Cosmographic reconstruction to discriminate between Modified Gravity and Dark Energy

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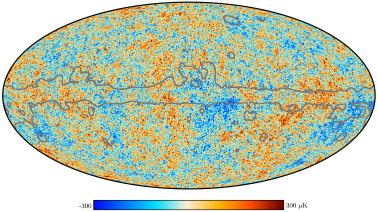
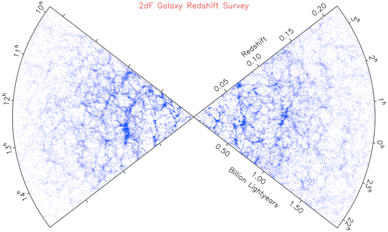
Theoretical foundation: **The Cosmological Principle**

- The Universe is spatially isotropic:



[Planck Collaboration (2018)]

- The Universe is homogeneous at large scales:



- The Universe is expanding.

The standard cosmological model

- Friedman equations:

$$H^2 \equiv \left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho - \frac{k}{a^2}$$
$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}(\rho + 3p)$$

- Energy conservation:

$$\dot{\rho} + 3H(\rho + p) = 0 \ , \qquad \rho = \sum_i \rho_i$$

- Equation of state (EoS): $w = p/\rho$

- Densities of the cosmic species: $\rho_i \propto a^{-3(1+w_i)}$

$$\dot{\rho_i} + 3H(1 + w_i)\rho_i = 0$$

- Normalized density parameters:

$$\Omega_i = \frac{8\pi G}{3H^2}\rho_i \ , \qquad \Omega_k = \frac{-k}{(aH)^2} \ , \qquad \sum_i \Omega_i = 1$$

The cosmological constant

- GR + cosmological constant:

$$G_{\mu\nu} + g_{\mu\nu}\Lambda = 8\pi G\; T_{\mu\nu}$$

- Friedmann equations (with Λ):

$$\begin{aligned} H^2 &= \frac{8\pi G}{3}\rho - \frac{k}{a^2} + \frac{\Lambda}{3} \\ \frac{\ddot{a}}{a} &= -\,\frac{4\pi G}{3}(\rho + 3p) + \frac{\Lambda}{3} \end{aligned}$$

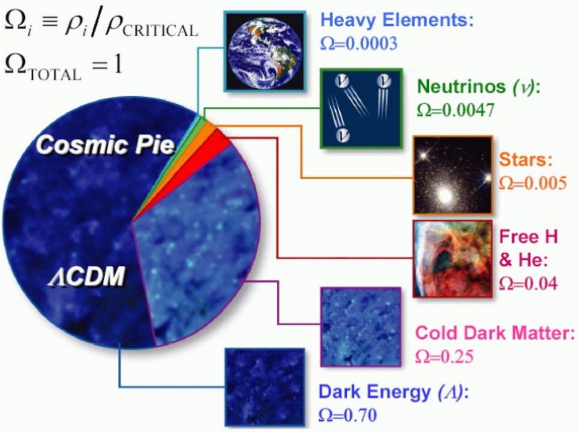
- Cosmological constant EoS:

$$w_\Lambda = -1\;, \quad \rho_\Lambda = \frac{\Lambda}{8\pi G} = -p_\Lambda$$

- Hubble expansion rate:

$$H(z) = H_0\sqrt{\Omega_r(1+z)^4 + \Omega_m(1+z)^3 + \Omega_k(1+z)^2 + \Omega_\Lambda}$$

The concordance paradigm



Huge numbers

- Energy scales (units of $c = \hbar = k_B = 1$):

$$M_{Pl} = G^{-1/2} \approx 10^{19} \text{ GeV} \,, \qquad H_0 \approx 10^{-42} \text{ GeV}$$

- FLRW cosmology:

$$\rho_\Lambda = \Lambda M_{Pl}^2 \simeq H_0^2 M_{Pl}^4 \approx 10^{-46} \text{ GeV}^4$$

- Quantum field theory:

$$\rho_{vac} \sim M_{Pl}^4 \approx 10^{76} \text{ GeV}^4 \quad \rho_{vac} \sim 10^{122} \rho_\Lambda$$

Coincidence

- Very different evolution histories:

$$\frac{\Omega_\Lambda}{\Omega_m} = \frac{\rho_\Lambda}{\rho_m} \propto a^3$$

- A fine tuning is needed to explain observations:

$$\Omega_\Lambda \simeq 0.7 \,, \qquad \Omega_m \simeq 0.3$$

Further issues with the standard cosmological model

Dark matter

- New particles seem to be elusive in laboratories and in direct detection.
- No WIMPs?
- No MACHOs?
- Standard Model of Particles extremely robust.

Dark Energy

- A new fundamental fluid?
- Modification of gravity at IR scales?
- Inflation at UV scales and DE at IR scales: acceleration at different scales.

The H_0 tension

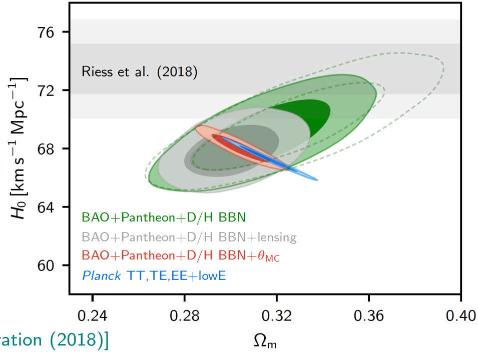
Another problem that compromises our understanding of the cosmic speed up concerns the discrepancy between the model-dependent and the direct measurements of the present expansion rate of the universe. Using the period-luminosity relation for Cepheids to calibrate a number of secondary distance indicators such as SNe Ia, Riess et al. (2019) estimate:

$$H_0 = (74.03 \pm 1.42)\ \text{km s}^{-1}\ \text{Mpc}^{-1}$$

This value is in 3.5σ tension with that of the CMB-Planck 2018 ΛCDM model:

$$H_0 = (66.88 \pm 0.92)\ \text{km s}^{-1}\ \text{Mpc}^{-1}$$

- The tension is not confined exclusively to the Planck results.
- The constraints on H_0 and Ω_m converge to the Planck values as more data are included.
- If the difference between Planck and the R19 measurements of H_0 is caused by new physics, then it is unlikely to be through some change to the late-time distance-redshift relation.



[Planck Collaboration (2018)]

How to go beyond? Two main possibilities

Barotropic unified models of dark energy and dark matter

- A first prototype: Chaplygin gas and its extensions:

$$P = -\frac{A}{\rho}, \quad P = B\rho^{\gamma} - \frac{A}{\rho^{\alpha}}.$$

- Logotropic dark energy models

$$P = -\sigma \log \rho.$$

Alternatives to General Relativity

- Extensions of Einstein's gravity

$$R \to f(R), \quad R \to f(R, G), \quad R \to f(R, \Box R), \quad R \to \text{Scalar-Tensor.}$$

- Modified gravity means choosing the "right invariant": curvature R, torsion T, non-metricity Q.

$$R \to T,\ Q \quad T \to f(T), \quad Q \to f(Q).$$

Big issue: Solving the concordance paradigm at UV and IR scales.

A minimal approach: A time-evolving equation of state. The CPL model

$$H(a) = H_0 \sqrt{\frac{\Omega_m}{a^3} + \Omega_{DE} \exp\left\{-3 \int_1^a [1 + w_{DE}(a')]\, d\ln a'\right\}}$$

A simple parametrization of $w(a)$ is obtained by a first-order Taylor expansion:

$$w_{DE} = w_0 + w_a(1 - a)$$

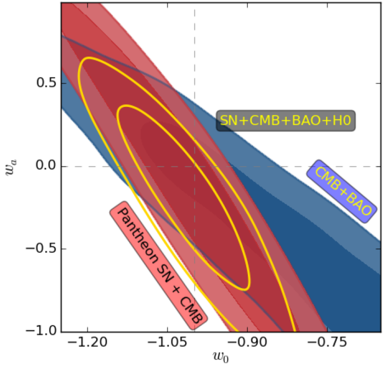
This is the Chevallier-Polarski-Linder (CPL) model, which well-behaves from very high redshift $(w(1) = w_0 + w_a)$ to the present epoch $(w(0) = w_0)$. Such a parametrization is capable of reproducing with high accuracy the EoS of many scalar fields, as well as the resulting distance-redshift relations.

Combining CMB, SN, BAO and local H_0 measurements:

$$w_0 = -1.007 \pm 0.089$$
$$w_a = -0.222 \pm 0.407$$

These results are consistent with the cosmological constant model $(w_0 = -1,\ w_a = 0)$, indicating no evidence for evolution of the dark energy equation of state.

[Scolnic et al., ApJ, 859, 101 (2018)]



Constraints on the wCDM model

Sample	w	Ω_m	H_0
CMB+BAO	-0.991 ± 0.074	0.312 ± 0.013	67.508 ± 1.633
CMB+H0	-1.188 ± 0.062	0.265 ± 0.013	73.332 ± 1.729
CMB+BAO+H0	-1.119 ± 0.068	0.289 ± 0.011	70.539 ± 1.425
SN+CMB	-1.026 ± 0.041	0.307 ± 0.012	68.183 ± 1.114
SN+CMB+BAO	-1.014 ± 0.040	0.307 ± 0.008	68.027 ± 0.859
SN+CMB+H0	-1.056 ± 0.038	0.293 ± 0.010	69.618 ± 0.969
SN+CMB+BAO+H0	-1.047 ± 0.038	0.299 ± 0.007	69.013 ± 0.791

Constraints on the CPL model

Sample	w_0	w_a	Ω_m	H_0
CMB+BAO	-0.616 ± 0.262	-1.108 ± 0.771	0.343 ± 0.025	64.614 ± 2.447
CMB+H0	-1.024 ± 0.347	-0.789 ± 1.338	0.265 ± 0.015	73.397 ± 1.961
CMB+BAO+H0	-0.619 ± 0.270	-1.098 ± 0.781	0.343 ± 0.026	64.666 ± 2.526
SN+CMB	-1.009 ± 0.159	-0.129 ± 0.755	0.308 ± 0.018	68.188 ± 1.768
SN+CMB+BAO	-0.993 ± 0.087	-0.126 ± 0.384	0.308 ± 0.008	68.076 ± 0.858
SN+CMB+H0	-0.905 ± 0.101	-0.742 ± 0.465	0.287 ± 0.011	70.393 ± 1.079
SN+CMB+BAO+H0	-1.007 ± 0.089	-0.222 ± 0.407	0.300 ± 0.008	69.057 ± 0.796

[Scolnic et al., ApJ, 859, 101 (2018)]

Unified Dark Energy Models

- The idea is to combine Dark Matter and Dark Energy behaviours under the same standard **without** asking for their fundamental counterparts.
- **Dark Matter** means the **clustering** properties of large scale structure.
- **Dark Energy** means reproducing the **accelerated** behaviour of the Hubble flow.
- The goal is reconstructing the cosmic history matching decelerated (matter dominance) and accelerated (dark energy dominance) behaviours at any redshift.
- Using cosmography at late $(z \simeq 0)$ and early $(z \gg 0)$ epochs.

The case of unified Anton-Schmidt dark energy

Consider crystalline solid's pressure under isotropic deformation in the Debye approximation:

$$P(V) = -\beta \left(\frac{V}{V_0}\right)^{-\frac{1}{6}-\gamma_G} \ln\left(\frac{V}{V_0}\right)$$

- V_0 is the equilibrium volume of the crystal;
- $\beta = -V_0 \left(\frac{dP}{dV}\right)_{V=V_0}$ is the bulk modulus at V_0;
- $\gamma_G = \frac{\partial \ln \theta_D}{\partial \ln V}$ is the Grüneisen parameter;
- $\theta_D = \frac{\hbar\omega_D}{k_B}$ is the Debye temperature, ω_D is the maximum vibrational frequency of a solid's atoms.

$$\gamma_G < -\frac{1}{6} : \begin{cases} V < V_0\ , & \text{vanishing pressure, matter-dominated phase} \\ V = V_0\ , & \text{transition epoch} \\ V > V_0\ , & \text{negative pressure, accelerated phase.} \end{cases}$$

A **single fluid** obeying the Anton-Schmidt EoS can describe the whole universe's evolution without the need of the cosmological constant!

[Anton, Schmidt, Intermetallics, 5, 449 (1997)]

[Capozziello, D'Agostino, Luongo, PDU, 20, 1 (2018)]

The case of unified Anton-Schmidt dark energy

Recast Anton-Schmidt's EoS in cosmological quantities, i.e. $V \propto \rho^{-1}$

$$P(\rho) = A \left(\frac{\rho}{\rho_*}\right)^{-n} \ln\left(\frac{\rho}{\rho_*}\right)$$

- $A \propto \beta > 0$;
- $n = -\frac{1}{6} - \gamma_G$;
- ρ_* is a reference density;
- $n = 0 \Longrightarrow$ Logotropic cosmological model [Chavanis, PLB, 758, 59 (2016)]

Integrating the first law of thermodynamics for an adiabatic fluid:

$$\epsilon = \rho c^2 - \left[\frac{A}{n+1}\left(\frac{\rho}{\rho_*}\right)^{-n} \ln\left(\frac{\rho}{\rho_*}\right) + \frac{A}{(n+1)^2}\left(\frac{\rho}{\rho_*}\right)^{-n}\right]$$

- First term: rest-mass energy, mimics (baryonic + dark) matter (ϵ_m).
- Second term: internal energy, mimics dark energy.

$$\epsilon_m = \rho c^2$$

$$\epsilon_{de} = -\frac{A}{n+1}\left(\frac{\rho}{\rho_*}\right)^{-n} \ln\left(\frac{\rho}{\rho_*}\right) - \frac{A}{(n+1)^2}\left(\frac{\rho}{\rho_*}\right)^{-n}$$

- $\rho \gg 1$: ϵ_m dominates and, for $n < 0$, $P \ll \epsilon$
- $\rho \ll 1$: ϵ_{de} dominates and, for $n < 0$, $P \to -K$ $(K > 0)$

[Capozziello, D'Agostino, Luongo, PDU, 20, 1 (2018)]

The case of unified Anton-Schmidt dark energy

- Evolution of the energy density terms ($\rho \propto a^3$):

$$\epsilon_m = \epsilon_{m0} a^{-3}$$

$$\epsilon_{de} = \epsilon_{de,0} a^{3n} + \frac{3A}{n+1} \left(\frac{\rho_{m0}}{\rho_*} \right)^{-n} a^{3n} \ln a$$

with

$$\epsilon_{m0} = \rho_{m0} c^2$$

$$\epsilon_{de,0} = -\frac{A}{n+1} \left(\frac{\rho_{m0}}{\rho_*} \right)^{-n} \ln \left(\frac{\rho_{m0}}{\rho_*} \right) - \frac{A}{(n+1)^2} \left(\frac{\rho_{m0}}{\rho_*} \right)^{-n}$$

- Hubble expansion rate:

$$H^2(a) = H_0^2 \left[\frac{\Omega_{m0}}{a^3} + (1 - \Omega_{m0})(1 + 3B \ln a) a^{3n} \right]$$

where

$$B = \frac{A}{n+1} \left(\frac{\rho_{m0}}{\rho_*} \right)^{-n} \frac{1}{\epsilon_c (1 - \Omega_{m0})}$$

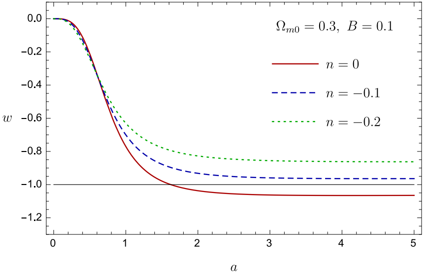
- $n = 0 \Longrightarrow B$ is the logotropic temperature.
- $n = B = 0 \Longrightarrow \Lambda$CDM model.

[Capozziello, D'Agostino, Luongo, PDU, 20, 1 (2018)]

The case of unified Anton-Schmidt dark energy

- Effective EoS parameter:

$$w = -\frac{(1-\Omega_{m0})\left[B+(n+1)(1+3B\ln a)\right]a^{3n}}{\Omega_{m0}a^{-3}+(1-\Omega_{m0})\left(1+3B\ln a\right)a^{3n}}$$

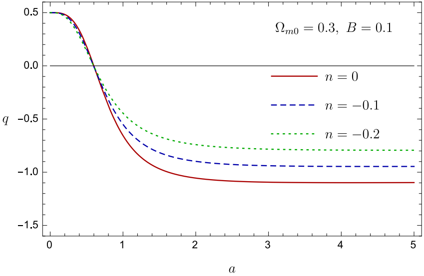


- Deceleration parameter:

$$q = \frac{\Omega_{m0}a^{-3}-(1-\Omega_{m0})\eta a^{3n}}{2\left[\Omega_{m0}a^{-3}+(1-\Omega_{m0})(1+3B\ln a)a^{3n}\right]}$$

where

$$\eta = 3(n+B)+3B(3n+2)\ln a+2$$



[Capozziello, D'Agostino, Luongo, PDU, 20, 1 (2018)]

The case of unified Anton-Schmidt dark energy

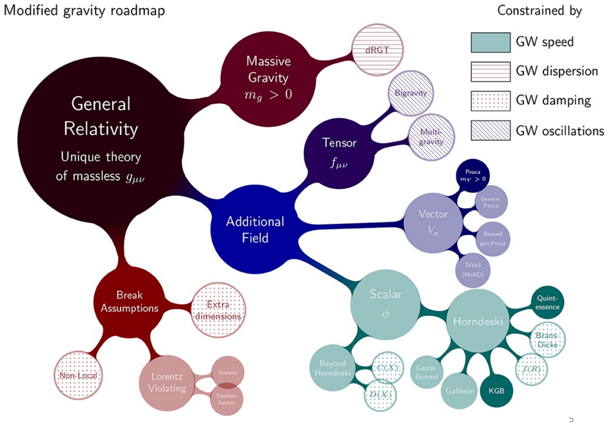
Parameter	H_0+SN	OHD	BAO	SN+OHD+BAO
H_0	70	$64.53^{+8.86}_{-6.81}$	$62.37^{+4.09}_{-3.80}$	$65.67^{+1.75}_{-1.78}$
Ω_{m0}	$0.107^{+0.111}_{-0.128}$	$0.242^{+0.065}_{-0.061}$	$0.272^{+0.051}_{-0.056}$	$0.286^{+0.034}_{-0.036}$
n	$-0.382^{+0.239}_{-0.170}$	$-0.251^{+0.699}_{-0.590}$	$-0.336^{+0.315}_{-0.283}$	$-0.147^{+0.113}_{-0.107}$
r_d	-	-	$142.9^{+6.9}_{-6.6}$	$144.6^{+3.5}_{-3.3}$
M	$-19.07^{+0.03}_{-0.02}$	-	-	$-19.18^{+0.05}_{-0.06}$
Δ_M	$-0.075^{+0.021}_{-0.021}$	-	-	$-0.077^{+0.021}_{-0.019}$
α	$0.121^{+0.006}_{-0.006}$	-	-	$0.121^{+0.006}_{-0.006}$
β	$2.559^{+0.067}_{-0.068}$	-	-	$2.565^{+0.068}_{-0.066}$

Figure: 68% confidence level constraints on the Anton-Schmidt's parameters.

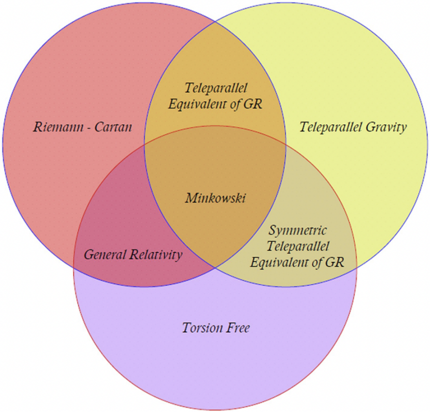
[Capozziello, D'Agostino, Luongo, PDU, 20, 1 (2018)]]

- Instead of searching for new particles, we can extend or modify GR.
- Dark Energy and Dark Matter as geometric effects at infrared scales.
- Extended Gravity means that GR is reproduced in a given regime, e.g. $f(R) \to R$.
- Modified Gravity means that standard GR could not be reproduced.
- Teleparallel Equivalent General Relativity (TEGR), gravitational field is represented by torsion T instead of curvature R, e.g. $f(T) \to T$.
- Symmetric Teleparallel Equivalent General Relativity (STEGR), gravitational field is represented by non-metricity Q instead of curvature R, e.g. $f(Q) \to Q$.
- Cosmography + GWs could discriminate for New Physics.

A roadmap from GWs



The geometrical Trinity of Gravity

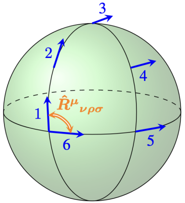


- Teleparallel geometry: $R^{\alpha}{}_{\beta\mu\nu} \equiv 0$
- Torsion-free geometry: $T^{\alpha}{}_{\mu\nu} \equiv 0$
- Riemann-Cartan geometry: $Q_{\alpha\mu\nu} \equiv 0$
- GR: $Q_{\alpha\mu\nu} \equiv 0,\ T^{\alpha}{}_{\mu\nu} \equiv 0$
- TEGR: $R^{\alpha}{}_{\beta\mu\nu} \equiv 0,\ Q_{\alpha\mu\nu} \equiv 0$
- STEGR: $R^{\alpha}{}_{\beta\mu\nu} \equiv 0,\ T^{\alpha}{}_{\mu\nu} \equiv 0$
- Minkowski space:
  $R^{\alpha}{}_{\beta\mu\nu} \equiv 0,\ T^{\alpha}{}_{\mu\nu} \equiv 0,\ Q_{\alpha\mu\nu} \equiv 0$

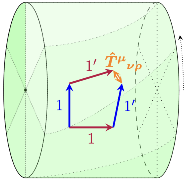
[Beltran Jimenez, Heisemberg, Koivisto, Universe, 5, 173 (2019)]
[Capozziello, De Falco, Ferrara, arXiv:2208.03011, to appear in EPJC (2022)]

The geometrical Trinity of Gravity

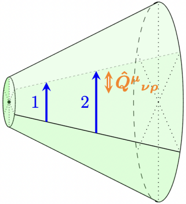
- **Curvature**: causes the parallel transport along a closed curve to be non-trivial, i.e., to change the transported vector.



- **Torsion**: the parallel transport is not symmetric under exchanging the transported vector and the direction of transport.



- **Non-metricity**: the length of the vector, as measured by the metric, changes along the transport.



[Bahamonde et al., arXiv:2106.13793 (2021)]

The case of $f(R)$ gravity

- Action:

$$S = \int d^4x\sqrt{-g}\left[\frac{f(R)}{2} + \mathcal{L}_m\right]$$

- Varying the action with respect to $g_{\mu\nu}$:

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = T^{(m)}_{\mu\nu} + T^{(curv)}_{\mu\nu}$$

- Matter energy-momentum tensor:

$$T^{(m)}_{\mu\nu} = \frac{-2}{f'\sqrt{-g}}\frac{\delta(\sqrt{-g}\ \mathcal{L}_m)}{\delta g^{\mu\nu}}\ , \qquad f' \equiv \frac{df}{dR}$$

- Effective curvature energy-momentum tensor:

$$T^{(curv)}_{\mu\nu} = \frac{1}{f'}\left[\frac{1}{2}g_{\mu\nu}(f - Rf') + (\nabla_\mu\nabla_\nu - g_{\mu\nu}\Box)f'\right]$$

- Flat FLRW metric:

$$ds^2 = dt^2 - a(t)^2\delta_{ij}dx^i dx^j$$

- Relation between the Ricci scalar and the Hubble parameter:

$$R = -6(\dot{H} + 2H^2)$$

The case of $f(R)$ gravity

- Matter energy-momentum tensor for a perfect fluid:

$$T_{\mu\nu}^{(m)} = \mathrm{diag}(\rho, -p, -p, -p)$$

- Modified Friedmann equations:

$$H^2 = \frac{1}{3}\left[\frac{1}{f'}\rho_m + \rho_{curv}\right]$$

$$2\dot{H} + 3H^2 = -\frac{p_m}{f'} - p_{curv}$$

$$\rho_{curv} = \frac{1}{f'}\left[\frac{1}{2}(f - Rf') - 3H\dot{R}f''\right]$$

$$p_{curv} = \frac{1}{f'}\left[2H\dot{R}f'' + \ddot{R}f'' + \dot{R}^2 f''' - \frac{1}{2}(f - Rf')\right]$$

- Effective dark energy given by curvature:

$$w_{de} \equiv \frac{p_{curv}}{\rho_{curv}} = -1 + \frac{\ddot{R}f'' + \dot{R}^2 f''' - H\dot{R}f''}{(f - Rf')/2 - 3H\dot{R}f''}$$

- Assuming matter as dust:

$$p_m = 0\ , \qquad \rho_m = \frac{\rho_{m0}}{a^3} = 3H_0^2\Omega_{m0}(1+z)^3$$

Among these several possibilities, the problem of cosmic evolution should be addressed by a model-independent approach. Cosmography could be useful to this aim because it is based only on the convergence of polynomials.

[see S. Weinberg, "Gravitation" (1972)]

A model-independent approach: The cosmography

- Taylor expansion of the scale factor (assuming flat FLRW universe):

$$a(t) = 1 + \sum_{k=1}^{\infty} \frac{1}{k!} \frac{d^k a}{dt^k}\bigg|_{t=t_0} (t - t_0)^k$$

- Cosmographic series:

$$H(t) \equiv \frac{1}{a}\frac{da}{dt}\,, \qquad q(t) \equiv -\frac{1}{aH^2}\frac{d^2a}{dt^2}$$

$$j(t) \equiv \frac{1}{aH^3}\frac{d^3a}{dt^3}\,, \qquad s(t) \equiv \frac{1}{aH^4}\frac{d^4a}{dt^4}$$

- Luminosity distance:

$$d_L(z) = (1+z)\int_0^z \frac{dz'}{H} = \frac{1}{H_0}\left(c_1 z + c_2 z^2 + c_3 z^3 + c_4 z^4\right) + \mathcal{O}(z^5)$$

- Hubble expansion rate:

$$H(z) = \left[\frac{d}{dz}\left(\frac{d_L(z)}{1+z}\right)\right]^{-1} = H_0\left[1 + H^{(1)} z + H^{(2)}\frac{z^2}{2} + H^{(3)}\frac{z^3}{6}\right] + \mathcal{O}(z^4)$$

$$H^{(1)} = 1 + q_0\,,\ H^{(2)} = j_0 - q_0^2\,,\ H^{(3)} = 3q_0^2 + 3q_0^3 - j_0(3 + 4q_0) - s_0$$

[Cattoen, Visser, PRD, 78, 063501 (2008)]

[Capozziello, Lazkoz, Salzano, PRD, 84, 124061 (2011)]

Standard cosmography vs rational polynomials

- Limits of standard cosmography:
  - the radius of convergence of the Taylor series is restricted to $|z| < 1$;
  - if cosmological data for $z > 1$ are used, the Taylor series does not provide a good approximation of the luminosity distance due to its divergent behaviour;
  - finite truncations cause errors propagation that may result in possible misleading outcomes.

- Advantages of rational polynomials:
  - they extend the radius of convergence of Taylor series;
  - they can better approximate situations at high-redishift domains;
  - the series can be modelled by choosing appropriate orders depending on each case of interest.

[Capozziello, D'Agostino, Luongo, MNRAS, 494, 2576 (2020)]

Cosmography with Padé polynomials

- Series expansion of a generic function: $f(z) = \sum_{k=0}^{\infty} c_k z^k$, $c_k = f^{(k)}(0)/k!$
- (N, M) Padé polynomial:

$$P_{N,M}(z) = \frac{\displaystyle\sum_{n=0}^{N} a_n z^n}{\displaystyle 1 + \sum_{m=1}^{M} b_m z^m}, \qquad \begin{cases} P_{N,M}(0) = f(0) \\ P'_{N,M}(0) = f'(0) \\ \vdots \\ P_{N,M}^{(N+M)}(0) = f^{(N+M)}(0) \end{cases}$$

- $N + M + 1$ unknown coefficients:

$$\sum_{k=0}^{\infty} c_k z^k = \frac{\sum_{n=0}^{N} a_n z^n}{1 + \sum_{m=1}^{M} b_m z^m} + \mathcal{O}(z^{N+M+1})$$

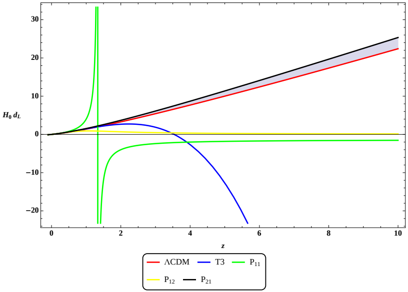
$$(1 + b_1 z + \ldots + b_M z^M)(c_0 + c_1 z + \ldots) = a_0 + a_1 z + \ldots + a_N z^N + \mathcal{O}(z^{N+M+1})$$

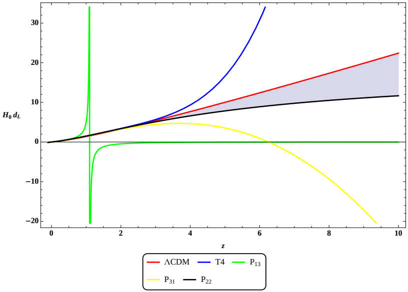
- (N, M) Padé approximation of the luminosity distance:

$$d_L(z) \approx P_{N,M}(z, H_0, q_0, j_0, s_0, \ldots)$$

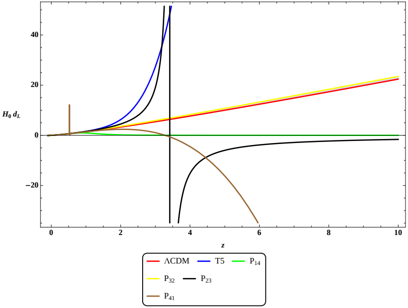
[Capozziello, Ruchika, Sen, MNRAS, 484, 4484 (2019)]

Taylor vs Padé





- The Taylor polynomials T_3, T_4 and T_5 rapidly diverge from the exact ΛCDM curve as $z > 2$.
- Padé polynomials P_{11}, P_{13} and P_{23} give spurious singularities when used to approximate the ΛCDM model.
- The Padé functions P_{21}, P_{22} and P_{32} fairly approximate the exact ΛCDM luminosity distance over the whole interval considered.



[Aviles et al., PRD, 87, 064025 (2014)]

Cosmography with Chebyshev polynomials

- Chebyshev polynomials of the first kind:

$$T_n(z) = \cos(n\theta)\ , \qquad n \in \mathbb{N}_0\ ,\ \theta = \arccos(z)$$

- They form an orthogonal set with respect to the weighting function $w(z) = (1 - z^2)^{-1/2}$ in the domain $|z| \leq 1$:

$$\int_{-1}^{1} T_n(z)\ T_m(z)\ w(z)\ dz = \begin{cases} \pi\ , & n = m = 0 \\ \dfrac{\pi}{2}\delta_{nm}\ , & \text{otherwise} \end{cases}$$

- Recurrence relation:

$$T_{n+1}(z) = 2zT_n(z) - T_{n-1}(z)$$

- Chebyshev series of a generic function $f(z)$:

$$f(z) = \sum_{k=0}^{\infty}{}' c_k T_k(z)$$

where $\sum'$ means that the first term in the sum must be divided by 2, and

$$c_k = \frac{2}{\pi}\int_{-1}^{1} g(z)\ T(z)\ w(z)\ dz$$

being $g(z)$ the Taylor series of $f(z)$ around $z = 0$.

- Construct the (n,m) rational Chebyshev approximation of $f(z)$:

$$R_{n,m}(z) = \frac{\displaystyle\sum_{i=0}^{n}{}' \; a_i T_i(z)}{\displaystyle\sum_{j=0}^{m}{}' \; b_j T_j(z)}$$

- Requiring $f(z) - R_{n,m}(z) = \mathcal{O}(T_{n+m+1})$:

$$\begin{cases} a_i = \dfrac{1}{2}\displaystyle\sum_{j=0}^{m}{}' \; b_j(c_{i+j} + c_{|i-j|}) = 0 \;, \qquad i = 0, \ldots, n \\ \displaystyle\sum_{j=0}^{m}{}' \; b_j(c_{i+j} + c_{|i-j|}) = 0 \;, \qquad i = n+1, \ldots, n+m \end{cases}$$

- Generalization to $z \in [a,b]$: $z = \dfrac{a(1-\cos\theta) + b(1+\cos\theta)}{2}$

$$T_n^{[a,b]}(z) = T_n\left(\frac{2z-(a+b)}{b-a}\right)$$

which are orthogonal with respect to $w_{[a,b]} = \left[(z-b)(b-z)\right]^{-1/2}$.

[Capozziello, D'Agostino, Luongo, MNRAS, 476, 3924 (2018)]

Comparison among different cosmographic techniques

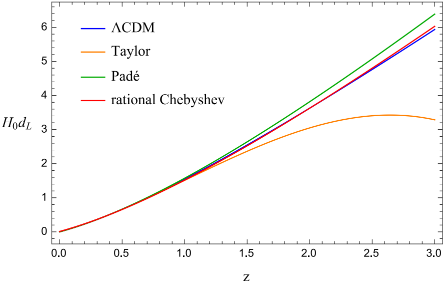


Figure: (2,1) rational Chebyshev approximation of the luminosity distance for the ΛCDM model with the correspondent Padé and Taylor approximations.

Parameter	Taylor			Padé			Rational Chebyshev		
	Mean	1σ	R.E.	Mean	1σ	R.E.	Mean	1σ	R.E.
H_0	65.80	$^{+2.09}_{-2.11}$	3.19%	64.94	$^{+2.11}_{-2.02}$	3.17%	64.95	$^{+1.89}_{-1.94}$	2.95%
q_0	-0.276	$^{+0.043}_{-0.049}$	16.8%	-0.285	$^{+0.040}_{-0.046}$	15.1%	-0.278	$^{+0.021}_{-0.021}$	7.66%
j_0	-0.023	$^{+0.317}_{-0.397}$	1534%	0.545	$^{+0.463}_{-0.652}$	102%	1.585	$^{+0.497}_{-0.914}$	44.5%

Table: 68% confidence level constraints and relative errors from the MCMC analysis of SN+OHD+BAO data for the fourth-order Taylor, (2,2) Padé and (2,1) rational Chebyshev polynomial approximations of the luminosity distance.

[Capozziello, D'Agostino, Luongo, MNRAS, 476, 3924 (2018)]

- Convert the derivatives:

$$\frac{d}{dt} \longrightarrow -(1+z)H\frac{d}{dz} \ ,$$
$$\frac{\partial}{\partial R} \longrightarrow \frac{1}{6}\Big[(1+z)H_z^2 + H\left(-3H_z + (1+z)H_{zz}\right)\Big]^{-1}\frac{d}{dz} \ .$$

- Combine first Friedmann equation and $R = -6(\dot{H} + 2H^2)$:

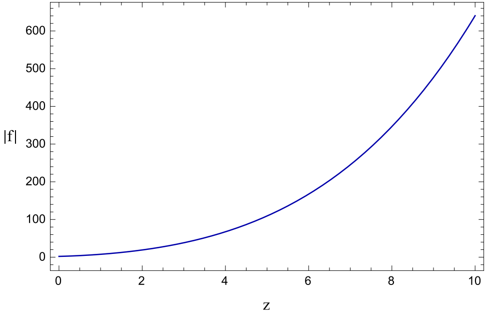
$$H^2 f_z = \Big[-(1+z)H_z^2 + H\big(3H_z - (1+z)H_{zz}\big)\Big] \times \Bigg[-6H_0^2(1+z)^3\Omega_{m0} - f$$
$$-\frac{Hf_z\left(2H-(1+z)H_z\right)}{(1+z)H_z^2 + H\left(-3H_z + (1+z)H_{zz}^2\right)} - \frac{(1+z)H^2}{\left[(1+z)H_z^2 + H\big(-3H_z + (1+z)H_{zz}\big)\right]^2}\times$$
$$\Big(f_{zz}\big((1+z)H_z^2 + H(-3H_z + (1+z)H_{zz})\big) + f_z\big(2H_z^2 - 3(1+z)H_zH_{zz}$$
$$+ H(2H_{zz} - (1+z)H_{zzz})\big)\Big)\Bigg].$$

- Assuming $f'(R_0) = 1$ ($G_{\text{eff}} = G_N/f'(R)$), the initial conditions are:

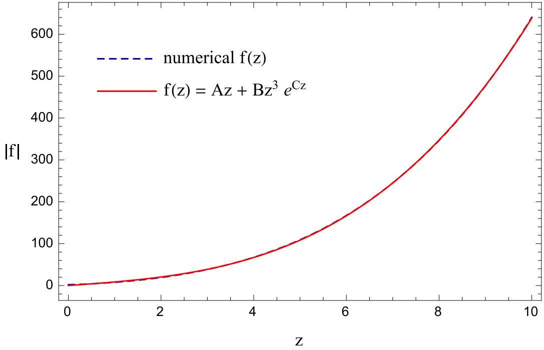
$$f_0 = R_0 + 6H_0^2(\Omega_{m0} - 1) \ , \quad f_z\big|_{z=0} = R_z\big|_{z=0} \ .$$

Bounds on cosmographic parameters

- (2,1) Padé approximation: $\begin{cases} h = 0.7064^{+0.0277}_{-0.0263} \\ q_0 = -0.4712^{+0.1224}_{-0.1106} \\ j_0 = 0.593^{+0.216}_{-0.210} \end{cases}$
- We fix $\Omega_{m0} = 0.3$
- $R < 0 \implies f(R) < 0 \implies f(z) < 0$ consistent with upper bounds values of cosmographic parameters.



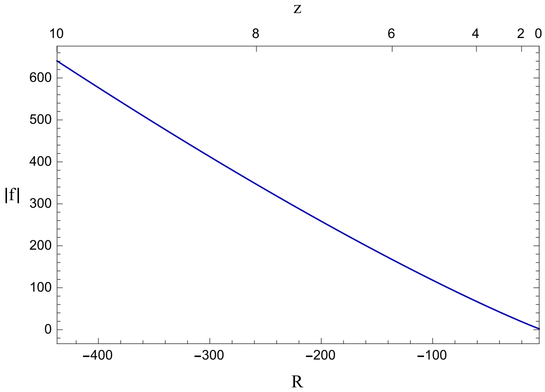
- Best analytical match for $f(z)$:



[Capozziello, D'Agostino, Luongo, JCAP, 1805, 008 (2018)]

Cosmographic reconstruction of $f(R)$ gravity

- Use $R = -6(\dot{H} + 2H^2)$ with $H_{2,1}(z)$ to get $R(z)$.
- Invert $R(z)$ and plug into $f(z) = \mathcal{A}z + \mathcal{B}z^3 e^{\mathcal{C}z}$ to obtain $f(R)$.



[Capozziello, D'Agostino, Luongo, JCAP, 1805, 008 (2018)]

Viability conditions for $f(R)$ models

- Avoid negative values of $G_{\text{eff}} = \dfrac{G_N}{f'(R)}$:

$$f'(R) > 0\ , \quad R \geq R_0 > 0$$

- Constraints from tests of gravity in the solar system, consistency with matter-dominated epoch and stability of cosmological perturbations:

$$f''(R) > 0\ , \quad R \geq R_0 > 0$$

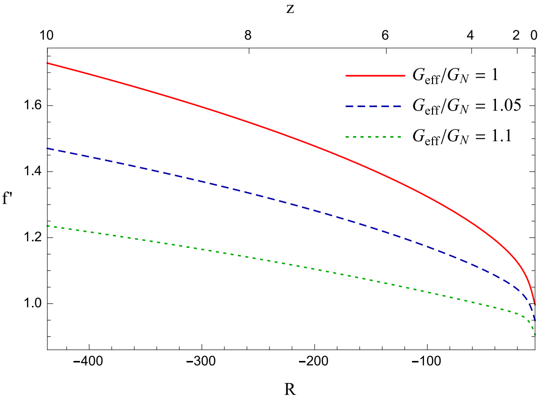
- Constraints from CMB observations:

$$f'(R) \longrightarrow 1\ , \quad R \gg 1$$

[Olmo, PRD, 72, 083505 (2005)]
[Hu, Sawicki, PRD, 76, 064004 (2007)]
[Amendola, Gannouji, Polarski, Tsujikawa, PRD, 75, 083504 (2007)]

- Relaxing the assumption $f'(R_0) = 1$:

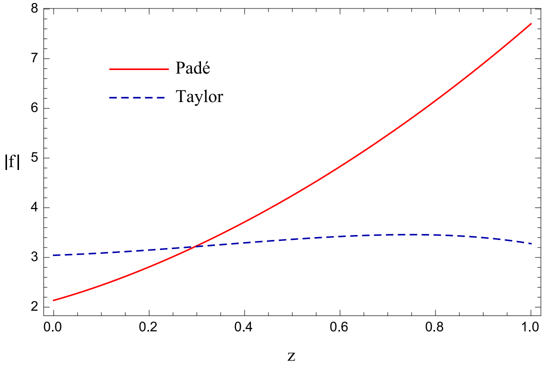
$$f_0 = f'(R_0)(6H_0^2 + R_0) - 6H_0^2\Omega_{m0}$$
$$f_z\big|_{z=0} = f'(R_0)\ R_z\big|_{z=0}$$



[Capozziello, D'Agostino, Luongo, JCAP, 1805, 008 (2018)]

Taylor vs Padé

- 3rd-order Taylor approximation: $\begin{cases} h = 0.7253^{+0.0353}_{-0.0351} \\ q_0 = -0.6642^{+0.2050}_{-0.1963} \\ j_0 = 1.223^{+0.644}_{-0.664} \\ s_0 = 0.394^{+1.335}_{-0.731} \end{cases}$



[Capozziello, D'Agostino, Luongo, JCAP, 1805, 008 (2018)]

Comparison between $f(R)$ gravity and ΛCDM

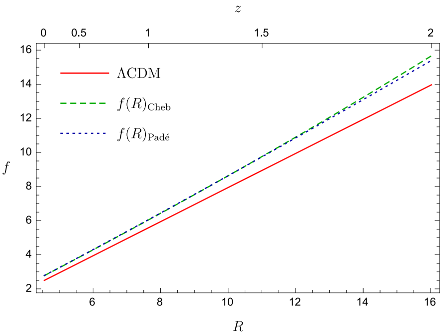


Figure: Comparison among the ΛCDM action and the $f(R)$ reconstructed actions using the Padé and the rational Chebyshev approximations.

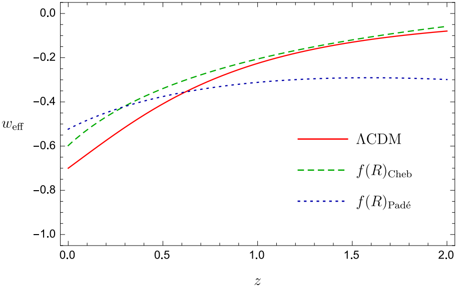


Figure: Comparison among the effective equation of state parameter for the ΛCDM model, the Padé and the rational Chebyshev reconstructions.

[Capozziello, D'Agostino, Luongo, GRG, 51, 2 (2019)]

Conclusions and perspectives

- **Cosmography is a procedure to reconstruct the Universe expansion in a model-independent way. The ΛCDM can be assumed as a "prior" model [Capozziello, Nesseris, Perivolaropoulos , JCAP, 0712, 009 (2007)].**
- **Adopting rational polynomials in cosmography allows us to frame the late-time accelerated expansion of the Universe with an accuracy greater than the standard Taylor approach.**
- **Calibration orders of Padé polynomials and rational Chebyshev polynomials are compared with data: Chebyshev reduces systematics.**
- **MOG cosmography indicates departures from the standard ΛCDM model, showing that the EoS is slightly evolving with respect to cosmic time.**
- **Cosmography as a IR tool to discriminate theories. UV probes from Lorentz Invariance and Equivalence Principle. Main role of GWs and Multimessengers.**
- **What next? Extensions to very high z: High-redshift cosmography.**
- **What next? Comparisons with the Cosmic Microwave Background observations.**
- **What next? The issue of Hubble tension. New Physics or lack of data?**
- **What next? Cosmography by GWs and standard sirens.**