The Critical $O(N)$ CFT

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"The critical $O(N)$ CFT: Methods and conformal data" [2201.09520]

Why the $O(N)$ CFT?

ε-expansion 50 years!

[Submitted on 24 Jan 2022]

The critical O(N) CFT: Methods and conformal data

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The critical $O(N)$ CFT in spacetime dimensions $2 < d < 4$ is one of the most important examples of a conformal field theory, with the Ising CFT at $N = 1$, $2 \leq d < 4$, as a notable special case. Apart from numerous physical applications, it serves frequently as a concrete testing ground for new approaches and techniques based on conformal symmetry. In the perturbative limits - the $4 - \varepsilon$ expansion, the large N expansion and the $2 + \tilde{\varepsilon}$ expansion - a lot of conformal data have been computed over the years. In this report, we give an overview of the critical $O(N)$ CFT, including some methods to study it, and present a large collection of conformal data. The data, extracted from the literature and supplemented by many additional computations of order ε anomalous dimensions, is made available through an ancillary data file.

Comments: 87+13 pages, many tables, one ancillary data file. Comments are welcome
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Milestone

Critical Exponents in 3.99 Dimensions

Kenneth G. Wilson and Michael E. Fisher
Phys. Rev. Lett. **28**, 240 – Published 24 January 1972

An article within the collection: Letters from the Past - A PRL Retrospective

Article | References | Citing Articles (1,093) | PDF | Export Citation

ABSTRACT

Critical exponents are calculated for dimension $d = 4 - \varepsilon$ with ε small, using renormalization-group techniques. To order ε the exponent γ is $1 + \frac{1}{6}\varepsilon$ for an Ising-like model and $1 + \frac{1}{5}\varepsilon$ for an XY model.

- Conformal invariance
- Simple formulation: $\mathcal{L} = \frac{1}{2}(\partial\phi)^2 + \frac{\lambda}{24}\phi^4$
- Experimental applications in $d = 3$
- Weakly coupled in $d = 4 - \varepsilon$ and at large N
- Exact results in $d = 2$
- Testing ground for CFT methods – "bootstrap"
- A lot of data available

Outline

Outline

The RG perspective

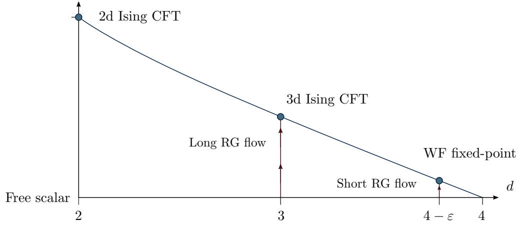
Universality – many UV systems can flow to the same IR behaviour
Ising universality class ($N = 1$)

$$H = -J \sum_{<I,J>} \sigma_I \sigma_J - h \sum_I \sigma_I. \tag{1}$$

For $h = 0$ and fine-tuned βJ, (1) $\to$ Ising CFT ($d > 1$)

$$\mathcal{L}_{\mathrm{UV}} = \frac{1}{2}(\partial\phi)^2 + \frac{m^2}{2}\phi^2 + \frac{\lambda}{24}\phi^4 \tag{2}$$

For fine-tuned m^2, (2) $\to$ Ising CFT



$$\beta(\bar{g}) = -\varepsilon\bar{g} + 3\bar{g}^2 + O(\bar{g}^3)$$

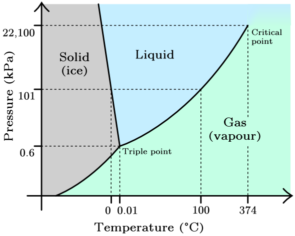
$$\beta(\bar{g}^*) = 0$$

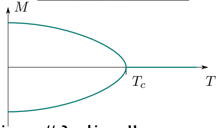
$$\bar{g}^* = \frac{\varepsilon}{3} + O(\varepsilon^2)$$

Experimental realisation – 2nd order phase transitions

$N = 1$: Ising CFT

- Liquid–vapour transitions
- Binary fluid mixtures
- Uniaxial magnets





$N = 2$: XY CFT

- XY magnets
- Helium superfluid transition "λ-line"

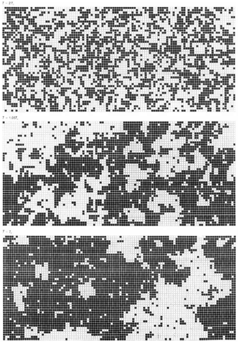
$N = 3$: Heisenberg CFT

- Isotropic magnets

$N \to 0$: Self-avoiding random walks

- Dilute polymers

$N \to -2$: Loop-erased random walks



Also: Quantum-critical systems at zero temperture: $(2+1)$d

Experimental data – focus: Ising ($N = 1$)

Critical exponents:

Note: $\phi \leftrightarrow \sigma \leftrightarrow \varphi$, $\quad \phi^2 \leftrightarrow \epsilon \leftrightarrow \varphi_S^2$

$$\eta : \quad \langle \phi(x)\phi(0)\rangle\Big|_{T=T_c} \propto \frac{1}{|x|^{2+\eta}} \qquad \Delta_\phi = \frac{d-2}{2} + \frac{1}{2}\eta$$

$$\nu : \quad \text{corr. length } \xi \propto |T - T_c|^{\nu} \qquad \Delta_{\phi^2} = d - \frac{1}{\nu}$$

	η	ν
Liquid-vapour transition	0.042(6) (CO_2)	0.62(3) (D_2O)
Binary liquid mixtures	0.041(5)	0.632(2)
Uniaxial magnet FeF_2		0.64(1)
Quantum phase transition		0.62(4)
ε-expansion $O(\varepsilon^7)$ resummation	0.03653(65)	0.62977(22)
High temperature expansion	0.03639(15)	0.63012(16)
MC simulations	0.036284(40)	0.62998(5)
Conformal bootstrap	0.0362978(20)	0.629971(4)

The CFT perspective

$O(N)$ model at criticality conjectured to be conformally invariant

- Distinguished set of operators: conformal primaries $\mathcal{O}$: $[K_\mu, \mathcal{O}(0)] = 0$
- Scaling dimension $[D, \mathcal{O}(0)] = i\Delta_{\mathcal{O}}\mathcal{O}(0)$. $\langle \mathcal{O}_i(x)\mathcal{O}_j(0)\rangle = |x|^{-2\Delta_{\mathcal{O}_i}}\delta_{ij}$
- OPE coefficients λ_{ijk}: $\langle \mathcal{O}_i(x_1)\mathcal{O}_j(x_2)\mathcal{O}_k(x_3)\rangle = \lambda_{ijk}|x_{12}|^{\Delta_k-\Delta_i-\Delta_j}\ldots$

CFT-data = spectrum of Δ_i + OPE coefficients

Determines all (flat-space) observables

- Critical exponents $\eta = 2\Delta_\phi - (d-2)$, $\nu = \frac{1}{d-\Delta_{\phi^2}}$, $\omega = \Delta_{\phi^4} - d$.
- n-point functions via OPE $\mathcal{O}_1 \times \mathcal{O}_2 = \sum_k \lambda_{12k}\mathcal{O}_k$

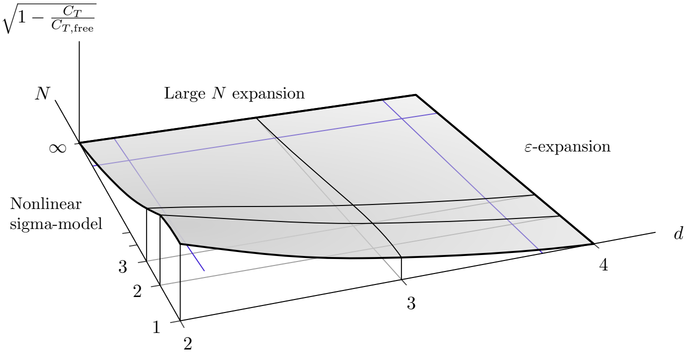
  E.g. 4-pt function

$$\langle \mathcal{O}\mathcal{O}\mathcal{O}\mathcal{O}\rangle = \frac{1}{x_{12}^{2\Delta_{\mathcal{O}}}x_{34}^{2\Delta_{\mathcal{O}}}}\sum_{\mathcal{O}'}\lambda^2_{\mathcal{O}\mathcal{O}\mathcal{O}'}G_{\Delta_{\mathcal{O}'},\ell_{\mathcal{O}'}}(u,v), \quad u = \frac{x_{12}^2x_{34}^2}{x_{13}^2x_{24}^2}, v = \frac{x_{14}^2x_{23}^2}{x_{13}^2x_{24}^2}$$

- Conserved currents J^μ, $T^{\mu\nu}$: C_J, C_T. *e.g.* $\lambda^2_{\mathcal{O}\mathcal{O}T} = \frac{d^2\Delta^2_{\mathcal{O}}}{4(d-1)^2C_T}$

A two-parameter family of CFTs

Conjecture: For any $2 < d < 4$ and $1 \leqslant N < \infty$, an interacting CFT

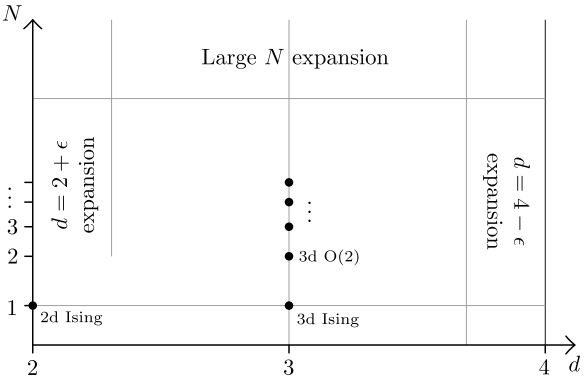


"Definition": "The unique gapless IR fixed-point of the theory of N scalars φ^i perturbed by $\lambda(\varphi^2)^2$ and tuning $m^2\varphi^2$"

Note: Not unitary for non-integer N, d [Hogervorst, Rychkov, Van Rees 2015]

Outline

Expansion limits



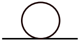
The ε-expansion

Bare action, $d = 4 - \varepsilon$ $\qquad \mathcal{L}_b = \frac{1}{2}(\partial^\mu \phi_b)^2 + \frac{1}{2} m_b^2 \phi_b^2 + \frac{\lambda_b}{24}\phi_b^4$

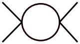
Let $\phi_b = \sqrt{Z}\phi$, $m_b^2 = Z_{m^2}/Z m^2$, $g = Z_g/Z^2 \mu^{-\varepsilon}\lambda_b$.

$$\Delta_\phi = \frac{d-2}{2} + \gamma_\phi(g^*), \quad \Delta_{\phi^2} = d - 2 + \gamma_{m^2}(g^*), \quad \Delta_{\phi^4} = d + \frac{\partial \beta}{\partial g}(g^*)$$

with $\gamma_\phi = \frac{\partial}{\partial \ln \mu} \ln \sqrt{Z}\big|_{m_b^2,\lambda_b}$, $\gamma_{m^2} = \frac{\partial \ln m^2}{\partial \ln \mu}\big|_{m_b^2,\lambda_b}$, $\beta(g) = \frac{\partial g}{\partial \ln \mu}\big|_{m_b^2,\lambda_b}$.
Leading-order computation:



$$: \quad Z_{m^2} = 1 + \frac{g}{(4\pi)^2}\frac{1}{\varepsilon} + \ldots, \qquad Z = 1 + O(g^2)$$



$$: \quad Z_g = 1 + \frac{g}{(4\pi)^2}\frac{3}{\varepsilon} + \ldots, \qquad \beta(g) = -\varepsilon g + 3\frac{g^2}{(4\pi)^2}$$

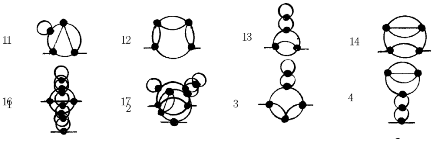
So at $\beta(g^*) = 0$, $g^* = (4\pi)^2 \varepsilon/3 + \ldots$,

$$\Delta_\phi = 1 - \frac{\varepsilon}{2} + O(\varepsilon^2), \quad \Delta_{\phi^2} = 2 - \varepsilon + \frac{\varepsilon}{3} + O(\varepsilon^2), \quad \Delta_{\phi^4} = 4 - \varepsilon + \varepsilon + O(\varepsilon^2)$$

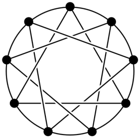
The ε-expansion – continued

Higher-order results

5-loop:



7-loop:



- ε^4: 1970's [Brezin, Guillou, Zinn-Justin, Nickel; Vladimirov, Kazakov, Tarasov; etc.]
- ε^5: 1990's [Kleinert, Schulte-Frohlinde; Chetyrkin, Larin, Neu; etc.]
- ε^6–ε^7: 2010's [Panzer; Batkovich, Chetyrkin, Kompaniets; Schnetz]
- ϵ^8 for Δ_ϕ [Schnetz 2021]

$$\Delta_{\phi^2} = 2 - \varepsilon + \tfrac{1}{3}\varepsilon + \tfrac{19}{162}\varepsilon^2 + \left(-\tfrac{4}{27}\zeta_3 + \ldots\right)\varepsilon^3 + \ldots \\ + \left(\tfrac{244}{243}\zeta_{5,3} + \tfrac{736}{729}\zeta_3\zeta_5 + \ldots\right)\varepsilon^6 + \left(-\tfrac{8}{2187}P_{7,11} + \ldots\right)\varepsilon^7 + O(\varepsilon^8)$$

More complicated composite operators computed by diagrams with insertions, and unmixing degenerate operators.

Large N expansion

[Ma; Vasiliev, Pismak, Honkonen; Derkachov, Manashov; Gracey; *etc.*]

At large N for any $2 < d < 4$, the $O(N)$ CFT is described by a theory with fields φ^i and $\sigma \leftarrow \varphi^2_S = \varphi^i\varphi^i$:

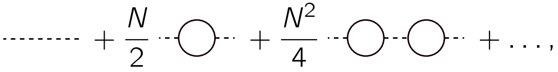
$$\Delta_\varphi = \frac{d-2}{2} + \frac{\gamma_\varphi^{(1)}}{N} + \ldots, \quad \Delta_\sigma = 2 + \frac{\gamma_\sigma^{(1)}}{N} + \ldots$$

Note $\Delta_{\varphi^2_S} = 2 - \varepsilon + \frac{N+2}{N+8}\varepsilon + \ldots = 2 + O(N^{-1})$
Large N action

$$\mathcal{L} = \frac{1}{2}(\partial\varphi)^2 + \frac{1}{2\sqrt{N}}\varphi^2\sigma - \frac{3}{2}\frac{\sigma^2}{\lambda N}$$

Integrating out σ gives $\lambda(\varphi^2)^2/24$

σ propagator dynamically generated

$G_\sigma(p) =$ 

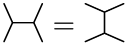
3d $O(N)$ CFT

Methods to estimate critical exponents and other CFT-data

- Monte Carlo simulations
- Resummation techniques
- Fixed-dimension, high-temperature expansions
- Non-perturbative/functional RG
- Numerical conformal bootstrap

Conformal bootstrap: consistency of OPE + unitarity: [Rattazzi *et al* 2008; review Poland *et al* 2018]

$$\mathcal{G}(u,v)=\sum_{\mathcal{O}'}\lambda^2_{\mathcal{O}\mathcal{O}\mathcal{O}'}G_{\Delta_{\mathcal{O}'},\ell_{\mathcal{O}'}}(u,v)=\left(\frac{u}{v}\right)^{\Delta_{\mathcal{O}}}\mathcal{G}(v,u)$$



- Correlators of φ, φ^2_S, $\varphi^2_T=\varphi^{\{i}\varphi^{j\}}$ + gap assumptions $\to$ small unitary islands [Kos *et al* 2015; Chester *et al* 2019; 2020]
- Extremal functional method [Poland, Simmons-Duffin 2010; El-Showk *et al* 2012]: estimate of low-lying spectrum

3d Ising spectrum

Extremal functional method + sampling in the 3d Ising island

$\mathbb{Z}_2$-even

7.25

6.90

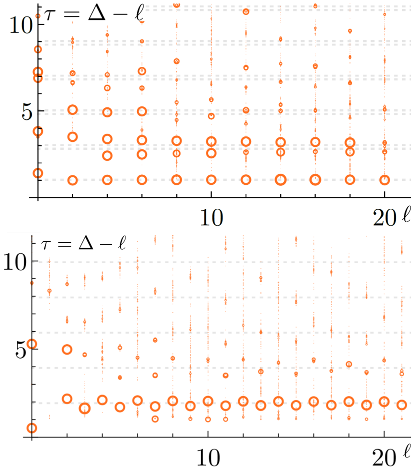
3.83

1.41

$\mathbb{Z}_2$-odd

5.29

.518



(even spins only)

[Simmons-Duffin 2016]

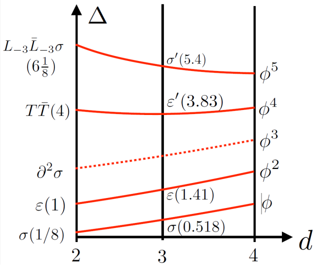
3d $O(2)$ spectrum [Liu *et al* 2020]

Outline

Spectrum continuity

Spectrum continuity [Hogervorst, Rychkov, Van Rees 2015]

Strong spectrum continuity: All conformal data (scaling dimensions, OPE coefficients) depend continuously on d and N. "Analytic" in d and N



Non-perturbative definition: In each representation of Lorentz $\times O(N)$, a list of operators with increasing scaling dimension.

Issues

- Representation theory (bonus slide)
- Non-unitarity [Hogervorst, Rychkov, Van Rees 2015]

One-loop dilatation operator

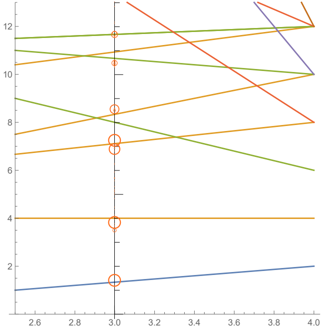
Order ε anomalous dimensions from conformal perturbation theory

$$-i[D, \mathcal{O}_i] = \Delta^{(0)}\mathcal{O}_i + \Gamma_{ij}\,\varepsilon\,\mathcal{O}_j + O(\varepsilon^2), \qquad \Gamma_{ij} \propto \lambda_{\mathcal{O}_i\mathcal{O}_j\phi^4}\big|_{\text{free}}$$

Systematic method [Kehrein, Wegner 1992; 1994; Hogervorst, Rychkov, Van Rees 2015]

Result in the scalar $\mathbb{Z}_2$-even sector

i	$\mathcal{O}_i$	$\Delta_{\mathcal{O}_i}$	known to
1	ϕ^2	$[2-\varepsilon] + \frac{1}{3}\varepsilon + \dots$	$O(\varepsilon^7)$
2	ϕ^4	$[4-2\varepsilon] + 2\varepsilon + \dots$	$O(\varepsilon^7)$
3	ϕ^6	$[6-3\varepsilon] + 5\varepsilon + \dots$	$O(\varepsilon^2)$
4	$\Box^2\phi^4$	$[8-2\varepsilon] + \frac{10}{9}\varepsilon + \dots$	$O(\varepsilon)$
5	ϕ^8	$[8-4\varepsilon] + \frac{28}{3}\varepsilon + \dots$	$O(\varepsilon^2)$
6	$\Box^3\phi^4$	$[10-2\varepsilon] + \frac{1}{3}\varepsilon + \dots$	$O(\varepsilon)$
7	$\Box^2\phi^6$	$[10-3\varepsilon] + \frac{11}{3}\varepsilon + \dots$	$O(\varepsilon)$
8	ϕ^{10}	$[10-5\varepsilon] + 15\varepsilon + \dots$	$O(\varepsilon^2)$



Testing spectrum continuity with bootstrap

Conformal bootstrap at intermediate d [El-Showk *et al* 2013; Cappelli *et al* 2018]

Work in progress [JH, Kousvos, Reehorst]: EFM at "most allowed point" using the "navigator" [Reehorst *et al* 2021]

Figure with preliminary data omitted

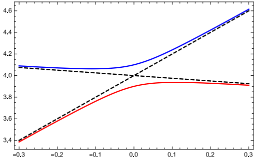
ϕ^{10}	$[10-5\varepsilon]+15\varepsilon+\dots$
$\Box^2\phi^6$	$[10-3\varepsilon]+\frac{11}{3}\varepsilon+\dots$
$\Box^3\phi^4$	$[10-2\varepsilon]+\frac{1}{3}\varepsilon+\dots$
ϕ^8	$[8-4\varepsilon]+\frac{28}{3}\varepsilon+\dots$
$\Box^2\phi^4$	$[8-2\varepsilon]+\frac{10}{9}\varepsilon+\dots$
ϕ^6	$[6-3\varepsilon]+5\varepsilon+\dots$
ϕ^4	$[4-2\varepsilon]+2\varepsilon+\dots$
ϕ^2	$[2-\varepsilon]+\frac{1}{3}\varepsilon+\dots$

$N=2$, $N=3$ [Sirois 2022]

Testing spectrum continuity with bootstrap: level repulsion

General expectation [von Neumann, Wigner 1929]: energy levels do not cross

Conformal field theory
[Korchemsky 2015]
($\mathcal{N} = 4$ SYM, other examples)



PRELIMINARY
[JH, Kousvos, Reehorst:
Work in progress]

Figure with preliminary data omitted

Outline

Outlook

Critical $O(N)$ CFT as a prototype example of a CFT in $d > 2$
Work in progress: *Mixed correlators from large spin perturbation theory* [Bertucci, JH, McPeak] $\langle \varphi\varphi\varphi\varphi \rangle$, $\langle \varphi\varphi\varphi^2_S\varphi^2_S \rangle$, $\langle \varphi^2_S\varphi^2_S\varphi^2_S\varphi^2_S \rangle$, ...
Open questions

- Further perturbative data:
  - Low-lying operators at large N: $(\partial\varphi)^4$, $\Delta = 2d+ ??/N$
  - Other parameters: $\langle JJT \rangle = C_J\sqrt{C_T}\,(\mathbf{K}_1 + \gamma\mathbf{K}_2)$
- Bootstrap of larger systems, and including spinning externals
- Are there exact values for $\eta \leftrightarrow \Delta_\phi$, $\nu \leftrightarrow \Delta_{\phi^2}$, etc.?
- Spectrum continuity near $d = 2$
  - Reorganization into Virasoro multiplets $N = 1$: [Cappelli *et al* 2018; Li 2021]
- Spectrum continuity in other critical models
- Answer general questions in this well-controlled context
  - Can we make sense of the asymptotic expansions?
  - Do all operators lie on Regge trajectories?

Bonus slides

The critical $O(N)$ CFT: Methods and conformal data

[2201.09520, to be submitted to *Physics Reports*] + `ONdata.m`

Aim: Collect all known results for (local) perturbative CFT-data for the $O(N)$ CFT

- High-order diagram results
- One-loop dilatation operator [Kehrein, Wegner 1992; 1994; Hogervorst, Rychkov, Van Rees 2015; Liendo 2017]
- Analytic bootstrap: dimensions and OPE coefficients of spinning operators
  Mellin space [Gopakumar *et al* 2016] Large spin perturbation theory/Lorentzian inversion formula [Caron-Huot 2017; Alday, JH, Van Loon 2017]

$\mathbb{Z}_2$-even scalars

Scalars in the $N = 1$ theory

Table 8: $\mathbb{Z}_2$ even scalar operators for $N = 1$. The table includes operators with $\Delta^{4d} \leqslant 12$. The fact that there are two operators of dimension $\frac{8}{3}$ of the form $\Box^3\phi^6$ is not a typo, for instance both operators are reported in table 5 of [41].

$\mathcal{O}$	$\mathcal{O}\|\varepsilon$	$\Delta^{(1)}_{4-\varepsilon}$	$\Delta(\varepsilon)$	Δ_{3d}	Family
`Op[E,0,0]`	$\mathbb{1}$	0	exact	0	
`Op[E,0,1]`	ϕ^2	$[2-\varepsilon]+\frac{1}{3}\varepsilon$	ε^7[82]	1.412625(10)[17]	1_2
`Op[E,0,2]`	ϕ^4	$[4-2\varepsilon]+2\varepsilon$	ε^7[82]	3.82951(61)[146]	1_4
`Op[E,0,3]`	ϕ^6	$[6-3\varepsilon]+5\varepsilon$	ε^2[240]	6.8956(43)[128]	1_6
`Op[E,0,4]`	$\Box^2\phi^4$	$[8-2\varepsilon]+\frac{10}{9}\varepsilon$	ε^1	7.2535(51)[128]	10_4
`Op[E,0,5]`	ϕ^8	$[8-4\varepsilon]+\frac{28}{3}\varepsilon$	ε^2[240]		1_8
`Op[E,0,6]`	$\Box^3\phi^4$	$[10-2\varepsilon]+\frac{1}{3}\varepsilon$	ε^1		
`Op[E,0,7]`	$\Box^2\phi^6$	$[10-3\varepsilon]+\frac{11}{3}\varepsilon$	ε^1		10_6
`Op[E,0,8]`	ϕ^{10}	$[10-5\varepsilon]+15\varepsilon$	ε^2[240]		1_{10}
`Op[E,0,9]`	$\Box^4\phi^4$	$[12-2\varepsilon]+\frac{14}{15}\varepsilon$	ε^1		
`Op[E,0,10]`	$\Box^3\phi^6$	$[12-3\varepsilon]+\frac{8}{3}\varepsilon$	ε^1		
`Op[E,0,11]`	$\Box^3\phi^6$	$[12-3\varepsilon]+\frac{8}{3}\varepsilon$	ε^1		
`Op[E,0,12]`	$\Box^2\phi^8$	$[12-4\varepsilon]+\frac{68}{9}\varepsilon$	ε^1		10_8
`Op[E,0,13]`	ϕ^{12}	$[12-6\varepsilon]+22\varepsilon$	ε^2[240]		1_{12}

At/near $d = 2$

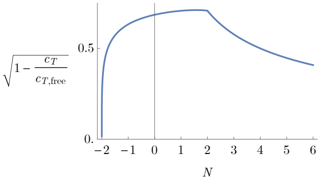
Mermin–Wagner theorem

$N > 2$, $d = 2 + \epsilon > 2$: Non-linear sigma model $\mathcal{L} = \frac{1}{2}(\partial\varphi)^2 + \alpha(\varphi^2 - f^2)$

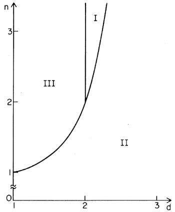
$$\Delta_\varphi = \frac{\epsilon}{2} + \frac{\epsilon}{2(N-2)} - \frac{N-1}{(N-2)^2}\epsilon^2 + \ldots, \qquad \Delta_{\varphi_S^2} = 2 - \frac{\epsilon^2}{N-2} + \ldots.$$

$d = 2$, $-2 \leqslant N \leqslant 2$: Exact results for quantum loop gas model [Nienhuis]

$$\Delta_\varphi = 1 - \frac{3g}{8} - \frac{1}{2g}, \qquad \Delta_{\varphi_S^2} = \frac{4}{g} - 2, \qquad g = 1 + \frac{1}{\pi}\arccos(N/2)$$



[Cardy, Hamber 1980] "In the (d, N) plane there is a line passing through $(2, 2)$ across which the critical exponents are nonanalytic." Disputed [Chlebicki, Jakubczyk 2020]



Representation theory

Lorentz $SO(d)$ plus parity $= O(d)$.

Analytic continuation of $O(d)$ and $O(N)$ representations: [Binder, Rychkov 2019; Gräns-Samuelsson *et al* 2021]

- One-row Young Tableaux: $Y_\ell = \square\square\square\cdots\square$: spin ℓ operators for any d
- Two-row Young Tableaux: Ex $Y_{n-1,1} = \begin{array}{l}\square\square\square\cdots\square\\ \square\end{array}$:
  - $d=4$: $(j,\bar{j}) = (\frac{n+1}{2},\frac{n-1}{2}) \oplus (\frac{n-1}{2},\frac{n+1}{2})$
  - $d=3$: parity-odd, spin $\ell = n-1$.
- Three-row Young Tableaux: Ex $Y_{1,1,1} = \begin{array}{c}\square\\ \square\\ \square\end{array}$
  - $d=4$: parity-odd, spin $\ell = 1$ “pseudovector”
  - $d=3$: parity-odd scalar
  - $d=2$: vanishes identically

One-loop diagonalisation

Master formula [Hogervorst, Rychkov, Van Rees 2015]

$$-i[D, \mathcal{O}_i] = \Delta^{(0)}\mathcal{O}_i + \varepsilon\Gamma_{ij}\mathcal{O}_j, \quad \Gamma_{ij} = \frac{1}{36}\lambda_{\mathcal{O}_i\mathcal{O}_j\hat{\phi}^4},$$

Four fields, spin zero:

$$\mathcal{O}_1 = \phi^4, \quad \lambda_{\phi^4\phi^4\hat{\phi}^4} = 72, \quad \Gamma_{11} = 2, \quad \Delta = 4 - 2\varepsilon + 2\varepsilon$$

Four fields, spin two:

$$\mathcal{O}_1 = \phi^3\partial^{\{\mu}\partial^{\nu\}}\phi, \quad \mathcal{O}_2 = \phi^2\partial^{\{\mu}\phi\partial^{\nu\}}\phi, \quad \Gamma = \frac{1}{36}\begin{pmatrix} 60 & 24 \\ 4 & 64 \end{pmatrix}$$

Eigenvalues and left eigenvectors

$$\gamma = 2: \ \mathcal{O}_1 + 3\mathcal{O}_2 \propto \partial^{\{\mu}\partial^{\nu\}}(\phi^4), \qquad \gamma = \frac{13}{9}: \ \mathcal{O}_1 - 2\mathcal{O}_2$$