Exact results in a $\mathcal{N} = 2$ SCFT at strong coupling

M.Beccaria, M.Billò, M.Frau, A.Lerda and AP arXiv:2105.1511 [hep-th]

M.Billò, M.Frau, F.Galvagno, A.Lerda and AP arXiv:2109.00559 [hep-th]

M.Billò, M.Frau, A.Lerda, AP and P.Vallarino arXiv:2202.06990 [hep-th]

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Introduction and motivations

4d SCFTs $\mathcal{N} = 4$, $\mathcal{N} = 2$ $\Rightarrow$ **Holographic** and **localization** techniques.

Mostly focus on SQCD or quiver gauge theories in the large N-limit

- BPS Wilson loop, partition function. [F. Passerini and K.Zarembo (2011)], [J.G. Russo and K.Zarembo (2012)], [M. Beccaria and A.A.Tsytlin (2021)]
- Correlators among Chiral and Anti-Chiral, Chiral and Wilson loops. [M. Baggio, V. Niarchos and K.Papadodimas (2014), (2015)], [F. Galvagno and M. Preti (2021)], [E. Gerchkovitz. J. Gomis, N. Ishtiaque, A. Karasik, Z. Komargodski and S. Pufu (2016)], [D. Rodríguez-Gómez and J.G. Russo (2016)],[B. Fiol and A.R. Fukelman (2019), (2020)]

For $\mathcal{N} = 2$ finding exact results at **strong coupling** is **difficult**

Introduction and motivations

Specific $4d$ $\mathcal{N}=2$ ("close" to $\mathcal{N}=4$) : type-E theory $\mathrm{AdS}_5 \times \mathbb{S}^5/\mathbb{Z}_2$

[I.G.Koh and S. Rajpoot (1984)]

New powerful tool: the X-**matrix**
[M. Beccaria, M. Billò, F. Galvagno, A. Hassan and A. Lerda (2020)]

Sum up the ⇓ perturbative series

Exact results in the 't Hooft limit

Contents of the talk

Contents of the talk

Type-E theory

[I.G.Koh and S. Rajpoot (1984)]

Matter content

$G = SU(N)$, $\mathcal{V}_{\mathcal{N}=2}$ Adj, $\mathcal{H}_{\mathcal{N}=2}$ Sym, $\mathcal{H}_{\mathcal{N}=2}$ Anti-Sym,

$4d\ \mathcal{N}=2$ $SU(2)_R \times U(1)_r$ $\beta = 0 \Rightarrow$ **CFT**

Chiral Operator

$$\varphi(x) \in \mathcal{V}_{\mathcal{N}=2} \quad \Rightarrow \quad \mathcal{O}_{\mathbf{n}}(x) \equiv \mathrm{tr}\varphi^{n_1}(x) \cdots \mathrm{tr}\varphi^{n_\ell}(x) \quad \mathbf{n} = \{n_1, \cdots, n_\ell\}$$

chiral C.P.O.

$\overline{\mathcal{Q}}^a_{\dot{\alpha}} \mathcal{O}_{\mathbf{n}}(x) = 0,$

Conformal dimension

$\Delta = n_1 + n_2 + \cdots + n_\ell$

Correlation functions

Extremal Correlator

$$\langle \mathcal{O}_{\mathbf{n_1}}(x_1) \cdots \mathcal{O}_{\mathbf{n}_k}(x_k) \overline{\mathcal{O}}_{\mathbf{m}}(y) \rangle = \frac{\mathcal{G}_{\mathbf{n_1},\cdots,\mathbf{n}_k;\mathbf{m}}}{(4\pi^2(x_1-y)^2)^{|\mathbf{n_1}|} \cdots (4\pi^2(x_k-y)^2)^{|\mathbf{n}_k|}}$$

$U(1)_r$ conservation $\Rightarrow$ $\mathcal{G}_{\mathbf{n_1},\cdots,\mathbf{n}_k;\mathbf{m}}(\lambda) \propto \delta_{|\mathbf{n_1}|+\cdots+|\mathbf{n}_k|;|\mathbf{m}|}$

[E. Gerchkovitz. J. Gomis, N. Ishtiaque, A. Karasik, Z. Komargodski and S. Pufu (2016)]

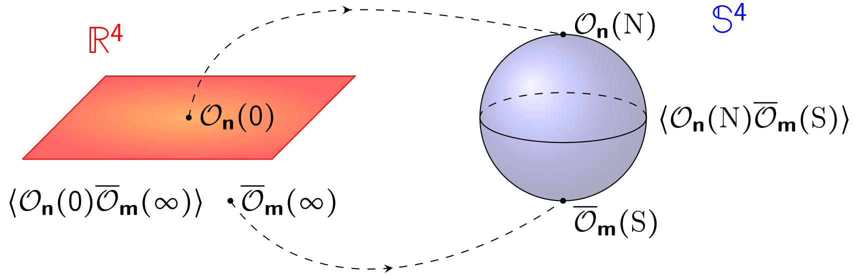
We focus on

$$\langle \mathcal{O}_n(x) \overline{\mathcal{O}_n}(y) \rangle \qquad \langle \mathcal{O}_{n_1}(x_1) \mathcal{O}_{n_2}(x_2) \overline{\mathcal{O}}_{n_1+n_2}(y) \rangle$$

Contents of the talk

Localization

[E. Gerchkovitz. J. Gomis, N. Ishtiaque, A. Karasik, Z. Komargodski and S. Pufu (2016)]



Large N-limit

Matrix Model $a \equiv a_b T^b$ [V.Pestun (2012)]

$$\mathcal{Z} = \int da\ \mathrm{e}^{-\mathrm{tr}a^2 - S_{\mathrm{int}}(a) - \cancel{S_{\mathrm{inst}}(a)}}\ , \langle f(a)\rangle = \mathcal{Z}^{-1}\int da\ f(a)\ \mathrm{e}^{-\mathrm{tr}a^2 - S_{\mathrm{int}}(a)}$$

[M. Billò, F. Fucito, A. Lerda, J.F. Morales, Ya.S. Stanev and Congkao Wen (2017)]

Gram-Schmidt Orthogonalization

Mixing with lower dimensional operators

$$\mathcal{O}_\Delta \ \rightarrow \ \mathcal{O}_\Delta + c_1\mathcal{O}_{\Delta-2} + c_2\mathcal{O}_{\Delta-4} + \cdots$$

$\mathbb{S}^4$ **Matrix Model** ⇒ $\mathbb{R}^4$ **CFT**

Orthogonalization procedure

$$\mathcal{O}_{\mathbf{n}} = \Omega_{\mathbf{n}} - \sum_{|\mathbf{m}|<|\mathbf{n}|} \mathcal{C}_{\mathbf{n}}^{\mathbf{m}} \mathcal{O}_{\mathbf{m}}, \qquad \Omega_{\mathbf{n}} \equiv \mathrm{tr}\, a^{n_1} \mathrm{tr}\, a^{n_2} \cdots \mathrm{tr}\, a^{n_k}$$

$$\langle \mathcal{O}_{\mathbf{n}} \mathcal{O}_{\mathbf{m}} \rangle = 0 \quad \forall \quad |\mathbf{m}| < |\mathbf{n}| \quad \Rightarrow \quad \mathcal{C}_{\mathbf{n}}^{\mathbf{m}}$$

[E. Gerchkovitz. J. Gomis, N. Ishtiaque, A. Karasik, Z. Komargodski and S. Pufu (2016)]

Type-E theory properties [M. Beccaria, M. Billò, F. Galvagno, A. Hassan and A. Lerda (2020)]

$$S_{\rm int}(a) = \sum_{\ell,m=1}^{\infty} g_{\ell,m}(\lambda)\ \boxed{\mathrm{tr}\,a^{2\ell+1}\,\mathrm{tr}\,a^{2m+1}}$$

only odd powers

Large N ⇓ limit

$$\mathcal{Z} = \int da\ \mathrm{e}^{-\mathrm{tr}\,a^2 - S_{\rm int}(a)} = \int D\omega\ \mathrm{e}^{-\frac{1}{2}\omega^T(\mathbb{1}-\mathrm{X})\omega} = \det{}^{-\frac{1}{2}}(\mathbb{1}-\mathrm{X})$$

X-matrix

$$(\mathrm{X})_{i,j} \propto \int_0^\infty \frac{dt}{t}\frac{\mathrm{e}^t}{(\mathrm{e}^t-1)^2}\ \boxed{J_{2i+1}(t\sqrt{\lambda})\ J_{2j+1}(t\sqrt{\lambda})}$$

convolution

2-point function

[M. Beccaria, M. Billò, M. Frau, A. Lerda and AP (2021)]

Explicit expression

$$\langle \mathrm{tr} a^{2k+1} \mathrm{tr} a^{2\ell+1} \rangle \simeq \left(\frac{N}{2}\right)^{k+\ell+1} \sum_{i=0}^{k-1} \sum_{j=0}^{\ell-1} c_{k,i} c_{\ell,j} \mathrm{D}_{k-i,\ell-j}(\lambda)$$

$$\mathrm{D}_{k,\ell}(\lambda) \equiv \delta_{k,\ell} + \mathrm{X}_{k,\ell} + \mathrm{X}^2_{k,\ell} + \mathrm{X}^3_{k,\ell} + \cdots$$

Weak coupling $\lambda << 1$

- Small λ-expansion of the Bessel functions $\Rightarrow$ analytic computation of the integral over t.
- Very **efficient way** to generate the perturbative expansion.

Sum up the series ⇒ heuristic argument

$$\delta_{k,\ell} + \mathrm{X}_{k,\ell} + \mathrm{X}^2_{k,\ell} + \mathrm{X}^3_{k,\ell} + \cdots \text{ “ = ” } \left(\frac{1}{\cancel{\mathbb{1}} - \mathrm{X}(\lambda)} \right)_{k,\ell} \underset{\lambda\to\infty}{\propto} \frac{1}{\lambda}$$

$\lambda >> 1$ [M. Beccaria, A. A. Tseytlin and G.V.Dunne (2021)], [Y.Ikebe, Y. Kikuchi and I. Fujishiro (1991)]

Strong coupling $\lambda >> 1$

$$\langle \mathrm{tr} a^{2k+1} \mathrm{tr} a^{2\ell+1} \rangle \simeq \frac{4\pi^2}{\boxed{\lambda}} \left(\frac{N}{2}\right)^{k+\ell+1} \frac{(2k+1)!}{k!(k-1)!} \frac{(2\ell+1)!}{\ell!(\ell-1)!} \frac{1}{k+\ell}$$

Building blocks ⇒ 2-point and 3-point functions on $\mathbb{R}^4$

Contents of the talk

Numerical checks

[M. Beccaria, M. Billò, M. Frau, A. Lerda and AP (2021)]

$$\frac{\langle \mathcal{O}_n \overline{\mathcal{O}}_n \rangle}{\langle \mathcal{O}_n \overline{\mathcal{O}}_n \rangle_0} = \sum_k d_k \left(\frac{\lambda}{\pi^2} \right)^k \quad \text{for} \quad |\lambda| < \lambda_c$$

Radius of convergence $\lambda_c = \pi^2 \quad \Rightarrow \quad$ Behaviour outside λ_c ??

[M. Beccaria, M. Billò, F. Galvagno, A. Hassan and A. Lerda (2020)]

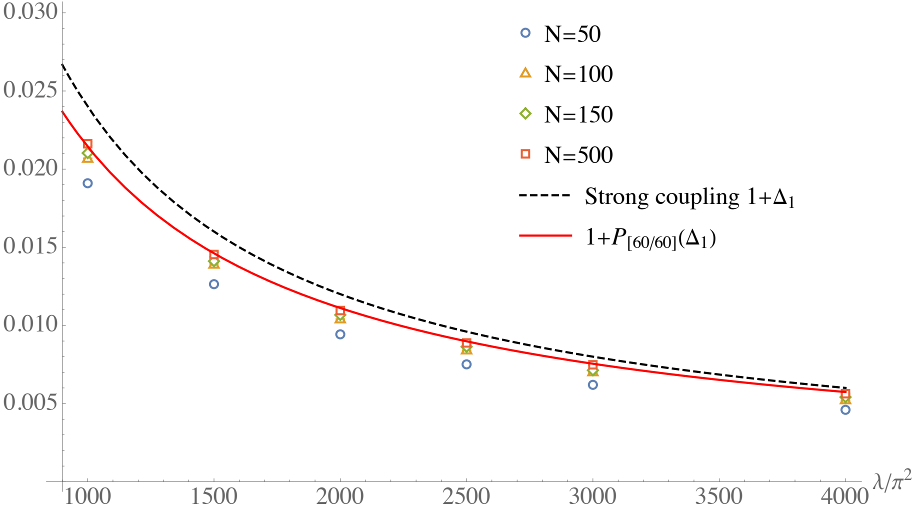
Two **independent** numerical methods

1 Padé approximant $P_{[N,M]}$ 2 MCMC simulation

Numerical checks

[M. Beccaria, M. Billò, M. Frau, A. Lerda and AP (2021)]

$$\gamma_3 \equiv \frac{\langle \mathcal{O}_3 \overline{\mathcal{O}}_3 \rangle}{\langle \mathcal{O}_3 \overline{\mathcal{O}}_3 \rangle_0} \simeq 1 + \Delta_1(\lambda)$$



Numerical checks

[M. Beccaria, M. Billò, M. Frau, A. Lerda and AP (2021)]

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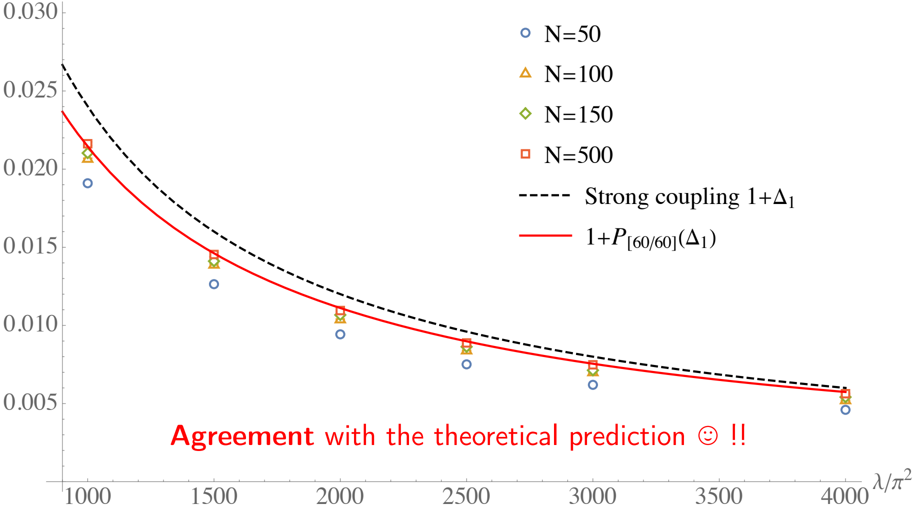


Agreement with the theoretical prediction ☺ !!

$$\langle \mathcal{O}_n(x)\overline{\mathcal{O}}_n(y)\rangle = \frac{G_n}{(4\pi^2(x-y)^2)^n},$$

$$\langle \mathcal{O}_{n_1}(x_1)\mathcal{O}_{n_2}(x_2)\overline{\mathcal{O}}_{n_1+n_2}(y)\rangle = \frac{G_{n_1,n_2}}{(4\pi^2(x_1-y)^2)^{n_1}(4\pi^2(x_2-y)^2)^{n_2}}$$

2-point functions

$\mathcal{N}=4$

$$G_{2k} \simeq G_{2k}^{(0)}, \qquad G_{2k+1} \simeq G_{2k+1}^{(0)}\frac{4\pi^2}{\lambda}2k(2k+1)$$

3-point functions

$\mathcal{N}=4$

$$G_{2k,2\ell+1} \simeq G^{(0)}_{2k,2\ell+1}\,\frac{16\pi^2}{\lambda}\,\ell(k+\ell),$$

$$G_{2k,2\ell} \simeq G^{(0)}_{2k,2\ell}, \qquad G_{2k+1,2\ell+1} \simeq G^{(0)}_{2k+1,2\ell+1}\,\frac{16\pi^2}{\lambda}\,k\ell$$

Normalized operator $\hat{\mathcal{O}}_n(x) \equiv \mathcal{O}_n(x)/\sqrt{G_n}$

$$\langle \hat{\mathcal{O}}_n(x)\bar{\hat{\mathcal{O}}}_n(y)\rangle = \frac{1}{(4\pi^2(x-y)^2)^n},$$

$$\langle \hat{\mathcal{O}}_{n_1}(x_1)\hat{\mathcal{O}}_{n_2}(x_2)\bar{\hat{\mathcal{O}}}_{n_1+n_2}(y)\rangle = \frac{\hat{G}_{n_1,n_2}}{(4\pi^2(x_1-y)^2)^{n_1}(4\pi^2(x_2-y)^2)^{n_2}}$$

Analytic results

[M. Billò, M. Frau, A. Lerda, AP and P.Vallarino (2022)]

Normalized coefficient

$$\hat{G}_{n_1,n_2} = \frac{G_{n_1,n_2}}{\sqrt{G_{n_1}\,G_{n_2}\,G_{n_1+n_2}}}$$

$\hat{\mathcal{O}}_{2k}(x) \to$ **Untwisted** sector Δ_U, $\hat{\mathcal{O}}_{2k+1}(x) \to$ **Twisted** sector Δ_T

$$C_{U_1,U_2,U_3} \simeq \frac{1}{N}\sqrt{\Delta_{U_1}\Delta_{U_2}\Delta_{U_3}},\;\; C_{U_1,T_2,T_3} \simeq \frac{1}{N}\sqrt{\Delta_{U_1}(\Delta_{T_2}-1)(\Delta_{T_3}-1)}$$

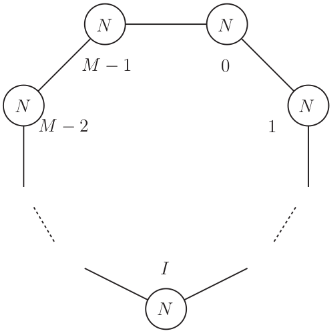
$\mathcal{N} = 4$ behaviour

New strong coupling result

[S.Lee, S. Minwalla, M. Rangamani and S. Seiberg (1998)]

Contents of the talk

The $\mathcal{N} = 2$ quiver theory



- $I = 0, \cdots, M-1$
- node $\mapsto$ $\mathcal{V}^{(I)}_{\mathcal{N}=2}$ $SU(N)_I$
- line $\mapsto$ $\mathcal{H}_{\mathcal{N}=2}$

Single trace chiral operators

$$\Phi_I \in \mathcal{V}^{(I)}_{\mathcal{N}=2} \qquad O^{(I)}_n(\vec{x}) \equiv \mathrm{tr}\ \Phi_I(\vec{x})^n$$

$\mathcal{N} = 2$ quiver - operators

[M. Billò, M. Frau, F. Galvagno, A. Lerda and AP (2021)]

Untwisted operators

$$U_n(\vec{x}) \equiv \frac{1}{\sqrt{M}} \sum_{I=0}^{M-1} O_n^{(I)}(\vec{x}),$$

Twisted operators

$$T_{\alpha,n}(\vec{x}) \equiv \frac{1}{\sqrt{M}} \sum_{I=0}^{M-1} \rho^{-\alpha I} O_n^{(I)}(\vec{x}) \qquad \rho \equiv e^{\frac{2\pi i}{M}}$$

Normalized coefficients

$$C_{U_1,U_2,U_3} \simeq \frac{\sqrt{\Delta_{U_1}\Delta_{U_2}\Delta_{U_3}}}{N\sqrt{M}}, \quad C_{U_1,T_2,T_3} \simeq \frac{\sqrt{\Delta_{U_1}(\Delta_{T_2}-1)(\Delta_{T_3}-1)}}{N\sqrt{M}}$$

[M. Billò, M. Frau, A. Lerda, AP and P.Vallarino (TO APPEAR 2022)]

Contents of the talk

Conclusions and outlook

- Exploiting the properties of the X-matrix we summed up the perturbative series.
- We found the explicit expressions of 2-point and 3-point functions in the 't Hooft limit for the type-E theory.
- Generalization of these results for new class of 4d $\mathcal{N} = 2$ theories. Circular quiver $G = SU(N)^M$.
- Known gravity dual $\mathrm{AdS}_5 \times \mathbb{S}^5/\mathbb{Z}_M \Rightarrow$ holographic analysis. [M. Billò, M. Frau, A. Lerda, AP and P.Vallarino (TO APPEAR 2022)]

THANKS FOR YOUR ATTENTION

Example 2-point function

[M. Beccaria, M. Billò, M. Frau, A. Lerda and AP (2021)]

Example - weak coupling $\lambda << 1$ expansion

$$\langle \mathrm{tr} a^3 \mathrm{tr} a^3 \rangle \simeq \frac{3N^3}{8} \mathrm{D}_{1,1} = \frac{3N^3}{8} \left(1 - \frac{5\lambda^3 \zeta(5)}{256\pi^6} + \frac{105\lambda^4 \zeta(7)}{4096\pi^8} - \frac{1701\lambda^5 \zeta(9)}{65536\pi^{10}} + \left(\frac{25\zeta(5)^2}{65536\pi^{12}} + \frac{12705\zeta(11)}{524288\pi^{12}} \right) \lambda^6 + \cdots + O\left(\lambda^{160}\right) \right)$$

Very high order of the series expansion

The X-matrix

[M. Beccaria, M. Billò, F. Galvagno, A. Hassan and A. Lerda (2020)]

The interaction action $S_{\rm int}(a)$

$$4\sum_{\ell,m=1}^{\infty}(-1)^{\ell+m}\left(\frac{g^2}{8\pi^2}\right)^{\ell+m+1}\frac{(2\ell+2m+1)!}{(2\ell+1)!(2m+1)!}\zeta(2\ell+2m+1)\mathrm{tr}a^{2\ell+1}\mathrm{tr}a^{2m+1}$$

The matrix element $X_{k,\ell}$

$$-8(-1)^{k+\ell}\sqrt{(2k+1)(2\ell+1)}\int_0^{\infty}\frac{dt}{t}\frac{e^t}{(e^t-1)^2}J_{2k+1}\left(\frac{t\sqrt{\lambda}}{2\pi}\right)J_{2\ell+1}\left(\frac{t\sqrt{\lambda}}{2\pi}\right)$$

The full Lie algebra approach [B. Fiol, J. Martinez-Montoya and A. Rios Fukelman (2019)]

[M. Beccaria, M. Billò, F. Galvagno, A. Hassan and A. Lerda (2020)]

$$\mathcal{Z}_{\mathbb{S}^4} = \int \prod_{u=1}^{N} dm_u \; \Delta(m) |Z(im, g)|^2 \delta\left(\sum_u m_u\right) = \int dM \; \mathrm{e}^{-S(M)} \delta(\mathrm{tr} M)$$

Hermitean traceless matrix M with eigenvalues m_u

We introduce the matrix a, $\quad \mathrm{tr}\; T_b T_c = \frac{1}{2}\delta_{bc} \quad b, c = 1, \cdots, N^2 - 1$

$$a \equiv \sqrt{\frac{8\pi^2}{g^2}} M \quad \Rightarrow \quad \mathcal{Z}_{\mathbb{S}^4} = \left(\frac{g^2}{8\pi^2}\right)^{\frac{N^2-1}{2}} \int da \; \mathrm{e}^{-\mathrm{tr} a^2 - S_{\mathrm{int}}(a)}$$

The full Lie algebra approach [B. Fiol, J. Martinez-Montoya and A. Rios Fukelman (2019)]

$$da \equiv \prod_b \frac{da^b}{\sqrt{2\pi}} \qquad \langle f(a) \rangle_{(0)} \equiv \int da \; \mathrm{e}^{-\mathrm{tr} a^2} f(a)$$

Expectation value in the interacting model

$$\langle f(a) \rangle \equiv \frac{\int da \; f(a) e^{-\mathrm{tr} a^2 - S_{\mathrm{int}}(a)}}{\int da \; e^{-\mathrm{tr} a^2 - S_{\mathrm{int}}(a)}} = \frac{\left\langle f(a) \mathrm{e}^{-S_{\mathrm{int}}(a)} \right\rangle_{(0)}}{\left\langle \mathrm{e}^{-S_{\mathrm{int}}(a)} \right\rangle_{(0)}}$$

The free-variables representation in the large N-limit

single trace operator on $\mathbb{R}^4$ for $\mathcal{N} = 4$ SYM

$$\langle O_n^{(0)} O_m^{(0)} \rangle_{(0)} = n \left(\frac{N}{2} \right)^n \delta_{n,m} \equiv G_n^{(0)} \delta_{n,m}$$

[M. Beccaria, M. Billò, F. Galvagno, A. Hassan and A. Lerda (2020)]

the free variables ω

$$\omega_i(a) \equiv \frac{O_{2i+1}^{(0)}(a)}{\sqrt{G_{2i+1}^{(0)}}} \qquad \Rightarrow \qquad \langle \omega_i(a) \omega_j(a) \rangle_{(0)} = \delta_{i,j}$$

The free-variables representation in the large N-limit

[M. Beccaria, M. Billò, F. Galvagno, A. Hassan and A. Lerda (2020)]

Wick's theorem for odd correlation function ⇒

$$\langle \omega_{i_1}(a)\omega_{i_2}(a)\cdots\omega_{i_n}(a)\rangle_{(0)} = \int D\omega\ \omega_{i_1}\omega_{i_2}\cdots\omega_{i_n}\mathrm{e}^{-\frac{1}{2}\omega^T\omega} \qquad D\omega \equiv \prod_{i=1}^{\infty}\frac{d\omega_i}{\sqrt{2\pi}}$$

interacting theory

$$S_{\mathrm{int}} = -\frac{1}{2}\omega^T \mathrm{X}\omega$$

$$\langle f(\omega)\rangle = \frac{1}{\mathcal{Z}}\int D\omega\ f(\omega)\mathrm{e}^{-\frac{1}{2}\omega^T(\mathbb{1}-\mathrm{X})\omega}, \qquad \mathcal{Z} = \det{}^{-\frac{1}{2}}(\mathbb{1}-\mathrm{X})$$

3-point function at Large-N

[M. Baggio, V. Niarchos and K.Papadodimas (2015)]

Planar limit $\Rightarrow$ Gram-Schmidt **simpler** set **single**-trace $\{O_n\}$

$$O_n = \Omega_n - \sum_{m<n} C_{n,m} O_m$$

$$\mathcal{O}_n = O_n + \frac{1}{N}(\text{single and multi trace})$$

$\mathcal{O}_n$: orthogonal to **all** lower

O_n: orthogonal **only** to single-trace

3-point

$$G_{n_1,n_2} = \langle \mathcal{O}_{n_1} \mathcal{O}_{n_2} \mathcal{O}_{n_1+n_2} \rangle = \langle O_{n_1} O_{n_2} O_{n_1+n_2} \rangle + O(1/N)$$

Results at strong coupling 1

[M. Beccaria, M. Billò, M. Frau, A. Lerda and AP (2021)]

Product of Bessel functions $\Rightarrow$ **inverse Mellin transform**

$$J_{2k+1}\left(\frac{t\sqrt{\lambda}}{2\pi}\right)J_{2\ell+1}\left(\frac{t\sqrt{\lambda}}{2\pi}\right)$$
$$=\int_{-i\infty}^{+i\infty}\frac{ds}{2\pi i}\frac{\Gamma(-s)\Gamma(2s+2k+2\ell+3)}{\Gamma(s+2k+2)\Gamma(s+2\ell+2)\Gamma(s+2k+2\ell+3)}\left(\frac{t\sqrt{\lambda}}{4\pi}\right)^{2(s+k+\ell+1)}$$

$$\mathrm{X}_{k\ell}=-8(-1)^{k+\ell}\sqrt{(2k+1)(2\ell+1)}\int_{-i\infty}^{+i\infty}\frac{ds}{2\pi i}\Bigg[\left(\frac{t\sqrt{\lambda}}{4\pi}\right)^{2(s+k+\ell+1)}$$
$$\frac{\Gamma(-s)\Gamma(2s+2k+2\ell+3)\Gamma(2s+2k+2\ell+2)}{\Gamma(s+2k+2)\Gamma(s+2\ell+2)\Gamma(s+2k+2\ell+3)}\zeta(2s+2k2\ell+1)\Bigg]$$

Results at strong coupling 2 [M. Beccaria, M. Billò, M. Frau, A. Lerda and AP (2021)]

Contributions ⇒ **poles** on the negative real axis

$$\mathrm{X}_{k\ell} = -8(-1)^{k+\ell}\sqrt{(2k+1)(2\ell+1)}\left[\frac{\lambda}{16\pi^2}\left(\frac{\delta_{k-1,\ell}}{2(2k-1)2k(2k+1)}+\frac{\delta_{k,\ell}}{2k(2k+1)(2k+2)}+\frac{\delta_{k+1,\ell}}{2(2k+1)(2k+2)(2k+3)}\right)-\frac{\delta_{k\ell}}{24(2k+1)}+O(\lambda^{-1/2})\right]$$

At strong coupling [M.Beccaria, A.A.Tseytlin and G.V. Dunne (2021)]

$$\mathrm{X} \underset{\lambda\to\infty}{\sim} -\frac{\lambda}{2\pi^2}\mathrm{S}$$

$$\mathrm{S}_{k\ell} = \frac{1}{4}(-1)^{k+\ell}\sqrt{\frac{2\ell+1}{2k+1}}\left(\frac{\delta_{k-1,\ell}}{k(2k-1)}+\frac{\delta_{k,\ell}}{k(k+1)}+\frac{\delta_{k+1,\ell}}{(k+1)(2k+3)}\right)$$

[Y. Ikebe, Y.Kikuchi and I.Fujishiro (1991)]

$$U_n \mapsto A_n = \frac{1}{\sqrt{M}} \left(\mathrm{tr} a_0^n + \mathrm{tr} a_1^n + \cdots + \mathrm{tr} a_{M-1}^n \right)$$

$$T_{\alpha,n} \mapsto A_{\alpha,n} = \frac{1}{\sqrt{M}} \sum_{I=0}^{M-1} \rho^{-\alpha I} \mathrm{tr} a_I^n$$

Interaction action

$$S_{int} = \sum_{I=0}^{M-1} \left[\sum_{m=2}^{\infty} \sum_{k=2}^{2m} \left(\frac{\lambda}{8\pi^2 N} \right)^m f_{m,k} (\mathrm{tr} a_I^{2m-k} - \mathrm{tr} a_{I+1}^{2m-k})(\mathrm{tr} a_I^k - \mathrm{tr} a_{I+1}^k) \right]$$

$$f_{m,k} = (-1)^{m+k} \binom{2m}{k} \frac{\zeta(2m-1)}{2m}$$

$$\mathrm{X}_{k,\ell} = \mathrm{X}_{\ell,k}\,, \qquad \mathrm{X}_{2k+1,2\ell} = 0$$

$$\mathrm{X}^{odd}_{k,\ell} \equiv \mathrm{X}_{2k+1,2\ell+1}$$

$$-8(-1)^{k+\ell}\sqrt{(2k+1)(2\ell+1)}\int_0^\infty \frac{dt}{t}\,\frac{e^t}{e^t-1}\,J_{2k+1}\left(\frac{t\sqrt{\lambda}}{2\pi}\right)J_{2\ell+1}\left(\frac{t\sqrt{\lambda}}{2\pi}\right)$$

$$\mathrm{X}^{even}_{k,\ell} \equiv \mathrm{X}_{2k,2\ell}$$

$$-8(-1)^{k+\ell}\sqrt{(2k)(2\ell)}\int_0^\infty \frac{dt}{t}\,\frac{e^t}{e^t-1}\,J_{2k}\left(\frac{t\sqrt{\lambda}}{2\pi}\right)J_{2\ell}\left(\frac{t\sqrt{\lambda}}{2\pi}\right)$$

Gram-Schimdt $\Rightarrow$ Change of basis $A_{\alpha,n} \mapsto \mathcal{P}_{\alpha,n}$

$$\mathcal{P}_{\alpha,n} \equiv \sum_k c_{n,k} A_{\alpha,n-2k} \quad \Rightarrow \quad \langle \mathcal{P}_{\alpha,n} \mathcal{P}_{\beta,m} \rangle_0 = \delta_{n,m} \delta_{\alpha+\beta,0}$$

Explicit expression

Expansion with the X-matrix $s_\alpha \equiv \sin^2\left(\frac{\pi\alpha}{M}\right)$

$$\langle \mathcal{P}_{\alpha,2k} \mathcal{P}^\dagger_{\alpha,2\ell} \rangle = \left(\frac{1}{1 - s_\alpha \mathrm{X}^{\mathrm{even}}} \right)_{k,\ell}$$

$$\langle \mathcal{P}_{\alpha,2k+1} \mathcal{P}^\dagger_{\alpha,2\ell+1} \rangle = \left(\frac{1}{1 - s_\alpha \mathrm{X}^{\mathrm{odd}}} \right)_{k,\ell}$$

Untwisted 2-point

$$\frac{\langle U_n(\vec{x}) U_n^\dagger(\vec{0}) \rangle}{\langle O_n(\vec{x}) O_n(\vec{0}) \rangle_0} = 1 + O\left(\frac{1}{N^2}\right)$$

Twisted 2-point

$$\frac{\langle T_{\alpha,n}(\vec{x}) T_{\alpha,n}^\dagger(\vec{0}) \rangle}{\langle O_n(\vec{x}) O_n(\vec{0}) \rangle_0} = 1 + \Delta_{\alpha,n}(\lambda) + O\left(\frac{1}{N^2}\right)$$

$$\Delta_{\alpha,n}(\lambda) \underset{\lambda\to\infty}{\sim} \frac{4\pi^2}{\lambda\, s_\alpha} n(n-1) - 1 \qquad s_\alpha \equiv \sin^2\left(\frac{\pi\alpha}{M}\right)$$

Geometry | **$\mathbb{Z}_M$ Orbifold**

$$\mathrm{AdS}_5 \times \mathbb{S}^5/\mathbb{Z}_M \simeq \mathrm{AdS}_5 \times \mathbb{S}^1 \times \mathbb{C}^2/\mathbb{Z}_M \qquad (w_1, w_2) \sim (\mathrm{e}^{\frac{2\pi i}{M}} w_1, \mathrm{e}^{-\frac{2\pi i}{M}} W_2)$$

fractional D3 $\mapsto$ D5 wrapped around exceptional 2-cycles $e_i \mapsto \omega_i$

$$\int_{e_i} \omega^j = \delta_i{}^j \qquad \int_{\mathcal{M}} \omega^i \wedge \omega^j = -(C^{-1})^{ij}$$

anti-self dual 2-form (ω^j) — ALE space ($\mathcal{M}$) — su(M) Cartan Matrix (C)

$M-1$ 2-cycles $e_i \quad e_0 = -\sum_i e_i \quad i, j = 1, \cdots, M-1$

 [S. Gukov (1998)]

Scalar Fields

$$\hat{b}_i = \frac{1}{2\pi\alpha'} \int_{e_i} B_{(2)}, \qquad \hat{c}_i = \frac{1}{2\pi\alpha'} \int_{e_i} C_{(2)} \qquad i = 0, \cdots, M$$

NS/NS 2-form

R/R 2-form

Twisted and **Untwisted** combinations

$$b = \frac{1}{2\sqrt{M}} \left(\hat{b}_0 + \hat{b}_1 + \cdots + \hat{b}_{M-1} \right), \qquad c = \frac{1}{2\sqrt{M}} \left(\hat{c}_0 + \hat{c}_1 + \cdots + \hat{c}_{M-1} \right),$$

$$b_\alpha = \frac{1}{2\sqrt{M}} \sum_{I=0}^{M-1} \rho^{-\alpha I} \hat{b}_I, \qquad c_\alpha = \frac{1}{2\sqrt{M}} \sum_{I=0}^{M-1} \rho^{-\alpha I} \hat{c}_I$$

10d Type II-B supergravity action

$$\frac{1}{2k_{10}^2}\left[\int d^{10}x\sqrt{G_{10}}\left(\frac{1}{12}(dB_{(2)})^2+\frac{1}{12}(dC_{(2)})^2\right)-\int 4C_{(4)}\wedge dB_{(2)}\wedge dC_{(2)}\right]$$

Wrapping the 2-forms ⇩ on the e_i cycles of the $\mathbb{Z}_M$ orbifold

6d supergravity action

$$\frac{1}{2}\sum_{\alpha=1}^{M-1}\frac{1}{2k_6^2}\frac{1}{s_\alpha}\left[\int d^6x\sqrt{G_6}(\partial b_\alpha^\star\cdot\partial b_\alpha+\partial c_\alpha^\star\cdot\partial c_\alpha)+\int 8F_5\wedge db_\alpha^\star\wedge c_\alpha\right]$$

Holographic description

[S. Gukov (1998)]

Space on which b_α, c_α propagate $\quad\Rightarrow\quad \mathrm{AdS}_5 \times \mathbb{S}^1$

$$b_\alpha = \frac{1}{\sqrt{2\pi}} \sum_{n\in\mathbb{Z}} b_{\alpha,n} e^{in\theta} \qquad c_\alpha = \frac{1}{\sqrt{2\pi}} \sum_{n\in\mathbb{Z}} c_{\alpha,n} e^{in\theta}$$

KK compactification on $\mathbb{S}^1$

$$S_{\mathrm{AdS}_5} = \frac{1}{2} \sum_{\alpha=1}^{M-1} \sum_{n\in\mathbb{Z}} \int_{\mathrm{AdS}_5} d^5x \sqrt{G_{\mathrm{AdS}_5}}\, \mathcal{L}_{\alpha,n}$$

$$\mathcal{L}_{\alpha,n} = \frac{1}{2k_6^2} \frac{1}{\sin^2\left(\frac{\pi\alpha}{M}\right)} \left(b^\star_{\alpha,n}, c^\star_{\alpha,n}\right) \cdot \begin{pmatrix} -\Delta + n^2 & -4in \\ 4in & -\Delta + n^2 \end{pmatrix} \cdot \begin{pmatrix} b_{\alpha,n} \\ c_{\alpha,n} \end{pmatrix}$$

Holographic description

[S. Gukov (1998)]

$$\gamma_{a,n} = c_{\alpha,n} + i b_{\alpha,n} \qquad m^2_{\gamma_{a,n}} = n(n+4)$$

$$\eta_{\alpha,n} = c_{\alpha,n} - i b_{\alpha,n} \qquad m_{\eta^2_{a,n}} = n(n-4)$$

dual to $T_{\alpha,n}(x)$ $\Delta = n$

$S[\eta]$-action and boundary term

$$\frac{1}{2}\sum_{\alpha=1}^{M-1}\sum_{n=2}^{\infty}\int_{\mathrm{AdS}_5} d^5x\sqrt{G_{AdS_5}}\left[\frac{1}{2k_6^2}\frac{1}{\sin^2\left(\frac{\pi\alpha}{M}\right)}\left(\partial\eta^{\star}_{\alpha,n}\cdot\partial\eta_{\alpha,n} + n(n-4)\eta^{\star}_{\alpha,n}\eta_{\alpha,n}\right)\right]$$

$$\propto \sum_{\alpha=1}^{M-1}\sum_{n=2}^{\infty}\int_{\partial(\mathrm{AdS}_5)} d^4x\left[T_{\alpha,n}(\vec{x})\eta_{\alpha,n}(\vec{x}) + \mathrm{c.c.}\right]$$

CFT 2-point function $\xleftrightarrow{\text{AdS/CFT}}$ Normalization of S_d

d-dimensional SUGRA action

$\mathcal{N} = 4$

$$\frac{1}{2k_{10}^2} = \frac{4N^2}{(2\pi)^5 R^8}$$

$\mathcal{N} = 2$ twisted operators

$$\frac{1}{2k_6^2}\frac{1}{\sin^2\left(\frac{\pi\alpha}{M}\right)} = \frac{4N^2}{(2\pi)^5 R^4 \frac{4\pi^2}{\lambda s_\alpha}}$$

Ratio in units of the AdS radius R ⇒ $\frac{4\pi^2}{\lambda s_\alpha}$