2HKK

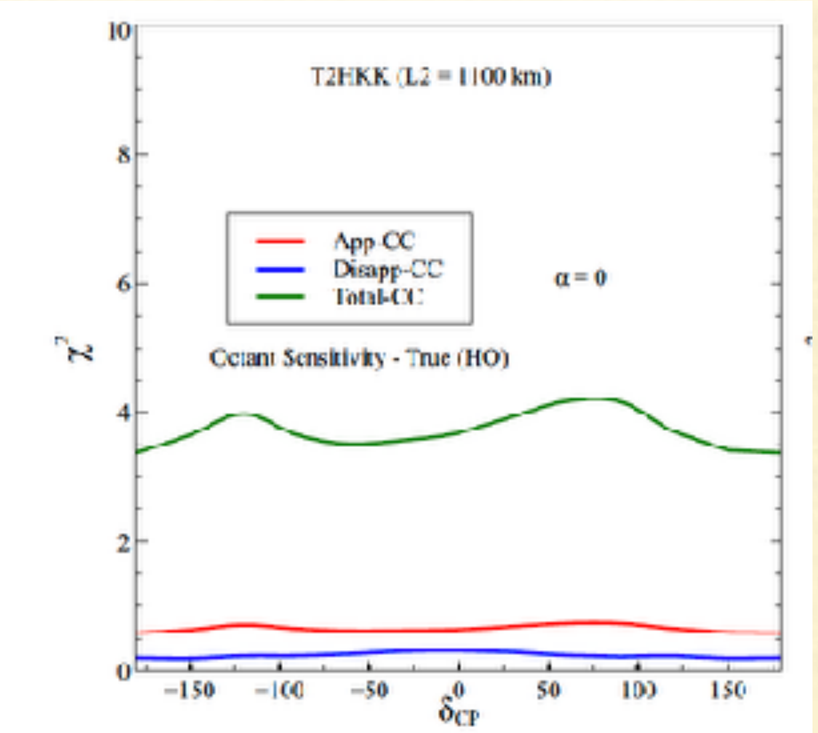
Octant sensitivity from $P_{\mu\mu}$

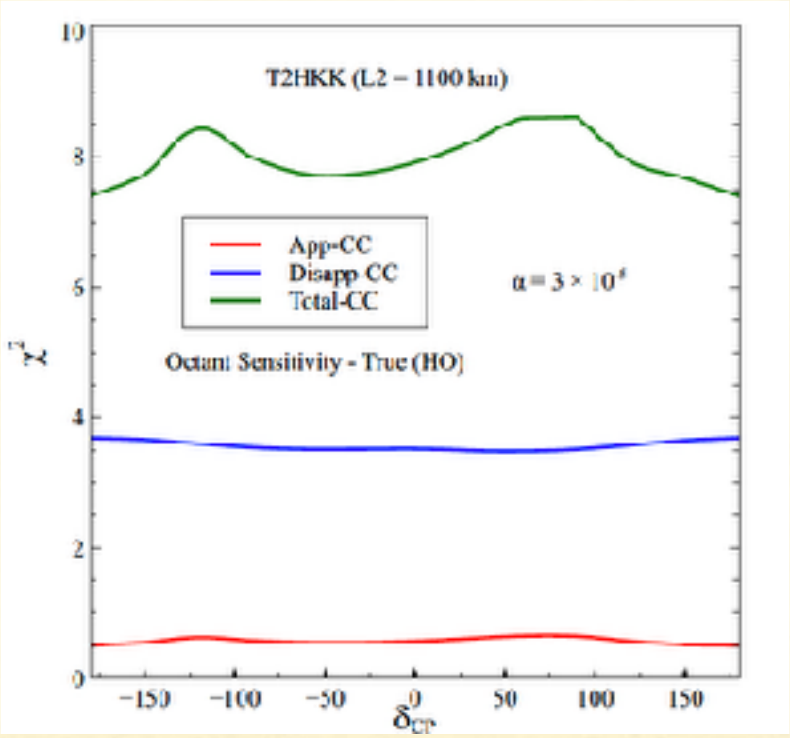
- No octant sensitivity in absence of decay

$$P_{\mu\mu}^{\text{leading}}(\text{dip}) = 1 - \sin^2 2\theta_{23} - s_{23}^4\left(1 - e^{-4\gamma_3\Delta}\right) + 2s_{23}^2 c_{23}^2\left(1 - e^{-2\gamma_3\Delta}\right)$$



Octant sensitivity in presence of decay





K. Chakraborty, D. Dutta, SG, D. Pramanik. JHEP05 (2021)

Future Possibilites

- Calculation of probabilities in varying density, relevant for astrophysical sources of neutrinos like Sun, Supernova.
- Studying the implications at a phenomenological level, extra degeneracies ..
- Decay and other new physics ..


Thank
you!!

Back up slides

Formalism

$$\mathscr{H}_m = H_m - i\,\Gamma_m/2$$

- Defining: $d_i \equiv a_i - ib_i$ $\quad\Delta_a \equiv a_2 - a_1,\ \Delta_b \equiv b_2 - b_1,\ \Delta_d \equiv d_2 - d_1$

$$\bar{\gamma} \equiv \frac{\gamma}{|\Delta_d|}\,, \quad \bar{\Delta}_a \equiv \frac{\Delta_a}{|\Delta_d|}\,, \quad \bar{\Delta}_b \equiv \frac{\Delta_b}{|\Delta_d|}\,.$$

$$-i\mathscr{H}_m t = -\frac{it}{2}(d_1 + d_2)\mathbb{I} + \mathbb{X} + \mathbb{Y}\,.$$

$$\mathbb{X} \equiv -\frac{i\,\Delta_d t}{2}\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \qquad \mathbb{Y} \equiv -\frac{\gamma t}{2}\begin{pmatrix} 0 & e^{i\chi} \\ e^{-i\chi} & 0 \end{pmatrix}$$

- The commutator of $\mathbb{X}$ and $\mathbb{Y}$ is

$$\mathscr{L}_{\mathbb{X}}\mathbb{Y} \equiv [\mathbb{X}, \mathbb{Y}] = i\frac{\gamma\,\Delta_d t^2}{2}\begin{pmatrix} 0 & -e^{i\chi} \\ e^{-i\chi} & 0 \end{pmatrix}\,.$$

Zassenhaus Expansion

- The inverse Baker-Campbell-Hausdraff expansion

$$e^{\mathbb{X}+\mathbb{Y}} = e^{\mathbb{X}}\, e^{\mathbb{Y}}\, e^{-\frac{1}{2}[\mathbb{X},\mathbb{Y}]}\, e^{\frac{1}{6}(2[\mathbb{Y},[\mathbb{X},\mathbb{Y}]]+[\mathbb{X},[\mathbb{X},\mathbb{Y}]])} \dots$$

- Note that $|\mathbb{Y}| \sim \bar{\gamma}|\mathbb{X}|$ $\quad \mathscr{L}_{\mathbb{X}}\mathbb{Y} \sim \bar{\gamma}|\mathbb{X}|^2$

- $$e^{\mathbb{X}+\mathbb{Y}} = \left(1 + \sum_{p=1}^{\infty} \sum_{n_1,\dots,n_p=1}^{\infty} \frac{n_p \dots n_1}{n_p(n_p+n_{p-1})\dots(n_p+\dots+n_1)} \mathscr{Y}_{n_p} \dots \mathscr{Y}_{n_1}\right) e^{\mathbb{X}}$$

$$\mathscr{Y}_n = \frac{1}{n!} \mathscr{L}_{\mathbb{X}}^{n-1} \mathbb{Y}$$

- To obtain expression upto $\mathscr{O}(\bar{\gamma})$ we need to consider p=1

$$e^{\mathbb{X}+\mathbb{Y}} = \left(1 + \frac{\sin(\Delta_d t)}{\Delta_d t}\mathbb{Y} - \frac{\cos(\Delta_d t) - 1}{\Delta_d t} i\sigma_3 \mathbb{Y}\right) e^{\mathbb{X}}.$$

Probability calculation $\mathcal{O}(\bar{\gamma})$

- The amplitude matrix in mass basis in matter is given by

$$\mathscr{A}_m \equiv e^{-i\mathscr{H}_m t} = \begin{pmatrix} e^{-id_1 t} & -i\frac{\gamma e^{i\chi} g_-(t)}{\Delta_d} \\ -i\frac{\gamma e^{-i\chi} g_-(t)}{\Delta_d} & e^{-id_2 t} \end{pmatrix},$$

$$g_\pm(t) = \frac{1}{2}(e^{-id_2 t} \pm e^{-id_1 t})$$

$$[\mathscr{A}_f]_{\alpha\beta} = \left[U_m\, e^{-i\mathscr{H}_m t}\, U_m^\dagger\right]_{\alpha\beta} \qquad P_{\beta\alpha} = |\mathscr{A}_{\alpha\beta}|^2$$