Lower-dimensional BPS sectors in 4d SCFTs

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Introduction

4d SCFTs observables can be related to lower-dimensional systems via supersymmetric or omega deformations and/or compactifications/reductions on e.g. S_1 or T_2.

AGT duality, ...

Today...

At zero-temperature, the thermal partition function of 4d SCFTs $\longrightarrow$ asymptotic expansion of lower dimensional topological (superselection) sectors (quantum phases).

What is the motivation to study these asymptotic expansions -- and try to understand "their completion"--?

They offer ways of improving our understanding of holography

Motivation

Asymptotic expansions → expansions of the conjectured Euclidean Gravitational path integral

Superselection sectors (α) → saddles of the "Euclidean Gravitational Path integral"

Topological degrees of freedom (quantum phases)→ "near horizon (stringy hairs) excitations around the gravitational saddles"

A concrete set up

SU(N) $\mathcal{N} = 4$ SCFT on $\mathbb{R} \times S_3 \rightarrow$ Strings on $AdS_5 \times S_5$

Start with the thermal partition function: $Z[\beta, \Omega, \Phi]$

Pick up a $\frac{1}{16}$ – BPS sector:

$$Q\,,\ Q^{\dagger} = S$$

Key: Semi-positivity condition: $\left\{ Q\,, Q^{\dagger} \right\} = E + J + Q \geq 0$

A "zero-temperature" observable

$$Z := \mathrm{Tr}_{\mathrm{Hilb}(S_3)}\, e^{-\beta\,(E+J+Q)}\, e^{-\beta\,(\Omega-1)\,J}\, e^{-\beta\,(\Phi-1)\,Q}$$

$$-\beta\,(\Omega - 1) \to 2\pi\, \mathrm{i}\,(\tau \pm 1), \quad -\beta\,(\Phi - 1) \to -2\pi\, \mathrm{i}\,\tau,$$

$$\tau = \tau_1 + \mathrm{i}\,\tau_2, \quad \tau_2 > 0$$

[CB, Cassani, Martelli, Murthy, 18]

Zero-temperature limit (given the semi-positivity condition on E + J + Q)

$$Z \underset{\beta\to\infty,\ \text{fixed}\ \tau}{\longrightarrow} \underset{\text{or}\ \frac{1}{16}-\text{BPS}}{\mathrm{Tr}_{\mathrm{Hilb}(S_3)}}\ (-1)^F\, e^{2\pi i \tau\left(Q+\frac{J}{2}\right)} =: I(\tau)$$

The index

The index equates to an N-dimensional integral

$$I(\tau) \;=\; \# \int_0^1 du_1 \ldots \int_0^1 du_N \; e^{-S_{\text{eff}}(u;\tau)}$$

$$e^{-S_{\text{eff}}(u;\tau)} := \prod_{I \in \text{matter}} \prod_{i,j=1}^{N} \Gamma_e(u_{ij} + \Delta_I; \tau, \tau)$$

$$\Gamma_e(z;\tau,\tau) := \prod_{n=0}^{\infty} \frac{\left(1 - e^{-2\pi i z + 2\pi i (n+2)\tau}\right)^{n+1}}{\left(1 - e^{2\pi i (z + n\tau)}\right)^{n+1}} \quad = \Gamma_e(z+1;\tau,\tau)$$

(Elliptic Gamma function)

Again, the scope is ...

$I(\tau)$ $\rightarrow$ asymptotic sums over superselection sectors α
(→ "zero-temperature topological phases")

(i) The top. order emerges in the limits $\tau \rightarrow -\frac{n}{m}$

(ii) The α-sectors themselves are "sums" and "products" of partition functions of 3d Chern -Simons and A-models)

The strategy

There are two ways to proceed:

Solve $I(\tau)$ and expand it in $\tau \to -\frac{n}{m}$

→ Expand the integrand of $I(\tau)$ and integrate to recover an asymptotic expansion for $I(\tau)$ ←

Today

→ Expand the integrand of $I(\tau)$ and integrate to recover an asymptotic expansion for $I(\tau)$ ←

$$I(\tau) \underset{\tau \to -\frac{n}{m}}{\longrightarrow} \sum_\alpha e^{\pi i \frac{P_3^{(\alpha)}(\tau)}{m(m\tau+n)^2}} \times Z^{(\alpha)}{}_{\text{top}}$$

$$Z^{(\alpha)}{}_{\text{top}} = O((m\tau + n)^0)$$

The expectation is that one can recover the full $I(\tau)$ from the asymptotic expansions around any of the limits $\tau \to -\frac{n}{m}$.

A step back

That expectation comes from solving first $I(\tau)$ and expand it in $\tau \to -\frac{n}{m}$ (equivariant integration = Bethe ansatz formula)

[Closset, Kim, Willet; Benini, Milan] [CB 22, to appear]

$$I(\tau) = \sum^{N-1}_{\substack{(p=0,q=0)\\ \gcd(p,q)=1\\ (p,q)\neq(0,0)}} I^{(p,q)}(\tau) + \sum_{\alpha_{\text{cont}}} I^{(\alpha_{\text{cont}})}(\tau)$$

$$I^{(\dots)}(\tau) \underset{\tau \to -\frac{n}{m}}{\to} e^{\pi i \frac{P_3{}^{(\dots)}(\tau)}{m\,(m\,\tau+n)^2}} \times O((m\,\tau+n)^0)$$

Educated guess: α ~ Bethe roots Checked for $N = 2$ [CB 21]

However, in this talk...

→ Expand the integrand of $I(\tau)$ and integrate to recover an asymptotic expansions for $I(\tau)$ ←

$$I(\tau) \underset{\tau \to -\frac{n}{m}}{\to} \sum_\alpha e^{\pi i \frac{P_3^{(\alpha)}(\tau)}{m(m\tau+n)^2}} \times Z^{(\alpha)}{}_{\text{top}}$$

$$Z^{(\alpha)}{}_{\text{top}} = O((m\tau + n)^0)$$

The expectation is that one can recover the full $I(\tau)$ from the asymptotic expansions around any of the limits $\tau \to -\frac{n}{m}$.

“SL(2,$\mathbb{Z}$) - closed” set of identities

[CB, Murthy 19][CB 21]

For any two co-primes *(m, n)* with $m \geq 1$

$$\Gamma_e(z;\tau,\tau) = e^{\pi i \frac{P_3^{(m,n)}(z,\tau)}{m(m\tau+n)^2}} e^{\pi i L^{(m,n)}(z,\tau)}$$

Piecewise rational in z and τ : $\pi i \frac{P_3^{(m,n)}(z,\tau)}{m(m\tau+n)^2}$

Transcendental in z and τ : $L^{(m,n)}(z,\tau)$

On the transcendental part

[CB, Murthy 19][CB 21]

Just to give a schematic idea of the transcendental part:

$$L^{(m,n)}(z,\ \tau) \sim \sum^{\infty}_{j=1} \frac{1}{j\,\mathrm{Sin}\left[\frac{\pi j}{m\tau+n}\right]}\,\mathrm{Cos}\left[\frac{\pi j\,[2z+1]}{m\tau+n}\right],$$

$$\sum^{\infty}_{j=1} \frac{z}{j\,\mathrm{Sin}\left[\frac{\pi j}{m\tau+n}\right]}\,\mathrm{Cos}\left[\frac{\pi j\,[2z+1]}{m\tau+n}\right],$$

$$\sum^{\infty}_{j=1} \frac{1}{j^2\,\mathrm{Sin}\left[\frac{\pi j}{m\tau+n}\right]}\,\mathrm{Cos}\left[\frac{\pi j\,[2z+1]}{m\tau+n}\right],\ \ldots.$$

$$2z+1 = x_1 + x_2\,(m\tau+n) \qquad x_{1,2} \in \mathbb{R}$$

$$[2z+1] := x_1 - \mathrm{floor}[x_1] + \ldots \qquad \{x_1\} = 1^{-}\ !$$

Limits to roots of unity $\tau \to -\frac{n}{m}$

[CB 21]

$$\mathrm{Re}\left[\pi\, i\, \frac{P_3{}^{(m,n)}(u+\Delta,\tau)}{m\,(m\,\tau+n)^2}\right] \underset{\tau\to-\frac{n}{m}}{\nearrow}$$

$$L^{(m,n)}(u+\Delta,\ \tau) \underset{\tau\to-\frac{n}{m}}{\rightarrow} 0$$ for $u \in \mathbb{C}$ except for.........

.... at $\frac{1}{2}$ - dimensional walls || to $(m\,\tau + n) \in \mathbb{C}$

These walls intersect the segment [0,1] at m regularly distributed points which we will denote by $l = 0, ..., m\text{-}1$.

The walls of non-analyticity

[CB 21]

“The profile” of the transcendental function L along the walls:

$\mathrm{Li}_1[\mathrm{Exp}[2\pi I(\pm v+\Delta_2)]]$, v $\mathrm{Li}_1[\mathrm{Exp}[2\pi I(\pm v+\Delta_2)]]$,
$\mathrm{Li}_2[\mathrm{Exp}[2\pi I(\pm v+\Delta_2)]]$,

v := $\frac{u-p}{m\tau+n}$ is the natural coordinate along the wall.

A toy example

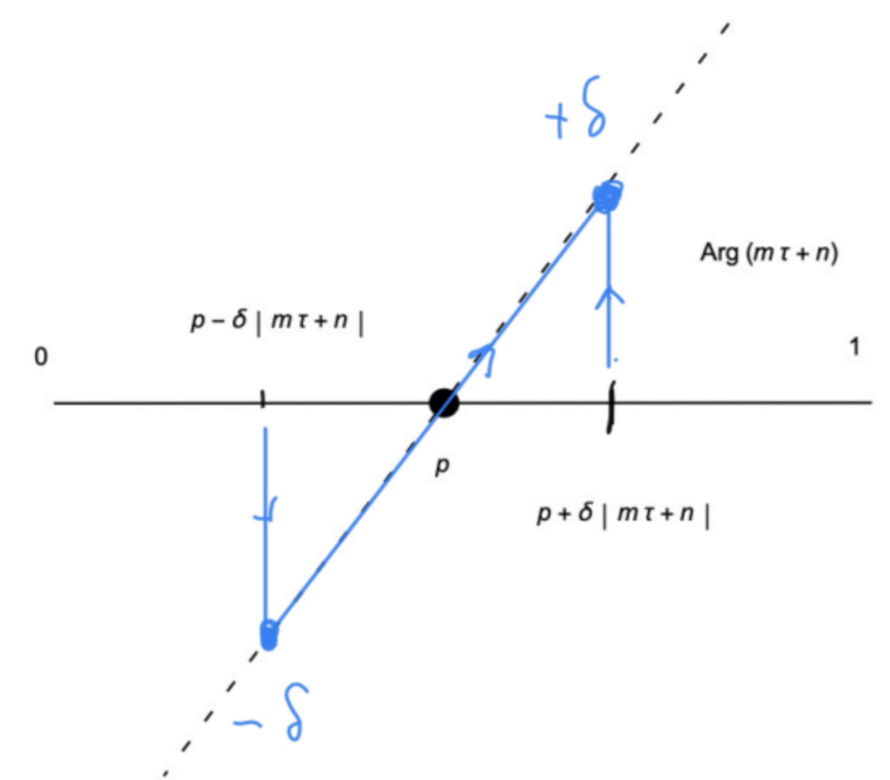
[CB 21]

$$\int_0^1 du\, \Gamma_e(u+\Delta;\tau)\,\Gamma_e(-u+\Delta;\ \tau)\ \ (=: \Gamma_e(\pm u+\Delta;\tau))$$

Splitting the contour of integration

[CB 21] [Similar construction presented in Ardehali, Hong 21]

We split the contour of integration into small segments covering the intersection between walls of non-analyticities



In equations...

[CB 21] [Similar construction presented in Ardehali, Hong 21]

$$\int_0^1 du \ \ldots \underset{\tau \to -\frac{n}{m}}{\to}$$

$$\int_{p-\delta}^{p+\delta} dv\, \Gamma_e\left(\pm p \pm v\,(m\,\tau+n)\right) + \Delta) \ + \left(\int_0^{p-\delta|m\,\tau+n|} + \int_{p-\delta|m\,\tau+n|}^{1}\right) du \ \ldots$$

$$\underset{\substack{\tau \to -\frac{n}{m} \\ \delta \to \infty}}{\to}$$

$$(I_{\text{Walls}} :=) \int_{-\infty}^{\infty} dv \ e^{\text{Piecewise Rat.} + \text{Transcendental}} +$$

(Bulk:=) Gaussian integral

Computing I_{Walls} (Vectors)

[Gonzalez Lezcano, Hong, Liu, Pando Zayas, 20] [Amariti, Fazzi, Segati, 21] [Cassani, Komargodski, 21][Ardehali, Murthy, 21][CB, 21]

For $\Delta = 2\tau \bmod 1$

$$I_{\text{Walls}}{}^{(m,n)} = \exp[\pi i \frac{P_{3\,\text{vector}}[\tau]}{m\,(m\tau+n)^2}] \quad O((m\tau+n)^0)$$

For $n = m - 1 = 0$

$$O((m\tau+n)^0) \to \int_{-\infty}^{+\infty} dv \quad e^{\pi i k_{\text{eff}} \frac{v^2}{2} - \text{Li}_1\left(e^{2\pi i(\pm v+\Delta_2)}\right)}$$

$\propto$ SU(2) Chern-Simons path integral on S_3

Other (m,n)'s $\to$ Lens spaces [Ardehali, Murthy, 21]

Computing I_{Walls} (Chirals)

[CB 21, and work to appear]

For generic Δ

$$I_{\text{walls}}^{(m,n)} := \exp[\pi \, \mathrm{i} \, \frac{P_{3\,\text{chiral}}[\tau]}{m\,(m\,\tau+n)^2}] \; \mathrm{O}((m\,\tau+n)^0)$$

$$O\left((m\,\tau+n)^0\right) := \sum^{m-1}{}_{l=0} \int_{\Gamma_l} \mathrm{dv} \;\; e^{2\pi i\,(g-1)\,\Omega + 2\pi i\,p\,(\mathcal{W} - v\,\partial_v\,\mathcal{W} - (\Delta_2 - 1)\,\partial_{\Delta_2}\,\mathcal{W})}$$

The original theory reduces to a sum of lower-dimensional (quiver) theories over Riemann surfaces of genus *g* with *p* punctures (String theory interpretation?)!

Computing I_{Walls} (Chirals)

[CB 21, and work to appear]

Concretely: $1 \leq g = m - l \leq m$ and $p = m \geq 1$

$$2\pi i\, \Omega(v) = -\operatorname{Li}_1\left(e^{2\pi i(\pm v + \Delta_2)}\right)$$

$$2\pi i\, \mathcal{W}(v) = \frac{k_{\text{eff}}\, v^2}{2} + \operatorname{Li}_2\left(e^{2\pi i(\pm v + \Delta_2)}\right)$$

[Closset, Willet, Kim]

A-twisted 2d SU(2) (2,2) GLSM with twisted superpotential $\mathcal{W}$ and dilaton profile Ω placed over a Riemann surface of genus g with p punctures.

More complicated integrals e.g. $I(\tau)$

Involve products of many Γ_e's . That makes things interesting:

1) $\Delta_I = 2\tau \bmod 1$ → Chern-Simons

2) generic Δ_I → A-models

3) Non-abelian gauge groups N → $\otimes$'s of $\oplus$'s of contributions from walls and bulk contributions

Symmetry-breaking classification

The superselection sectors α are related to symmetry-breaking patterns (← contour decomposition)

$$\mathrm{SU(N)} \underset{\text{symm. breaking patterns}}{\longrightarrow} \prod \mathrm{SU}(N_i) \times \prod \mathrm{U(1)}$$

To each of the symmetry factors one can assign either

$\bigoplus_{l=0}^{m-1}$ walls, or bulk contributions

For example

$$\mathrm{I}(\tau) \to$$

$$\sum_{\lambda \in P(N-1)} e^{\pi i \frac{\hat{P}_3{}^{(\alpha)}(\tau)}{m\,(m\,\tau+n)^2}}$$

$$\prod_{\lambda_i} \left(\sum_{l=0}^{m-1} Z^{(l)}{}_{\mathrm{CS}}[\lambda_i] + \sum_{l=0}^{m-1} Z^{(l)}{}_{A}[\lambda_i] + \mathrm{Bulk} \right)$$

$$\to \qquad \sum_{\alpha} e^{\pi i \frac{P_3{}^{(\alpha)}(\tau)}{m\,(m\,\tau+n)^2}} \quad Z^{(\alpha)}{}_{\mathrm{top}}$$

$$Z^{(\alpha)}{}_{\mathrm{top}} := \prod_{p_\alpha} \left(\sum_{l=0}^{m-1} Z^{l}{}_{\mathrm{CS}}[p_\alpha] \right) \prod_{q_\alpha} \left(\sum_{l=0}^{m-1} Z^{l}{}_{A}[q_\alpha] \right)$$

$\alpha \;\; \leftrightarrow$ cubic polynomial (~ anomaly pol., central charges)

The labels of superselection sectors

$\{\alpha\} \rightarrow$ set of non-negative integers $\{\{\{p_\alpha\}, \{q_\alpha\}\}\}$ s.t.

$$\sum p_\alpha + \sum q_\alpha < N$$

For instance for N = 2 $\longrightarrow$

α = { {1,0} (vector) , {0,1} (chiral), {0,0} (bulk)}

In one-to-one relation with Bethe roots [CB 21]

Questions and comments

- Relation to the asymptotic expansions of the Bethe ansatz formula for N > 2 (tool to infer information about continuum families of saddles and subleading corrections)

- Relation to the giant graviton expansion [Lee, Gaiotto 21; Lee 22; Murthy 22].

- Do similar expansions exist for the ABJM index? [Benetti Genolini, CB, Murthy, to appear]

More ambitious question

Can these tools be useful to explain (from first principles) certain universal small-temperature properties of black holes?