Integrability in the ODE/IM correspondence approach for new exact results on $\mathcal{N} = 2$ supersymmetric gauge theories and black holes

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Based on arXiv:1908.08030, arXiv:2112.11434 with Davide Fioravanti and arXiv:22**.***** with also Hongfei Shu

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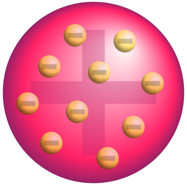
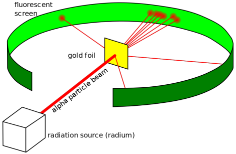
Introduction

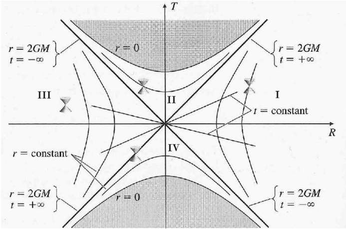
Lessons from history



- **Thomson**'s "plum-pudding" atomic model was **carefully constructed** and **tested theoretically** for inconsistencies.
- E. **Rutherford** set up his famous experiment of scattering of α particles **NOT with the aim of disproving the model**, BUT just of testing its accuracy.



- Black holes can be regarded as the **"elementary particles" of gravity** (simplest and indivisible in GR).

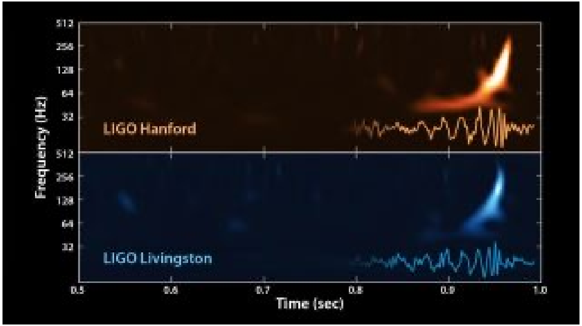


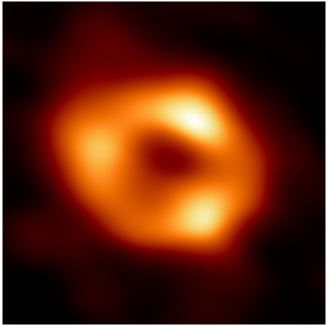
Could BHs hold the same surprises that the electron and the hydrogen atom did when they started to be experimentally probed?

[Cardoso,Pani:2017]

Gravitational Phenomenology

- In the last few years, **gravitational waves** detections and **black hole imaging** have opened the doors of **gravitational phenomenology**. [Mayerson:2020]

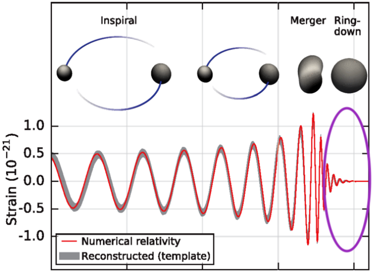




- Finally, we can fully scientifically investigate whether real astrophysical black holes show **deviations from general relativity (GR)**, such as **horizon scale structure**.

Colliding BHs and quasinormal modes

- A **black hole collision** can be divided in 3 phases: **inspiral, merger** and **ringdown**.



- The **quasinormal modes (QNMs)** are responsible for the **damped oscillations** appearing, for example, in the **ringdown phase** of two colliding BH and have a direct connection to **gravitational waves observations**.

Alternative models of BHs

- GR BHs present fundamental theoretical problems (e.g. information paradox).
- Also to solve such problems, theoretical models of **Exotic Compact Objects (ECOs)** in alternative theories of gravity have been developed. They have **horizon scale structure**.
- For subtype of ECOs, called **Clean Photosphere Objects (ClePhOs)**, the later stage ringdown signal shows a peculiar train of **echoes**, with **significant deviations from GR**.
- An example of ClePhoS are **fuzzballs** in **String Theory**, with neither horizon nor central singularity and which may solve also the information paradox.

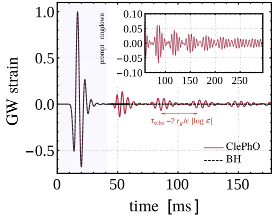


Figure: [Cardoso,Pani:2017]

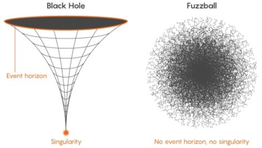


Figure: Cr. Quanta Magazine

From ($\mathcal{N} = 2$) gauge to gravity and back

Classical Seiberg-Witten curve and gauge periods

- The **classical Seiberg-Witten (SW) curve** and **SW differential** are defined as

$$
\begin{aligned}
y_{SW}^2 &= x^3 + c_2 x^2 + c_1 x + c_0 \\
\frac{\partial \lambda}{\partial u} &= \frac{\sqrt{2}}{8\pi} \frac{2u - (4 - N_f)x + C_0}{y_{SW}}
\end{aligned}
\tag{1}
$$

- Define the **classical SW periods** by integrating over the cycles $\mathcal{A}, \mathcal{B}$ of the SW curve

$$
\begin{aligned}
a^{(0)}(u, m, \Lambda) &= \oint_{\mathcal{A}} \lambda(x, u, m, \Lambda)\, d\,x\,, \\
a_D^{(0)}(u, m, \Lambda) &= \oint_{\mathcal{B}} \lambda(x, u, m, \Lambda)\, d\,x\,.
\end{aligned}
\tag{2}
$$

- From them one can compute the **SW prepotential** $\mathcal{F}^{(0)}(u, m, \Lambda)$.

Quantum Seiberg-Witten curve and gauge periods from resummation

- To compute **instanton contributions** spacetime is deformed by two complex parameters ϵ_1, ϵ_2 into the Ω-**background**.
- Interesting for the connection to gravity is the **Nekrasov-Shatashvili limit** $\epsilon_2 \to 0, \epsilon_1 = \hbar \neq 0$

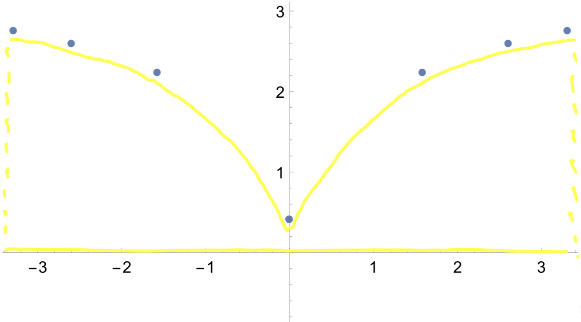
$$-\hbar^2 \frac{d^2}{dy^2}\psi(y) + \left[\frac{\Lambda_1^2}{4}(e^{2y} + e^{-y}) + \Lambda_1 m e^y + u\right]\psi(y) = 0\,, \tag{3}$$

(here for $SU(2)$ $N_f = 2$ theory)

Quantum SW periods I

- We can define **quantum exact periods** by **exact integrals** of $\mathcal{P}(y) = -i\frac{d}{dy}\ln\psi(y)$ as sums over residues at the poles which as $\hbar \to 0$ reduce to the classical cycles (branch cuts).

$$\begin{pmatrix} a(\hbar,u,m,\Lambda) \\ a_D(\hbar,u,m,\Lambda) \end{pmatrix} \doteq \oint_{A,B} \mathcal{P}(y,\hbar,u,m,\Lambda)\,dy = 2\pi i \sum_n \mathrm{Res}\mathcal{P}(y)\Big|_{y_n^{A,B}} \tag{4}$$



Quantum SW periods II

- Alternatively, one can define possibly different **quantum periods** by the **Nekrasov-Shatashvili prepotential** $\mathcal{F}(a,\hbar,m,\Lambda)$

$$A_D(u,\hbar,m,\Lambda)=\partial_a\mathcal{F}(u,\hbar,m,\Lambda)\Big|_{a=a(\hbar,u,m,\Lambda)}. \tag{5}$$

$$u=a^2-\frac{\Lambda}{4-N_f}\frac{\partial\mathcal{F}_{\text{inst}}(a;\hbar,m)}{\partial\Lambda}. \tag{6}$$

- In practice $\mathcal{F}$ is computed by **combinatorial calculus on Young Tableaux** of the gauge group representation.

A surprising application

- In the last two years, a **surprising connection** between $\mathcal{N} = 2$ supersymmetric (SUSY) $SU(2)$ gauge theories (Nekrasov-Shatashvili deformed) and black holes (BHs) perturbation theory has emerged [**arXiv:2006.06111**, 2105.04245, 2105.04483, arXiv:2109.09804].
- G. Aminov, A. Grassi and Y. Hatsuda first found that **quantization conditions on the gauge periods** a, A_D allow to **compute the (QNMs)** ω_n spectrum of black holes from gauge theory methods.

$$A_D(\hbar, u, m, \Lambda) = 2\pi \left(n + \frac{1}{2}\right) , \qquad n = 0, 1, 2, \dots . \tag{7}$$

A fruitful new field

The importance of this result is manifold.

1. It constitutes a **novel analytic characterization of QNMs**, for which previously very few were known [arXiv:2006.06111].
2. In the increasingly growing outflow of research on this topic, it has already allowed to **find new results** for the BHs theory, such as:
    1. an **isospectral simpler equation** to the perturbation ODE [arXiv:2007.07906];
    2. improved **theoretical proofs of BHs stability** [arXiv:2105.13329];
    3. **more accurate computations** of observable quantities such as **Love numbers**, describing tidal deformations [arXiv:2105.04483];
    4. an **simpler interpretation of Chandrasekhar transformation** as exchange of gauge mass parameters [arXiv:2111.05857];
    5. precise determination of the **conditions of invariance under (Couch-Torrence) transformations** which exchange inner horizon and null infinity [arXiv:2203.14900].
3. It constitutes an **unexpected application of Supersymmetry**, which was originally thought to describe elementary particles, but has not yet been found by experimentalists.

Integrability for $\mathcal{N} = 2$ gauge theory

ODE/IM correspondence

- In this classic approach to integrability [**arXiv:9812211,9812247,9906219**], the Q **function** is typically the wronskian of the regular solutions at different singular points

$$Q = W[\psi_+, \psi_-] \qquad \psi_\pm(y) \to 0 \quad y \to \pm\infty \tag{8}$$

of some ODE, like (for self-dual Liouville IM or $SU(2)$ $N_f = 0$ gauge theory) [**arXiv:1908.08030**]:

$$-\frac{d^2}{dy^2}\psi + \left[e^{2\theta}(e^y + e^{-y}) + P^2\right]\psi = 0\,. \tag{9}$$

- This **innovates ODE/IM correspondence** itself because such ODEs have 2 **irregular singularities** rather than just 1 as usual. One can derive also T, Y **functions** as well as the **functional and integral equations** they satisfy.
- ODE/IM is an **elegant approach** to integrability which **allows to apply it to very different physical theories!**

Basic gauge-integrability identifications

- Using ODE/IM correspondence, we **connected** the **basic integrability functions** - the Baxter's Q, T and Y functions - to the gauge exact quantum periods a, a_D (from which the prepotential can be obtained). We proved relations like

$$\begin{aligned} Q(\theta, P) &= \exp 2\pi i a_D(\hbar, u, \Lambda_0) \\ T(\theta, P) &= 2\cos 2\pi a(\hbar, u, \Lambda_0) \end{aligned} \tag{10}$$

  under the parameters correspondence

$$\frac{\hbar}{\Lambda_0} = \frac{\epsilon_1}{\Lambda_0} = e^{-\theta} \qquad \frac{u}{\Lambda_0^2} = \frac{1}{2}P^2 e^{-2\theta} \tag{11}$$

- These for the self-dual Liouville IM and $SU(2)$ $N_f = 0$ gauge theory [**arXiv:1908.08030**] but also similar ones for the Perturbed Hairpin IM and $SU(2)$ $N_f = 1, 2$ [**Fioravanti,Gregori,Shu-to appear**] and $SU(3)$ $N_f = 0$ gauge theories [**arXiv:1909.11100**].

New results for both gauge theory and integrability

- This fundamental identification allowed us to find several **new interesting results for both sides** of this new kind of Integrability/Gauge correspondence, for instance:
  1. an exact non linear integral equation (**Thermodynamic Bethe Ansatz, TBA**) for the **gauge (dual) periods**;
  2. an interpretation of the integrability functional relations as **new exact R-symmetry relations for the periods**;
  3. **new formulas for the local integrals of motion** in terms of gauge periods.
- For instance, the **Baxter's TQ relation**

$$T(\theta,u)=\frac{Q(\theta-i\pi/2,-u)+Q(\theta+i\pi/2,-u)}{Q(\theta)} \tag{12}$$

turns out to be a new exact R-symmetry relation for the periods, reducing to the known asymptotic ones in the limit $\theta\to\infty$ (for the gauge periods expansion modes $a^{(n)}$, $a_D^{(n)}$)

$$a_D^{(n)}(-u)=i(-1)^n\left[-\mathrm{sgn}(\mathrm{Im}\,u)a_D^{(n)}(u)+a^{(n)}(u)\right] \tag{13}$$

Integrability for black holes

Mathematical definition of quasinormal modes

- **Perturbations of the BH** metric or fields turns out to be **solutions Φ of some PDEs** of the form

$$\left\{+\frac{\partial^2}{\partial t^2}-\frac{\partial^2}{\partial x^2}+U(x)\right\}\Phi(t,x)=0\,, \tag{14}$$

(with coordinate x such that the BH horizon is put at $x\to-\infty$ and spacetime infinity at $x\to+\infty$)

- By ordinary DE techniques (Laplace tr. $\to$ non-hom. ODE $\to$hom. ODE) we can express the **perturbation ϕ as an expansion over some frequencies ω_n**

$$\Phi(t,x)=\sum_n e^{i\omega_n t}\text{Res}\left(\frac{1}{W(s)}\right)\Bigg|_{\omega_n}\int_{-\infty}^{\infty}\Psi_-(\omega_n,x_<)\Psi_+(\omega_n,x_>)\mathcal{I}(\omega_n,x')\,dx'\,. \tag{15}$$

ω_n are the **quasinormal modes (QNMs)** and we see that they are defined as the **zeros of wronskian of the fundamental regular solutions** at $x\to\pm\infty$ (20):

$$W[\Psi_+,\Psi_-](\omega_n)=0\,,\qquad \Psi_\pm(x)\to 0\quad x\to\pm\infty\,. \tag{16}$$

Details

- If we take the Laplace transform of Φ

$$\hat{\Psi}(s,x) = \int_0^\infty e^{-st}\Phi(t,x)\,dt\,, \tag{17}$$

$$\left\{-\frac{\partial^2}{\partial x^2} + U(x) + s^2\right\}\hat{\Psi}(s,x) = -\mathcal{I}(s,x)\,, \qquad \mathcal{I}(s,x) = -s\Psi(t,x)\big|_{t=0} - \frac{\partial\Psi(t,x)}{\partial t}\big|_{t=0}\,. \tag{18}$$

- The corresponding homogeneous equation is exactly the ODE we are going to study in the next sections

$$\left\{-\frac{\partial^2}{\partial x^2} + U(x) + s^2\right\}\Psi(s,x) = 0\,. \tag{19}$$

  Its solutions bounded at $x \to \pm\infty$, for $\mathrm{Re}\, s > 0$, are

$$\Psi_+(s,x) \sim e^{-sx}\,, \qquad x \to +\infty \qquad \Psi_-(s,x) \sim e^{sx}\,, \qquad x \to -\infty\,. \tag{20}$$

- The solution of the homogenous equation is then found to be given by the Green function G as

$$\hat{\Psi}(s,x) = \int_{-\infty}^{\infty} G(s,x,x')\mathcal{I}(s,x')dx'\,, \qquad G(s,x,x') = \frac{1}{W[\Psi_-,\Psi_+]}\Psi_-(s,x_<)\Psi_+(s,x_>)\,, \tag{21}$$

  with $x_< = \min(x',x)$, $x_< = \max(x',x)$.
- Then taking the antiplace transform of $\hat{\Psi}$ and setting $s = i\omega$ we get the original perturbation as

$$\Phi(t,x) = \frac{1}{2\pi i}\int_{\epsilon-i\infty}^{\epsilon+i\infty} e^{st}\hat{\Psi}(s,x)\,ds = \sum_n e^{i\omega_n t}\mathrm{Res}\left(\frac{1}{W(s)}\right)\Bigg|_{\omega_n}\int_{-\infty}^{\infty}\Psi_-(\omega_n,x_<)\Psi_+(\omega_n,x_>)\mathcal{I}(\omega_n,x')\,dx'\,. \tag{22}$$

ODE for the perturbation of generalized RN BHs

- Line element for intersection of four stacks of D3-branes in type IIB supergravity:

$$ds^2 = -f(r)dt^2 + f(r)^{-1}[dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2)]\,, \tag{23}$$

with

$$f(r) = \prod_{i=1}^{4}\left(1 + \frac{\mathcal{Q}_i}{r}\right)^{-\frac{1}{2}}\,. \tag{24}$$

If the charges $\mathcal{Q}_i = Q = M$ are all equal, it leads to an **extremal Reissner Nordström (charged) BH** with $f(r) = 1 - \frac{2M}{r} + \frac{\mathcal{Q}^2}{r^2}$.

- The **ODE for the scalar perturbation** is

$$\frac{d^2\phi}{dr^2} + \left\{-\frac{(l+\frac{1}{2})^2 - \frac{1}{4}}{r^2} + \frac{\omega^2}{r^4}\left[\Sigma_4 + \Sigma_3 r + \Sigma_2 r^2 + \Sigma_1 r^3 + r^4\right]\right\}\phi = 0 \tag{25}$$

where we defined $\Sigma_k = \sum_{i_1<\ldots<i_k}^{4} \mathcal{Q}_{i_1}\cdots\mathcal{Q}_{i_k}$.

ODE for generalized Perturbed Hairpin Integrable Model

- **Changing variables** as

$$r = \sqrt[4]{\Sigma_4} e^y \,, \qquad \omega \sqrt[4]{\Sigma_4} = -i e^{\theta} \,, \tag{26}$$

$$\frac{\Sigma_1}{\sqrt[4]{\Sigma_4}} = 2M_1 e^{-\theta} \,, \quad \frac{\Sigma_3}{\sqrt[4]{\Sigma_4^3}} = 2M_2 e^{-\theta} \,, \quad P^2 = (l + \frac{1}{2})^2 - \omega^2 \Sigma_2 \,. \tag{27}$$

the ODE takes the form

$$-\frac{d^2}{dy^2}\psi + \left[e^{2\theta}(e^{2y} + e^{-2y}) + 2e^{\theta}(M_1 e^y + M_2 e^{-y}) + P^2\right]\psi = 0 \,. \tag{28}$$

- In this form, this ODE is a **generalization** of the one for the **Perturbed Hairpin Integrable Model**:

Asymptotic solutions and discrete symmetries

- The **regular solutions** of (28) at $y \to \pm\infty$ ($j = 1, 2$) have boundary conditions

$$\begin{aligned} \psi_{-,0}(y) &\simeq 2^{-\frac{1}{2}-M_2} e^{-(\frac{1}{2}+M_2)\theta+(\frac{1}{2}+M_2)y} e^{-e^{\theta-y}} , \quad \mathrm{Re}\, y \to -\infty . \\ \psi_{+,0}(y) &\simeq 2^{-\frac{1}{2}-M_1} e^{-(\frac{1}{2}+M_1)\theta-(\frac{1}{2}+M_1)y} e^{-e^{\theta+y}} , \quad \mathrm{Re}\, y \to +\infty . \end{aligned} \tag{29}$$

- Equation (28) enjoys the **discrete symmetries**

$$\Omega_\pm : \ \theta \to \theta + i\pi/2 , \ y \to y \pm i\pi/2 , \ M_1 \to \mp M_1 , \ M_2 \to \pm M_2 , \tag{30}$$

- One can **define other independent solutions** as

$$\psi_{-,k} = \Omega_-^k \psi_{-,0} , \qquad \psi_{+,k} = \Omega_+^k \psi_{+,0} . \tag{31}$$

We also have the **invariance properties**

$$\Omega_+^k \psi_{-,0} = \psi_{-,0} , \qquad \Omega_+^k \psi_{+,0} = \psi_{+,0} . \tag{32}$$

Baxter's Q function and quasinormal modes

- The **Baxter's Q function** is defined **precisely as the wronskian of the regular solutions**

$$Q_{+,+}(\theta) = W[\psi_{+,0}, \psi_{-,0}] \,. \tag{33}$$

(We will use the notation $Q_{\pm,\pm} = Q(\theta, P, \pm M_1, \pm M_2)$, $Q_{\pm,\mp} = Q(\theta, P, \pm M_1, \mp M_2)$).

- Crucially, the **QNMs condition** (16) translates into

$$Q(\theta_n) = 0 \,, \tag{34}$$

namely the **zeros of the Baxter's Q function** which are called **Bethe roots**.

Central connection relation as QQ system

- The solutions are normalised such that their wronskians are $W[\psi_{-,k+1},\psi_{-,k}] = -i\exp\{(-1)^k i\pi M_2\}$, $W[\psi_{+,k+1},\psi_{+,k}] = i\exp\{(-1)^k i\pi M_1\}$.
- By the properties of wronskians $W[\psi_{\pm,1},\psi_{\pm,1}] = W[\psi_{\pm,0},\psi_{\pm,0}] = 0$, we can write the **linear (central) connection relations** as

$$
\begin{aligned}
ie^{i\pi M_1}\psi_{-,0} &= Q_{-,+}(\theta + i\pi/2)\psi_{+,0} - Q_{+,+}(\theta)\psi_{+,1} \\
ie^{i\pi M_1}\psi_{-,1} &= Q_{-,-}(\theta + i\pi)\psi_{+,0} - Q_{+,-}(\theta + i\pi/2)\psi_{+,1}
\end{aligned}
\tag{35}
$$

- Taking their wronskian we get the QQ **system (quantum wronskian)**

$$
Q_{+,-}(\theta + \frac{i\pi}{2})Q_{-,+}(\theta - \frac{i\pi}{2}) = e^{-i\pi(M_1 - M_2)} + Q_{-,-}(\theta)Q_{+,+}(\theta)\,. \tag{36}
$$

Q function as non-compact integral

- From the central connection relation (35) we get an **integral formula for** Q

$$Q_{+,+}(\theta) = -ie^{i\pi M_1} \lim_{y\to+\infty} \frac{\psi_{-,0}(y)}{\psi_{+,1}(y)} \tag{37}$$

$$= \int_{-\infty}^{\infty} dy \left[\sqrt{2\cosh(2y)}\Pi(y) - 2e^{\theta}\cosh y - \left(\frac{M_1}{1+e^{-y/2}} + \frac{M_2}{1+e^{y/2}} \right) \right] + \left(\theta + \frac{1}{2}\ln 2 \right)(M_1 - M_2) \tag{38}$$

- The integrand $\Pi(y) = \frac{1}{\sqrt{2\cosh(2y)}} \frac{d}{dy} \ln(\sqrt[4]{-2\cosh(2y)}\psi(y))$ satisfies the **Riccati equation** equivalent to the 2° order linear ODE for $\psi(y)$

$$\Pi(y)^2 + \frac{1}{\sqrt{2\cosh(2y)}} \frac{d}{dy}\Pi(y)$$
$$= e^{2\theta} + e^{\theta}\mathrm{sech}(2y)\left(M_1 e^{y} + M_2 e^{-y}\right) + \frac{1}{2}\mathrm{sech}(2y)\left[P^2 + \mathrm{sech}(2y) - \frac{5}{4}\tanh^2(2y) \right] \tag{39}$$

Y function and quasinormal modes

- One can define a Y function as

$$Y_{+,+}(\theta) = e^{i\pi(M_1 - M_2)} Q_{+,+}(\theta) Q_{-,-}(\theta) \tag{40}$$

- From the QQ system it follows the Y **system** as

$$Y_{+,-}(\theta + \frac{i\pi}{2}) Y_{-,+}(\theta - \frac{i\pi}{2}) = [1 + Y_{+,+}(\theta)][1 + Y_{-,-}(\theta)]\,. \tag{41}$$

- Eventually, the QQ **system** written as

$$e^{i\pi(M_1 + M_2)} Q_{+,+}(\theta + \frac{i\pi}{2}) Q_{-,-}(\theta - \frac{i\pi}{2}) = 1 + Y_{+,-}(\theta)\,. \tag{42}$$

characterizes the **QNMs** with other **quantizations conditions**

$$Y_{+,-}(\theta_n - i\pi/2) = -1 \qquad Y_{-,+}(\theta_n + i\pi/2) = -1 \tag{43}$$

Thermodynamic Bethe Ansatz

- The Y **system can be inverted in the Thermodynamic Bethe Ansatz (TBA)** for $\varepsilon_{\pm,\pm}(\theta) = -\ln Y_{\pm,\pm}(\theta)$:

$$\varepsilon_{\pm,\pm}(\theta) = \frac{8\sqrt{\pi^3}}{\Gamma\left(\frac{1}{4}\right)^2} e^{\theta} \mp i\pi(M_1 - M_2) - \int_{-\infty}^{\infty} \frac{d\theta'}{2\pi} \frac{\ln[1+\exp(-\varepsilon_{\pm,\mp}(\theta'))] + \ln[1+\exp(-\varepsilon_{\mp,\pm}(\theta'))]}{\cosh(\theta-\theta')}$$
$$\varepsilon_{\pm,\mp}(\theta) = \frac{8\sqrt{\pi^3}}{\Gamma\left(\frac{1}{4}\right)^2} e^{\theta} \mp i\pi(M_1 + M_2) - \int_{-\infty}^{\infty} \frac{d\theta'}{2\pi} \frac{\ln[1+\exp(-\varepsilon_{\pm,\pm}(\theta'))] + \ln[1+\exp(-\varepsilon_{\mp,\mp}(\theta'))]}{\cosh(\theta-\theta')} \quad (44)$$

- P enters as the $\theta \to -\infty$ boundary condition (from perturbative expansion of ODE)

$$\varepsilon_{+,+}(\theta, P) \sim 4P\theta + 2C(P, M_1, M_2)\,, \qquad \theta \to -\infty \quad (45)$$

$$C(P, M_1, M_2) = \ln\left[\frac{2^{1-2P} P\,\Gamma(2P)^2}{\sqrt{\Gamma\left(P+\frac{1}{2}-M_1\right)\Gamma\left(P+\frac{1}{2}+M_1\right)\Gamma\left(P+\frac{1}{2}-M_2\right)\Gamma\left(P+\frac{1}{2}-M_2\right)}}\right] \quad (46)$$

Quasinormal modes from TBA

- The **QNMs condition** in gravity variables **reads alternatively as**

$$\bar{Y}_{+,+}(\theta_{n'} - i\pi/2) = -1\,, \quad Q_{+,+}(\theta_n) = 0 \tag{47}$$

or

$$\bar{\varepsilon}_{+,+}(\theta_{n'} - i\pi/2) = -i\pi(2n' + 1)\,. \tag{48}$$

- Through the quantization condition on $\bar{\varepsilon}_{+,+}$ we can **actually numerically compute QNMs from TBA**.

n	l	TBA	Leaver	WKB
0	1	0.869623 − 0.372022i	0.868932 − 0.372859i	0.89642 − 0.36596i
0	2	1.477990 − 0.368144i	1.477888 − 0.368240i	1.4940 − 0.36596i
0	3	2.080200 − 0.367076i	2.080168 − 0.367097i	2.0916 − 0.36596i
0	4	2.680363 − 0.366637i	2.680350 − 0.366642i	2.6893 − 0.36596i

Table: Comparison of QNMs in different methods ($n' = 0$, $\Sigma_1 = \Sigma_3 = 0.2$, $\Sigma_2 = 0.4$, $\Sigma_4 = 1$).

Comparison of methods of computation of quasinormal modes

n	l	TBA	Leaver	WKB
0	1	$0.896681 - 0.40069i$	N.A.	$0.93069 - 0.39458i$
0	2	$1.5308 - 0.39676i$	N.A.	$1.5511 - 0.39458i$
0	3	$2.15708 - 0.395689i$	N.A.	$2.1716 - 0.39458i$
0	4	$2.78077 - 0.39525i$	N.A.	$2.7921 - 0.39458i$

Table: Here $\Sigma_1 \neq \Sigma_3$ and the Leaver method seems not applicable (N.A.).

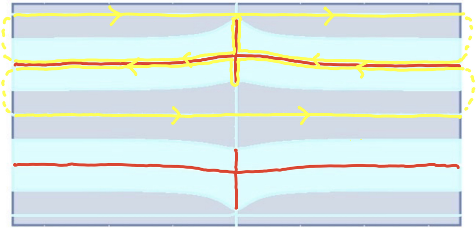
- Computing QNMs has been typically quite **laborious**, also because of their **few exact analytic characterizations**.
- The standard analytic method is the one with the **continued fractions** by Leaver we found **sometimes not applicable** (in its original form).
- Application of $\mathcal{N} = 2$ **gauge theory** is a new analytic characterization of QNMs, but it requires a **nontrivial re-summation procedure** for $\omega_n \sim \Lambda_n \sim 1$.
- Our **integrability** exact method is **direct and simple**, but now we have developed for just a **few models** (see below).

Quantum gauge B period from TBA

- Also for this $SU(2)$ $N_f = 2$ found a **relation between the pseudoenergy** $\epsilon = -\ln Y$ **and the gauge periods**

$$\begin{aligned} \varepsilon(\theta, -u, im_1, -im_2, \Lambda_2) &= \frac{8\sqrt{2}\pi}{\Lambda_2} a_D(\hbar, u, m_1, m_2, \Lambda_2) \\ \varepsilon(\theta, -u, -im_1, im_2, \Lambda_2) &= \frac{8\sqrt{2}\pi}{\Lambda_2} a_D(\hbar, u, m_1, m_2, \Lambda_2) + \frac{8\pi}{\Lambda_2}(m_1 + m_2)e^{\theta} \end{aligned} \tag{49}$$

- We can prove it analytically by **Cauchy theorem relating the different integration contours** in the complex plane (in red the branch cuts).



Quasinormal modes from gauge periods quantization conditions

- We see that from our formalism **it follows immediately that QNMs are given by quantization conditions on the gauge integral periods**

$$\frac{8\sqrt{2}\pi}{\Lambda_2} a_D(\hbar, u, m_1, m_2, \Lambda_2) = -i\pi(2n' + 1) \quad (50)$$

- Recently other authors found QNMs to be given by quantization conditions on the gauge instanton periods

$$A_D(\hbar, u, m_1, m_2, \Lambda_2) = i\pi n \quad (51)$$

but we know a consistent relations between the two kind of gauge periods. For instance for $N_f = 0$ $SU(2)$:

$$Q(\theta, P) = \exp\left\{\frac{2\pi i}{\hbar} a_D(u, \Lambda_0, \hbar)\right\} = i\frac{\sinh \frac{1}{\hbar} A_D(a, \Lambda_0, \hbar)}{\sinh \frac{2\pi i}{\hbar} a}, \quad (52)$$

Baxter's T functions and their relations

- Now, the presence of the irregular singularity of (28) at $y \to \pm\infty$ (Stokes phenomenon) plays a role for **defining the T functions**

$$T_{+,+}(\theta) = -iW[\psi_{-,-1}, \psi_{-,1}]\,, \;\; \tilde{T}_{+,+}(\theta) = iW[\psi_{+,-1}, \psi_{+,1}]\,. \tag{53}$$

- By expanding $\psi_{\pm,1}$ in terms of $\psi_{\pm,0}$, $\psi_{\pm,-1}$ as

$$\psi_{+,-1} = -e^{-2\pi i M_1}\psi_{+,1} + e^{-i\pi M_1}\tilde{T}_{+,+}(\theta)\psi_{+,0} \tag{54}$$

$$\tilde{T}_{+,+}(\theta)\psi_{+,0} = e^{i\pi M_1}\psi_{+,-1} + e^{-i\pi M_1}\psi_{+,+1} \tag{55}$$

we obtain the TQ **relations**

$$\begin{aligned} T_{+,+}(\theta)Q_{+,+}(\theta) &= e^{i\pi M_2}Q_{+,-}(\theta - \frac{i\pi}{2}) + e^{-i\pi M_2}Q_{+,-}(\theta + \frac{i\pi}{2}) \\ \tilde{T}_{+,+}(\theta)Q_{+,+}(\theta) &= e^{i\pi M_1}Q_{-,+}(\theta - \frac{i\pi}{2}) + e^{-i\pi M_1}Q_{-,+}(\theta + \frac{i\pi}{2})\,. \end{aligned} \tag{56}$$

- By the invariance under the symmetries $\Omega_\pm$ also the T **periodicities** follow

$$T_{-,+}(\theta + i\pi/2) = T_{+,+}(\theta)\,, \quad \tilde{T}_{+,-}(\theta + i\pi/2) = \tilde{T}_{+,+}(\theta) \tag{57}$$

Another basic gauge-integrability identification

- One can prove a **relation** between the T **function** for our doubly confluent Heun equation and its **Floquet exponent** (such that $\psi(y+2\pi i) = e^{2\pi i\nu}\psi(y)$)

$$
\begin{aligned}
2\cos 2\pi\nu + 2\cos 2\pi M_1 &= \tilde{T}_{+,+}(\theta)\,\tilde{T}_{-,+}(\theta + i\frac{\pi}{2}) \\
2\cos 2\pi\nu + 2\cos 2\pi M_2 &= T_{+,+}(\theta)\,T_{+,-}(\theta + i\frac{\pi}{2})\,.
\end{aligned}
\tag{58}
$$

- By comparing the Λ (instanton) expansions we find

$$\nu = \Pi_A = a \tag{59}$$

and therefore the afore relations constitutes **another basic link between integrability and gauge theory**.

Alternative QNMs quantization condition for $N_f = 0$

- From the relation between T and a it follows a proof of the **alternative quantization condition for quasinormal modes** for the $N_f = 0$ $SU(2)$ theory

$$a(\theta_n) = \frac{n}{2}\,, \qquad n \in \mathbb{Z}\,. \tag{60}$$

which was given by some authors for the D3 brane (corresponding to $SU(2)$ $N_f = 0$).

- Indeed, the TQ relation $T(\theta)Q(\theta) = Q(\theta - i\pi/2) + Q(\theta + i\pi/2)$ means $Q(\theta_n - i\pi/2) + Q(\theta_n + i\pi/2) = 0$. This and the QQ relation $Q(\theta + i\pi/2)Q(\theta - i\pi/2) = 1 + Q^2(\theta)$ actually fixes $Q(\theta_n + i\pi/2)Q(\theta_n - i\pi/2) = 1$ and then

$$Q(\theta_n \pm i\pi/2) = \pm i \tag{61}$$

are fixed, too. Again the QQ relation around θ_n forces $Q(\theta + i\pi/2) = i \pm Q(\theta) + \dots$ and $Q(\theta - i\pi/2) = -i \pm Q(\theta) + \dots$ up to smaller corrections (dots). Therefore, TQ relation imposes

$$T(\theta_n) = 2\cos 2\pi a(\theta_n) = \pm 2\,. \tag{62}$$

Partial generalization of alternative QNMs quantization condition

- By making considerations on these TQ systems and the QQ system (42) like in the previous slide, we are **not in general able to conclude that the T-s are quantized**, except in the case of **equal masses** $M_1 = M_2$ where we find

$$T_{+,+}(\theta_n)T_{-,-}(\theta_n) = 4\,, \qquad (M_1 = M_2)\,. \tag{63}$$

  that generalizes (62).

- It corresponds to $(M_1 = M_2 = M)$

$$\begin{aligned} T_{+,+}(\theta_n)T_{-,-}(\theta_n) &= 4 \\ [\cos 2\pi\nu + \cos 2\pi M]_{\theta=\theta_n} &= \pm 2\,. \end{aligned} \tag{64}$$

Interpretation of integrability duality

- Moreover $\psi_{+,0}(y) = \psi_{-,0}(-y)$ is just the **symmetry that exchange infinity ($y \to +\infty$) and the (analogue) horizon ($y \to -\infty$), leaving the photon sphere ($y = 0$) fixed** as in [BianchiRusso:2021] (thanks to identifications of certain scattering angles with the SW period). In this respect, under this symmetry we have the T **self-duality**

$$\tilde{T}_{+,+}(\theta) = T_{+,+}(\theta) \qquad (M_1 = M_2) \tag{65}$$

Greybody factor

- We notice that **much of the BH theory seems to go in parallel to the ODE/IM correspondence** construction and its 2D integrable field theory interpretation, beyond the determination of QNMs.
- For instance also the **greybody factor** that parametrizes the **Hawking radiation** seems to be **ratio of** Q**s**.
- This can be understood by considering its absorption coefficient role as viewed in 1D quantum mechanics.

Further developments and conclusions

Present limitations and possible future developments

- What we have described holds for the **generalization of extremal charged BHs** (intersection of 4 stacks of D3 branes) and it corresponds to $SU(2)$ $N_f = 2, 1, 0$ **gauge theories**.
- However, from the **generality of construction** it is manifest that our method **should apply many other theories**. We can list
  - the $SU(2)$ $N_f = (2, 0)$ gauge theory and the associated gravity counterparts, like D1D5 fuzzballs and CCLP 5D BHs;
  - as to $SU(2)$ $N_f = 3$ gauge theory and the general asymptotically flat Kerr-Newman BH. The associated integrable structure appears to be a generalization of the conformal minimal models.
  - Another simple but tricky case happens when all the (Heun) ODE singularities are regular, e.g. $SU(2)$ $N_f = 4$ or $SU(2)$ quivers that is asymptotically AdS BHs. The corresponding IM appear to be spin chains.

Present limitations and possible future developments (cont.)

- **Analytic continuations of the TBAs** in rapidity θ and the moduli are necessary to obtain **overtones** ω_n $n \geq 1$ and also some **particular gravitational systems**.
- Besides **model generalization**, it is very intriguing to study the application of the integrability structure **beyond the determination of the QNMs** for the study of gravitational solutions (greybody factor, etc.).
- For **AdS BHs** the 3-fold correspondence Integrability/Gauge/Gravity could become 4-fold through **holographic duality to CFT**.

Conclusions

- We have seen how **2D integrable models** when studied in **ODE/IM correspondence** approach can find
  - a natural connection to (deformed) $\mathcal{N} = 2$ **supersymmetric gauge theory**
  - as well as to **black hole perturbation theory**
  - and shed light on the **relation** recently found **between the two**.
- This allows to
  - **find new results** an all three sides of the correspondence
  - at the **non-perturbative exact level**.
- In these new directions **much extension work in either breadth and depth** can still be done.

Additional details on the new gauge/gravity correspondence

New result from $\mathcal{N}=2$ SYM: an alternative to Kerr Teukolsky equation

- Consider **Kerr BHs**. The **radial Teukolsky equation** is complicated and uncoventional. This fact often prevents us from understanding its analytic properties.
- Hatsuda found a **much simpler** description through a **new alternative equation**

$$\left[f(z)\frac{d}{dz}f(z)\frac{d}{dz}+(2M\omega)^2-V(z)\right]\phi(z)=0 \quad \text{with} \quad f(z)=1-\frac{1}{z} \tag{66}$$

$$V(z)=f(z)\left[4c^2+\frac{4c(m-c)}{z}+\frac{{}_sA_{lm}+s(s+1)-c(2m-c)}{z^2}-\frac{s^2-1}{z^3}\right] \tag{67}$$

- The alternative equation (66) is **isospectral** to the original one. Besides, it has the same form as the master equations for spherically symmetric BHs if c and ${}_sA_{lm}$ are regarded as given parameters.
- Once moving to the $\mathcal{N}=2$ SYM, we have a **symmetry for flavor masses** respected by the **QNM spectrum**. However, the same symmetry is **not manifest** at the level of differential equations.
- The **physical origin** of equation (66) in BH perturbation theory is **not known**.

General modal stability problem for Kerr(-dS) BHs

- As **Kerr(-de Sitter)** black holes are stationary spacetimes, they correspond to equilibrium states for Einsten equations, and one would like to **determine** whether they are **stable or unstable equilibria**.
- Outside very special classes, stability of Kerr(-de Sitter) is an **open question**.
- If Kerr-(dS) BHs are nonlinearly stable, the most basic statement one can hope to prove for Teukolsky equation is that it is **modally stable**, i.e. that there are no separable solutions to it which are exponentially growing or bounded and non-decaying in time. By separable solution we mean a solution of the form

$$\alpha^{[s]}(t,r,\theta,\phi) = e^{-i\omega t} e^{im\phi} S(\theta) R(r)\,, \tag{68}$$

  where S and R satisfy, respectively, the angular and radial Teukolsky ODE.
- For Kerr-(dS) BHs there are **many possible sources of mode instabilities** (such as *superradiance*).
  - It is remarkable that, in the $\Lambda = 0$ **case, mode stability holds for the entire Kerr black hole family** with angular momentum $|a| \leq M$.
  - In the $\Lambda > 0$ **Kerr-dS** case modal stability remains an **open problem** for general parameters.

New result from $\mathcal{N} = 2$ SYM: proofs of mode stability for Kerr(-dS) BHs

- **Thanks to the correspondence with $\mathcal{N} = 2$ theories** discovered first by Aminov, Grassi Hatsuda, Casals and Costa gave
  1. a new alternative proof of the classic mode stability result for Kerr BHs
  2. a **completely new (though partial in the parameter range) proof of mode stability for Kerr-dS BHs**.
- The proofs of Casals-Costa makes use of **previously unknown symmetries of the point spectrum** of the radial Teukolsky equation conjectured by Aminov-Grassi-Hatsuda and proven by Casals-Costa.
- The second proof of Casals-Costa is most useful in the case $s \in \mathbb{Z}$, where it rules out some of the modes in the superradiant range, where it is known from energy conservation that unstable and non-decaying modes, if they exist, must lie.

New result from 2D CFT: greybody factor

- The emission spectrum as measured by an observer at infinity is no longer thermal, but is given by

$$\frac{\sigma(\omega)}{\exp\frac{\omega - m\Omega}{T_H} - 1} \tag{69}$$

where $\sigma(\omega)$ is the so-called **greybody factor**, which can be defined as the flux going into the horizon normalized by the flux coming in from infinity $\sigma = \phi_{abs}/\phi_{in}$.

- From the **solution of the connection problem** for the radial Teukolsky / confluent Heun equation, Bonelli, Iossa, Lichtig and Tanzini obtain an exact expression of the **greybody factor** σ in terms of the instanton part of the **NS free energy** $\mathcal{F}^{inst}$.
  - Their result can be expressed in terms of only **BH parameters** by using **Matone's relation**.
  - They **checked their result with two different asymptotics**. One of them is the semiclassical result obtained by **WKB analysis of the ODE**, where it is given by the **dual SW period**

$$\sigma \simeq \exp\left\{-\frac{a_D}{\epsilon_1}\right\}, \qquad \epsilon_1 \to 0\,. \tag{70}$$

New result from 2D CFT: Love numbers

- Applying an **external gravitational field to a self-gravitating body** generically causes it to deform. The response of the body to the external gravitational tidal field is captured by the so-called **tidal response coefficients** or **Love numbers**.
- In general relativity, the tidal response coefficients are generally complex, and the **real part captures the conservative response** of the body, whereas the **imaginary part captures dissipative effects**. For four-dimensional Kerr black holes, the conservative (real part) of the response coefficient to static external perturbations has been found to vanish.
- Love numbers are **measurable quantities** that can be probed with **gravitational wave observations**. [?]
- **Using their 2D CFT approach** to the Teukolsky equation Bonelli, Iossa, Lichtig, Tanzini compute
  - the **static** $\omega = 0$ **Love number** which agrees with the result in the literature;
  - the **non-static Love number** as an expansion in ω, that improves the result in the literature by also adding instanton corrections.

Thank you

for your attention!