Novel exact results and new indices for supersymmetric theories in three dimensions

Based on arXiv 2006.06692 (with R. Panerai & K. Polydorou) and 220x.yyyy (with M. Inglese & D. Martelli)

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Outline

What the index is

Trace formula

- $\mathcal{I}_{d+1}(\varphi_i) = \mathrm{Tr}_{\mathbb{H}_{S^d}}(-)^F x^{\mathcal{H}} \varphi_i^{\Phi_i}, \qquad \mathcal{H} = \frac{1}{2}\left\{Q, \widetilde{Q}\right\}$
- Witten index of a $(d+1)$-dimensional (superconformal) field theory in radial quantization, flavoured by Φ_i, with $[\Phi_i, \mathcal{H}] = 0$
- $|\psi\rangle$ with $\mathcal{H} \neq 0$ are Bose/Fermi pairs $\rightarrow (-)^F \rightarrow \partial_x \mathcal{I}_{d+1} = 0$
- φ_i can be tuned, e.g. $E = 2R + J \rightarrow$ Schur index

Path integral representation

- $\mathcal{I}_{d+1} = \widehat{Z}_{S^d \times S^1}$ with suitable background fields, e.g. $A^{(R)}$
- In 3d: $\int_{S^2} \mathrm{d}A^{(R)} = 2\pi\mathfrak{c}$, e.g. $\mathfrak{c} = 0 \rightarrow$ (gen) sc; $\mathfrak{c} = \pm 1 \rightarrow$ twisted

Why the index

Path integral representation

- $\mathcal{I}_{d+1}$ for $SU(N)$ gauge theory at large $N \rightarrow$ bh entropy[1]
- Check non-perturbative dualities/ correspondences[2]
- S^1-reduction[3] of $\mathcal{I}_{d+1}$ provides Z_{S^d}
- Factorization, e.g. holomorphic-blocks story[4]

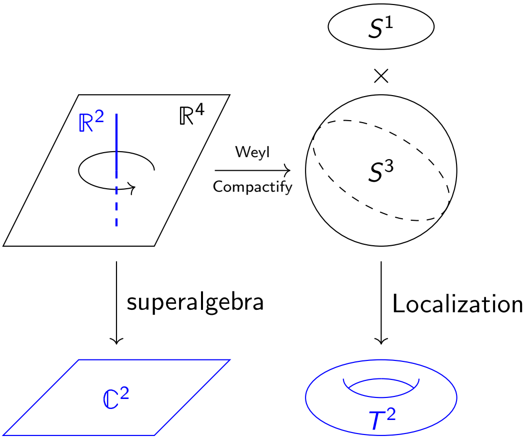
[1] Benini, Hristov, Zaffaroni, '16; Cabo-Bizet, Cassani, Martelli, Murthy, '18
[2] Kapustin, Willett, '11
[3] Benini, Cremonesi, '12
[4] Beem, Dimofte, Pasquetti, '14

Schur index and holomorphic correlators

Infinite-dimensional symmetry in 4d

- Schur ops in a CFT_4 have holomorphic correlation functions[a]
- $\mathcal{I}_{4d}^{\text{gen Schur}} = \widehat{Z}_{S^3\times S^1} = Z_{T^2}^{\text{holomorphic}}[\text{Schur ops}]$[b]

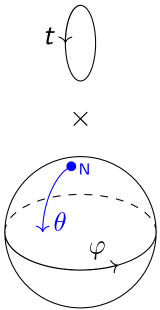
[a] Beem, Lemos, Liendo, Peelaers, Rastelli, van Rees, '13
[b] Pan, Peelaers, '19



$\mathcal{N} = 4$ matter theories on $S^2 \times S^1$

Setup

- Metric: $\mathrm{d}s^2 = \mathrm{d}\theta^2 + \sin^2\theta^2 \mathrm{d}\varphi^2 + \beta^2 \mathrm{d}t^2$
- Initial R-symmetry: $SU(2)_H \times SU(2)_C$
- R-sym bg fields: $A_H = -\frac{\mathrm{i}\beta}{2}\sigma^3 \mathrm{d}t$ and $A_C = 0$
- bg VM: $\left(A, \Phi_{\dot{a}\dot{b}}, D_{ab}\right)$
- dynamical HM: $\left(q^a, \widetilde{q}^a, \psi^{\dot{a}}, \widetilde{\psi}^{\dot{a}}, G^a, \widetilde{G}^a\right)$
- $\{\mathbb{Q}, \mathbb{S}\} \sim J_3 - R_{\mathrm{i}\dot{2}}$ with $J_3 \sim \partial_\varphi$ leaving $S^1_{N,S}$ fixed
- BPS VM: $A = a\mathrm{d}t$, $\Phi_{\dot{1}\dot{2}} = \sigma$. Set $z_{N,S} = a \mp \mathrm{i}\beta\sigma$
- BPS HM ops: $q_{N,S} = q^1 \pm q^2$, same for $\widetilde{q}_{N,S}$
- $SU(2)_C$-neutral ops on S^1 and defects wrapping S^2 are BPS (e.g. Higgs-branch ops)



$\mathcal{N}=4$ index and topological correlators

One-dimensional theory and exact correlators

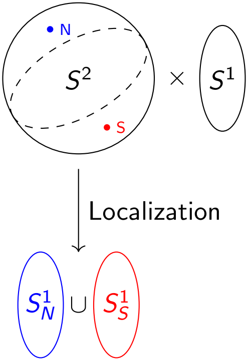
As for the three-sphere[a], 3d theory = topological quantum mechanics for local BPS ops:

$$Z_{3d}=Z_{1d}=\int[\mathrm{d}\widetilde{q}_{N,S}][\mathrm{d}q_{N,S}]e^{-S_{1d}},$$

$$\frac{S_{1d}}{2\pi}=\int_{S^1_N}\mathrm{d}t\,\widetilde{q}_N(\partial_t-\mathrm{i}z_N)q_N-\int_{S^1_S}\mathrm{d}t\,\widetilde{q}_S(\partial_t-\mathrm{i}z_S)q_S,$$

$$\langle\widetilde{q}_{N,S}(t)q_{N,S}(0)\rangle=\mp\frac{\mathrm{sign}(t)-\mathrm{i}\cot\left(\pi z_{N,S}\right)}{4\pi}e^{-\mathrm{i}z_{N,S}t}.$$

Defects wrapping S^2 or $S^1_{N,S}$ can be readily included.



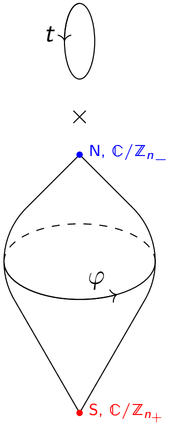
[a] Dedushenko, Pufu, Yacoby, '16

A spiky index from accelerating black-holes

Spindles at the horizon

- The metrics of accelerating black-holes have conical singularities[a]
- Conical singularities signal the presence of $\Sigma = \mathbb{WCP}^1_{[n_-,n_+]}$, namely *spindles*
- Field theory dual: $\mathcal{N} = 2$ QFT on $\Sigma \times S^1$ with $A^{(R)}$ such that $\int_\Sigma \mathrm{d}A^{(R)} = \frac{2\pi(n_- - n_+)}{2n_- n_+}$
- $\int_\Sigma \mathrm{d}A^{(R)} \neq \pi\chi \to \widehat{Z}_{\Sigma\times S^1} =$ anti-twisted index

[a] Ferrero, Gauntlett, Ipiña, Martelli, Sparks, '20



Geometry and bg fields on orbifolds admitting $Q, \widetilde{Q}$

Starting point

1. $\mathrm{d}s^2 = g_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu = f(x)^2\mathrm{d}x^2 + C_{ij}(x)\mathrm{d}\psi^i\mathrm{d}\psi^j \in \mathbb{C}$
2. $K = N_0(\mathrm{i}\omega_0\partial_{\psi^1} + \partial_{\psi^2}) \in \mathbb{C}$ and $\mathcal{L}_K g_{\mu\nu} = 0$
3. $\exists Q, \widetilde{Q} : \left\{Q, \widetilde{Q}\right\} \sim K$, no need $\partial_{\psi^1}, \partial_{\psi^2}$ separately

Construct:

- $(k_0, P, \widetilde{P}) : k_0^2 = \iota_K K^\flat, \iota_P \widetilde{P}^\flat \neq 0$, $\mathrm{d}s^2 = \left(K^\flat / k_0\right)^2 - P^\flat \widetilde{P}^\flat$
- (A, H, V), e.g.:
  $V = \frac{1}{k_0^2}\left(\mathrm{i}k_0 H - \iota_K \frac{\star \mathrm{d}K^\flat}{2k_0^2}\right)K^\flat + \frac{\mathrm{i}}{2}\left(\widetilde{P}^\flat \mathcal{L}_P - P^\flat \mathcal{L}_{\widetilde{P}}\right)\log k_0$

Supersymmetry on orbifolds admitting $Q, \widetilde{Q}$

Killing spinor

$$\zeta = e^{\frac{\mathrm{i}\alpha_1}{2}\psi^1 + \frac{\mathrm{i}\alpha_2}{2}\psi^2}(u_1, -u_2), \text{ with } u_{1,2} = \sqrt{k_0 \mp \mathrm{i} N_0 \omega_0 \sqrt{\frac{\det C}{C_{22}}}}$$

satisfies the KSE: $(\nabla_\mu - \mathrm{i}A_\mu)\zeta = -\frac{H}{2}\gamma_\mu\zeta - \mathrm{i}V_\mu\zeta - \frac{1}{2}\epsilon_{\mu\nu\rho}V^\nu\gamma^\rho\zeta$

Remarks

- $\widetilde{\zeta}$ satisfies the KSE with $(A, H, V) \to (-A, H, -V)$
- After fixing $g_{\mu\nu}$, twist/no-twist/ anti-twist are selected by ω_0
- $\left(\zeta, \widetilde{\zeta}\right) \to \left(Q, \widetilde{Q}\right) \to \delta = Q + \widetilde{Q} \to \delta^2 \sim K = \widetilde{\zeta}\gamma\zeta$

Localization and cohomology

Cohomological complex

Chiral multiplet: $(\phi, \psi, F) \in \mathcal{R}$. If $\psi \sim B\zeta + (C/k_0)\widetilde{\zeta}$, then

$$\delta\phi = C, \qquad \delta B = \Theta, \qquad \delta^2 = -2\mathrm{i}(L_K + \mathcal{G}_{\Phi_G}),$$
$$L_K = \mathcal{L}_K - \mathrm{i}q_R\Phi_R, \qquad \Phi_R = \iota_K[A - (V/2)] - \mathrm{i}k_0 H,$$
$$\mathcal{G}_{\Phi_G}X = -\mathrm{i}\Phi_G \circ_{\mathcal{R}} X, \qquad \Phi_G = \iota_K A_G - \mathrm{i}k_0\sigma,$$

Partition function

$$Z = \sum_{\circ} \int_{\bullet} Z_{\text{classical}} \times \frac{\det_{\mathrm{Ker} L_P}(L_K + \mathcal{G}_{\Phi_G})}{\det_{\mathrm{Ker} L_{\widetilde{P}}}(L_K + \mathcal{G}_{\Phi_G})}$$

Geometry on anti-twisted $\Sigma \times S^1$

Besse metric and frame

$\mathrm{d}s^2 = f^2\mathrm{d}x^2 + \left(1-x^2\right)\left(\mathrm{d}\psi^1 - \mathrm{i}\Omega\mathrm{d}\psi^2\right)^2 + \beta^2\left(\mathrm{d}\psi^2\right)^2$, with $\Omega \in \mathbb{C}$ and

$$e^1 = -f\mathrm{d}x, \quad e^2 = \beta\sqrt{\frac{1-x^2}{\beta^2-(1-x^2)\Omega^2}}\mathrm{d}\psi^1,$$

$$e^3 = \sqrt{\beta^2-(1-x^2)\Omega^2}\left(-\frac{\mathrm{i}\Omega\left(1-x^2\right)\mathrm{d}\psi^1}{\beta^2-(1-x^2)\Omega^2} + \mathrm{d}\psi^2\right),$$

$$\lim_{x\to\pm1} f \to \frac{n_{\mp}}{\sqrt{2(1\mp x)}}, \quad K = N_0\left[\mathrm{i}\omega_0\partial_{\psi^1} + \partial_{\psi^2}\right], \quad \omega_0 = \Omega - \beta$$

$$P^\flat = \mathrm{i}e^{\mathrm{i}\alpha_i\psi^i}\left\{f\mathrm{d}x + \frac{\sqrt{1-x^2}}{x}\left[\mathrm{i}\mathrm{d}\psi^1 + \omega_0\mathrm{d}\psi^2\right]\right\},$$

Supersymmetry on anti-twisted $\Sigma \times S^1$

Killing spinor

$\zeta = e^{\frac{\mathrm{i}\alpha_i \psi^i}{2}}(u_1, -u_2)$ satisfies the KSE with

$$u_{1,2} = \sqrt{\frac{\beta N_0}{2}} \sqrt{x \mp \mathrm{i}\omega_0 \sqrt{\frac{1-x^2}{\beta^2 - (1-x^2)\Omega^2}}}$$

$$A = \frac{3}{2}V + \frac{\mathrm{d}\psi^1 - \mathrm{i}(\beta+\Omega)\mathrm{d}\psi^2}{2f(x)\sqrt{1-x^2}} + \frac{\alpha_i}{2}\mathrm{d}\psi^i,$$

$$\int_\Sigma \frac{\mathrm{d}A}{2\pi} = \frac{n_- - n_+}{2n_- n_+} \to \text{anti-twist},$$

Partition function on anti-twisted $\Sigma \times S^1$

The anti-twisted index

Finally, given $\mathfrak{r} = r_{\text{eff}} = r + \mathfrak{n} + \mathfrak{m}(n_- + n_+)/(n_- - n_+)$,

$$Z^{\text{chi}}_{\Sigma \times S^1} = e^{-\mathrm{i}\pi\Psi(w,\mathfrak{m},\mathfrak{n},\omega_0)} \frac{\left(q^{1+|q_m \mathfrak{r}| - \mathfrak{m} q_m} e^{-2\pi\mathrm{i}(w+\gamma)}; q\right)}{\left(q^{1+|q_m(\mathfrak{r}-2)| + \mathfrak{m} q_m} e^{2\pi\mathrm{i}(w-\gamma)}; q\right)},$$

$$q = e^{-2\pi\omega_0}, \quad \mathfrak{m}, \mathfrak{n} \in \mathbb{Z}, \quad \gamma = \frac{\mathrm{i}\varphi}{2\pi} = \frac{n}{2} + \frac{\mathrm{i}\chi\omega_0}{4}, \quad n \in \mathbb{Z},$$

$$q_m = \frac{n_- - n_+}{4 n_- n_+}, \quad w = (r-1)\frac{\mathrm{i}\varphi}{2\pi} - \frac{u}{2\pi}, \quad u \in \mathbb{C},$$

Discussion

Results

- 3d SUSY index as a topological QM on $S^1 \cup S^1$
- exact correlators for BPS operators on $S^1 \subset S^2 \times S^1$
- a unique ζ for a large class of 3d orbifolds
- new SUSY index from exact partition functions on spindles

Outlook

- anti-twisted index as a topological QM on $S^1 \cup S^1$
- black-hole entropy from ABJM on Σ at large N
- localization of 5d gauge theories compactified on spindles

The end?

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