MATCHED ASYMPTOTIC EXPANSION FOR SPINNING BLACK HOLE MAGNETOSPHERES

14/06/2022

FILIPPO CAMILLONI

Blandford-Znajek monopole expansion revisited:
novel non-analytic contributions to the power emission

F. Camilloni, O. J. C. Dias, G. Grignani, T. Harmark, R. Oliveri,
M. Orselli, A. Placidi and J. E. Santos.

[2201.11068]
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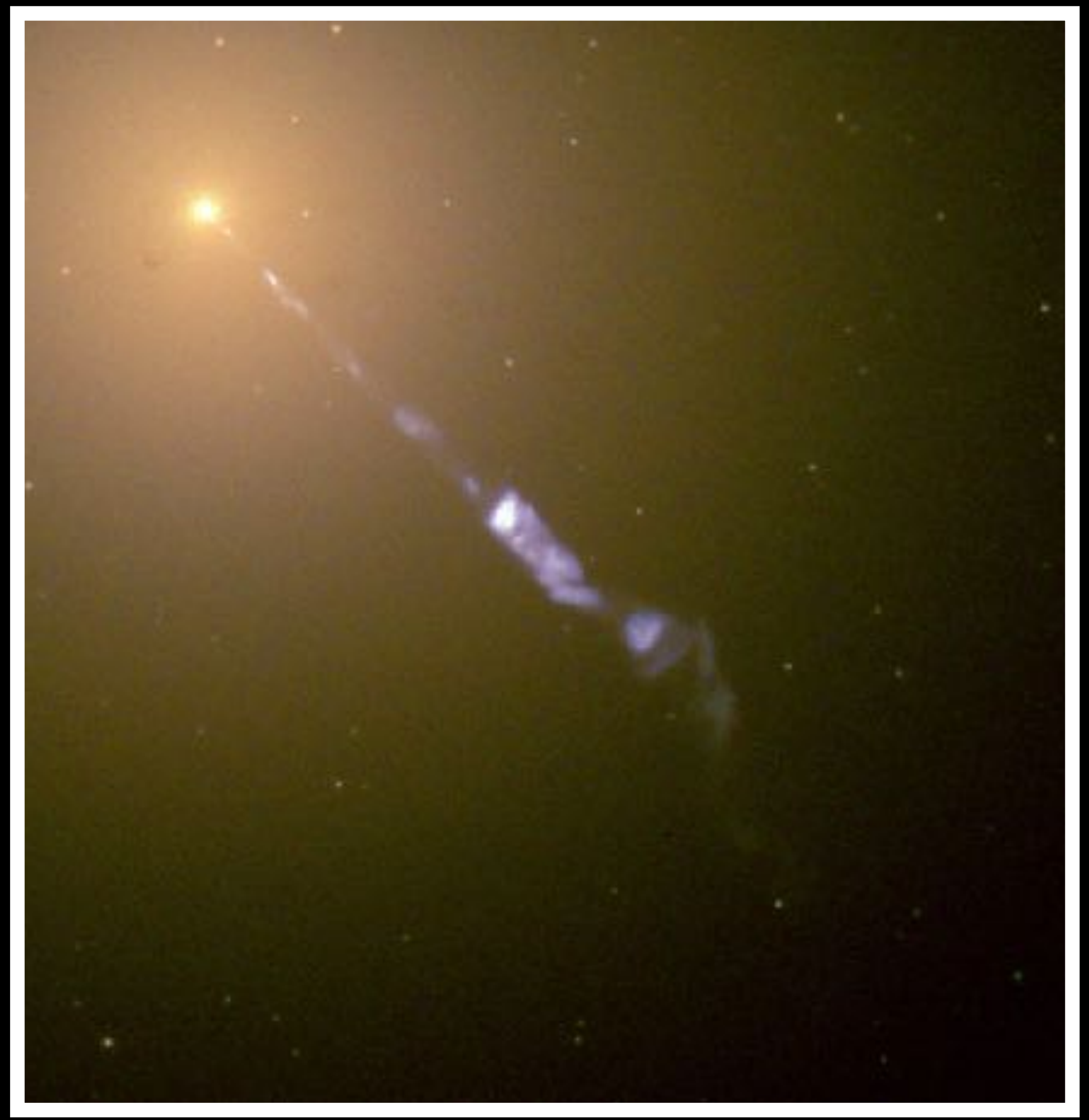
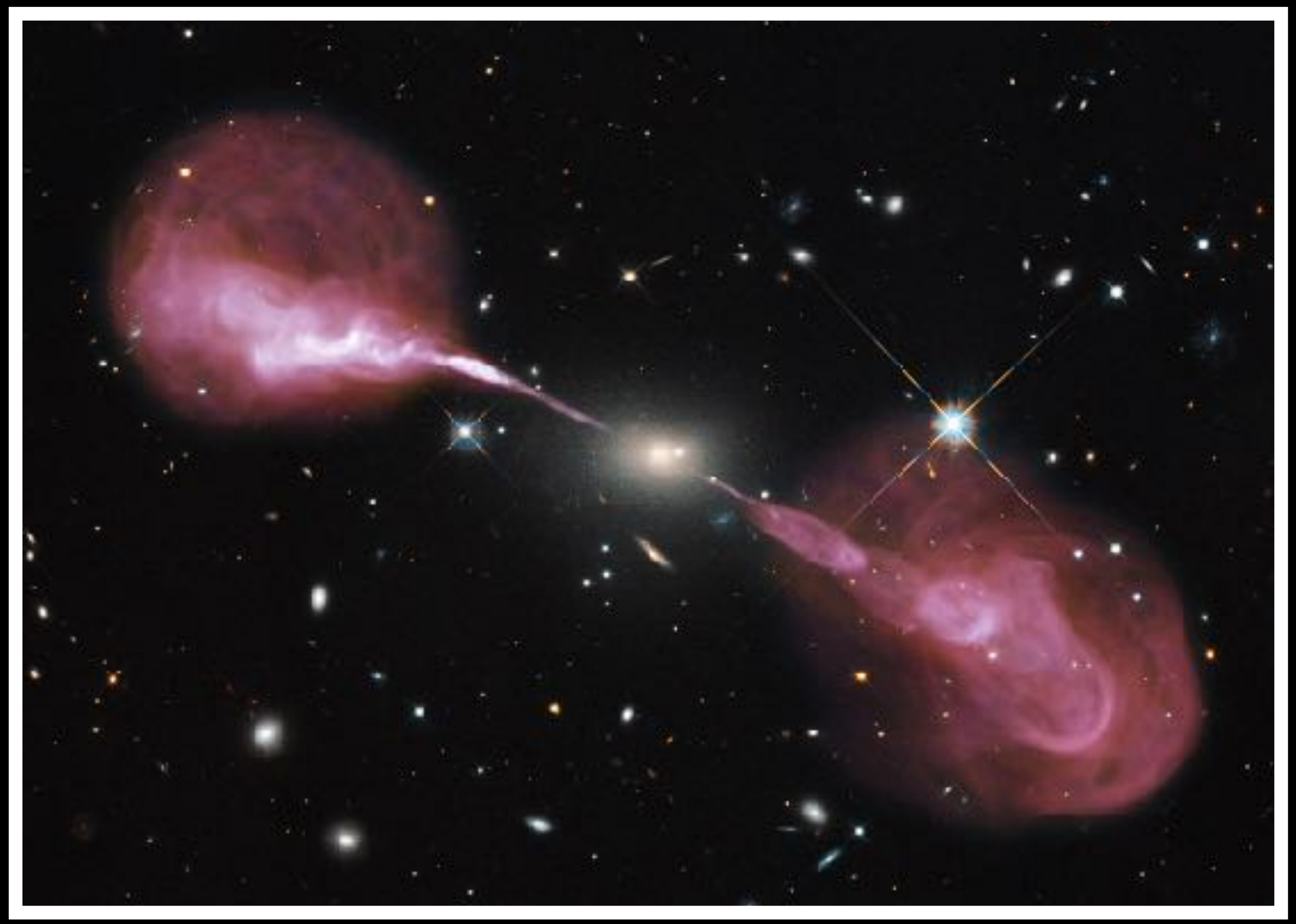
M87*

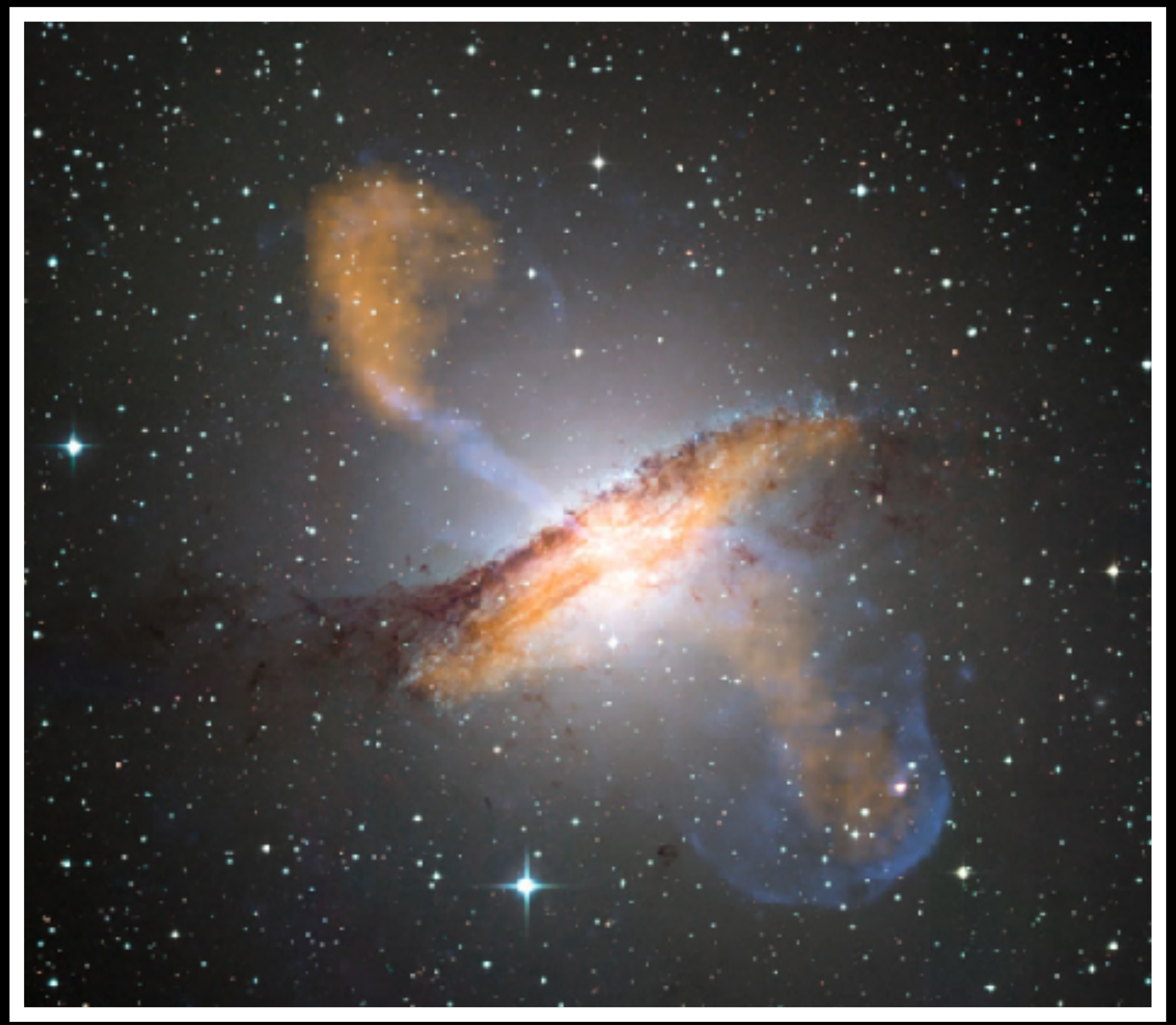


“DEATH STAR GALAXY” (3C321)

HERCULES A



CENTAURUS A



RELATIVISTIC JETS

- Highly energetic streams of plasma
- Ultra-relativistic
- Collimated
- Propagates for long distances

Most powerful jets emitted by rapidly spinning Black Holes (BHs) at the center of galaxies!

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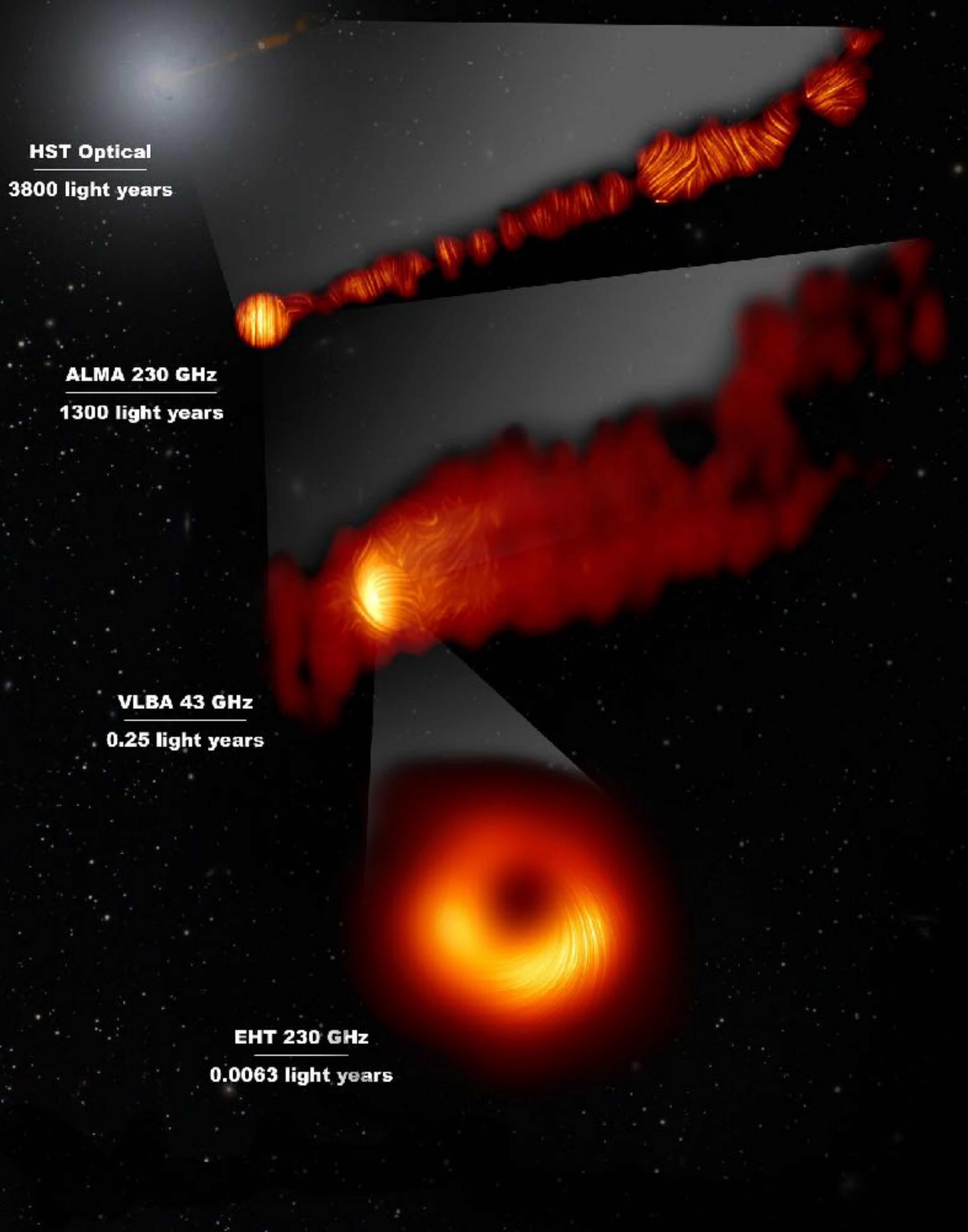
HST Optical
3800 light years
ALMA 230 GHz
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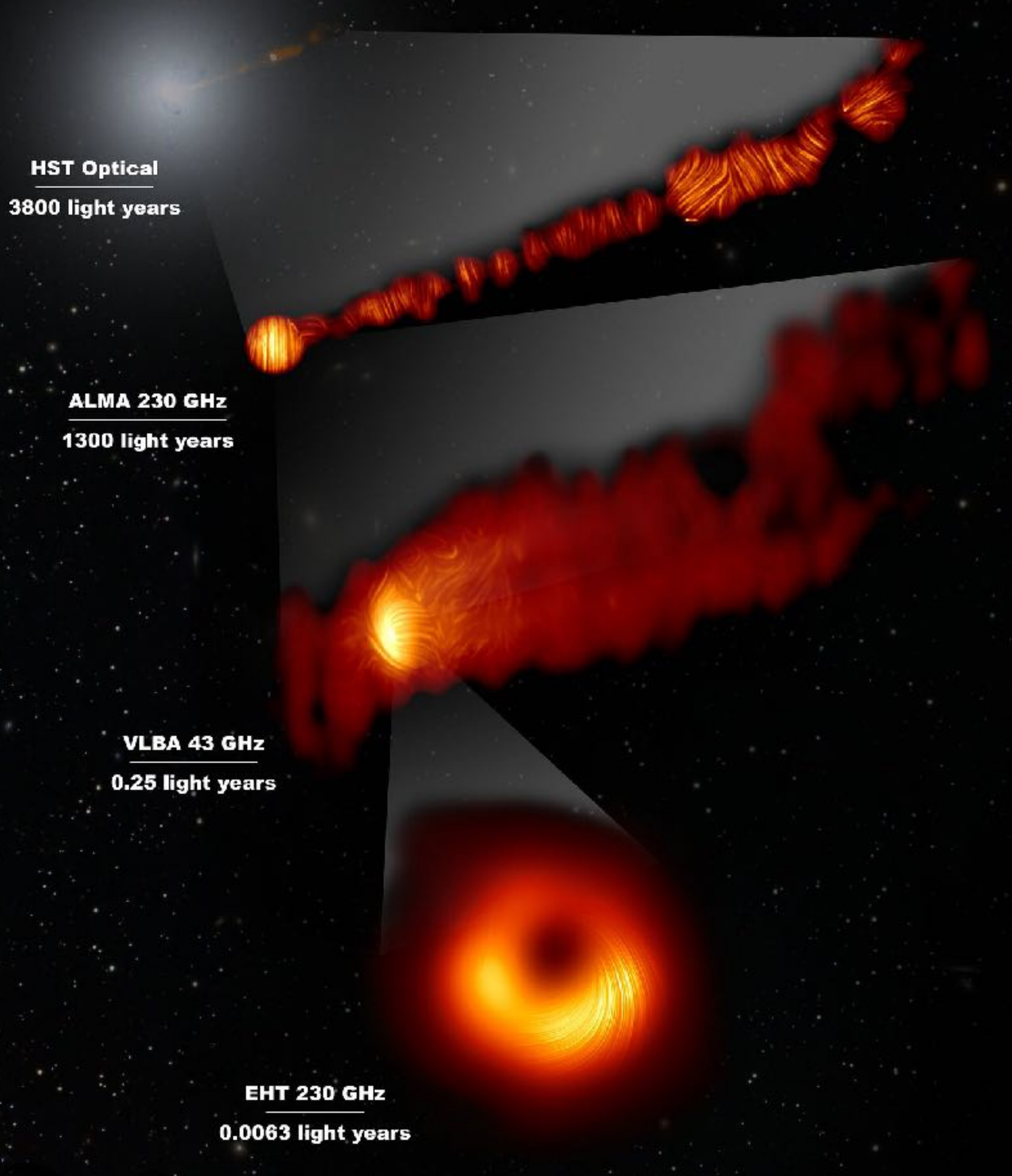
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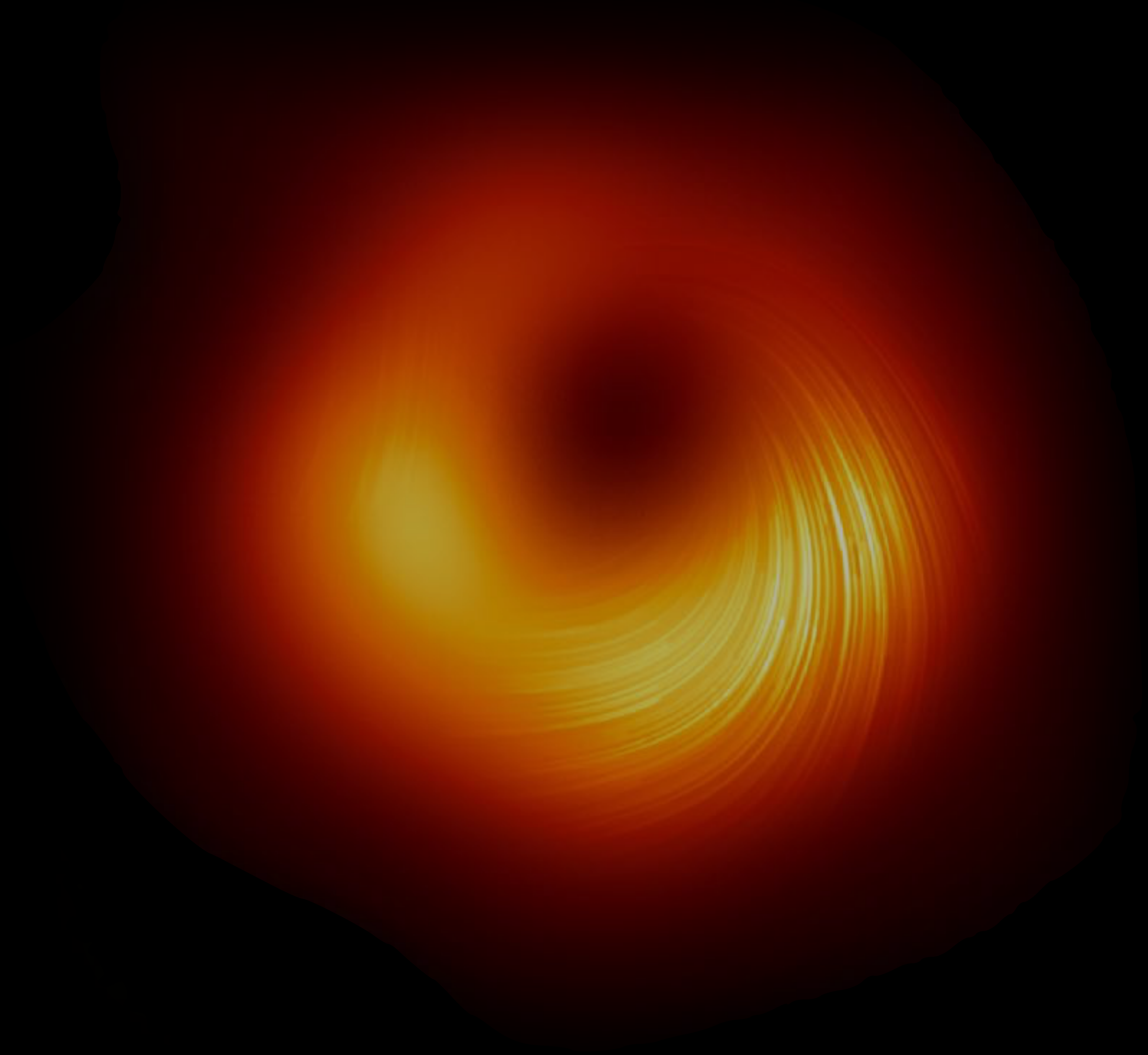

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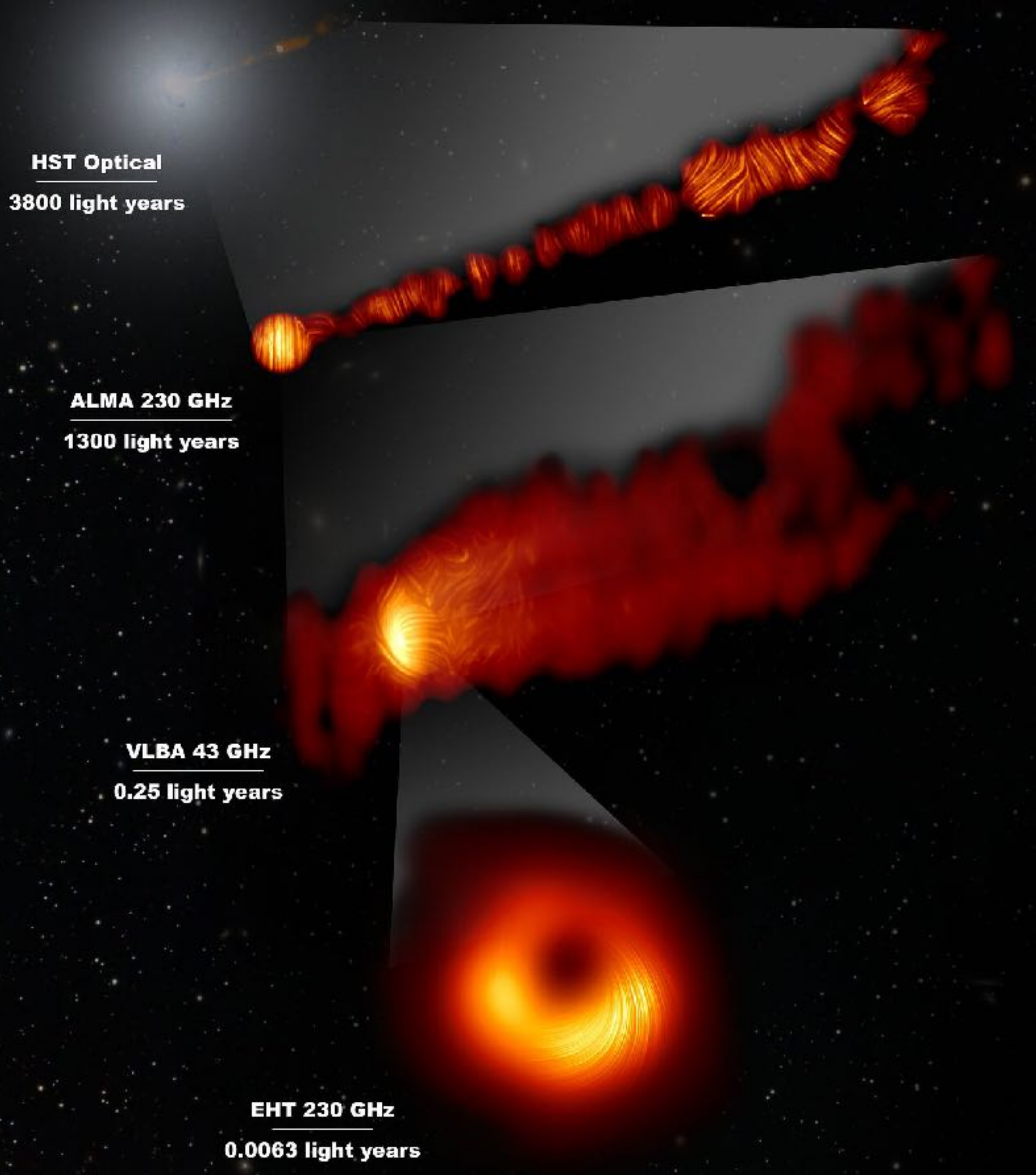

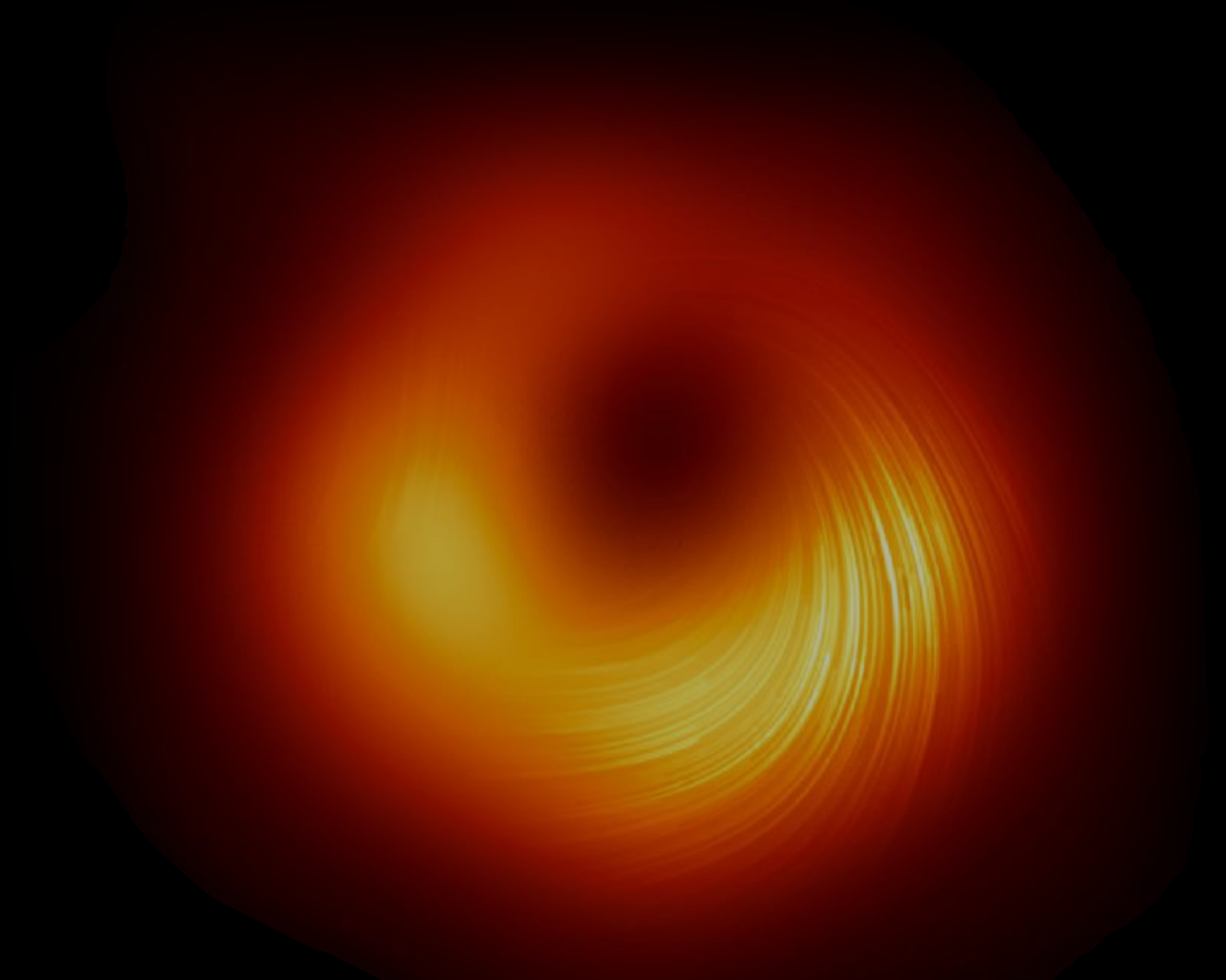

- Polarization is **magnetic field** signature!

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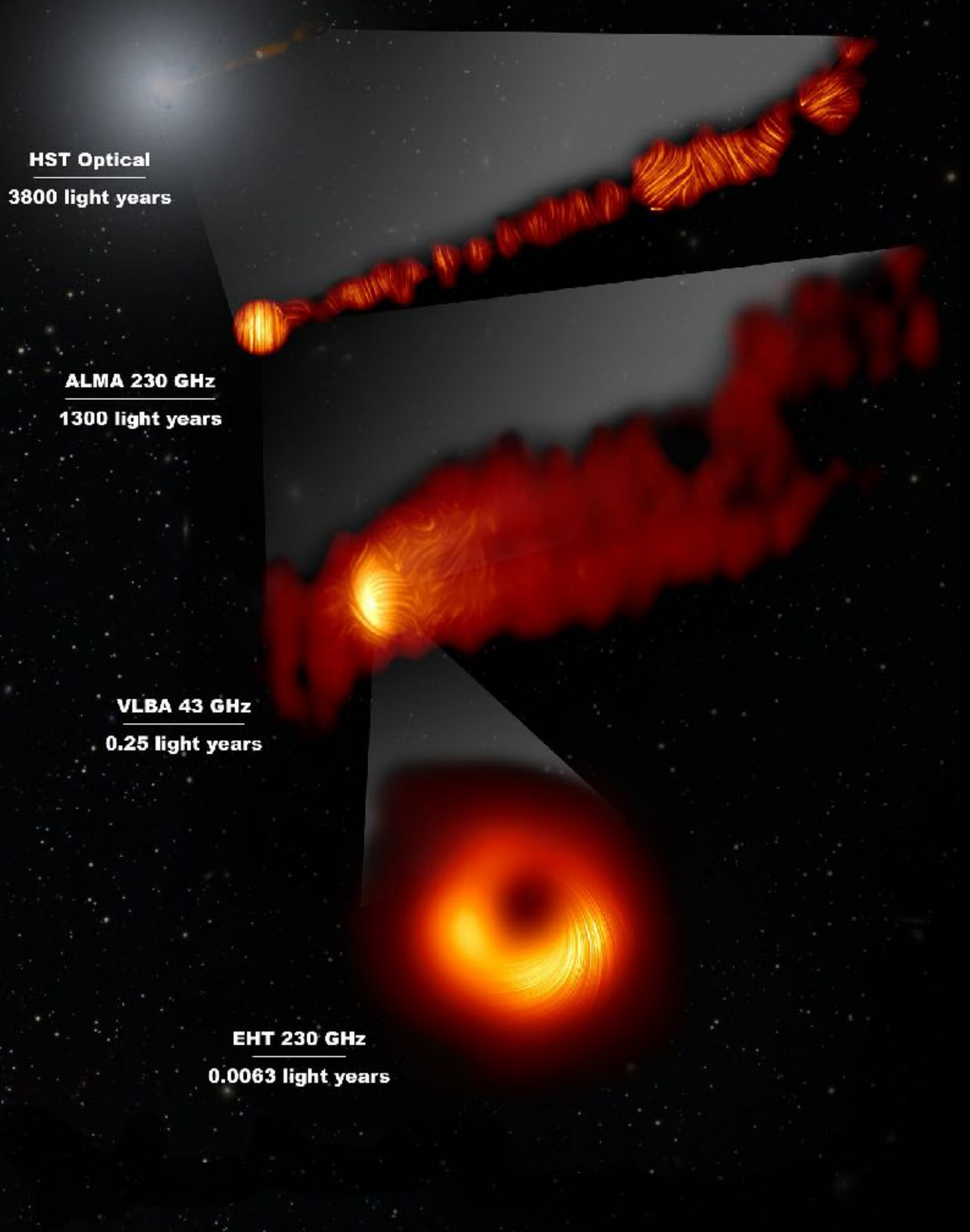

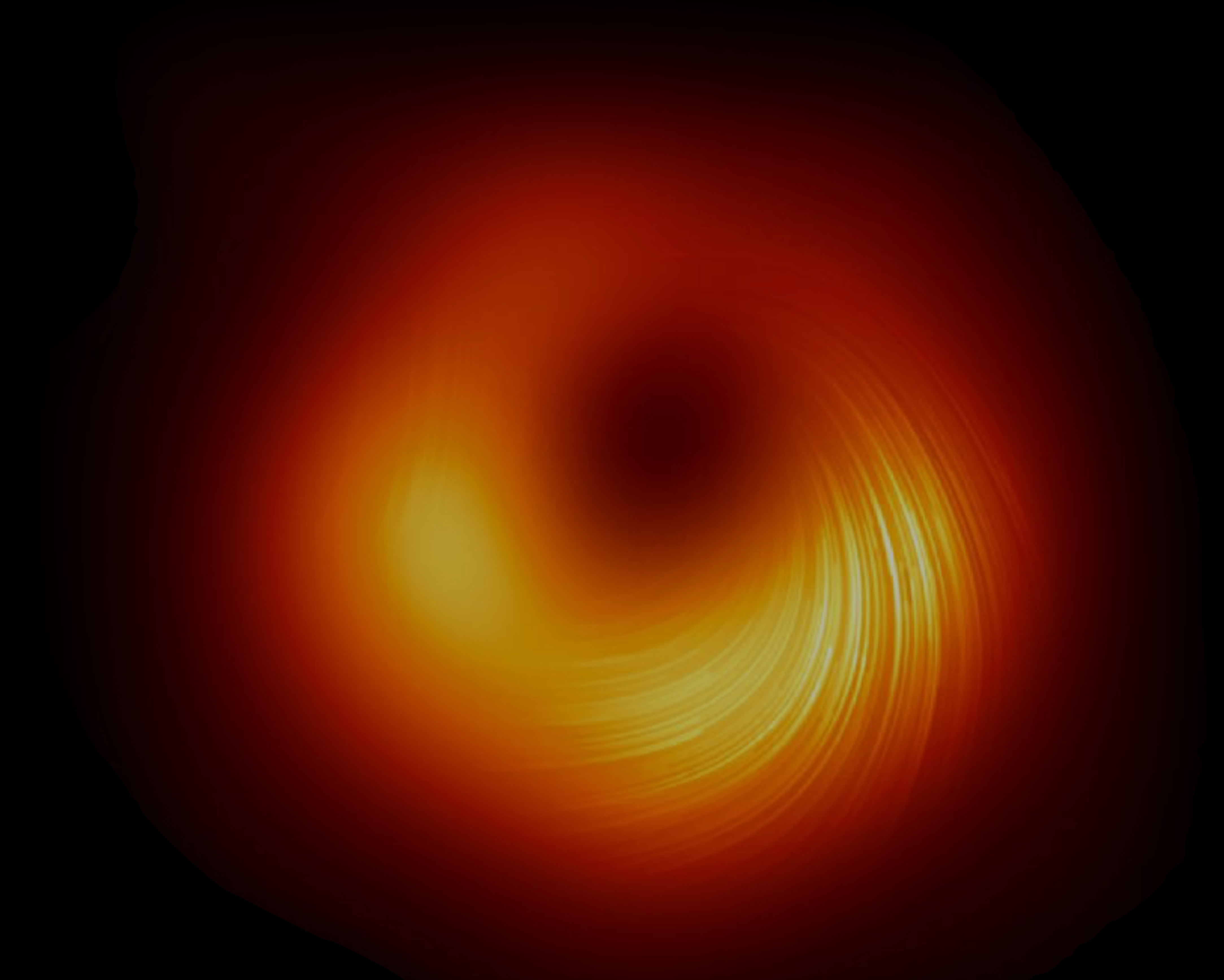

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- The observed Power emitted is consistent with the **Blandford & Znajek mechanism**!

EHT (2019)
Blandford, Znajek (1977)

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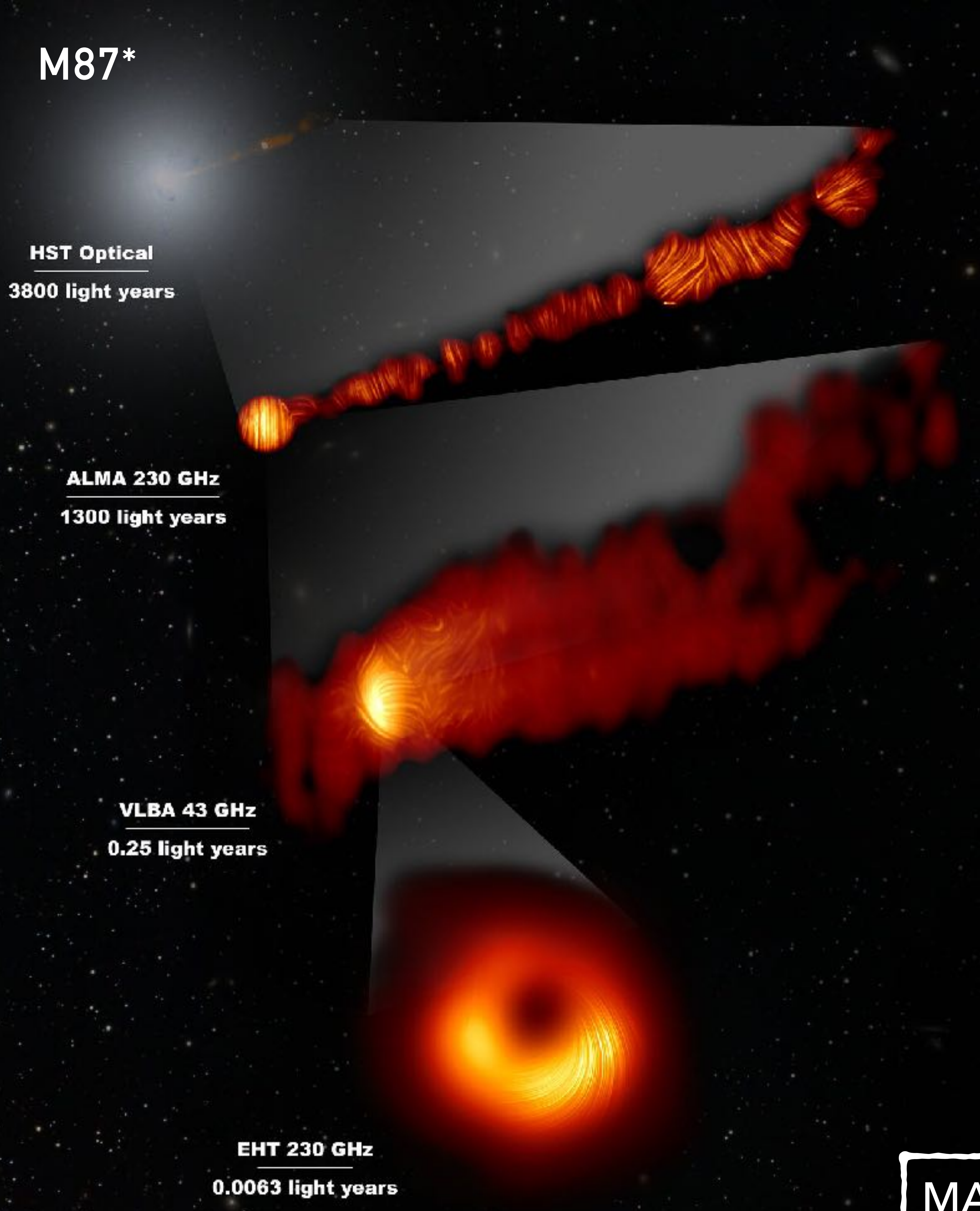

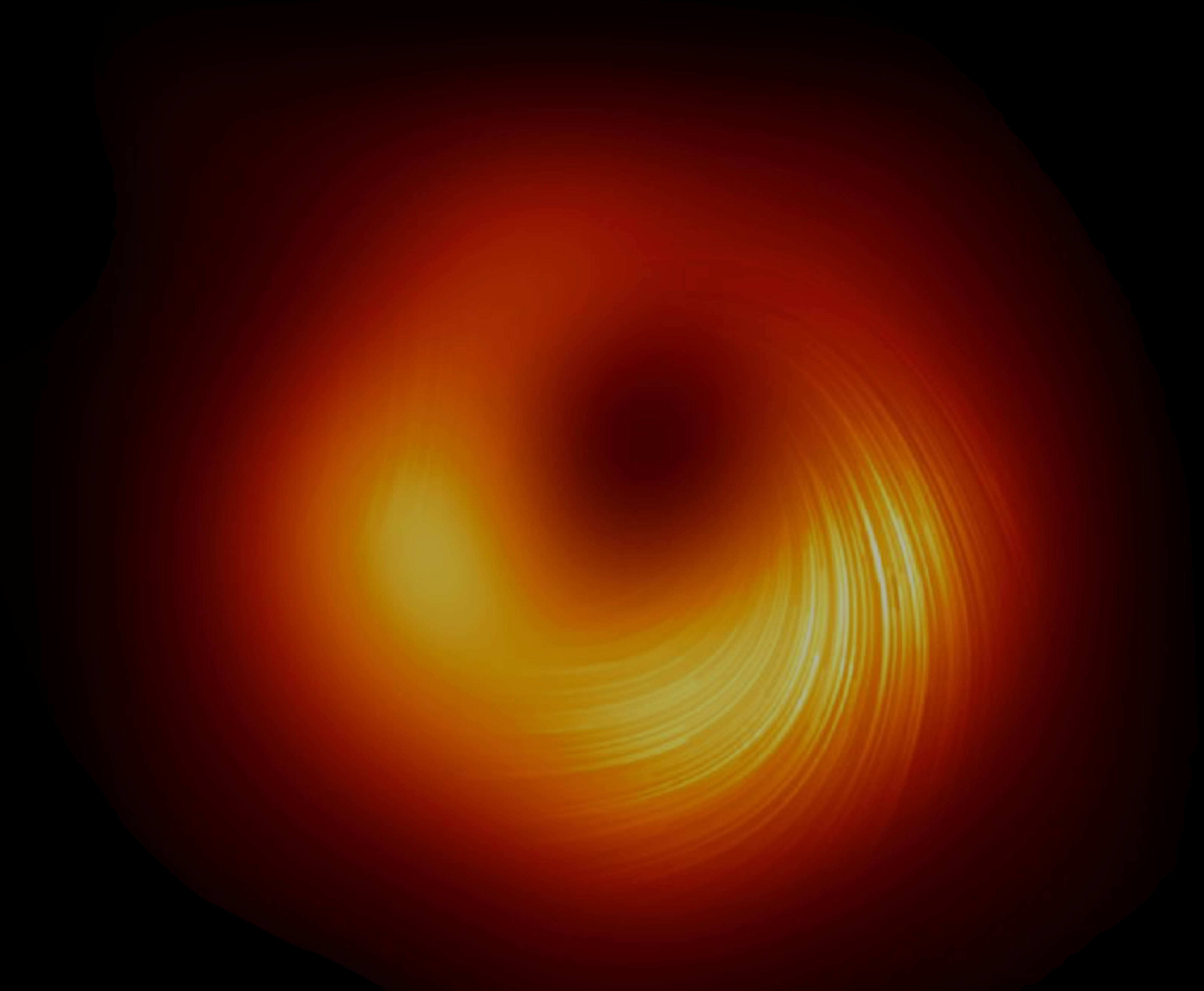

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MAGNETOSPHERE + SPINNING BH = ENERGY AND ANGULAR MOMENTUM EXTRACTION

SPINNING BLACK HOLES AND PENROSE PROCESS

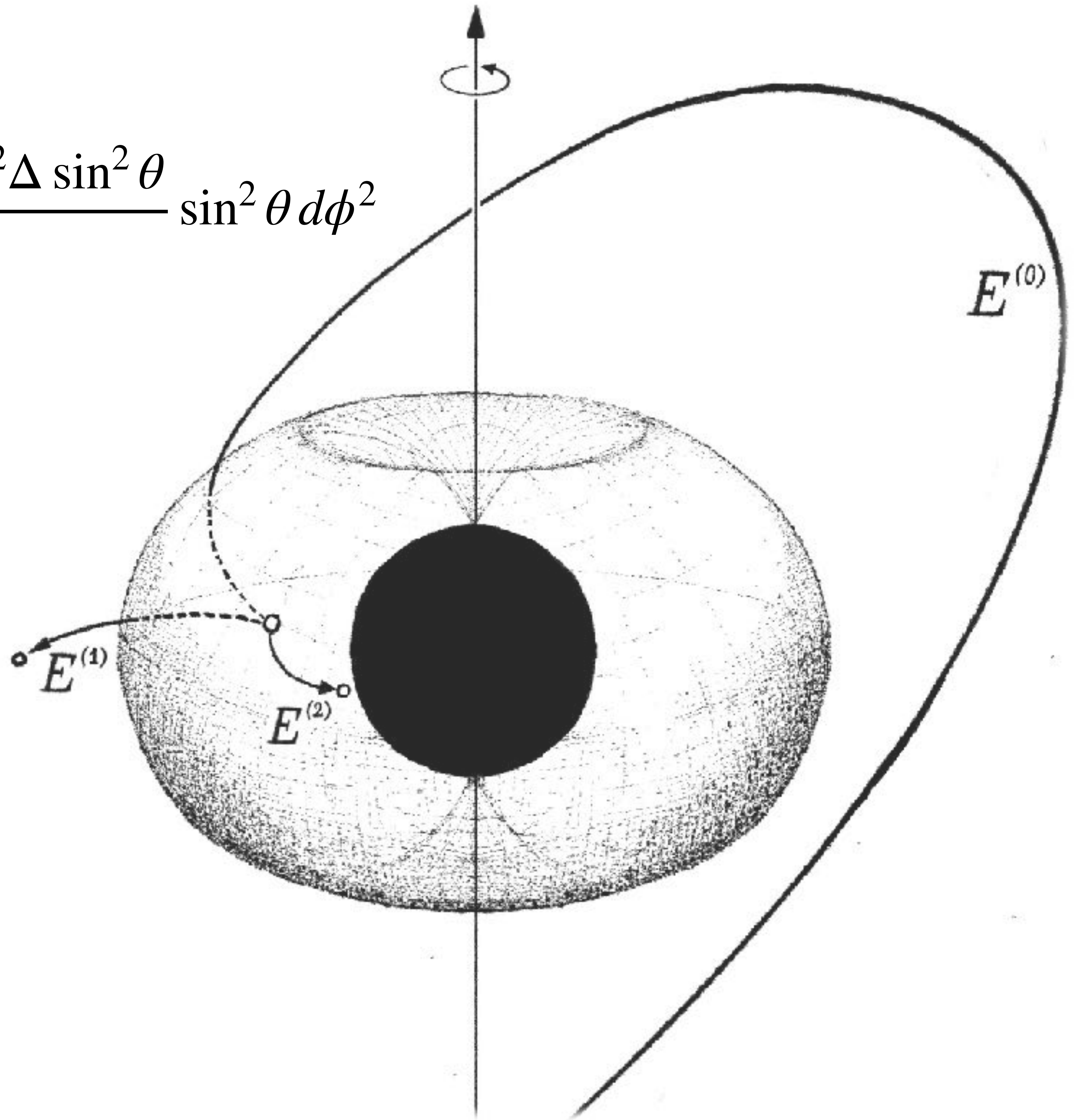
$$ds^2 = -\left(1 - \frac{r_0 r}{\Sigma}\right) dt^2 + \frac{\Sigma}{\Delta} dr^2 + \Sigma\, d\theta^2 - 2\frac{r_0 r}{\Sigma} a \sin^2\theta\, dt d\phi + \frac{(r^2+a^2)^2 - a^2\Delta \sin^2\theta}{\Sigma} \sin^2\theta\, d\phi^2$$

dimensionless spin parameter

$$\alpha = \frac{J}{M^2}$$

$$\boxed{1 \geq \alpha \geq 0}$$

- $\alpha \ll 1$ **Slowly spinning**
- $\alpha \approx 1$ **Nearly-Extreme**
- $\alpha = 1$ **Extreme**



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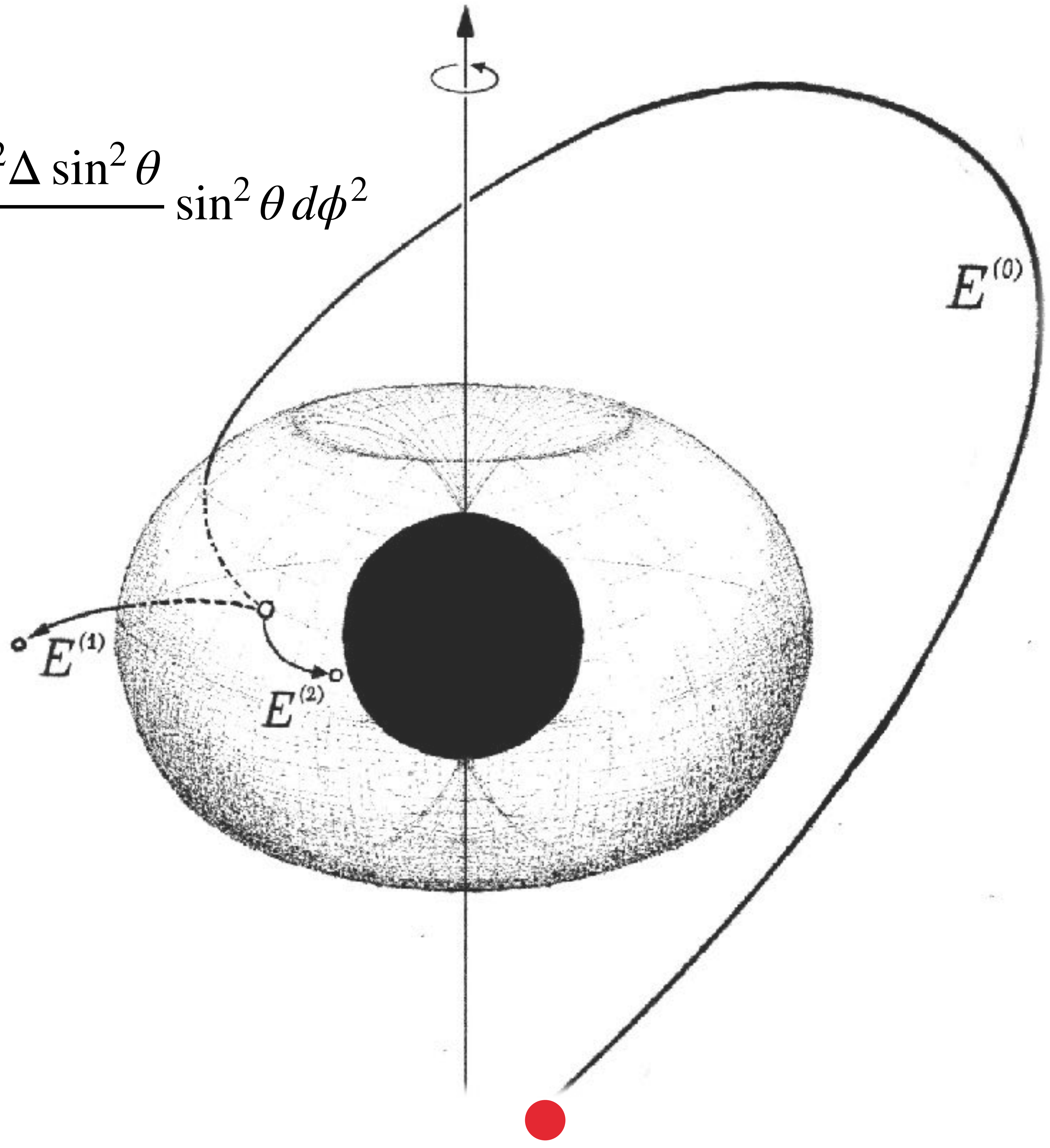

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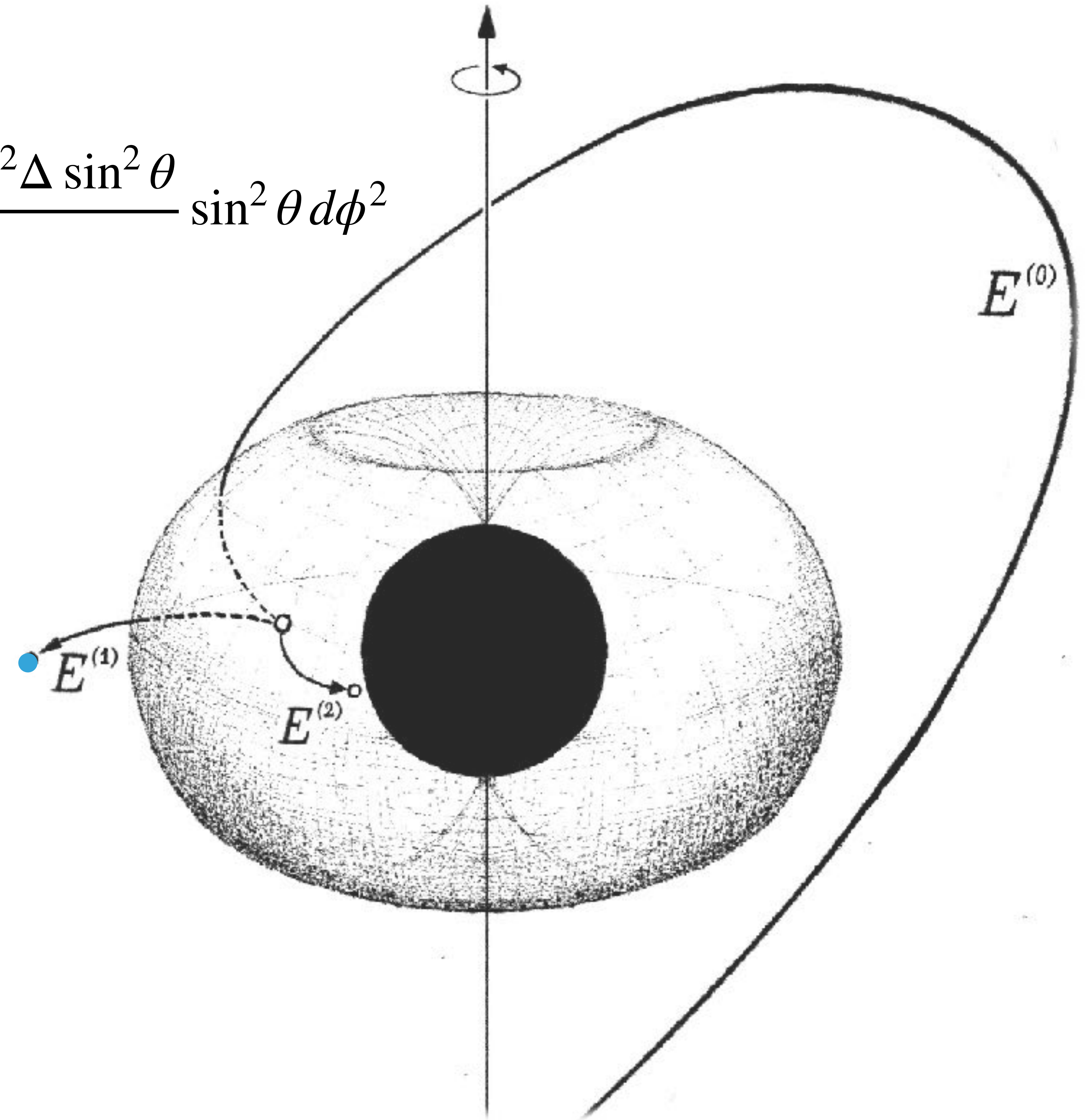

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GENERALISED PENROSE PROCESS

Lasota et Al (2014)

Energy and angular momentum **outflow horizon**

$$\Delta E_H = \int_{\Delta t \times S_+} T^r{}_t \sqrt{-g}\; dt d\theta d\phi$$

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Energy and angular momentum **inflow infinity**

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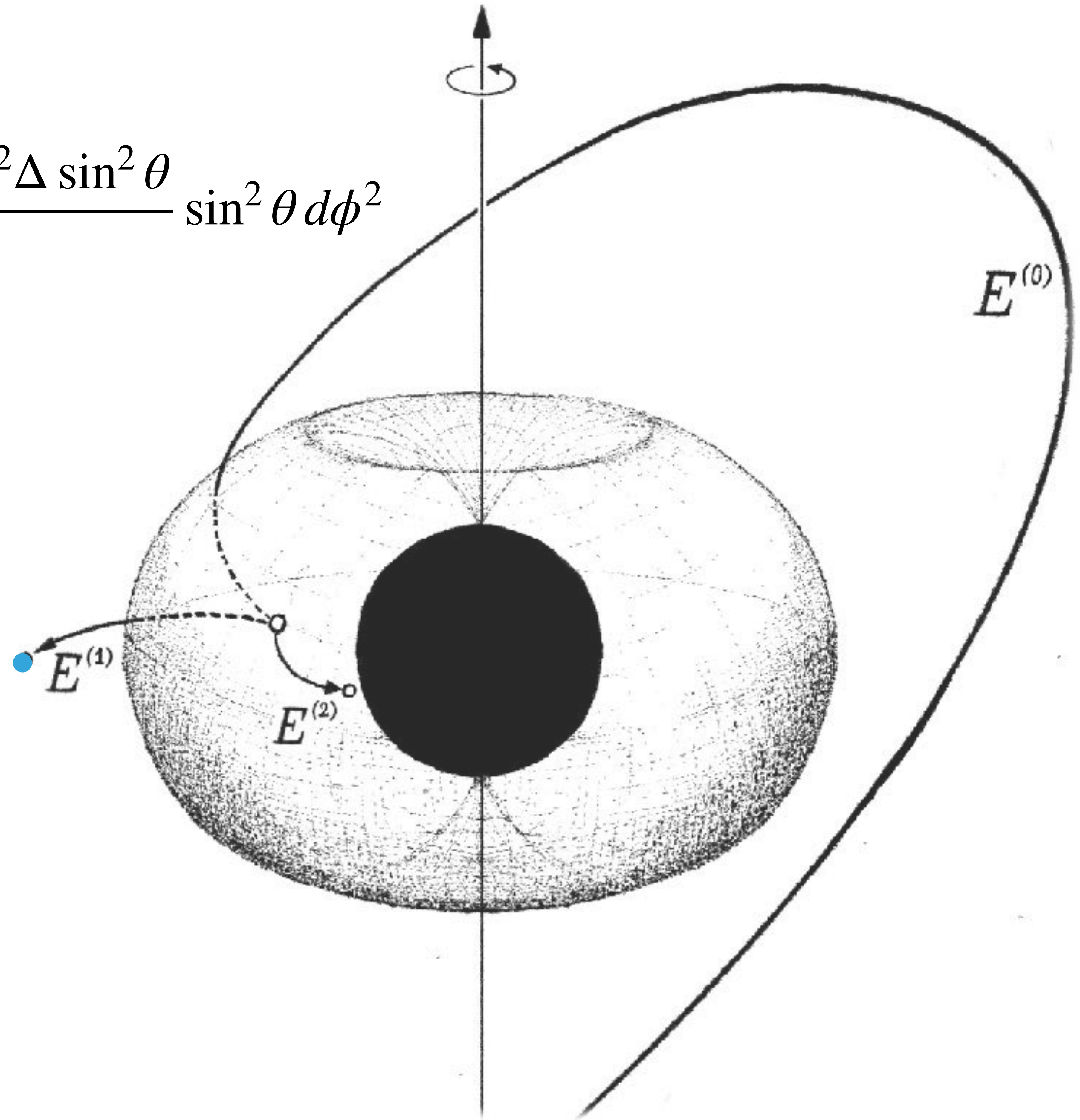

Blandford&Znajek MECHANISM Blandford, Znajek (1977)

FORCE-FREE ELECTRODYNAMICS (FFE)

- MHD: long-distance dynamics between **thermal degrees of freedom** and **electromagnetic fields**

$$T^{\mu\nu}_{\mathrm{MHD}} = T^{\mu\nu}_{mat} + T^{\mu\nu}_{em} \longrightarrow \nabla_\mu T^{\mu\nu}_{\mathrm{MHD}} = 0 \quad , \quad \nabla_\mu(\star F^{\mu\nu}) = 0$$

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- Highly **non-linear** electrodynamics regime in a **curved background**

(NO EXACT SOLUTION FOUND IN KERR)

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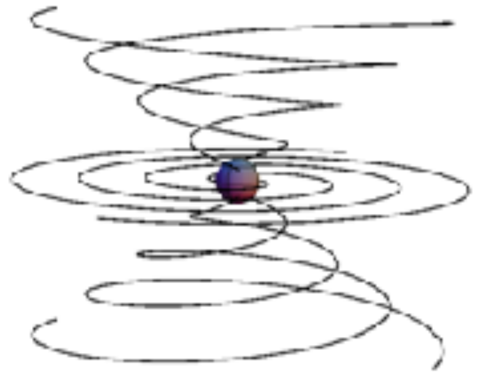

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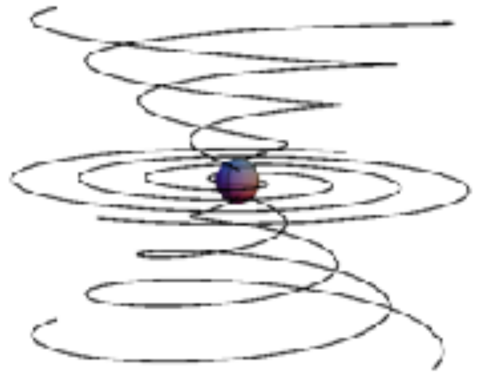

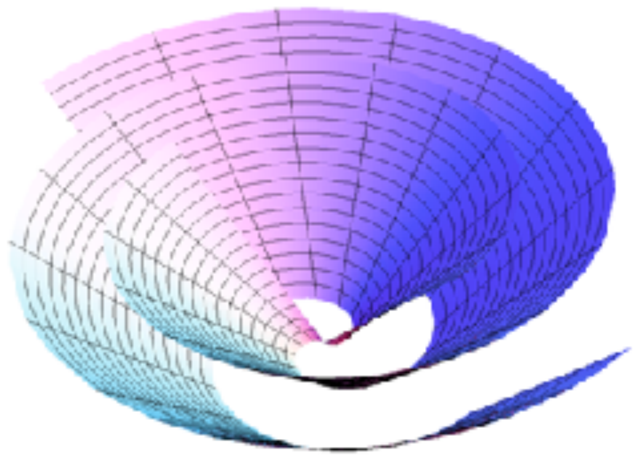

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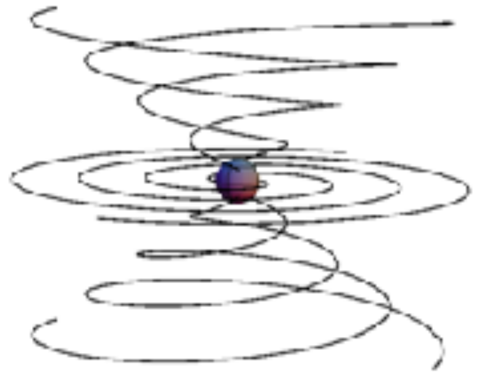

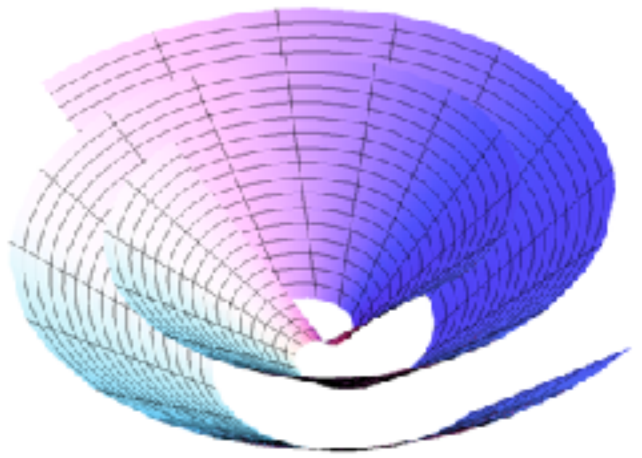

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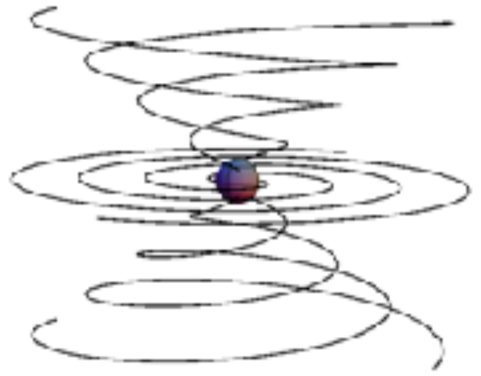

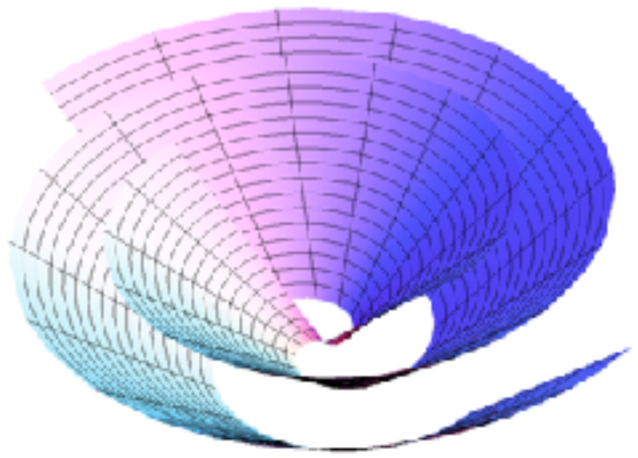

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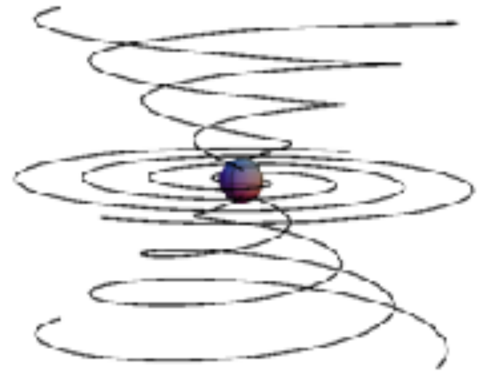

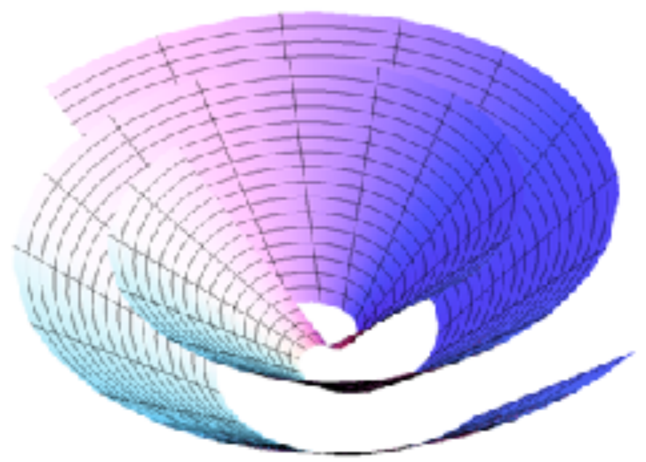

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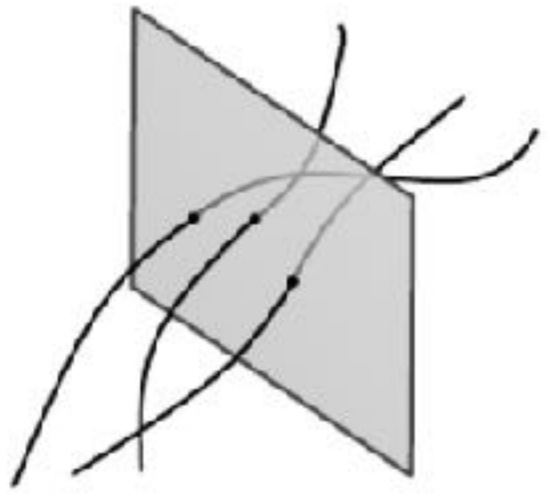
(1-form symmetry)



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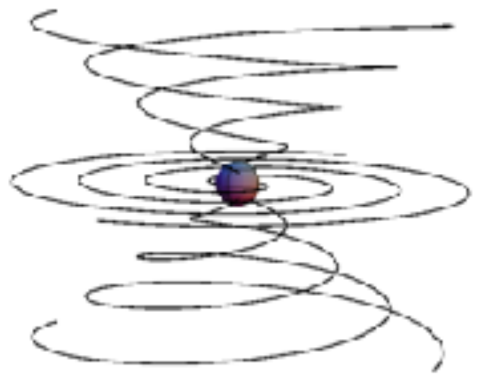

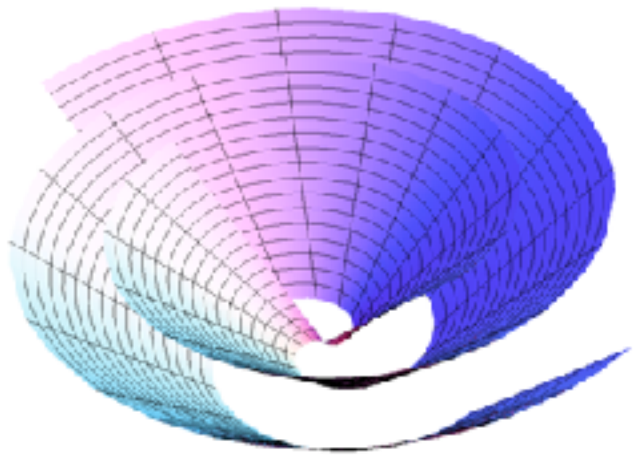

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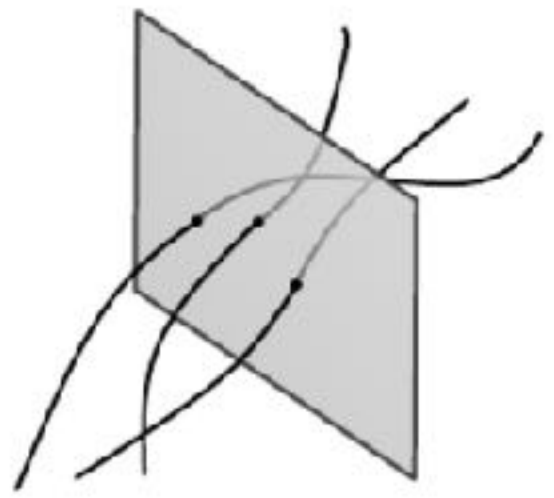

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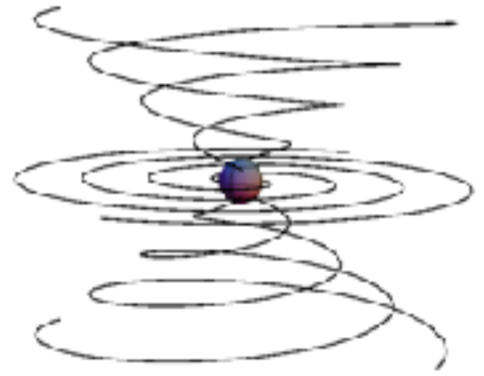

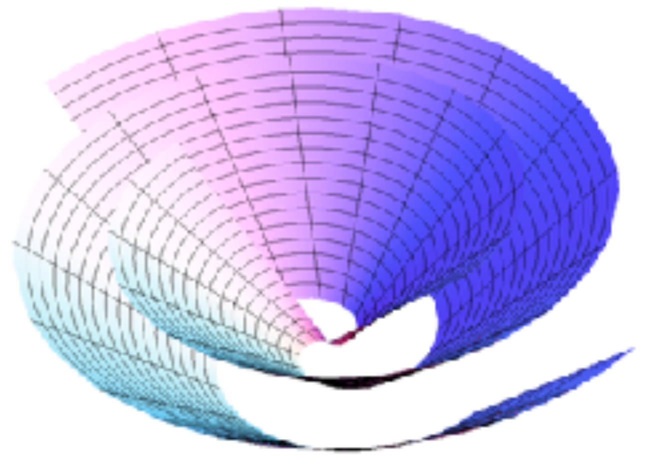

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- Hydrodynamics/EFT of cold string-fluid Grozdanov et Al (2016), Gralla Iqbal (2018)
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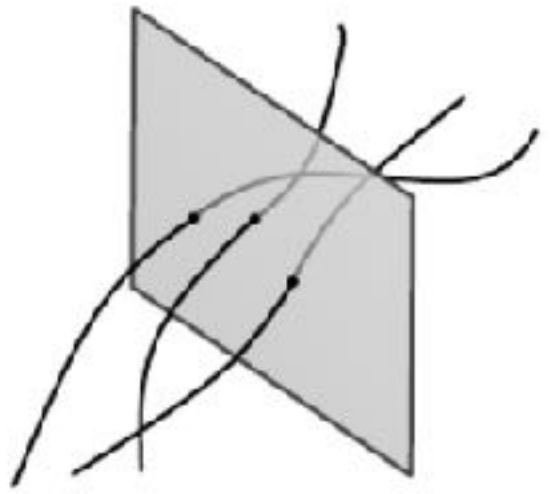

$$Q^{(1)}=\int_S \star J$$

Force-Free Magnetospheres
- Numerically: **GRMHD simulations**
- Analytically: **Perturbative approaches**

FORCE-FREE MAGNETOSPHERES

$$\boxed{F_{\rho\mu}\nabla_\nu F^{\mu\nu} = 0 \ \ , \ \ \nabla_\mu(\star F^{\mu\nu}) = 0 \ \ , \ \ j^\mu_{\text{mat}} \neq 0}$$

Stationary-Axisymmetric Kerr Magnetospheres $\mathscr{L}_{\partial_t} F = \mathscr{L}_{\partial_\phi} F = 0$

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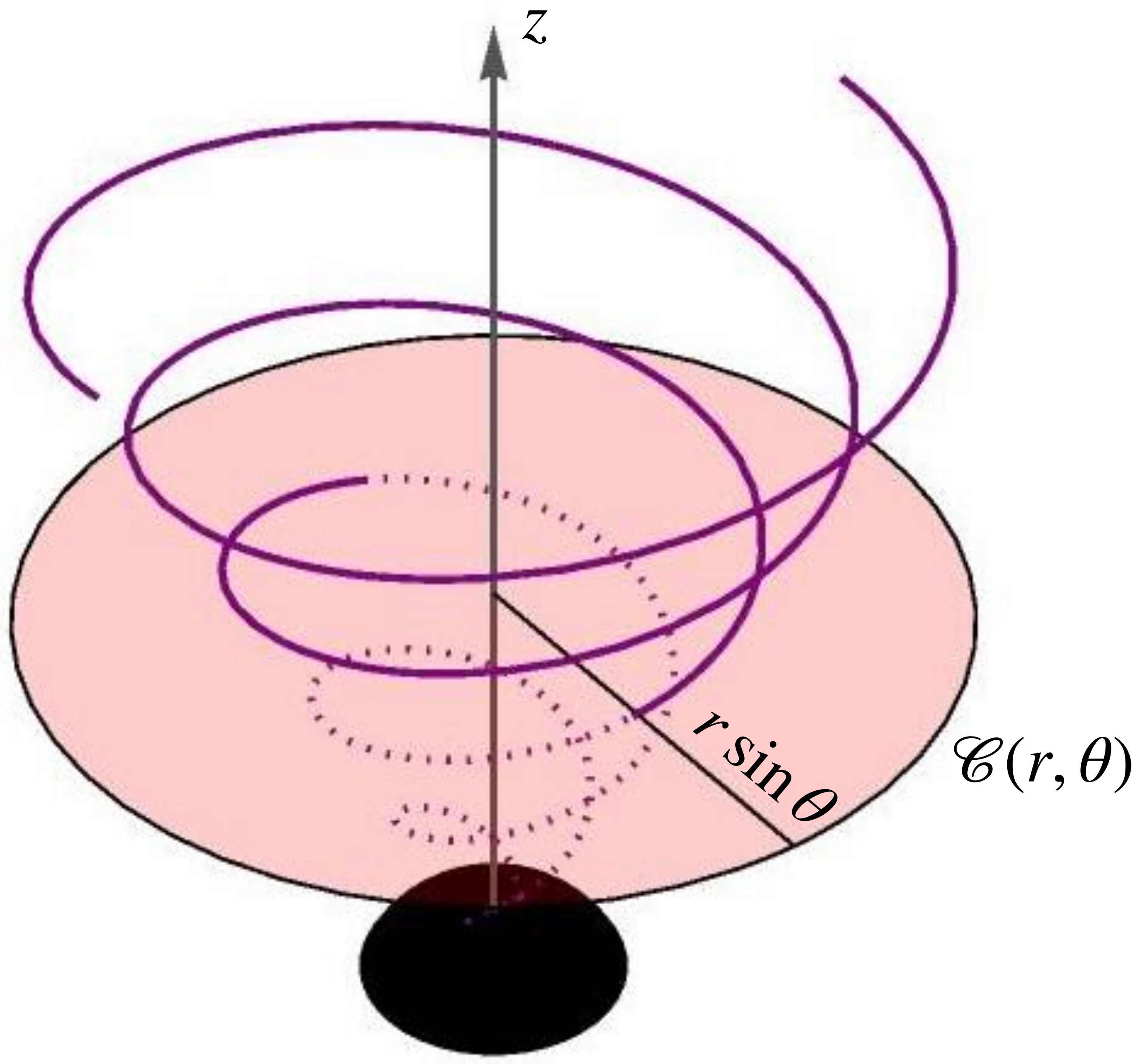

Stationary-Axisymmetric Kerr Magnetospheres $\mathscr{L}_{\partial_t}F = \mathscr{L}_{\partial_\phi}F = 0$

$$F = -I(\psi)\frac{\Sigma}{\Delta\sin\theta}dr\wedge d\theta + d\psi\wedge\left(d\phi - \Omega(\psi)dt\right)$$

Force-free field variables

$\psi(r,\theta)$	**MAGNETIC FLUX** through	$\mathscr{C}(r,\theta)$
$I(\psi)$	**POLOIDAL CURRENT** through	$\mathscr{C}(r,\theta)$
$\Omega(\psi)$	**ANGULAR VELOCITY OF FIELD LINES**	

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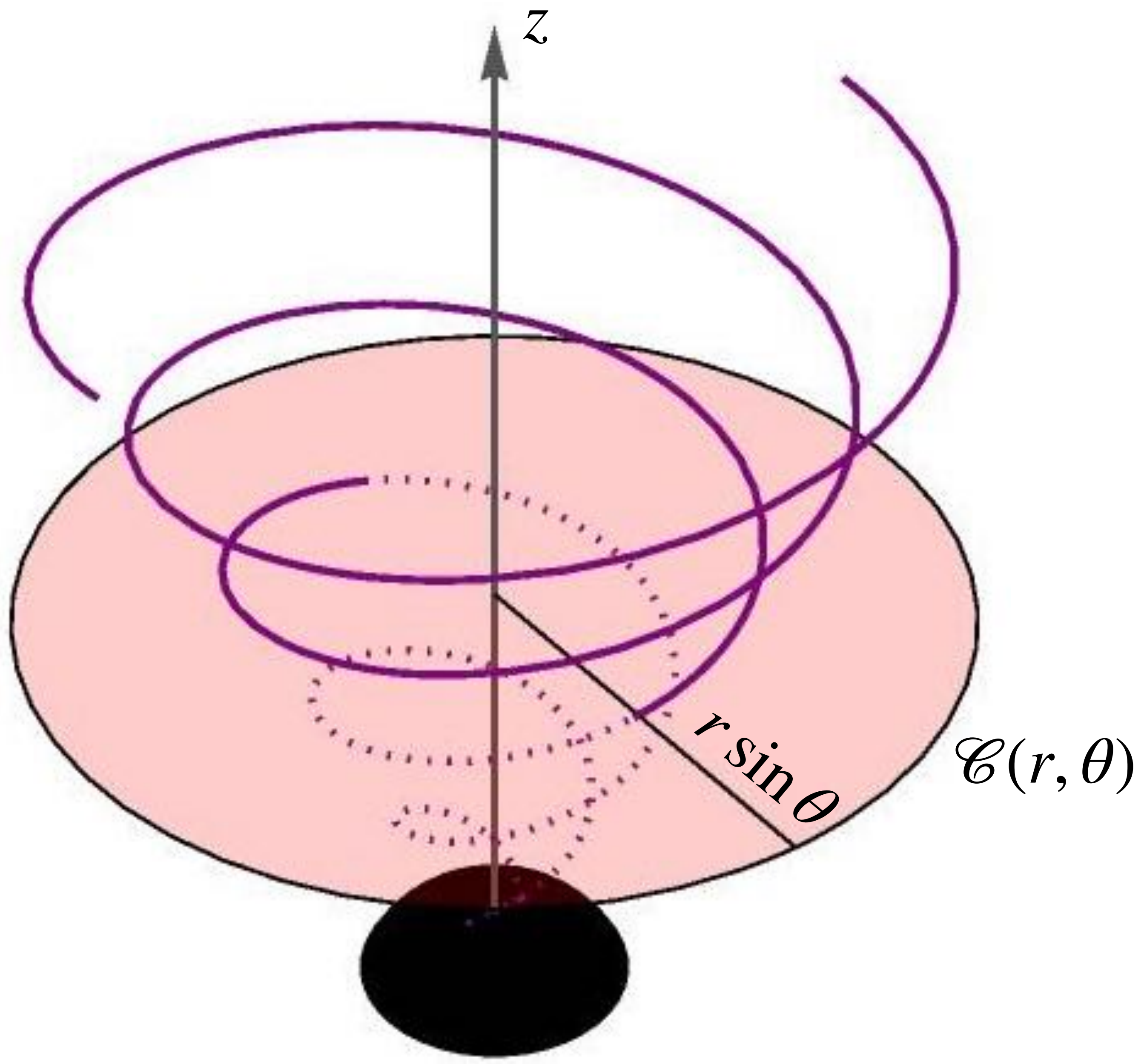

The FFE system give a relation among (ψ, I, Ω)

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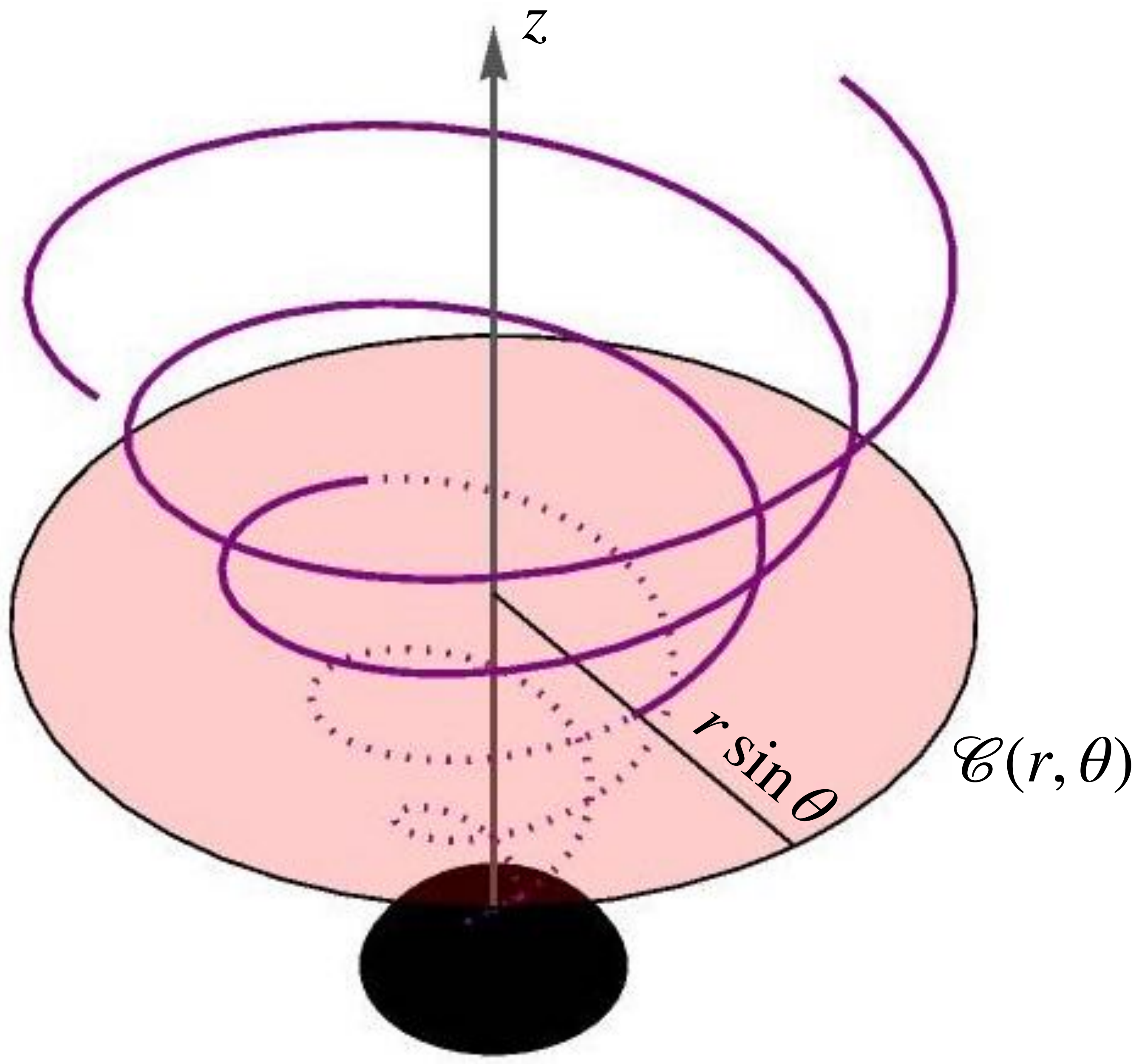

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STREAM EQUATION IN KERR

$$\eta_\mu\partial_r\left(\eta^\mu\Delta\sin\theta\,\partial_r\psi\right) + \eta_\mu\partial_\theta\left(\eta^\mu\sin\theta\,\partial_\theta\psi\right) + \frac{\Sigma}{\Delta\sin\theta}I\frac{dI}{d\psi} = 0$$

co-rotational 1-form

$$\eta = d\phi - \Omega(\psi)dt$$

STREAM EQUATION AND CRITICAL SURFACES

$$\boxed{\eta_\mu \partial_r\left(\eta^\mu \Delta \sin\theta\, \partial_r \psi\right) + \eta_\mu \partial_\theta\left(\eta^\mu \sin\theta\, \partial_\theta \psi\right) + \frac{\Sigma}{\Delta \sin\theta} I \frac{dI}{d\psi} = 0}$$

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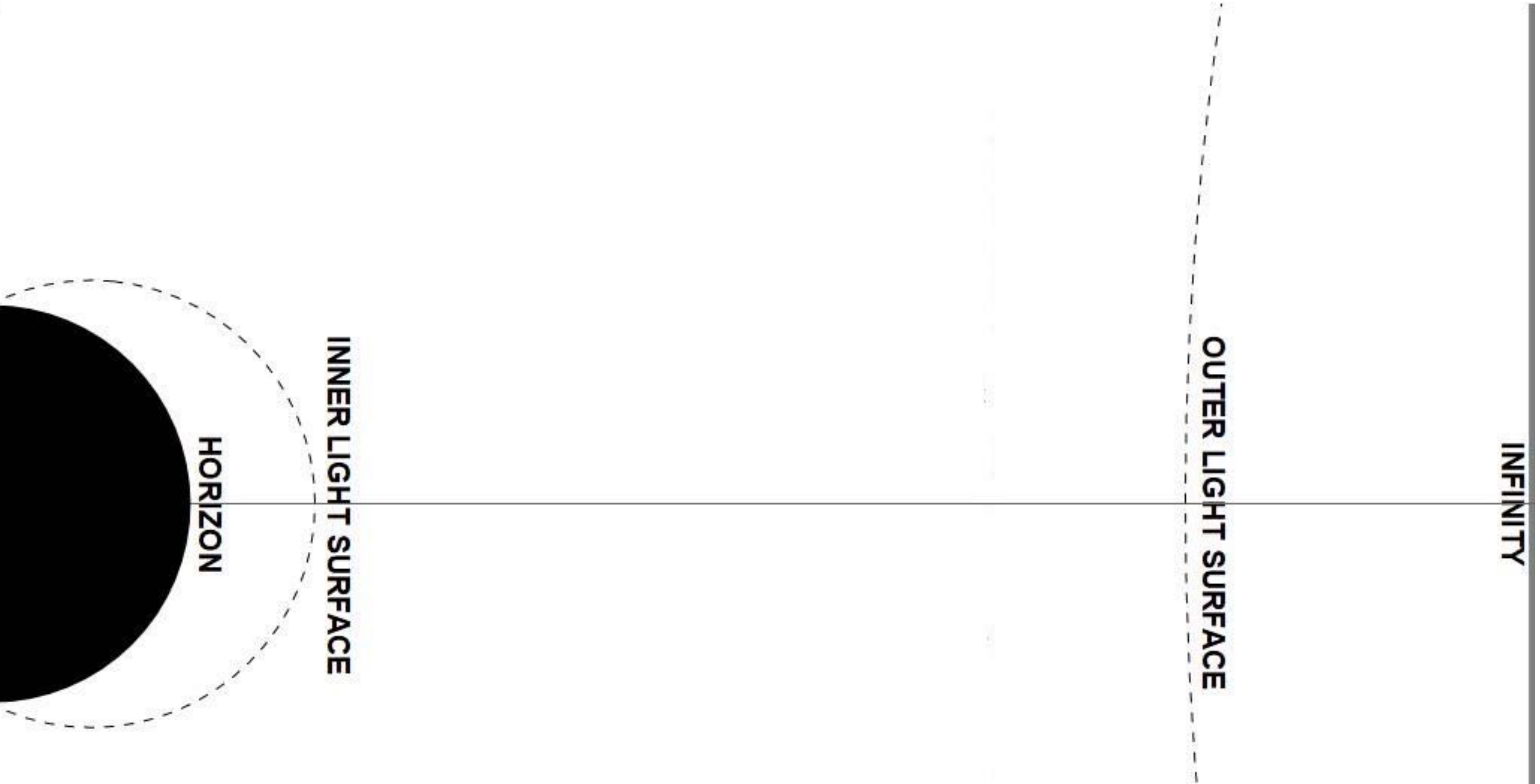
Critical surfaces reveal the structure of BH magnetospheres

Four critical surfaces, each with a proper regularity condition!



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Znajek Conditions

At the **HORIZON**, $r \to r_+$

$$\boxed{I(r_+, \theta) = \left[\left(\frac{r_0 r_+}{\Sigma} \sin\theta\right)(\Omega_H - \Omega)\, \partial_\theta \psi\right]\Bigg|_{r_+}}$$

At **INFINITY**, $r \to \infty$

$$\boxed{I^\infty(\theta) = \sin\theta\, \Omega^\infty(\theta) (\partial_\theta \psi)^\infty}$$

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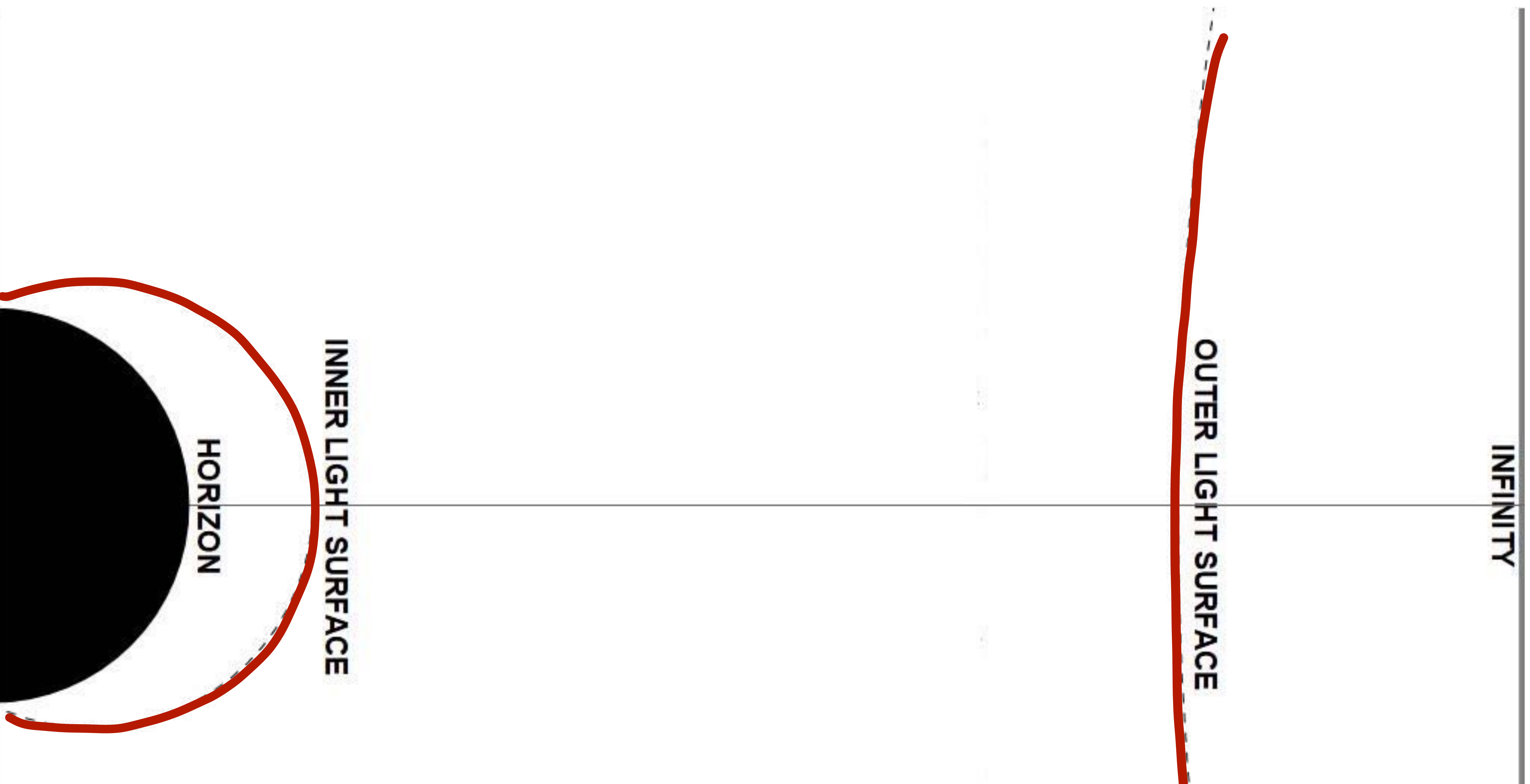

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Reduced Stream Equation

At the **LIGHT SURFACES**, $\eta^\mu \eta_\mu = 0$

$$\boxed{\Delta\, \eta_\mu \partial_r \eta^\mu \partial_r \psi + \eta_\mu \partial_\theta \eta^\mu \partial_\theta \psi + \frac{\Sigma}{\Delta \sin^2\theta} I \frac{dI}{d\psi} = 0}$$

BLANDFORD & ZNAJEK PERTURBATIVE APPROACH

Blandford, Znajek 1977

$$ds^2 = -\left(1 - \frac{r_0}{r}\right) dt^2 + \left(1 - \frac{r_0}{r}\right)^{-1} dr^2 + r^2 d\Omega^2 + \mathcal{O}(\alpha) \qquad \alpha \ll 1$$

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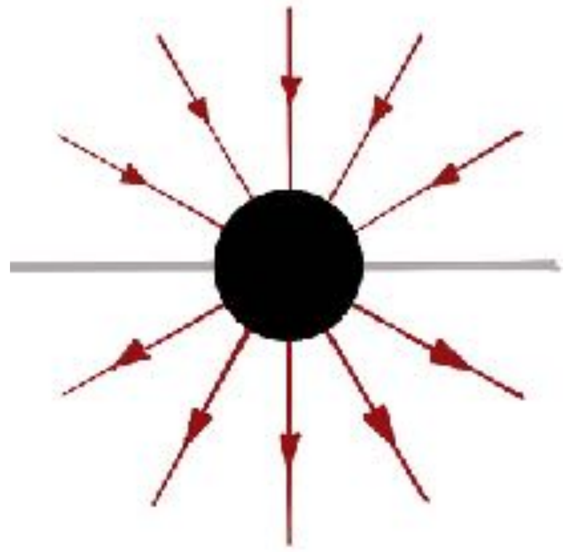
SPLIT MONOPOLE

$$\psi_0 = 1 - \cos\theta$$



$$\psi|_{\theta=0} = \psi|_{\theta=\pi/2} = \partial_\theta \psi|_{\theta=0} = 0,$$

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- Stream equation **perturbatively** to construct **corrections** to the field variables

$$\psi = \psi_0 + \alpha^2\psi_2 + \alpha^4\psi_4 + \mathcal{O}(\alpha^5)$$
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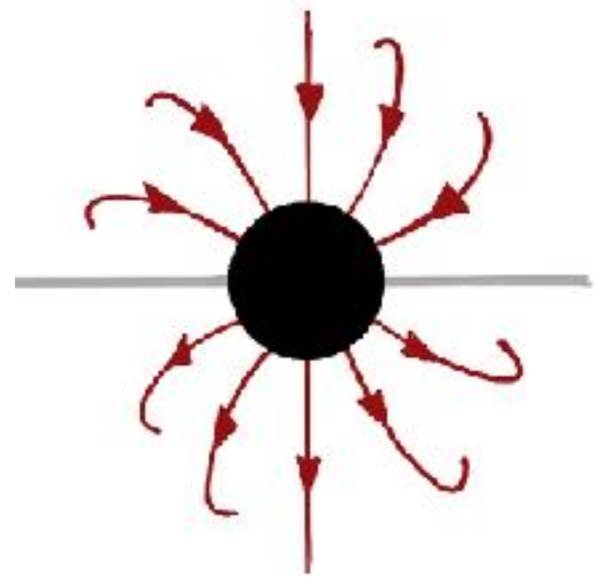

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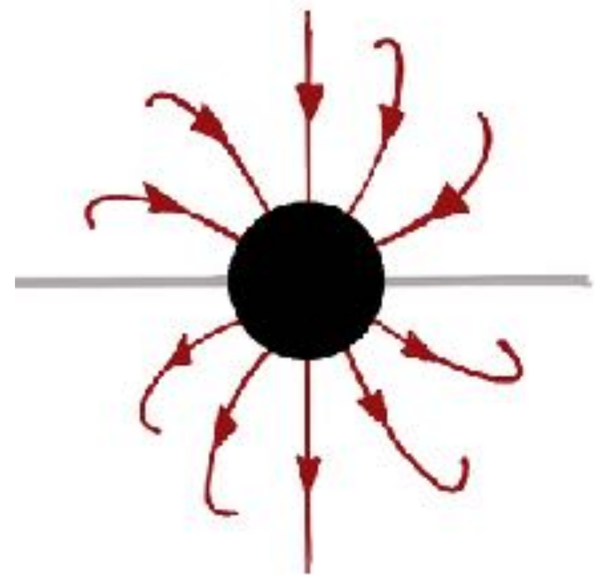
- Compute **energy and angular momentum extraction rates** for the **BZ process** at the horizon (at infinity)

$$\dot{E}_+ = 2\pi \int_0^\pi \Omega(r_+, \theta) I(r_+, \theta) \partial_\theta \psi(r_+, \theta)\, d\theta \longrightarrow \dot{E}_+ = \alpha^2\, \dot{E}_+^{(2)} + \alpha^4\, \dot{E}_+^{(4)} \dots$$

$$\dot{L}_+ = 2\pi \int_0^\pi I(r_+, \theta) \partial_\theta \psi(r_+, \theta)\, d\theta \longrightarrow \dot{L}_+ = \alpha\, \dot{L}_+^{(1)} + \alpha^3\, \dot{L}_+^{(3)} + \dots$$

SPLIT MONOPOLE

$$\psi_0 = 1 - \cos\theta$$



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$$\longrightarrow \boxed{\mathscr{L}\psi_n(r,\theta) = \mathcal{S}(r,\theta; \psi_{k<n}, i_{k<n}, \omega_{k<n})}$$

Stream Operator

$$\mathscr{L} = \frac{1}{\sin\theta}\partial_r\left[\left(1 - \frac{r_0}{r}\right)\partial_r\right] + \frac{1}{r^2}\partial_\theta\left(\frac{1}{\sin\theta}\partial_\theta\right)$$

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$$\left.\begin{aligned} \mathscr{L}^\theta_\ell[\Theta] &= \frac{d}{d\theta}\left(\frac{1}{\sin\theta}\frac{d\Theta}{d\theta}\right) + \frac{\ell(\ell+1)\Theta}{\sin\theta} = 0 \\ \mathscr{L}^r_\ell[R] &= \frac{d}{dr}\left[\left(1 - \frac{r_0}{r}\right)\frac{dR}{dr}\right] - \frac{\ell(\ell+1)R}{r^2} = 0 \end{aligned}\right\} \psi(r,\theta) = R(r)\Theta(\theta)$$

Sturm-Liouville problem

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Split-Monopole $\psi_0 = 1 - \cos\theta = \Theta_0(\theta)$

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Armas et Al (2020) $\vdots$

BLANDFORD & ZNAJEK PERTURBATIVE APPROACH

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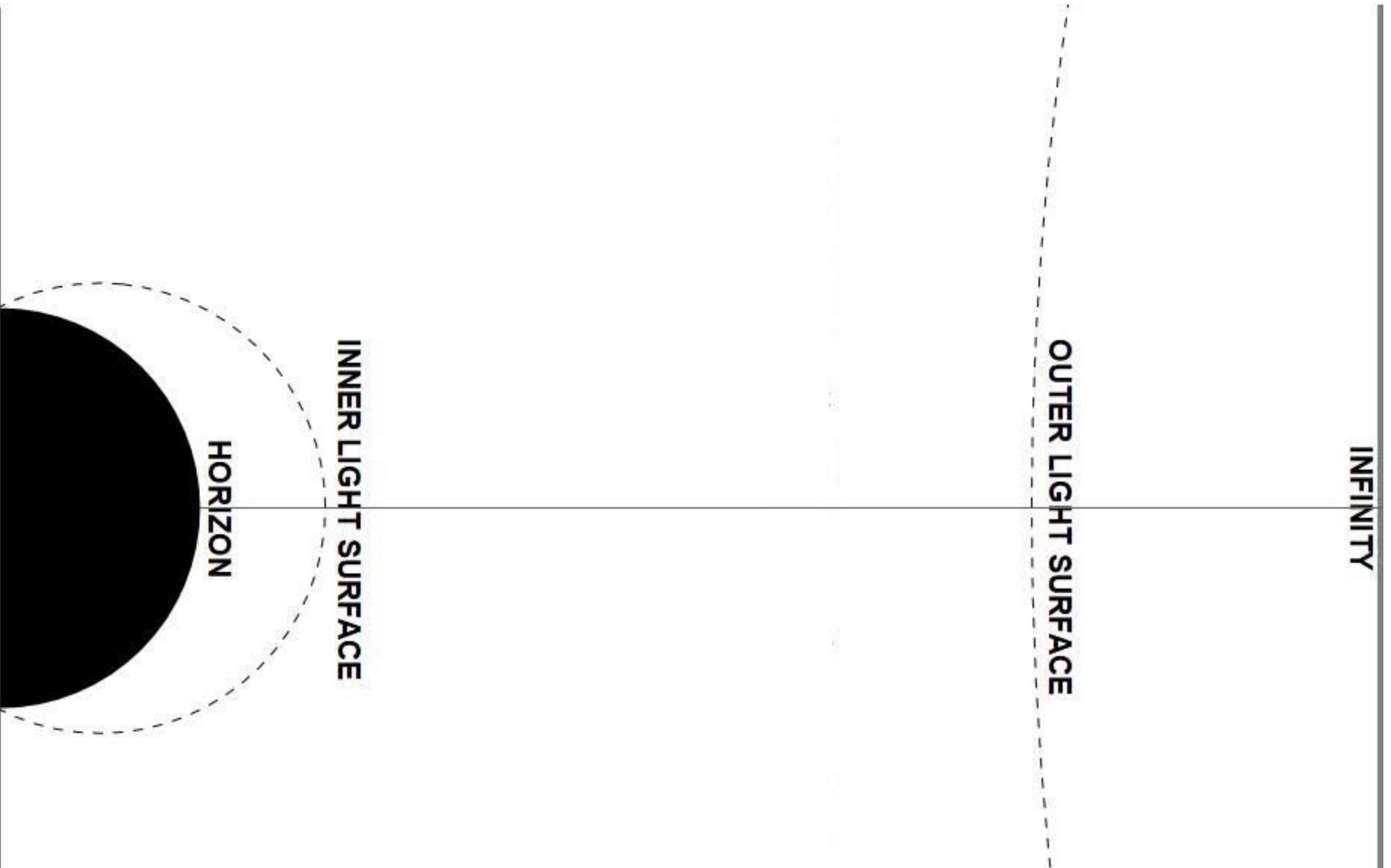
$\vdots$

BZ PERTURBATIVE APPROACH AND MATCHED ASYMPTOTIC EXPANSION (I)

From $\psi = \psi_0 + \mathcal{O}(\alpha)\ ,\ I(\psi) = \mathcal{O}(\alpha)\ ,\ \Omega(\psi) = \mathcal{O}(\alpha)$ the critical surfaces scale as

$$\frac{r_+}{r_0} = \frac{r_{\mathrm{ILS}}}{r_0} = 1 + \mathcal{O}(\alpha^2) \quad , \qquad \frac{r_{\mathrm{OLS}}}{r_0} \sim \frac{1}{\alpha} + \mathcal{O}(\alpha^0)$$



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NON PERTURBATIVE SCALING!
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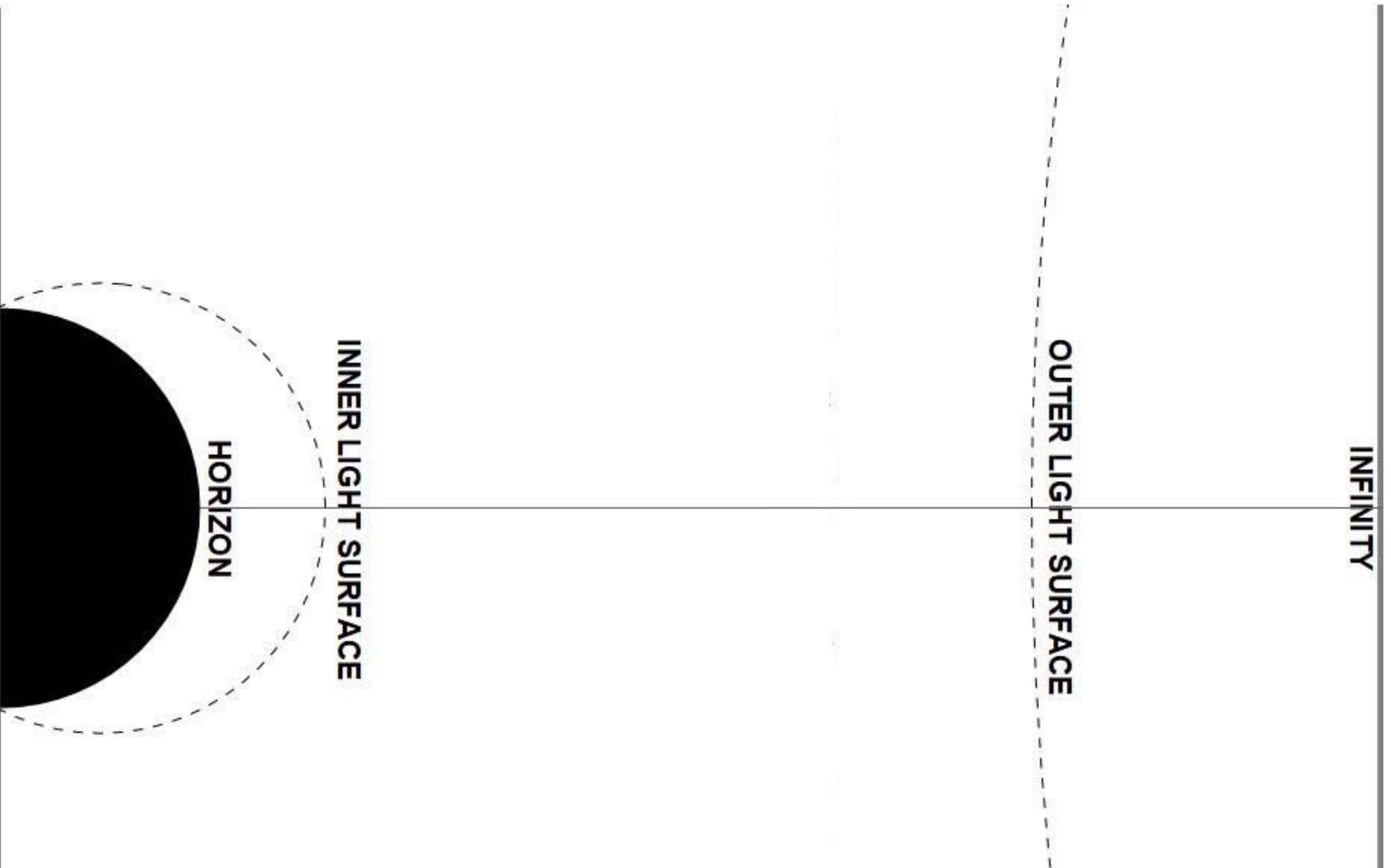
Armas et Al 2020



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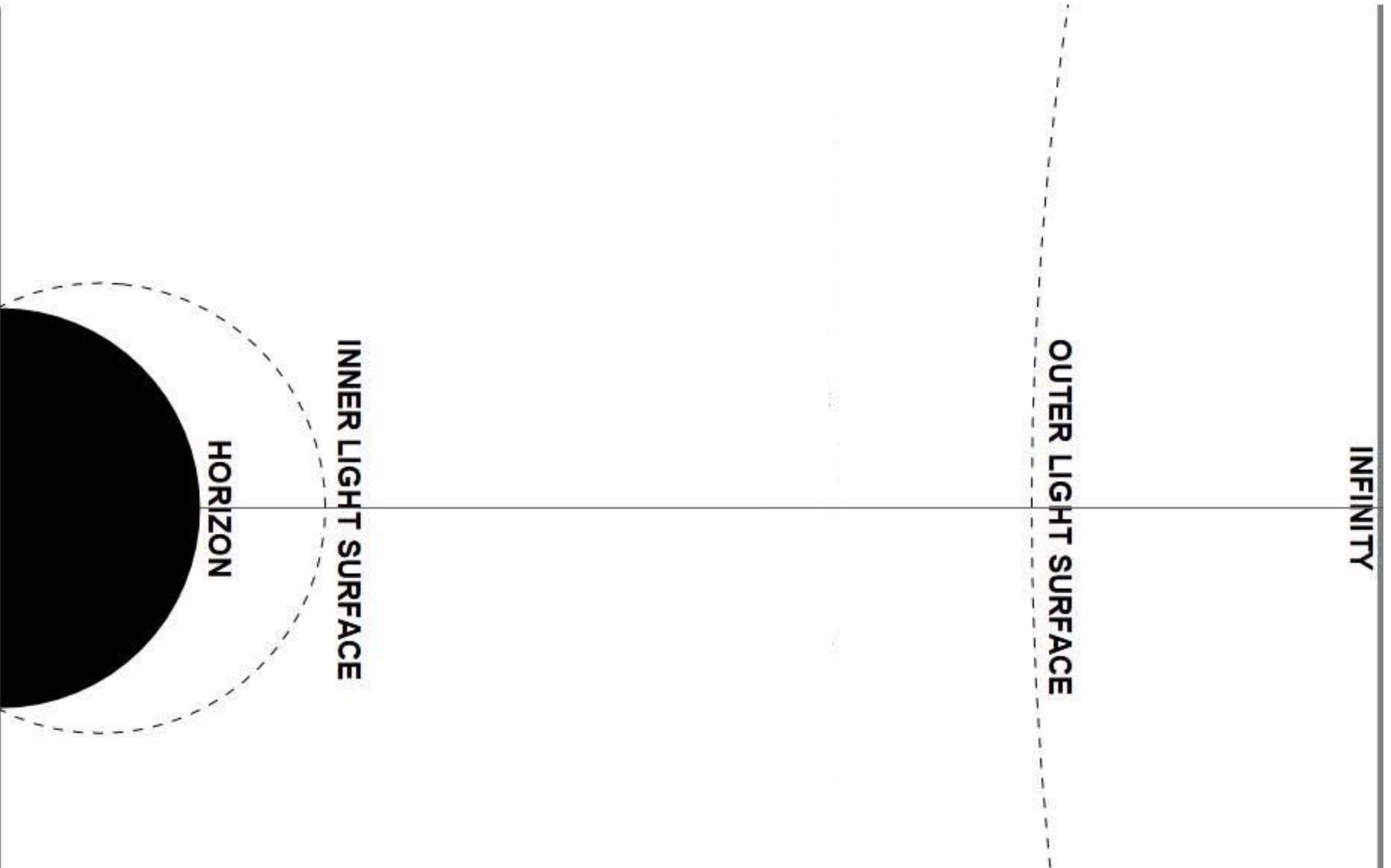
Armas et Al 2020

To resolve the OLS $\bar{r} = \alpha r$



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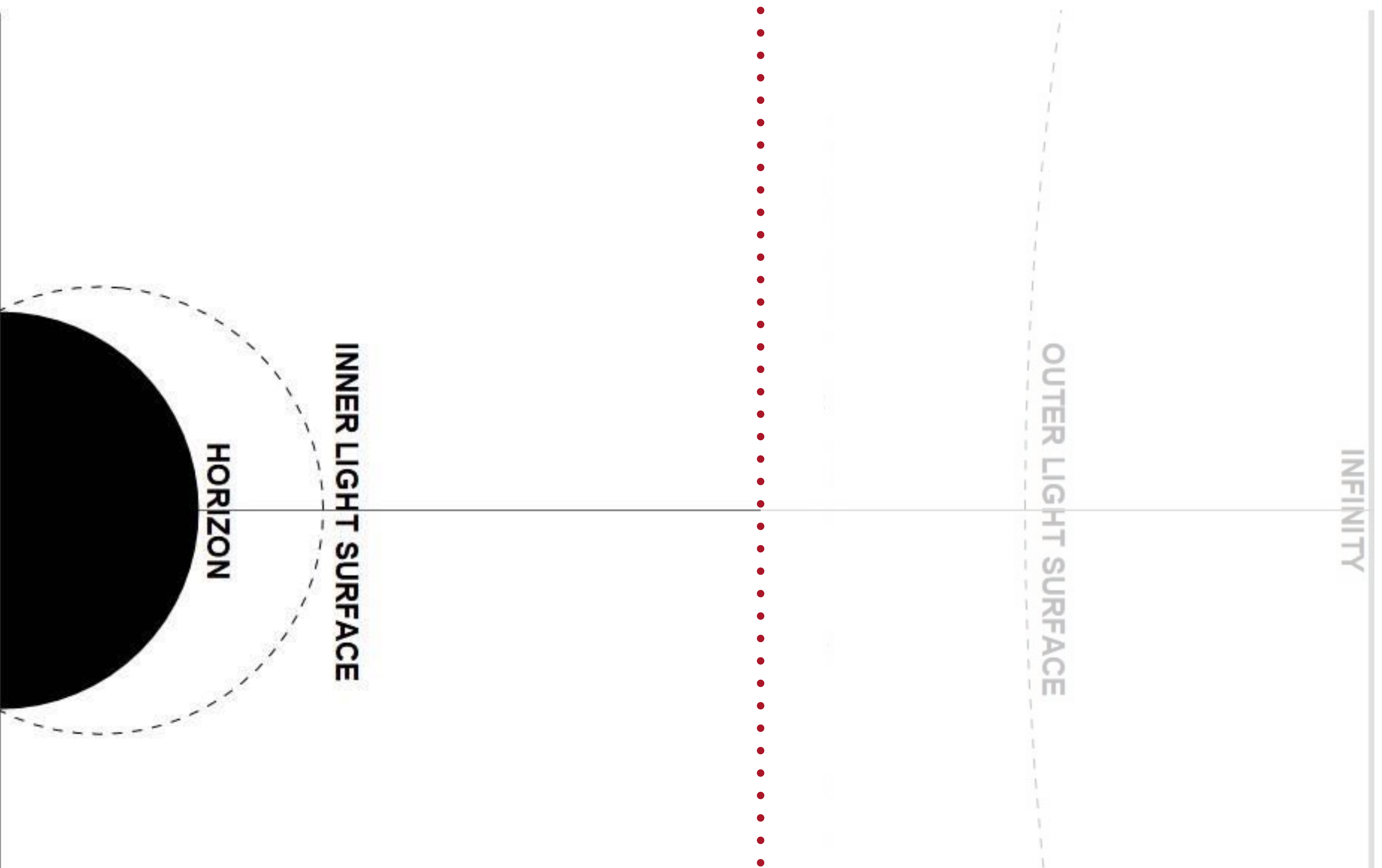

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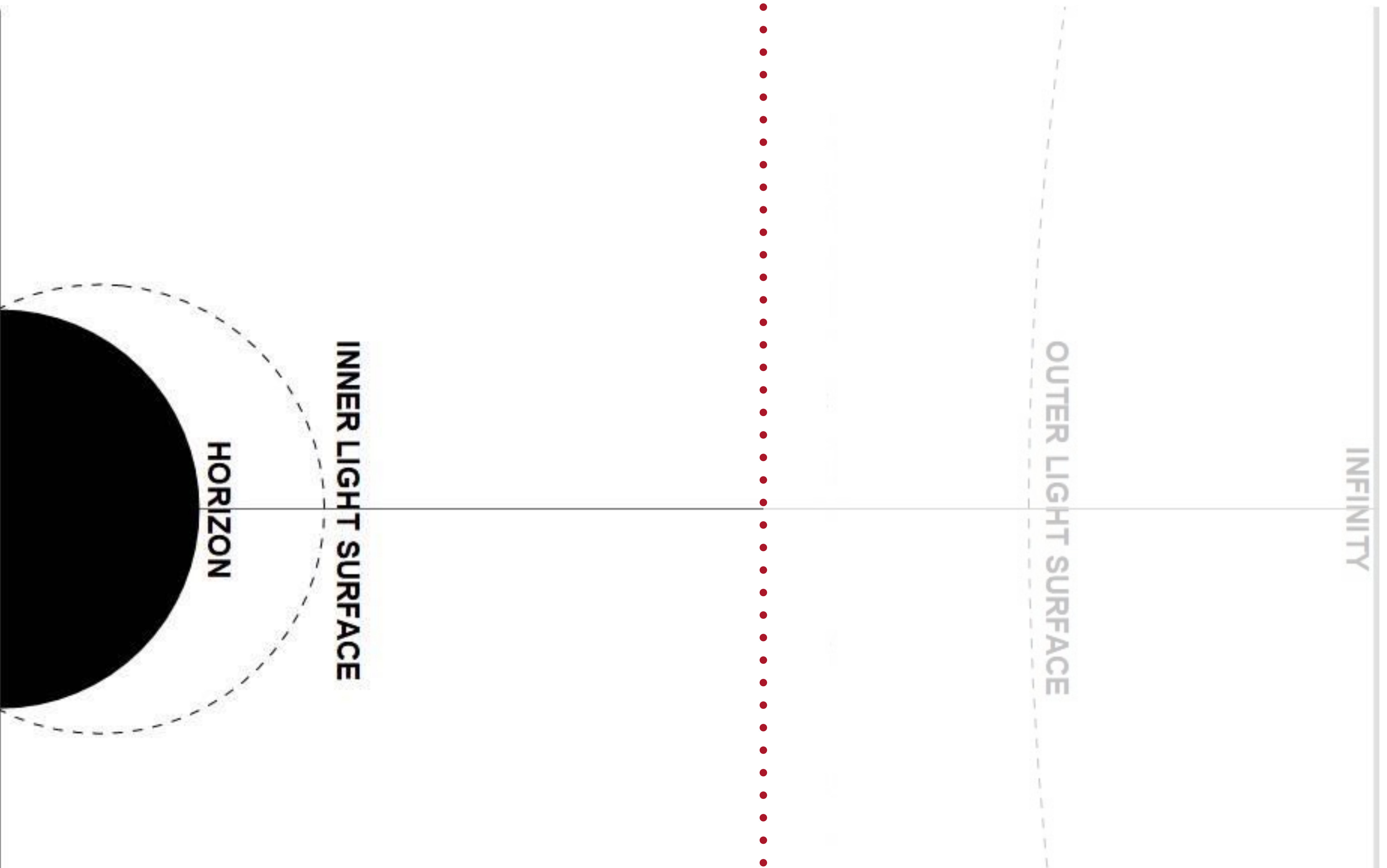

r-region $\quad \frac{r}{r_0} \ll \frac{1}{\alpha} \quad \Leftrightarrow \quad \frac{\bar{r}}{r_0} \ll 1$

- **α-EXPANSION** in r

$$\psi(r,\theta) = \psi_0 + \alpha^2\, \psi_2 + \mathcal{O}(\alpha^4)$$

$$I(\psi) = \alpha\; i_1 + \mathcal{O}(\alpha^3) \quad , \quad \Omega(\psi) = \alpha\; \omega_1 + \mathcal{O}(\alpha^3)$$

- **STREAM EQUATION** in r

$$\mathcal{L}\psi_n(r,\theta) = \mathcal{S}(r,\theta;\psi_{k<n}, i_{k<n}, \omega_{k<n})$$

- **ZNAJEK CONDITION at** $r = r_+$ & **ILS CONDITION**

BZ PERTURBATIVE APPROACH AND MATCHED ASYMPTOTIC EXPANSION (I)

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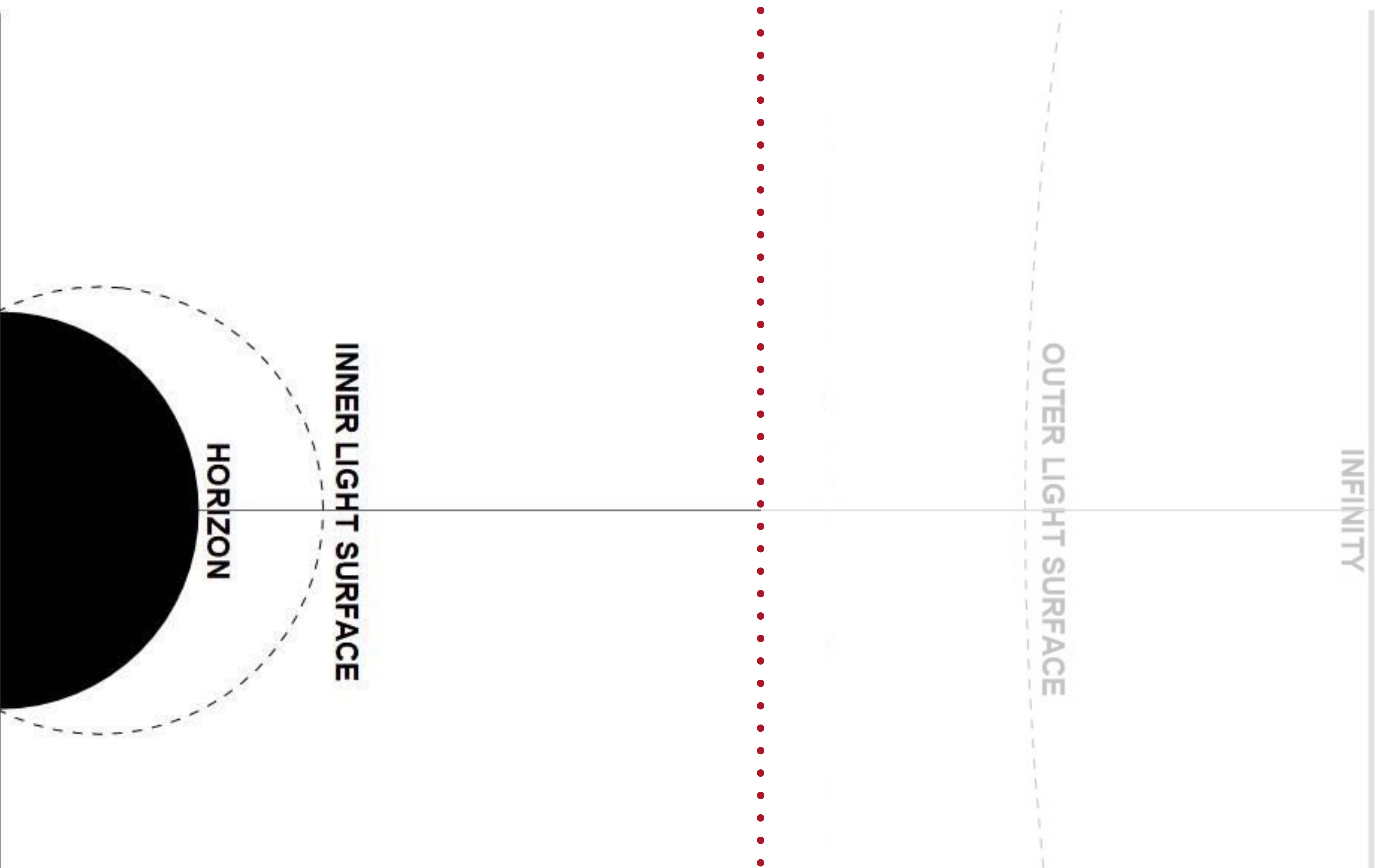
$$\frac{r_+}{r_0} = \frac{r_{\mathrm{ILS}}}{r_0} = 1 + \mathcal{O}(\alpha^2) \quad , \quad \boxed{\frac{r_{\mathrm{OLS}}}{r_0} \sim \frac{1}{\alpha} + \mathcal{O}(\alpha^0)}$$

NON PERTURBATIVE SCALING!

MATCHED ASYMPTOTIC EXPANSION

Armas et Al 2020

To resolve the OLS $\bar{r} = \alpha r$



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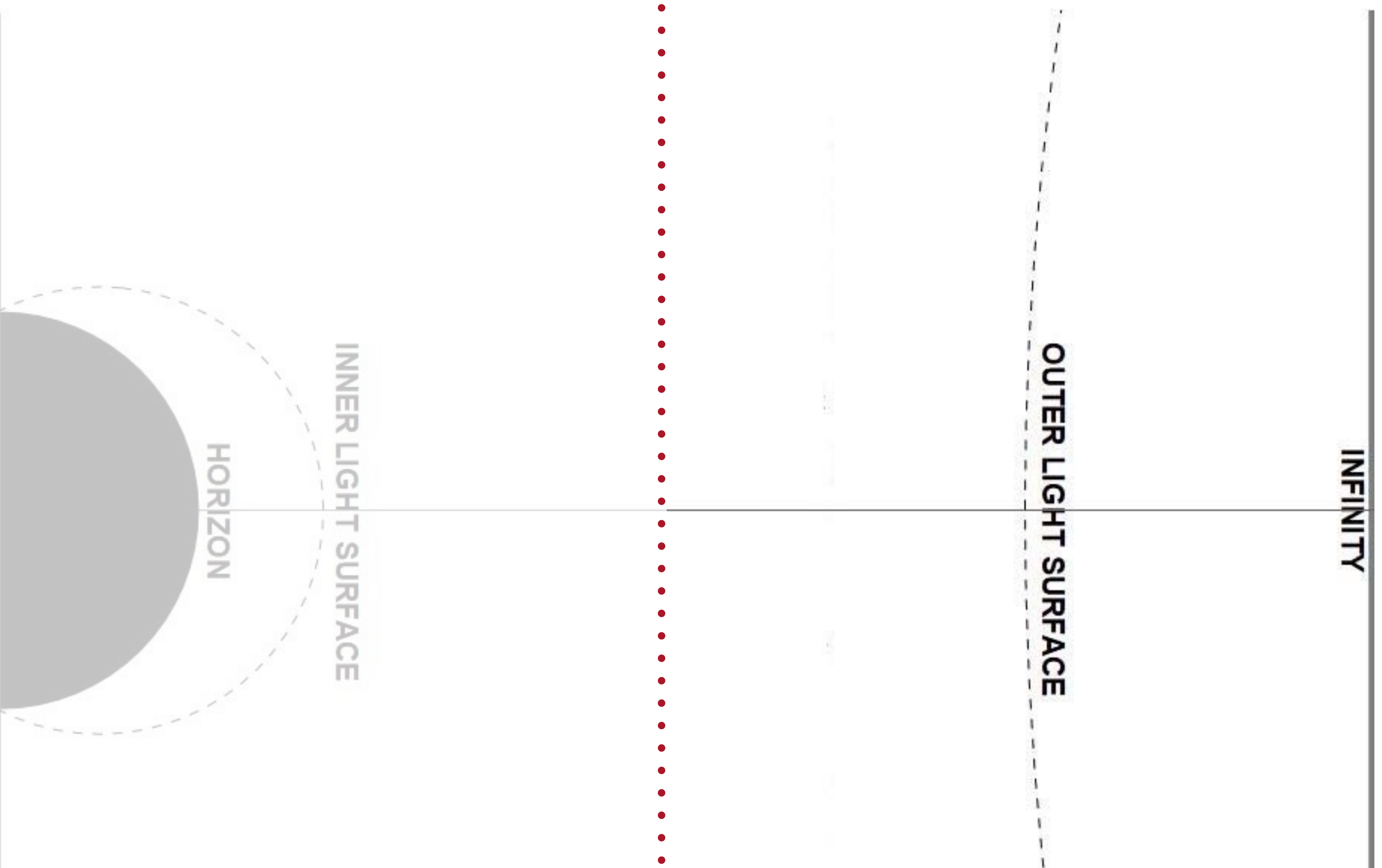

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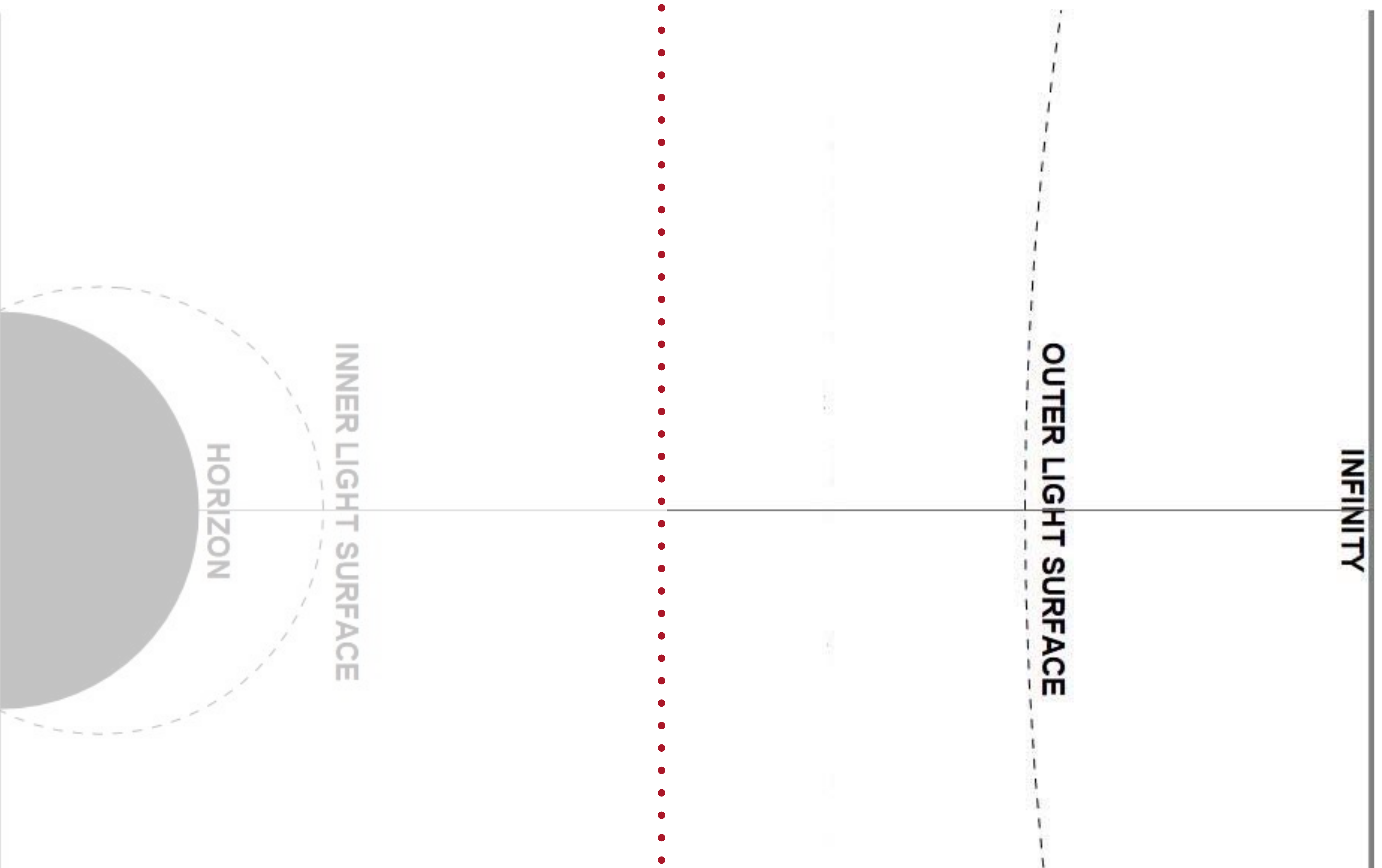

$\bar{r}$-region $\quad \frac{r}{r_0} \gg 1 \quad \Leftrightarrow \quad \frac{\bar{r}}{r_0} \gg \alpha$

- **α-EXPANSION** in $\bar{r}$

$$\psi(\bar{r}, \theta) = \psi_0 + \alpha^3 \, \bar{\psi}_3 + \mathcal{O}(\alpha^4)$$

$$I(\psi) = \alpha \, \bar{i}_1 + \mathcal{O}(\alpha^3) \quad , \quad \Omega(\psi) = \alpha \, \bar{\omega}_1 + \mathcal{O}(\alpha^3)$$

- **STREAM EQUATION** in $\bar{r}$

$$\bar{\mathcal{L}} \bar{\psi}_n(\bar{r}, \theta) = \mathcal{S}(\bar{r}, \theta; \; \bar{\psi}_{k<n}, \bar{i}_{k<n+1}, \bar{\omega}_{k<n+1})$$

- **ZNAJEK CONDITION at $\bar{r} = \infty$** & **OLS CONDITION**

BZ PERTURBATIVE APPROACH AND MATCHED ASYMPTOTIC EXPANSION (I)

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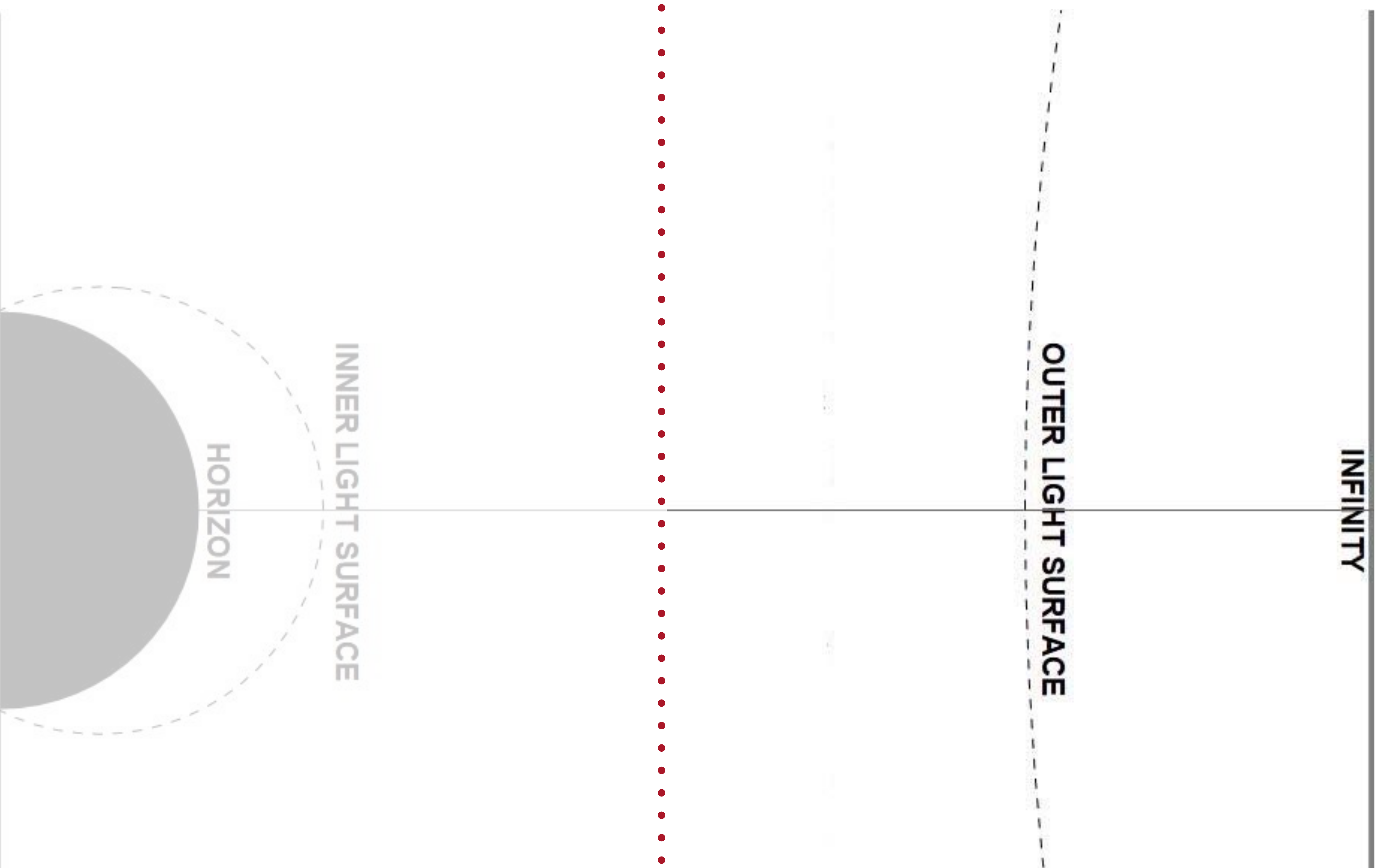

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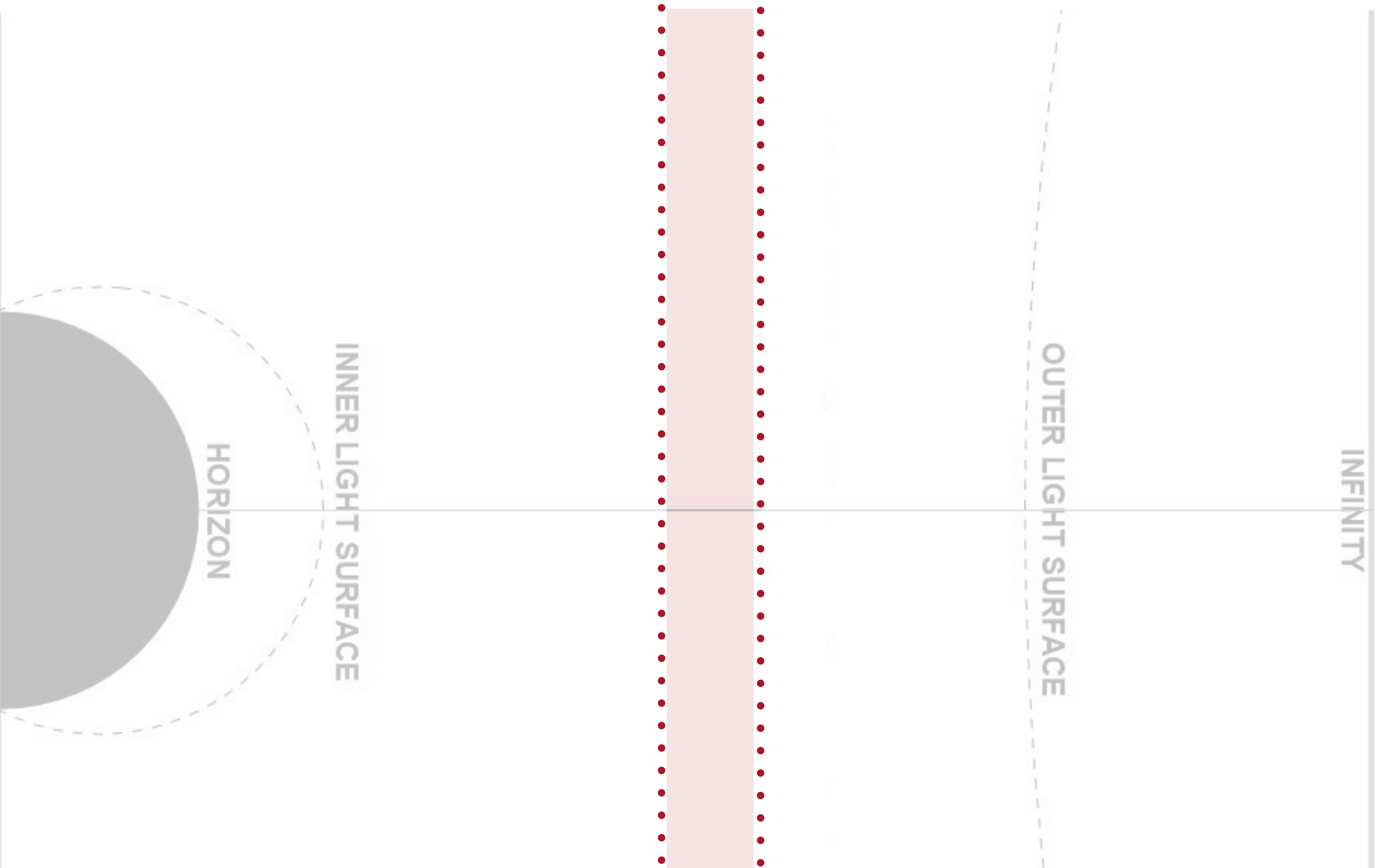

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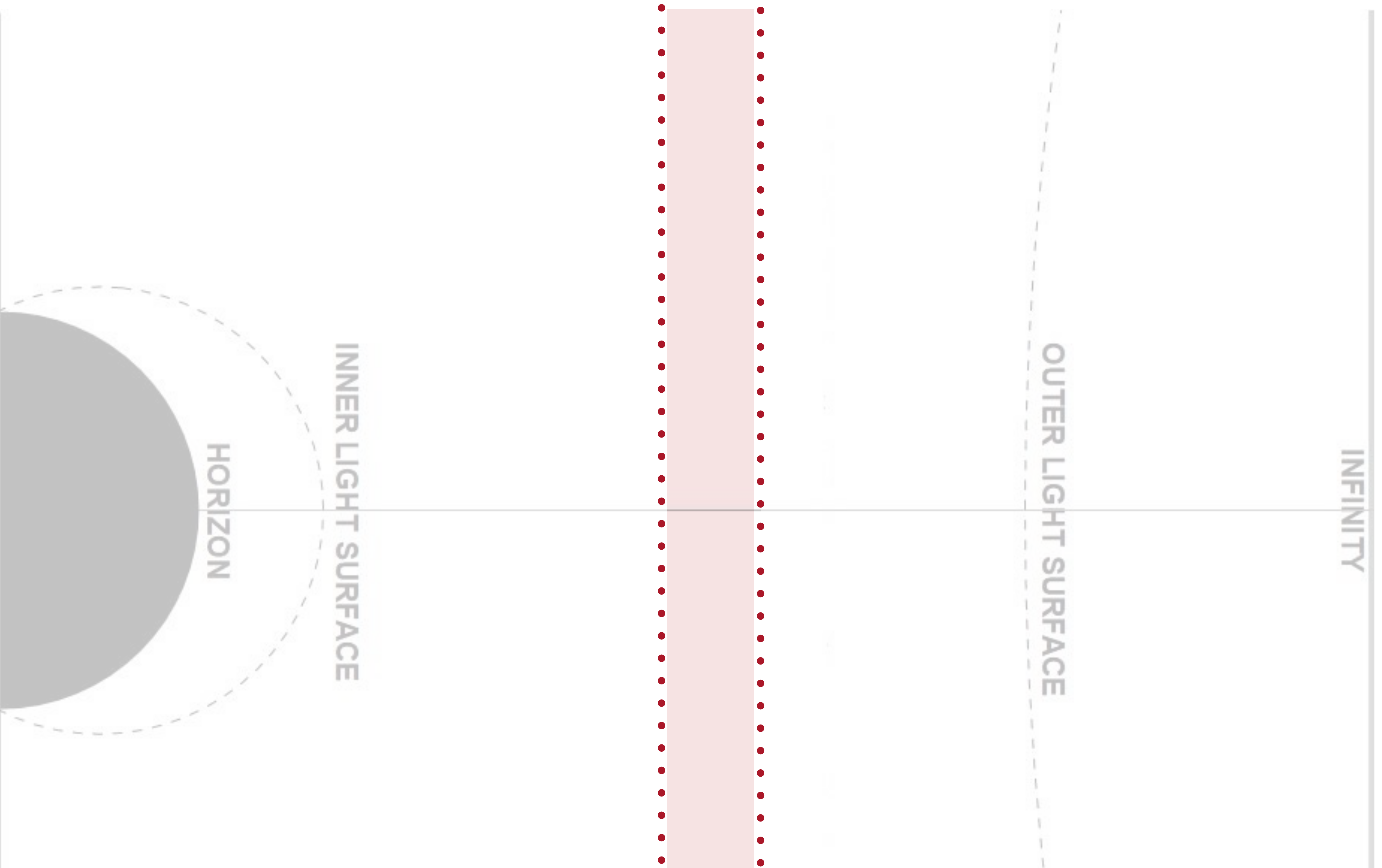

overlap region $1 \ll \frac{r}{r_0} \ll \frac{1}{\alpha} \quad \Leftrightarrow \quad \alpha \ll \frac{\bar{r}}{r_0} \ll 1$

- **BOTH EXPANSIONS** VALID!

$$\psi(r,\theta) = \psi_0 + \alpha^2\, \psi_2 + \mathcal{O}(\alpha^4)$$

$$\psi(\bar{r},\theta) = \psi_0 + \alpha^3\, \bar{\psi}_3 + \mathcal{O}(\alpha^4)$$

- $\forall\ \ \mathcal{O}(\alpha)$ **ASYMPTOTICS MATCH**!

$$\lim_{r\to\infty} \psi(r,\theta) = \lim_{\bar{r}\to 0} \psi(\bar{r},\theta)$$

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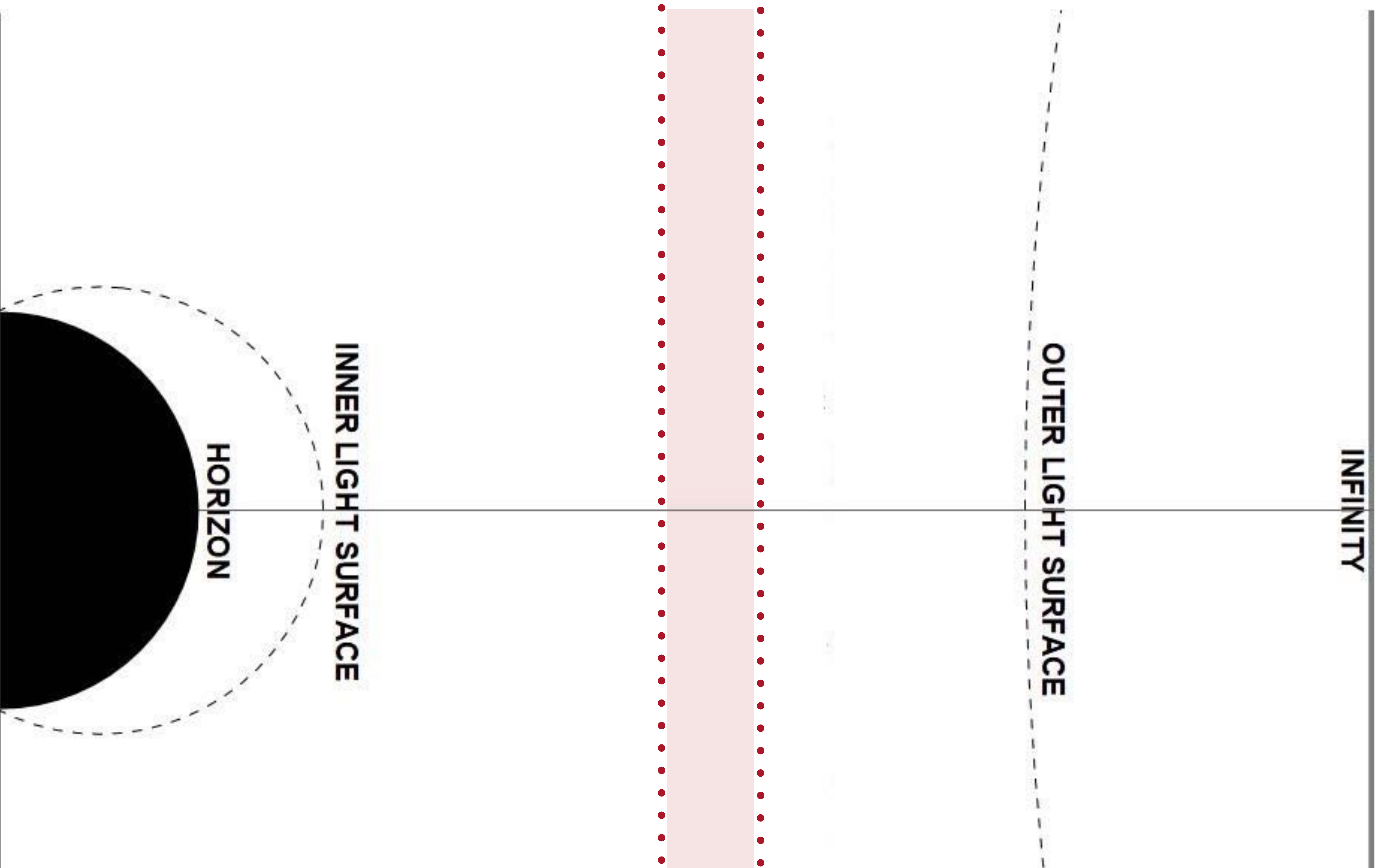

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BZ PERTURBATIVE APPROACH AND MATCHED ASYMPTOTIC EXPANSION (II)

r-REGION ANSATZ

$$\psi(r,\theta) = \psi_0(\theta) + \alpha^2\,\psi_2(r,\theta) + \alpha^4\,\psi_4(r,\theta) + \alpha^5\,\boldsymbol{\psi_5(r,\theta)} + \mathcal{O}(\alpha^6\log\alpha),$$

$$r_0 I(\psi) = \alpha\, i_1(\psi_0) + \alpha^3 i_3(\psi_2) + \alpha^4\, i_4(\psi_0) + \alpha^5\left[\boldsymbol{I_5(r,\theta) + \log\alpha\, I_{5L}(r,\theta)}\right] + \mathcal{O}(\alpha^6\log\alpha),$$

$$r_0\Omega(\psi) = \alpha\,\omega_1(\psi_0) + \alpha^3\,\omega_3(\psi_0) + \alpha^4\,\omega_4(\psi_0) + \alpha^5\left[\boldsymbol{\Omega_5(r,\theta) + \log\alpha\,\Omega_{5L}(r,\theta)}\right] + \mathcal{O}(\alpha^6\log\alpha)\,.$$

$\bar{r}$-REGION ANSATZ

$$\psi(\bar{r},\theta) = \psi_0(\theta) + \alpha^3\,\bar{\psi}_3(\bar{r},\theta) + \alpha^4\left[\boldsymbol{\bar{\psi}_4(\bar{r},\theta) + \log\alpha\,\bar{\psi}_{4L}(\bar{r},\theta)}\right] + \mathcal{O}(\alpha^5\log\alpha),$$

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r-REGION ANSATZ

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DETERMINE

$\psi_5 = ?$

$\bar{\psi}_4 = ?$

$\bar{\psi}_{4L} = ?$

$\bar{r}$-REGION ANSATZ

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Analytically:

$$\psi_5 = R_2^{(5)}(r)\Theta_2(\theta) + R_4^{(5)}(r)\Theta_4(\theta) + R_6^{(5)}(r)\Theta_6(\theta) + \ldots$$

$$\bar{\psi}_{4L} = -\frac{1}{40}\frac{r_0^2}{\bar{r}^2}\Theta_2(\theta) - \left[\frac{1}{1680}\Theta_2(\theta) + \frac{3}{22400}\Theta_4(\theta)\right]$$

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$\bar{r}$-REGION ANSATZ

$$\psi(\bar r,\theta) = \psi_0(\theta) + \alpha^3\,\bar\psi_3(\bar r,\theta) + \alpha^4\left[\boldsymbol{\bar\psi_4(\bar r,\theta) + \log\alpha\,\bar\psi_{4L}(\bar r,\theta)}\right] + \mathcal{O}(\alpha^5 \log\alpha),$$

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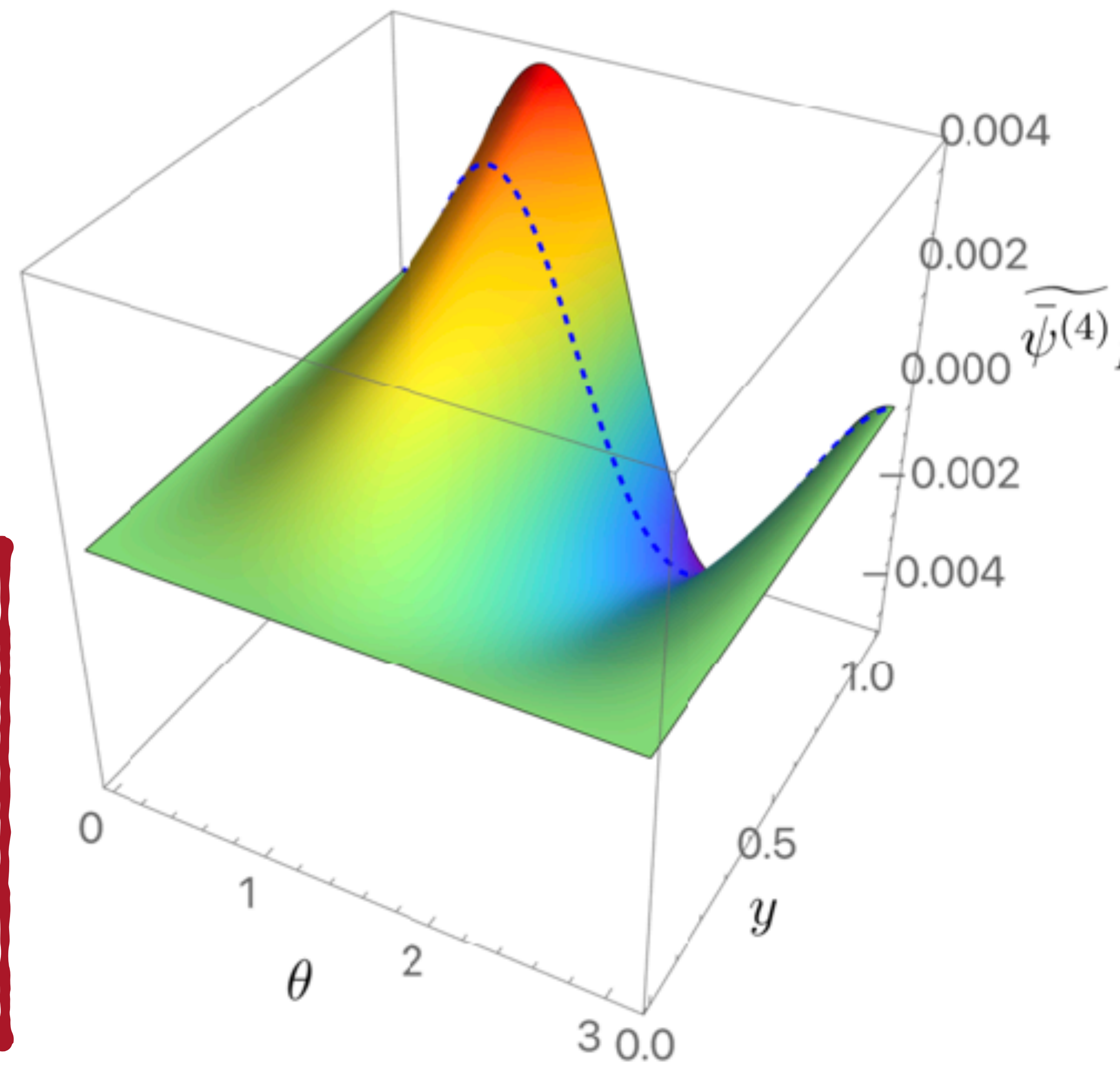

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Numerical integration, assuming

$$\bar\psi_4^{\infty}(\theta) = \bar c_2^{(4)}\Theta_2(\theta) + \bar c_4^{(4)}\Theta_4(\theta) + \bar c_6^{(4)}\Theta_6(\theta) + \dots$$

BZ PERTURBATIVE APPROACH AND MATCHED ASYMPTOTIC EXPANSION (III)

MATCHING IN THE OVERLAP REGION

- r-region

$$\psi_2(r,\theta) = \left[\frac{1}{8}\frac{r_0}{r} - \frac{11}{800}\frac{r_0^2}{r^2} + \frac{1}{40}\frac{r_0^2}{r^2}\log\frac{r}{r_0}\right]\Theta_2(\theta) + \mathcal{O}\left(\frac{r_0^3}{r^3}\log\frac{r}{r_0}\right)$$

$$\psi_4(r,\theta) = \left[\frac{1}{224}\frac{r}{r_0} + \frac{227}{100800} + \frac{1}{1680}\log\frac{r}{r_0}\right]\Theta_2(\theta) + \left[\frac{9}{8960}\frac{r}{r_0} + \frac{363}{896000} + \frac{3}{22400}\log\frac{r}{r_0}\right]\Theta_4(\theta) + \mathcal{O}\left(\frac{r_0^3}{r^3}\log\frac{r}{r_0}\right)$$

$$\psi_5(r,\theta) = \frac{r^2}{r_0^2}\mathcal{F}(\theta) + \mathcal{O}\left(\frac{r}{r_0}\right)$$

- $\bar{r}$-region

$$\bar{\psi}_3(\bar{r},\theta) = \frac{1}{8}\frac{r_0}{\bar{r}}\Theta_2(\theta) + \frac{\bar{r}}{r_0}\left[\frac{1}{224}\Theta_2(\theta) + \frac{9}{8960}\Theta_4(\theta)\right] + \frac{\bar{r}^2}{r_0^2}\mathcal{F}(\theta) + \mathcal{O}\left(\frac{\bar{r}^3}{r_0^3}\right)$$

$$\bar{\psi}_4(\bar{r},\theta) = -\frac{11}{800}\frac{r_0^2}{\bar{r}^2}\Theta_2(\theta) + \frac{1}{40}\frac{r_0^2}{\bar{r}^2}\log\frac{\bar{r}}{r_0}\Theta_2(\theta) + \frac{227}{100800}\Theta_2(\theta) + \frac{363}{896000}\Theta_4(\theta) + \left[\frac{1}{1680}\Theta_2(\theta) + \frac{3}{22400}\Theta_4(\theta)\right]\log\frac{\bar{r}}{r_0} + \mathcal{O}\left(\frac{\bar{r}}{r_0}\right)$$

$$\bar{\psi}_{4L}(\bar{r},\theta) = -\frac{1}{40}\frac{r_0^2}{\bar{r}^2}\Theta_2(\theta) - \left[\frac{1}{1680}\Theta_2(\theta) + \frac{3}{22400}\Theta_4(\theta)\right]$$

BZ PERTURBATIVE APPROACH AND MATCHED ASYMPTOTIC EXPANSION (III)

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$$\psi_5(r,\theta) = \frac{r^2}{r_0^2}\mathcal{F}(\theta) + \mathcal{O}\left(\frac{r}{r_0}\right)$$

- $\bar{r}$-region

$$\bar{\psi}_3(\bar{r},\theta) = \frac{1}{8}\frac{r_0}{\bar{r}}\Theta_2(\theta) + \frac{\bar{r}}{r_0}\left[\frac{1}{224}\Theta_2(\theta) + \frac{9}{8960}\Theta_4(\theta)\right] + \frac{\bar{r}^2}{r_0^2}\mathcal{F}(\theta) + \mathcal{O}\left(\frac{\bar{r}^3}{r_0^3}\right)$$

$$\bar{\psi}_4(\bar{r},\theta) = -\frac{11}{800}\frac{r_0^2}{\bar{r}^2}\Theta_2(\theta) + \frac{1}{40}\frac{r_0^2}{\bar{r}^2}\log\frac{\bar{r}}{r_0}\ \Theta_2(\theta) + \frac{227}{100800}\Theta_2(\theta) + \frac{363}{896000}\Theta_4(\theta) + \left[\frac{1}{1680}\Theta_2(\theta) + \frac{3}{22400}\Theta_4(\theta)\right]\log\frac{\bar{r}}{r_0} + \mathcal{O}\left(\frac{\bar{r}}{r_0}\right)$$

$$\bar{\psi}_{4L}(\bar{r},\theta) = -\frac{1}{40}\frac{r_0^2}{\bar{r}^2}\Theta_2(\theta) - \left[\frac{1}{1680}\Theta_2(\theta) + \frac{3}{22400}\Theta_4(\theta)\right]$$

$$\psi(\bar{r},\theta) = \psi_0(\theta) + \alpha^3\bar{\psi}_3(\bar{r},\theta) + \alpha^4\left[\boldsymbol{\bar{\psi}_4(\bar{r},\theta)} + \boldsymbol{\log\alpha\,\bar{\psi}_{4L}(\bar{r},\theta)}\right] + \mathcal{O}(\alpha^5\log\alpha)$$

BZ PERTURBATIVE APPROACH AND MATCHED ASYMPTOTIC EXPANSION (III)

MATCHING IN THE OVERLAP REGION

$$\log(r/r_0) \xrightarrow{r\to\alpha^{-1}\bar{r}} \log(\bar{r}/r_0) - \log\alpha$$

- r-region

$$\psi_2(r,\theta) = \left[\frac{1}{8}\frac{r_0}{r} - \frac{11}{800}\frac{r_0^2}{r^2} + \boxed{\frac{1}{40}\frac{r_0^2}{r^2}\log\frac{r}{r_0}}\right]\Theta_2(\theta) + \mathcal{O}\left(\frac{r_0^3}{r^3}\log\frac{r}{r_0}\right)$$

$$\psi_4(r,\theta) = \left[\frac{1}{224}\frac{r}{r_0} + \frac{227}{100800} + \frac{1}{1680}\log\frac{r}{r_0}\right]\Theta_2(\theta) + \left[\frac{9}{8960}\frac{r}{r_0} + \frac{363}{896000} + \frac{3}{22400}\log\frac{r}{r_0}\right]\Theta_4(\theta) + \mathcal{O}\left(\frac{r_0^3}{r^3}\log\frac{r}{r_0}\right)$$

$$\psi_5(r,\theta) = \frac{r^2}{r_0^2}\mathcal{F}(\theta) + \mathcal{O}\left(\frac{r}{r_0}\right)$$

- $\bar{r}$-region

$$\bar{\psi}_3(\bar{r},\theta) = \frac{1}{8}\frac{r_0}{\bar{r}}\Theta_2(\theta) + \frac{\bar{r}}{r_0}\left[\frac{1}{224}\Theta_2(\theta) + \frac{9}{8960}\Theta_4(\theta)\right] + \frac{\bar{r}^2}{r_0^2}\mathcal{F}(\theta) + \mathcal{O}\left(\frac{\bar{r}^3}{r_0^3}\right)$$

$$\bar{\psi}_4(\bar{r},\theta) = -\frac{11}{800}\frac{r_0^2}{\bar{r}^2}\Theta_2(\theta) + \frac{1}{40}\frac{r_0^2}{\bar{r}^2}\log\frac{\bar{r}}{r_0}\Theta_2(\theta) + \frac{227}{100800}\Theta_2(\theta) + \frac{363}{896000}\Theta_4(\theta) + \left[\frac{1}{1680}\Theta_2(\theta) + \frac{3}{22400}\Theta_4(\theta)\right]\log\frac{\bar{r}}{r_0} + \mathcal{O}\left(\frac{\bar{r}}{r_0}\right)$$

$$\bar{\psi}_{4L}(\bar{r},\theta) = -\frac{1}{40}\frac{r_0^2}{\bar{r}^2}\Theta_2(\theta) - \left[\frac{1}{1680}\Theta_2(\theta) + \frac{3}{22400}\Theta_4(\theta)\right]$$

$$\psi(\bar{r},\theta) = \psi_0(\theta) + \alpha^3\bar{\psi}_3(\bar{r},\theta) + \alpha^4\left[\boldsymbol{\bar{\psi}_4(\bar{r},\theta) + \log\alpha\,\bar{\psi}_{4L}(\bar{r},\theta)}\right] + \mathcal{O}(\alpha^5\log\alpha)$$

BZ PERTURBATIVE APPROACH AND MATCHED ASYMPTOTIC EXPANSION (III)

MATCHING IN THE OVERLAP REGION

$$\log(r/r_0) \xrightarrow{r\to\alpha^{-1}\bar{r}} \log(\bar{r}/r_0) - \log\alpha$$

- r-region

$$\psi_2(r,\theta) = \left[\frac{1}{8}\frac{r_0}{r} - \frac{11}{800}\frac{r_0^2}{r^2} + \boxed{\frac{1}{40}\frac{r_0^2}{r^2}\log\frac{r}{r_0}}\right]\Theta_2(\theta) + \mathcal{O}\left(\frac{r_0^3}{r^3}\log\frac{r}{r_0}\right)$$

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$$\bar{\psi}_3(\bar{r},\theta) = \frac{1}{8}\frac{r_0}{\bar{r}}\Theta_2(\theta) + \frac{\bar{r}}{r_0}\left[\frac{1}{224}\Theta_2(\theta) + \frac{9}{8960}\Theta_4(\theta)\right] + \frac{\bar{r}^2}{r_0^2}\mathcal{F}(\theta) + \mathcal{O}\left(\frac{\bar{r}^3}{r_0^3}\right)$$

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$$\bar{\psi}_{4L}(\bar{r},\theta) = -\frac{1}{40}\frac{r_0^2}{\bar{r}^2}\Theta_2(\theta) - \left[\frac{1}{1680}\Theta_2(\theta) + \frac{3}{22400}\Theta_4(\theta)\right]$$

$$\psi(\bar{r},\theta) = \psi_0(\theta) + \alpha^3\bar{\psi}_3(\bar{r},\theta) + \alpha^4\left[\bar{\psi}_4(\bar{r},\theta) + \log\alpha\,\bar{\psi}_{4L}(\bar{r},\theta)\right] + \mathcal{O}(\alpha^5\log\alpha)$$

BZ PERTURBATIVE APPROACH AND MATCHED ASYMPTOTIC EXPANSION (III)

MATCHING IN THE OVERLAP REGION

$$\log(r/r_0) \xrightarrow{r\to\alpha^{-1}\bar{r}} \log(\bar{r}/r_0) - \log\alpha$$

- r-region

$$\psi_2(r,\theta) = \left[\frac{1}{8}\frac{r_0}{r} - \frac{11}{800}\frac{r_0^2}{r^2} + \boxed{\frac{1}{40}\frac{r_0^2}{r^2}\log\frac{r}{r_0}}\right]\Theta_2(\theta) + \mathcal{O}\left(\frac{r_0^3}{r^3}\log\frac{r}{r_0}\right)$$

$$\psi_4(r,\theta) = \left[\frac{1}{224}\frac{r}{r_0} + \frac{227}{100800} + \frac{1}{1680}\log\frac{r}{r_0}\right]\Theta_2(\theta) + \left[\frac{9}{8960}\frac{r}{r_0} + \frac{363}{896000} + \frac{3}{22400}\log\frac{r}{r_0}\right]\Theta_4(\theta) + \mathcal{O}\left(\frac{r_0^3}{r^3}\log\frac{r}{r_0}\right)$$

$$\psi_5(r,\theta) = \frac{r^2}{r_0^2}\mathscr{F}(\theta) + \mathcal{O}\left(\frac{r}{r_0}\right)$$

- $\bar{r}$-region

$$\bar{\psi}_3(\bar{r},\theta) = \frac{1}{8}\frac{r_0}{\bar{r}}\Theta_2(\theta) + \frac{\bar{r}}{r_0}\left[\frac{1}{224}\Theta_2(\theta) + \frac{9}{8960}\Theta_4(\theta)\right] + \frac{\bar{r}^2}{r_0^2}\mathscr{F}(\theta) + \mathcal{O}\left(\frac{\bar{r}^3}{r_0^3}\right)$$

$$\bar{\psi}_4(\bar{r},\theta) = -\frac{11}{800}\frac{r_0^2}{\bar{r}^2}\Theta_2(\theta) + \boxed{\frac{1}{40}\frac{r_0^2}{\bar{r}^2}\log\frac{\bar{r}}{r_0}}\Theta_2(\theta) + \frac{227}{100800}\Theta_2(\theta) + \frac{363}{896000}\Theta_4(\theta) + \left[\frac{1}{1680}\Theta_2(\theta) + \frac{3}{22400}\Theta_4(\theta)\right]\log\frac{\bar{r}}{r_0} + \mathcal{O}\left(\frac{\bar{r}}{r_0}\right)$$

$$\bar{\psi}_{4L}(\bar{r},\theta) = -\boxed{\frac{1}{40}\frac{r_0^2}{\bar{r}^2}}\Theta_2(\theta) - \left[\frac{1}{1680}\Theta_2(\theta) + \frac{3}{22400}\Theta_4(\theta)\right]$$

$$\psi(\bar{r},\theta) = \psi_0(\theta) + \alpha^3\bar{\psi}_3(\bar{r},\theta) + \alpha^4\left[\boldsymbol{\bar{\psi}_4(\bar{r},\theta)} + \boldsymbol{\log\alpha}\,\boldsymbol{\bar{\psi}_{4L}(\bar{r},\theta)}\right] + \mathcal{O}(\alpha^5\log\alpha)$$

BZ PERTURBATIVE APPROACH AND MATCHED ASYMPTOTIC EXPANSION (III)

MATCHING IN THE OVERLAP REGION

$$\log(r/r_0) \xrightarrow{r\to\alpha^{-1}\bar{r}} \log(\bar{r}/r_0) - \log\alpha$$

- r-region

$$\psi_2(r,\theta) = \left[\frac{1}{8}\frac{r_0}{r} - \frac{11}{800}\frac{r_0^2}{r^2} + \boxed{\frac{1}{40}\frac{r_0^2}{r^2}\log\frac{r}{r_0}}\right]\Theta_2(\theta) + \mathcal{O}\left(\frac{r_0^3}{r^3}\log\frac{r}{r_0}\right)$$

$$\psi_4(r,\theta) = \left[\frac{1}{224}\frac{r}{r_0} + \frac{227}{100800} + \frac{1}{1680}\log\frac{r}{r_0}\right]\Theta_2(\theta) + \left[\frac{9}{8960}\frac{r}{r_0} + \frac{363}{896000} + \frac{3}{22400}\log\frac{r}{r_0}\right]\Theta_4(\theta) + \mathcal{O}\left(\frac{r_0^3}{r^3}\log\frac{r}{r_0}\right)$$

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$$\bar{\psi}_3(\bar{r},\theta) = \frac{1}{8}\frac{r_0}{\bar{r}}\Theta_2(\theta) + \frac{\bar{r}}{r_0}\left[\frac{1}{224}\Theta_2(\theta) + \frac{9}{8960}\Theta_4(\theta)\right] + \frac{\bar{r}^2}{r_0^2}\mathcal{F}(\theta) + \mathcal{O}\left(\frac{\bar{r}^3}{r_0^3}\right)$$

$$\bar{\psi}_4(\bar{r},\theta) = -\frac{11}{800}\frac{r_0^2}{\bar{r}^2}\Theta_2(\theta) + \boxed{\frac{1}{40}\frac{r_0^2}{\bar{r}^2}\log\frac{\bar{r}}{r_0}}\Theta_2(\theta) + \frac{227}{100800}\Theta_2(\theta) + \frac{363}{896000}\Theta_4(\theta) + \left[\frac{1}{1680}\Theta_2(\theta) + \frac{3}{22400}\Theta_4(\theta)\right]\log\frac{\bar{r}}{r_0} + \mathcal{O}\left(\frac{\bar{r}}{r_0}\right)$$

$$\bar{\psi}_{4L}(\bar{r},\theta) = -\boxed{\frac{1}{40}\frac{r_0^2}{\bar{r}^2}}\Theta_2(\theta) - \left[\frac{1}{1680}\Theta_2(\theta) + \frac{3}{22400}\Theta_4(\theta)\right]$$

$$\boxed{\alpha^6\left[\psi_6 + \log\alpha\ \psi_{6L}\right] \xrightarrow{r\to\alpha^{-1}\bar{r}} \alpha^3\bar{\psi}_3^{(\bar{r}^3)} + \alpha^4\bar{\psi}_4^{(\bar{r}^2)} + \mathcal{O}\left(\alpha^5\log\alpha\right)}$$

BZ PERTURBATIVE APPROACH AND MATCHED ASYMPTOTIC EXPANSION (III)

MATCHING IN THE OVERLAP REGION

$$\log(r/r_0) \xrightarrow{r\to\alpha^{-1}\bar{r}} \log(\bar{r}/r_0) - \log\alpha$$

- r-region

$$\psi_2(r,\theta) = \left[\frac{1}{8}\frac{r_0}{r} - \frac{11}{800}\frac{r_0^2}{r^2} + \boxed{\frac{1}{40}\frac{r_0^2}{r^2}\log\frac{r}{r_0}}\right]\Theta_2(\theta) + \mathcal{O}\left(\frac{r_0^3}{r^3}\log\frac{r}{r_0}\right)$$

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$$\bar{\psi}_3(\bar{r},\theta) = \frac{1}{8}\frac{r_0}{\bar{r}}\Theta_2(\theta) + \frac{\bar{r}}{r_0}\left[\frac{1}{224}\Theta_2(\theta) + \frac{9}{8960}\Theta_4(\theta)\right] + \frac{\bar{r}^2}{r_0^2}\mathcal{F}(\theta) + \mathcal{O}\left(\frac{\bar{r}^3}{r_0^3}\right)$$

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$$r_0\Omega_H = \frac{\alpha}{1+\sqrt{1-\alpha^2}}$$

$$\dot{E}_+ = \frac{2\pi}{3}\Omega_H^2\left[1 + 0.34588327\ r_0^2\Omega_H^2 - 0.70309718\ r_0^4\Omega_H^4 + 0.0483269(2)\ r_0^5|\Omega_H|^5 + \left[0.1837383(5) - 0.0027081(2)\log(r_0|\Omega_H|)\right]r_0^6\Omega_H^6 + \cdots\right]$$

BZ PERTURBATIVE APPROACH AND MATCHED ASYMPTOTIC EXPANSION (III)

MATCHING IN THE OVERLAP REGION

$$\log(r/r_0) \xrightarrow{r\to\alpha^{-1}\bar{r}} \log(\bar{r}/r_0) - \log\alpha$$

r-region

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$$\psi_5(r,\theta) = \frac{r^2}{r_0^2}\mathcal{F}(\theta) + \mathcal{O}\left(\frac{r}{r_0}\right)$$

$\bar{r}$-region

$$\bar{\psi}_3(\bar{r},\theta) = \frac{1}{8}\frac{r_0}{\bar{r}}\Theta_2(\theta) + \frac{\bar{r}}{r_0}\left[\frac{1}{224}\Theta_2(\theta) + \frac{9}{8960}\Theta_4(\theta)\right] + \frac{\bar{r}^2}{r_0^2}\mathcal{F}(\theta) + \mathcal{O}\left(\frac{\bar{r}^3}{r_0^3}\right)$$

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$$r_0\Omega_H = \frac{\alpha}{1+\sqrt{1-\alpha^2}}$$

BZ (1977) Tanabe, Nagataki (2008)

$$\dot{E}_+ = \frac{2\pi}{3}\Omega_H^2\left[1 + 0.34588327\ r_0^2\Omega_H^2 - 0.70309718\ r_0^4\Omega_H^4 + 0.0483269(2)\ r_0^5|\Omega_H|^5 + \left[0.1837383(5) - 0.0027081(2)\log(r_0|\Omega_H|)\right]r_0^6\Omega_H^6 + \cdots\right]$$

BZ PERTURBATIVE APPROACH AND MATCHED ASYMPTOTIC EXPANSION (III)

MATCHING IN THE OVERLAP REGION

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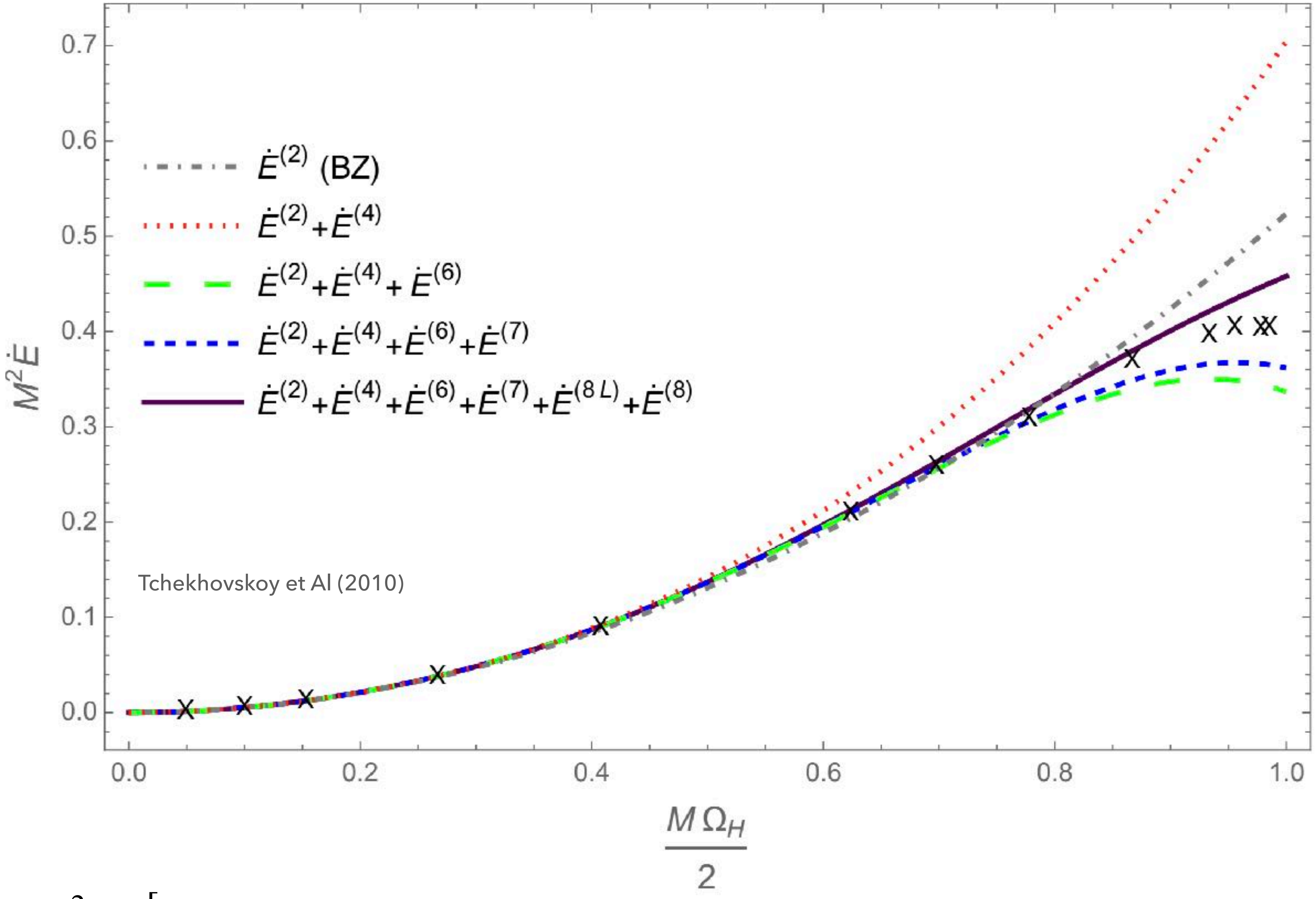
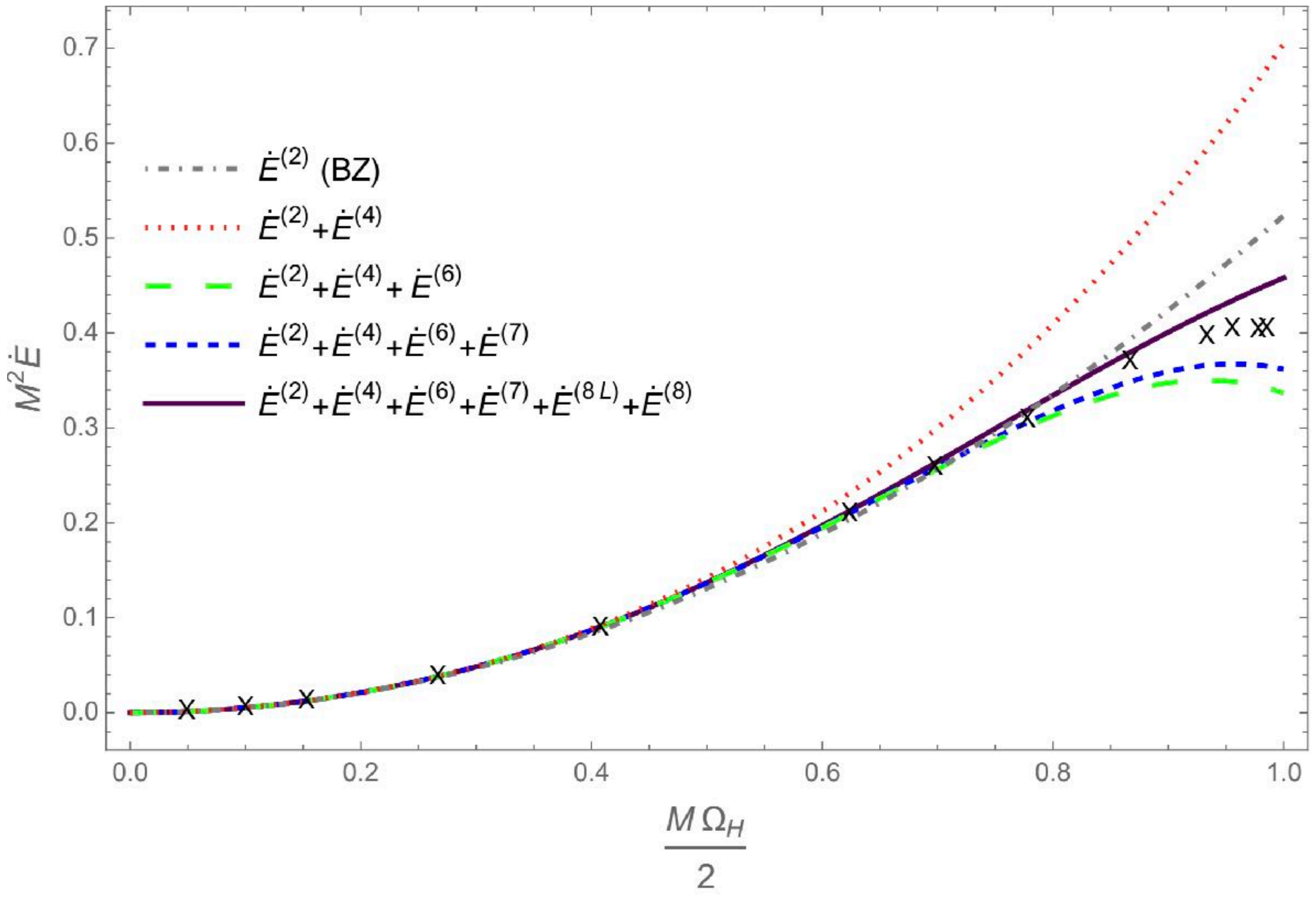
BZ ENERGY EXTRACTION RATE



$$\dot{E}_+ = \frac{2\pi}{3}\Omega_H^2\left[1 + 0.34588327\ r_0^2\Omega_H^2 - 0.70309718\ r_0^4\Omega_H^4 + 0.0483269(2)\ r_0^5|\Omega_H|^5 + \left[0.1837383(5) - 0.0027081(2)\log(r_0|\Omega_H|)\right] r_0^6\Omega_H^6 + \cdots\right]$$

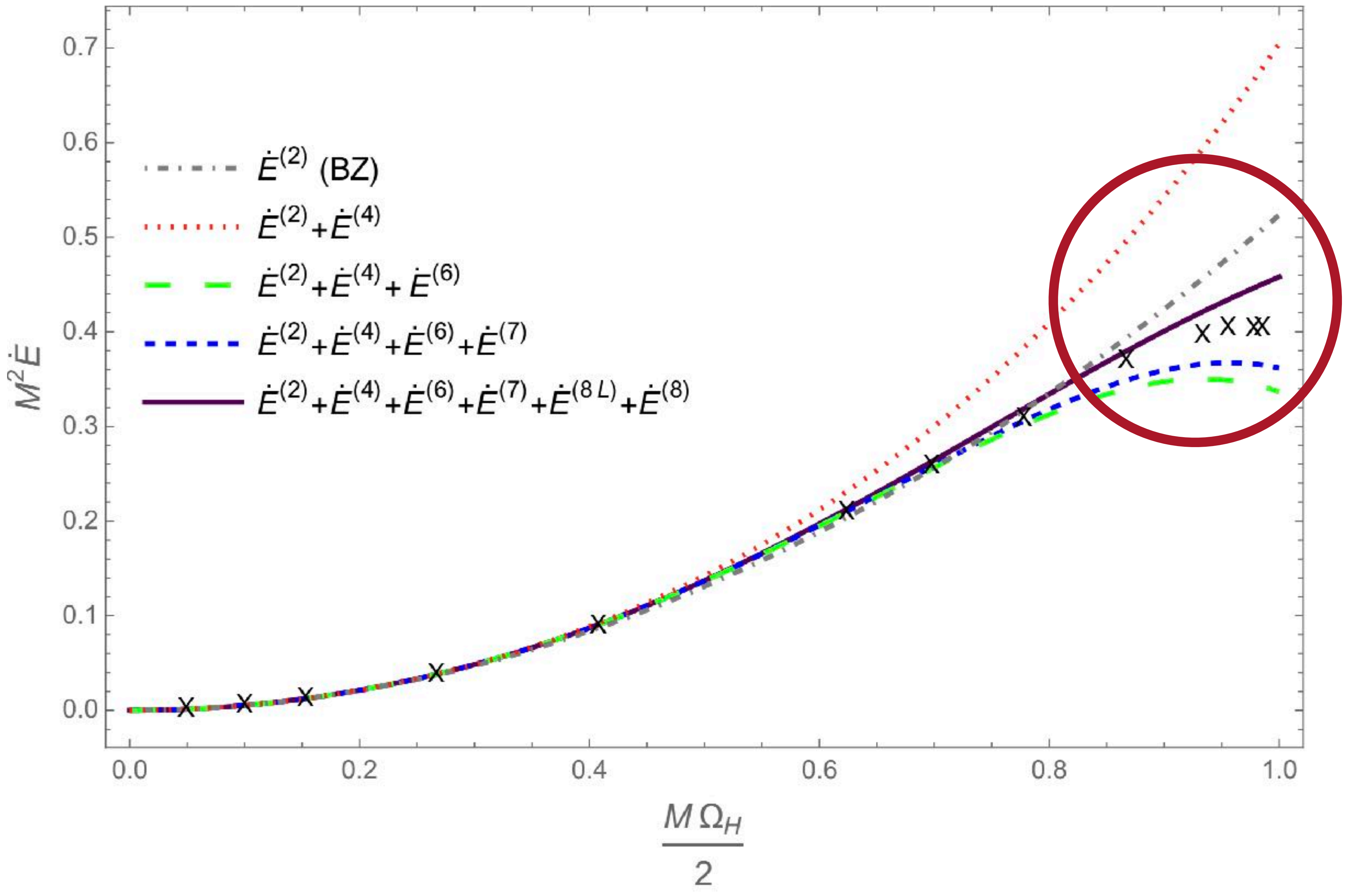
BZ ENERGY EXTRACTION RATE



Tchekhovskoy et Al (2010)

$$\dot{E}_+ = \frac{2\pi}{3}\Omega_H^2\left[1 + 0.34588327\, r_0^2\Omega_H^2 - 0.70309718\, r_0^4\Omega_H^4 + 0.0483269(2)\, r_0^5|\Omega_H|^5 + \left[0.1837383(5) - 0.0027081(2)\log(r_0|\Omega_H|)\right] r_0^6\Omega_H^6 + \cdots\right]$$

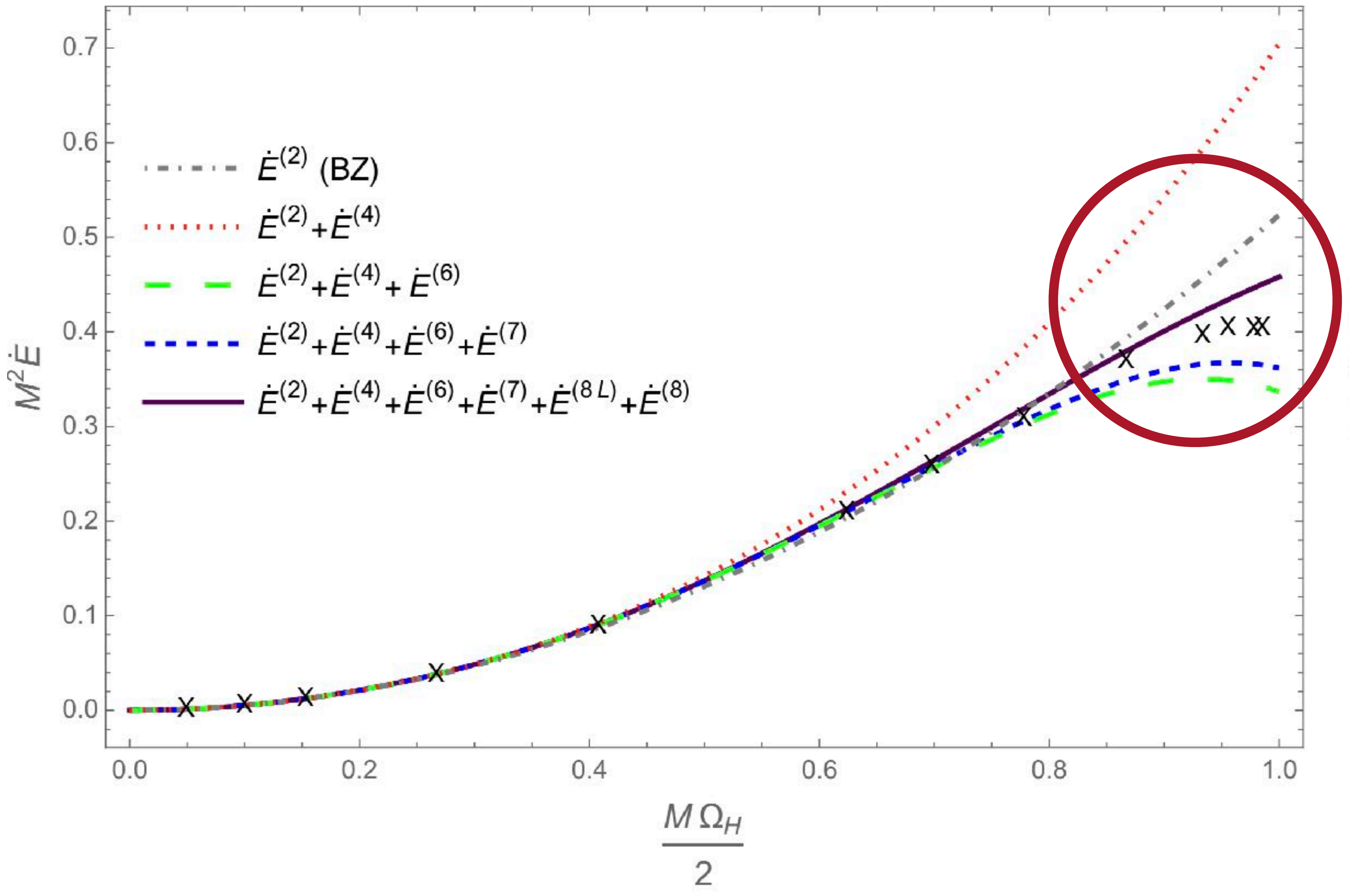
BZ ENERGY EXTRACTION RATE



Tchekhovskoy et Al (2010)

$$\dot{E}_+ = \frac{2\pi}{3}\Omega_H^2\left[1 + 0.34588327\ r_0^2\Omega_H^2 - 0.70309718\ r_0^4\Omega_H^4 + 0.0483269(2)\ r_0^5|\Omega_H|^5 + \left[0.1837383(5) - 0.0027081(2)\log(r_0|\Omega_H|)\right]r_0^6\Omega_H^6 + \cdots\right]$$

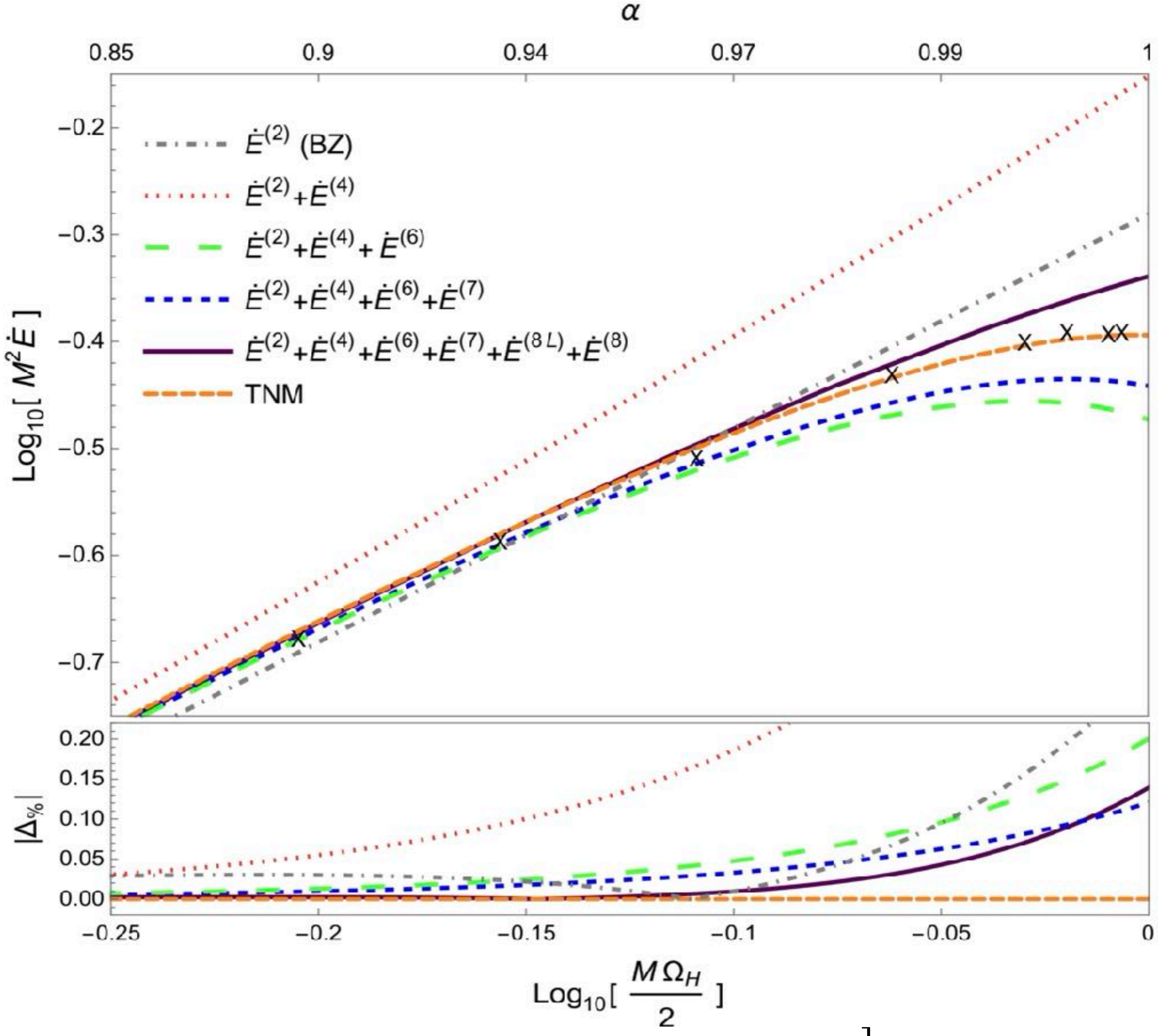
BZ ENERGY EXTRACTION RATE



Tchekhovskoy et Al (2010)

$$\dot{E}_{(\mathrm{TNM})} = \frac{2\pi}{3}\Omega_H^2\left[1 + 0.3459\ r_0^2\Omega_H^2 - 0.575\ r_0^4\Omega_H^4\right]$$

$|\Delta_\%| \approx 7.5\,\%$ for $\alpha = 0.998$



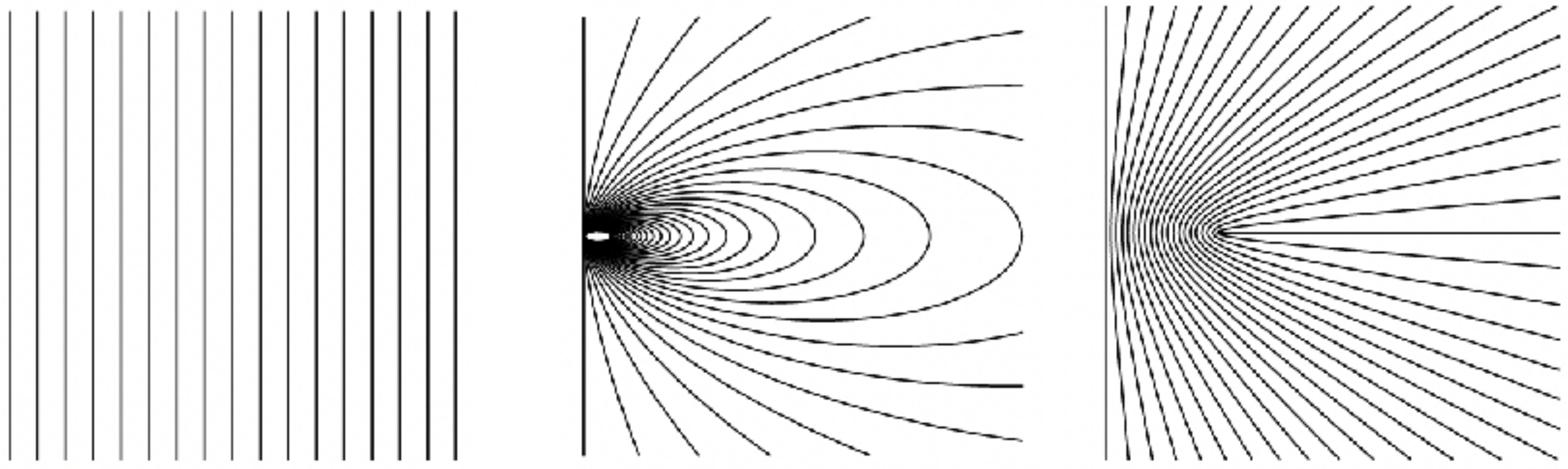
$$\dot{E}_+ = \frac{2\pi}{3}\Omega_H^2\left[1 + 0.34588327\ r_0^2\Omega_H^2 - 0.70309718\ r_0^4\Omega_H^4 + 0.0483269(2)\ r_0^5|\Omega_H|^5 + \left[0.1837383(5) - 0.0027081(2)\log(r_0|\Omega_H|)\right] r_0^6\Omega_H^6 + \cdots\right]$$

RESULTS AND OUTLOOK

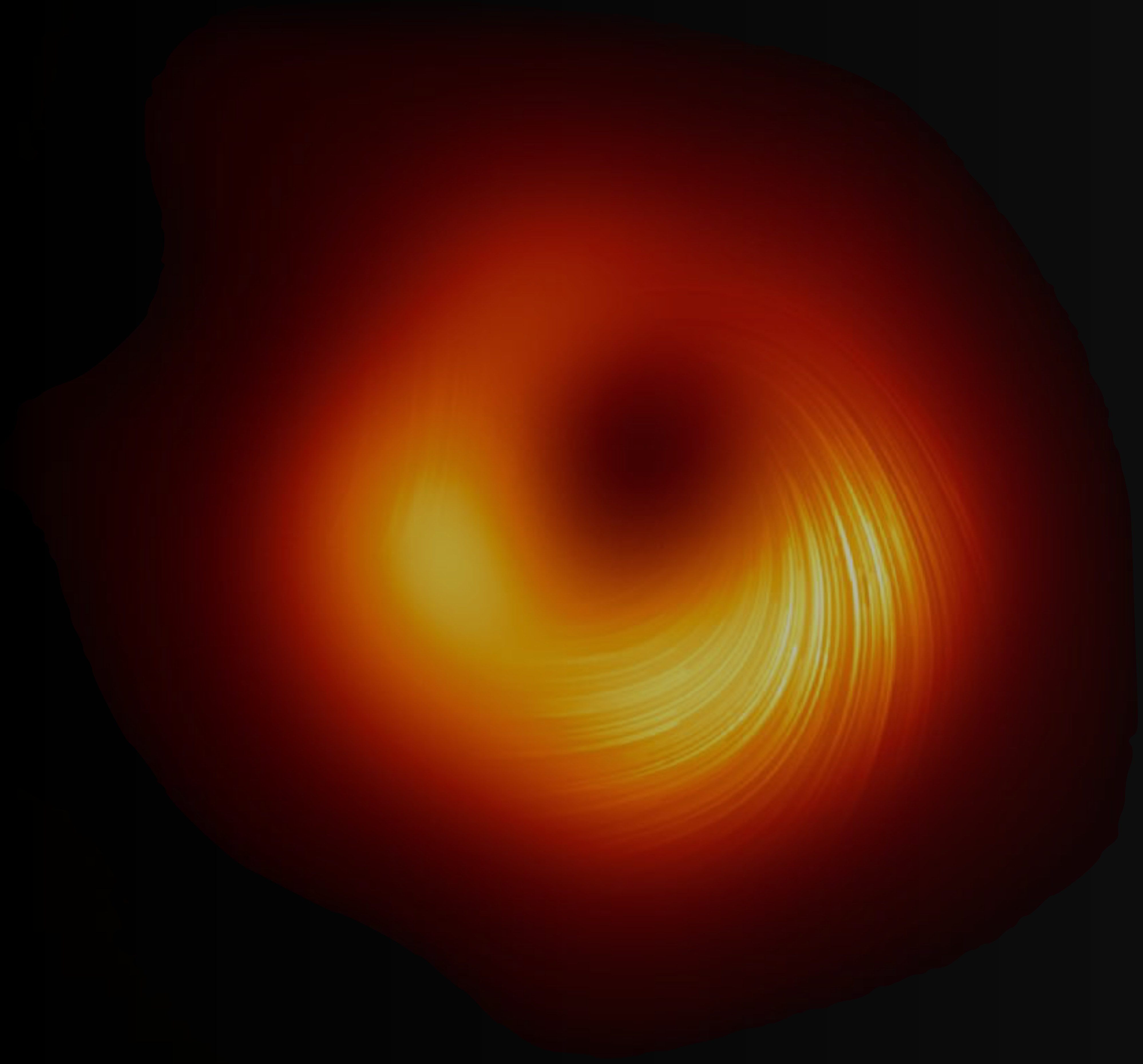
- **Higher-orders** and **non-analytic terms** in the BZ perturbative approach. Unprecedented a**greement with GRMHD simulations up to $\alpha \approx 0.9$** .
- **Unavoidable non-analytic** terms $\sim \log \alpha,\ |\alpha|$
- (Agreement at $\alpha \approx 0.9$) + ($\log \alpha$ terms) $\Longrightarrow$ **Non-perturbative structure** of BZ theory?
- Force-free analytic solutions are used to **calibrate GRMHD codes**, which are exploited to **constrain observations of the EHT**
- **Different topologies** for the starting solution?
- **String Fluid** perspective on FFE
  $\Longrightarrow$ 1° order theory of FFE: dissipative corrections!
  (Causality and Stability with BDNK method)

Bemfica, Disconci, Noronha (2017)

Kovtun (2018)



(a) vertical (d) dipolar (f) hyperbolic



THANKS!

NUMERICAL PROCEDURE

The functions $P_k(\theta) \equiv \Theta_{2k}(\theta)$ form a basis, we also rescale the functions by means of their divergent terms

$$\widetilde{\bar{\psi}^{(3)}}(\bar{r},\theta) = \bar{\psi}^{(3)}(\bar{r},\theta) - \frac{\sin^2\theta\cos\theta}{8}\frac{r_0}{\bar{r}} = \sum_{k=1}^{N} P_k(\theta)\bar{Q}^{(n)}_{k\,N}(\bar{r})$$

$$\widetilde{\bar{\psi}^{(4)}}(\bar{r},\theta) = \tilde{\bar{Q}}^{(4)}_{1\,N}(\bar{r})P_1(\theta) + \tilde{\bar{Q}}^{(4)}_{2\,N}(\bar{r})P_2(\theta) + \sum_{k=3}^{N} P_k(\theta)\bar{Q}^{(4)}_{k}(\bar{r})$$

Monopolar behavior at infinity

$$\bar{\psi}^{\infty}_n(\theta) = \sum_{k=1}^{+\infty} P_k(\theta)\bar{c}^{(n)}_k \quad , \qquad \lim_{\bar{r}\to+\infty} \bar{Q}^{(n)}_k(\bar{r}) = \bar{c}^{(n)}_k$$

By projecting the Stream Equations onto the functions $P_k(\theta)$

$$\int_0^{\pi} \frac{P_k(\theta)}{\sin\theta}\left[\bar{\mathscr{L}}(\bar{\psi}_n) - S_n(\bar{\psi}^{\infty})\right]d\theta = 0 \quad \Longrightarrow \quad \partial^2_{\bar{r}}\bar{Q}^{(n)}_k(\bar{r}) - \frac{1}{16}\left[\frac{(2k-1)(2k+1)}{(4k-3)(4k-1)}\partial_{\bar{r}}(\bar{r}^2\partial_{\bar{r}}\bar{Q}^{(3)}_{k-1}(\bar{r})) + \dots\right] = 0 \qquad k = 1,\dots,\infty$$

- Truncate these equations for a **finite number of harmonics** $\Longrightarrow$ **REGULATOR** N : $\bar{Q}^n_{k>N} = \bar{c}^{(n)}_{k>N} = 0$

  We have $2N$ ODEs to solve for $\bar{Q}^n_{k,N}$, each equation has an **OLS** located at $\bar{r}_{\mathrm{OLS}\;k,N} = \dfrac{2\sqrt{(4k-1)(4k+3)}}{\sqrt{k(2k+1)}}r_0$

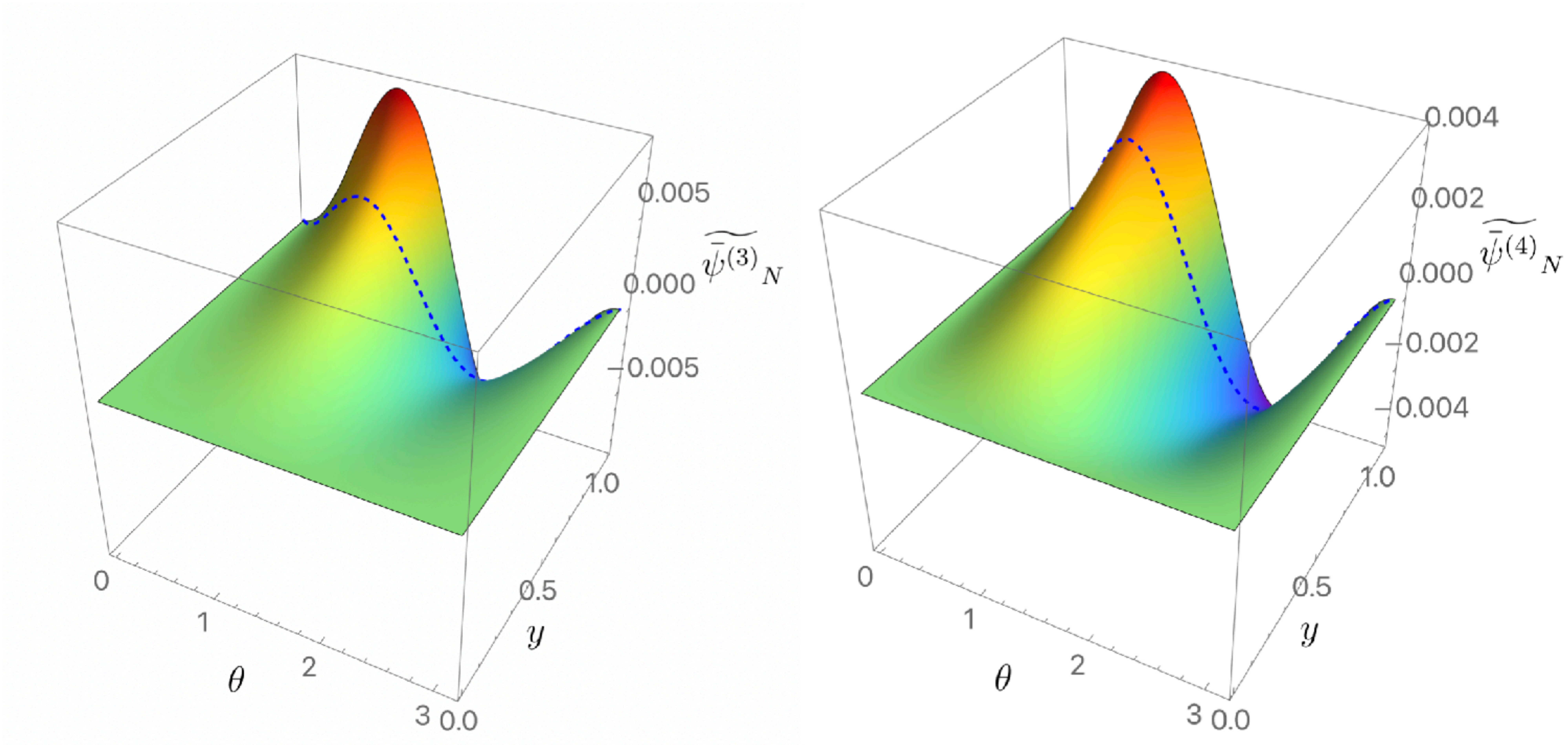
- Promote coefficients $\bar{c}^{(n)}_k$ to functions subject to the constraints: $\partial^2_{\bar{r}}\bar{c}^{(n)}_{k,N}(\bar{r}) = 0$, $\partial_{\bar{r}}\bar{c}^{(n)}_{k,N}(\bar{r})\,|_{\bar{r}=0} = 0 \Longrightarrow$ 2N + (2N) ODEs

- Introduce a **compact coordinate**: $y = \dfrac{\bar{r}/r_0}{1-\bar{r}/r_0} \in [0,1] \Longrightarrow$ **REGULATOR** N_y

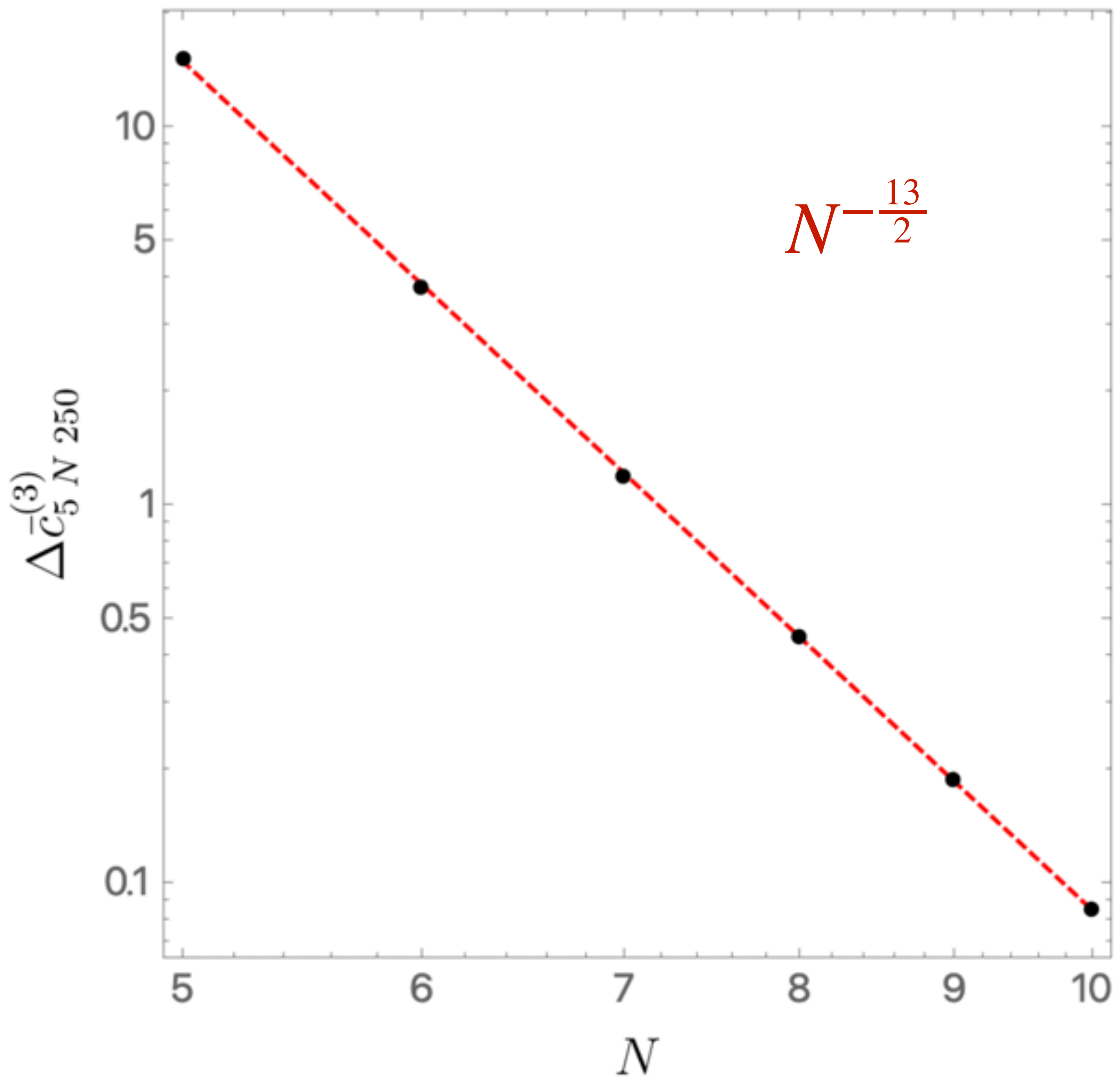
NUMERICAL PROCEDURE — SOLUTIONS

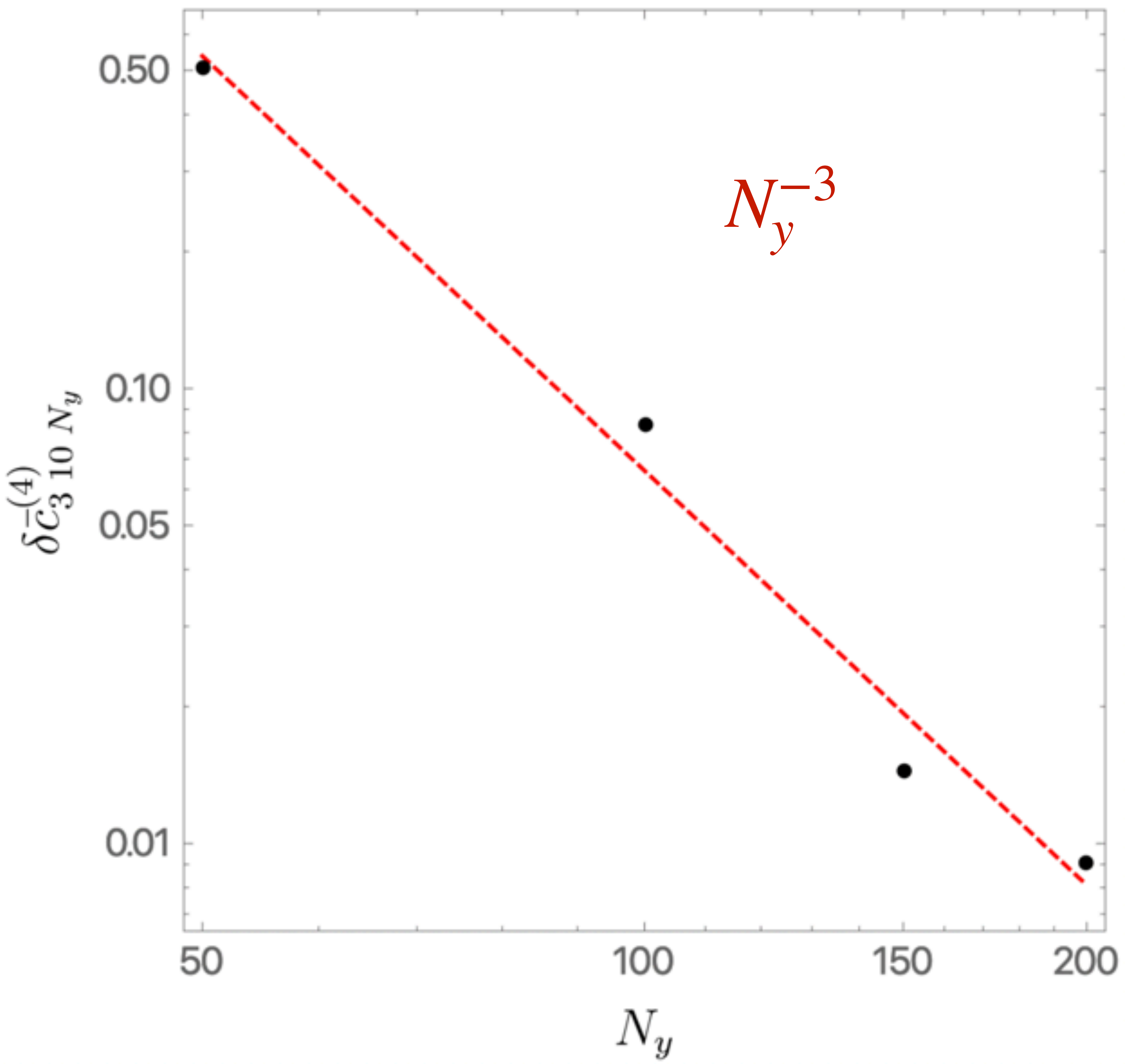
The two solutions are unique, antisymmetric with respect to $\theta \to \theta + \pi/2$ and smooth at the Outer Light Surface (- - -)

$$\psi(\bar{r},\theta) = \psi_0(\theta) + \alpha^3\,\bar{\psi}_3(\bar{r},\theta) + \alpha^4\left[\bar{\psi}_4(\bar{r},\theta) + \log\alpha\,\bar{\psi}_{4L}(\bar{r},\theta)\right] + \mathcal{O}(\alpha^5\log\alpha)$$



NUMERICAL PROCEDURE — CONVERGENCE





$$\bar{\psi}_3^\infty(\theta) = \bar{c}_2^{(3)}\Theta_2(\theta)+\bar{c}_4^{(3)}\Theta_4(\theta)+\bar{c}_6^{(3)}\Theta_6(\theta) + \dots$$

$$\bar{\psi}_4^\infty(\theta) = \bar{c}_2^{(4)}\Theta_2(\theta)+\bar{c}_4^{(4)}\Theta_4(\theta)+\bar{c}_6^{(4)}\Theta_6(\theta) + \dots$$

The more we increase the regulators, the less the coefficients are subject to variations

NO MINIMISATION REQUIRED

$$\Delta\bar{c}^{(n)}_{k\;N\;N_y} \equiv 100\left|1 - \frac{\bar{c}^{(n)}_{k\;N\;N_y}}{\bar{c}^{(n)}_{k\;N+1\;N_y}}\right|$$

Variation w.r.t. # of harmonic

$$\delta\bar{c}^{(n)}_{k\;N\;N_y} \equiv 100\left|1 - \frac{\bar{c}^{(n)}_{k\;N\;N_y}}{\bar{c}^{(n)}_{k\;N\;N_y+50}}\right|$$

Variation w.r.t. # Grid-points

STREAM EQUATION AND MATCHED ASYMPTOTIC EXPANSION

$$\boxed{\eta_\mu \partial_r\left(\eta^\mu \Delta \sin\theta\, \partial_r \psi\right) + \eta_\mu \partial_\theta\left(\eta^\mu \sin\theta\, \partial_\theta \psi\right) + \frac{\Sigma}{\Delta \sin\theta} I \frac{dI}{d\psi} = 0}$$

- **r-region**

$$\mathscr{L}\psi_n(r,\theta) = \mathscr{S}(r,\theta;\psi_{k<n}, i_{k<n}, \omega_{k<n})$$

$$\mathscr{L} = \frac{1}{\sin\theta}\partial_r\left[\left(1-\frac{r_0}{r}\right)\partial_r\right] + \frac{1}{r^2}\partial_\theta\left(\frac{1}{\sin\theta}\partial_\theta\right)$$

$$\left.\begin{aligned} \mathscr{L}^\theta_\ell[\Theta] &= \frac{d}{d\theta}\left(\frac{1}{\sin\theta}\frac{d\Theta}{d\theta}\right) + \frac{\ell(\ell+1)\Theta}{\sin\theta} = 0 \\ \mathscr{L}^r_\ell[R] &= \frac{d}{dr}\left[\left(1-\frac{r_0}{r}\right)\frac{dR}{dr}\right] - \frac{\ell(\ell+1)R}{r^2} = 0 \end{aligned}\right\} \quad \psi(r,\theta) = R(r)\Theta(\theta)$$

- **$\bar{r}$-region**

$$\bar{\mathscr{L}}\bar{\psi}_n(\bar{r},\theta) = \bar{\mathscr{S}}(\bar{r},\theta;\bar{\psi}_{k<n}, i_{k\le n+1}, \omega_{k\le n+1})$$

$$\bar{\mathscr{L}} = \frac{\sin\theta}{r_0^2}\partial_\theta\left[\sin\theta\left(\frac{r_0^2}{\bar{r}^2\sin^2\theta} - \frac{1}{16}\right)\partial_\theta\right] + \sin^2\theta\,\partial_{\bar{r}}\left[\frac{\bar{r}^2}{r_0^2}\left(\frac{r_0^2}{\bar{r}^2\sin^2\theta} - \frac{1}{16}\right)\partial_{\bar{r}}\right] + \frac{\left(2-3\sin^2\theta\right)}{8r_0^2}$$

STREAM EQUATION AND MATCHED ASYMPTOTIC EXPANSION

$$\boxed{\eta_\mu \partial_r\left(\eta^\mu \Delta \sin\theta\, \partial_r \psi\right) + \eta_\mu \partial_\theta\left(\eta^\mu \sin\theta\, \partial_\theta \psi\right) + \frac{\Sigma}{\Delta \sin\theta} I \frac{dI}{d\psi} = 0}$$

- ***r*-region**

$$\mathscr{L}\psi_n(r,\theta) = \mathscr{S}(r,\theta;\psi_{k<n}, i_{k<n}, \omega_{k<n})$$

$$\mathscr{L} = \frac{1}{\sin\theta}\partial_r\left[\boxed{\left(1-\frac{r_0}{r}\right)}\partial_r\right] + \frac{1}{r^2}\partial_\theta\left(\frac{1}{\sin\theta}\partial_\theta\right)$$

$$\frac{r_+}{r_0} = \frac{r_{\rm ILS}}{r_0} = 1 + \mathcal{O}(\alpha^2)$$

$$\left.\begin{aligned} \mathscr{L}^\theta_\ell[\Theta] &= \frac{d}{d\theta}\left(\frac{1}{\sin\theta}\frac{d\Theta}{d\theta}\right) + \frac{\ell(\ell+1)\Theta}{\sin\theta} = 0 \\ \mathscr{L}^r_\ell[R] &= \frac{d}{dr}\left[\left(1-\frac{r_0}{r}\right)\frac{dR}{dr}\right] - \frac{\ell(\ell+1)R}{r^2} = 0 \end{aligned}\right\} \quad \psi(r,\theta) = R(r)\Theta(\theta)$$

Singular point at the effective boundary

- ***$\bar{r}$*-region**

$$\bar{\mathscr{L}}\bar{\psi}_n(\bar{r},\theta) = \bar{\mathscr{S}}(\bar{r},\theta;\bar{\psi}_{k<n}, i_{k\le n+1}, \omega_{k\le n+1})$$

$$\bar{\mathscr{L}} = \frac{\sin\theta}{r_0^2}\partial_\theta\left[\sin\theta\left(\frac{r_0^2}{\bar{r}^2\sin^2\theta} - \frac{1}{16}\right)\partial_\theta\right] + \sin^2\theta\,\partial_{\bar{r}}\left[\frac{\bar{r}^2}{r_0^2}\left(\frac{r_0^2}{\bar{r}^2\sin^2\theta} - \frac{1}{16}\right)\partial_{\bar{r}}\right] + \frac{(2-3\sin^2\theta)}{8r_0^2}$$

STREAM EQUATION AND MATCHED ASYMPTOTIC EXPANSION

$$\boxed{\eta_\mu \partial_r\left(\eta^\mu \Delta \sin\theta\, \partial_r \psi\right) + \eta_\mu \partial_\theta\left(\eta^\mu \sin\theta\, \partial_\theta \psi\right) + \frac{\Sigma}{\Delta \sin\theta} I \frac{dI}{d\psi} = 0}$$

- ***r*-region**

$$\mathscr{L}\psi_n(r,\theta) = \mathscr{S}(r,\theta;\psi_{k<n}, i_{k<n}, \omega_{k<n})$$

$$\mathscr{L} = \frac{1}{\sin\theta}\partial_r\left[\boxed{\left(1-\frac{r_0}{r}\right)}\partial_r\right] + \frac{1}{r^2}\partial_\theta\left(\frac{1}{\sin\theta}\partial_\theta\right)$$

$$\left.\begin{aligned} \mathscr{L}^\theta_\ell[\Theta] &= \frac{d}{d\theta}\left(\frac{1}{\sin\theta}\frac{d\Theta}{d\theta}\right) + \frac{\ell(\ell+1)\Theta}{\sin\theta} = 0 \\ \mathscr{L}^r_\ell[R] &= \frac{d}{dr}\left[\left(1-\frac{r_0}{r}\right)\frac{dR}{dr}\right] - \frac{\ell(\ell+1)R}{r^2} = 0 \end{aligned}\right\} \psi(r,\theta) = R(r)\Theta(\theta)$$

$$\rightarrow \frac{r_+}{r_0} = \frac{r_{\rm ILS}}{r_0} = 1 + \mathcal{O}(\alpha^2)$$

Singular point at the effective boundary

- ***$\bar{r}$*-region**

$$\bar{\mathscr{L}}\bar{\psi}_n(\bar{r},\theta) = \bar{\mathscr{S}}(\bar{r},\theta;\bar{\psi}_{k<n}, i_{k\le n+1}, \omega_{k\le n+1})$$

$$\bar{\mathscr{L}} = \frac{\sin\theta}{r_0^2}\partial_\theta\left[\sin\theta\boxed{\left(\frac{r_0^2}{\bar{r}^2\sin^2\theta} - \frac{1}{16}\right)}\partial_\theta\right] + \sin^2\theta\,\partial_{\bar{r}}\left[\frac{\bar{r}^2}{r_0^2}\left(\frac{r_0^2}{\bar{r}^2\sin^2\theta} - \frac{1}{16}\right)\partial_{\bar{r}}\right] + \frac{\left(2-3\sin^2\theta\right)}{8r_0^2}$$

$$\rightarrow \frac{r_{\rm OLS}}{r_0} = \frac{4}{\alpha\sin\theta} + \mathcal{O}(\alpha^0)$$

Singular point in the bulk

ADDING SUBLEADING TERMS

$$\boxed{\alpha^6\left[\psi_6 + \log\alpha\ \psi_{6L}\right] \xrightarrow{r\to\alpha^{-1}\bar{r}} \alpha^3\bar{\psi}_3^{(\bar{r}^3)} + \alpha^4\bar{\psi}_4^{(\bar{r}^2)} + \mathcal{O}\left(\alpha^5\log\alpha\right)}$$

$$\psi_{6_L}(r,\theta) = R_2^{(6_L)}(r)\Theta_2(\theta) + R_4^{(6_L)}(r)\Theta_4(\theta) + R_6^{(6_L)}(r)\Theta_6(\theta)$$

$$R_2^{(6_L)}(r) = -\frac{3}{22400}\frac{r^3}{r_0^3} + \frac{3}{22400}\frac{r^2}{r_0^2} + \mathcal{O}\left(\frac{r}{r_0}\right),$$

$$R_4^{(6_L)}(r) = \frac{61}{5913600}\frac{r^2}{r_0^2} + \mathcal{O}\left(\frac{r}{r_0}\right),$$

$$R_6^{(6_L)}(r) = \frac{1}{563200}\frac{r^2}{r_0^2} + \mathcal{O}\left(\frac{r}{r_0}\right).$$

$$\psi_6(r,\theta) = R_2^{(6)}(r)\Theta_2(\theta) + R_4^{(6)}(r)\Theta_4(\theta) + R_6^{(6)}(r)\Theta_6(\theta) + \ldots$$

$$R_2^{(6)}(r) = -\frac{3}{22400}\frac{r^3}{r_0^3}\log\frac{r}{r_0} + \frac{3}{22400}\frac{r^2}{r_0^2}\log\frac{r}{r_0} + \ldots + \mathcal{O}\left(\frac{r}{r_0}\log\frac{r}{r_0}\right),$$

$$R_4^{(6_L)}(r) = \frac{61}{5913600}\frac{r^2}{r_0^2}\log\frac{r}{r_0} + \ldots + \mathcal{O}\left(\frac{r}{r_0}\log\frac{r}{r_0}\right),$$

$$R_6^{(6_L)}(r) = \frac{1}{563200}\frac{r^2}{r_0^2}\log\frac{r}{r_0} + \ldots + \mathcal{O}\left(\frac{r}{r_0}\log\frac{r}{r_0}\right).$$

ADDING SUBLEADING TERMS

$$\boxed{\alpha^6\left[\psi_6 + \log\alpha\ \psi_{6L}\right] \xrightarrow{r\to\alpha^{-1}\bar{r}} \alpha^3\bar{\psi}_3^{(\bar{r}^3)} + \alpha^4\bar{\psi}_4^{(\bar{r}^2)} + \mathcal{O}\left(\alpha^5\log\alpha\right)}$$

$$\psi_{6_L}(r,\theta) = R_2^{(6_L)}(r)\Theta_2(\theta) + R_4^{(6_L)}(r)\Theta_4(\theta) + R_6^{(6_L)}(r)\Theta_6(\theta)$$

$$R_2^{(6_L)}(r) = -\frac{3}{22400}\frac{r^3}{r_0^3} + \frac{3}{22400}\frac{r^2}{r_0^2} + \mathcal{O}\left(\frac{r}{r_0}\right),$$

$$R_4^{(6_L)}(r) = \frac{61}{5913600}\frac{r^2}{r_0^2} + \mathcal{O}\left(\frac{r}{r_0}\right),$$

$$R_6^{(6_L)}(r) = \frac{1}{563200}\frac{r^2}{r_0^2} + \mathcal{O}\left(\frac{r}{r_0}\right).$$

$$\psi_6(r,\theta) = R_2^{(6)}(r)\Theta_2(\theta) + R_4^{(6)}(r)\Theta_4(\theta) + R_6^{(6)}(r)\Theta_6(\theta) + \dots$$

$$R_2^{(6)}(r) = -\frac{3}{22400}\frac{r^3}{r_0^3}\log\frac{r}{r_0} + \frac{3}{22400}\frac{r^2}{r_0^2}\log\frac{r}{r_0} + \dots + \mathcal{O}\left(\frac{r}{r_0}\log\frac{r}{r_0}\right),$$

$$R_4^{(6_L)}(r) = \frac{61}{5913600}\frac{r^2}{r_0^2}\log\frac{r}{r_0} + \dots + \mathcal{O}\left(\frac{r}{r_0}\log\frac{r}{r_0}\right),$$

$$R_6^{(6_L)}(r) = \frac{1}{563200}\frac{r^2}{r_0^2}\log\frac{r}{r_0} + \dots + \mathcal{O}\left(\frac{r}{r_0}\log\frac{r}{r_0}\right).$$

EXAMPLE: red terms

$$\alpha^6\log\alpha\underbrace{\left(-\frac{3}{22400}\frac{r^3}{r_0^3} + \frac{3}{22400}\frac{r^2}{r_0^2}\right)}_{\psi_{6_L}} + \alpha^6\underbrace{\left(-\frac{3}{22400}\frac{r^3}{r_0^3}\log\frac{r}{r_0} + \frac{3}{22400}\frac{r^2}{r_0^2}\log\frac{r}{r_0}\right)}_{\psi_6} \xrightarrow{r\to\alpha^{-1}\bar{r}} \alpha^3\log\alpha\left(-\frac{3}{22400}\frac{\bar{r}^3}{r_0^3}\right) + \alpha^4\log\alpha\left(\frac{3}{22400}\frac{\bar{r}^2}{r_0^2}\right) +$$

$$\alpha^3\left(-\frac{3}{22400}\frac{\bar{r}^3}{r_0^3}\log\frac{\bar{r}}{r_0}\right) + \alpha^4\left(\frac{3}{22400}\frac{\bar{r}^2}{r_0^2}\log\frac{\bar{r}}{r_0}\right) +$$

$$\alpha^3\log\alpha\left(\frac{3}{22400}\frac{\bar{r}^3}{r_0^3}\right) + \alpha^4\log\alpha\left(-\frac{3}{22400}\frac{\bar{r}^2}{r_0^2}\right)$$

$$\log(r/r_0) \xrightarrow{r\to\alpha^{-1}\bar{r}} \log(\bar{r}/r_0) - \log\alpha$$

ADDING SUBLEADING TERMS

$$\boxed{\alpha^6\left[\psi_6 + \log\alpha\ \psi_{6L}\right] \xrightarrow{r\to\alpha^{-1}\bar r} \alpha^3\bar\psi_3^{(\bar r^3)} + \alpha^4\bar\psi_4^{(\bar r^2)} + \mathcal{O}\left(\alpha^5\log\alpha\right)}$$

$$\psi_{6_L}(r,\theta) = R_2^{(6_L)}(r)\Theta_2(\theta) + R_4^{(6_L)}(r)\Theta_4(\theta) + R_6^{(6_L)}(r)\Theta_6(\theta)$$

$$R_2^{(6_L)}(r) = -\frac{3}{22400}\frac{r^3}{r_0^3} + \frac{3}{22400}\frac{r^2}{r_0^2} + \mathcal{O}\left(\frac{r}{r_0}\right),$$

$$R_4^{(6_L)}(r) = \frac{61}{5913600}\frac{r^2}{r_0^2} + \mathcal{O}\left(\frac{r}{r_0}\right),$$

$$R_6^{(6_L)}(r) = \frac{1}{563200}\frac{r^2}{r_0^2} + \mathcal{O}\left(\frac{r}{r_0}\right).$$

$$\psi_6(r,\theta) = R_2^{(6)}(r)\Theta_2(\theta) + R_4^{(6)}(r)\Theta_4(\theta) + R_6^{(6)}(r)\Theta_6(\theta) + \dots$$

$$R_2^{(6)}(r) = -\frac{3}{22400}\frac{r^3}{r_0^3}\log\frac{r}{r_0} + \frac{3}{22400}\frac{r^2}{r_0^2}\log\frac{r}{r_0} + \dots + \mathcal{O}\left(\frac{r}{r_0}\log\frac{r}{r_0}\right),$$

$$R_4^{(6_L)}(r) = \frac{61}{5913600}\frac{r^2}{r_0^2}\log\frac{r}{r_0} + \dots + \mathcal{O}\left(\frac{r}{r_0}\log\frac{r}{r_0}\right),$$

$$R_6^{(6_L)}(r) = \frac{1}{563200}\frac{r^2}{r_0^2}\log\frac{r}{r_0} + \dots + \mathcal{O}\left(\frac{r}{r_0}\log\frac{r}{r_0}\right).$$

EXAMPLE: red terms

$$\alpha^6\log\alpha\underbrace{\left(-\frac{3}{22400}\frac{r^3}{r_0^3} + \frac{3}{22400}\frac{r^2}{r_0^2}\right)}_{\psi_{6_L}} + \alpha^6\underbrace{\left(-\frac{3}{22400}\frac{r^3}{r_0^3}\log\frac{r}{r_0} + \frac{3}{22400}\frac{r^2}{r_0^2}\log\frac{r}{r_0}\right)}_{\psi_6} \xrightarrow{r\to\alpha^{-1}\bar r} \alpha^3\log\alpha\left(\cancel{-\frac{3}{22400}\frac{\bar r^3}{r_0^3}}\right) + \alpha^4\log\alpha\left(\frac{3}{22400}\frac{\bar r^2}{r_0^2}\right) +$$

$$\alpha^3\left(-\frac{3}{22400}\frac{\bar r^3}{r_0^3}\log\frac{\bar r}{r_0}\right) + \alpha^4\left(\frac{3}{22400}\frac{\bar r^2}{r_0^2}\log\frac{\bar r}{r_0}\right) +$$

$$\alpha^3\log\alpha\left(\cancel{\frac{3}{22400}\frac{\bar r^3}{r_0^3}}\right) + \alpha^4\log\alpha\left(-\frac{3}{22400}\frac{\bar r^2}{r_0^2}\right)$$

$$\log(r/r_0) \xrightarrow{r\to\alpha^{-1}\bar r} \log(\bar r/r_0) - \log\alpha$$

ADDING SUBLEADING TERMS

$$\boxed{\alpha^6\left[\psi_6 + \log\alpha\ \psi_{6L}\right] \xrightarrow{r\to\alpha^{-1}\bar{r}} \alpha^3\bar{\psi}_3^{(\bar{r}^3)} + \alpha^4\bar{\psi}_4^{(\bar{r}^2)} + \mathcal{O}\left(\alpha^5\log\alpha\right)}$$

$$\psi_{6_L}(r,\theta) = R_2^{(6_L)}(r)\Theta_2(\theta) + R_4^{(6_L)}(r)\Theta_4(\theta) + R_6^{(6_L)}(r)\Theta_6(\theta)$$

$$R_2^{(6_L)}(r) = -\frac{3}{22400}\frac{r^3}{r_0^3} + \frac{3}{22400}\frac{r^2}{r_0^2} + \mathcal{O}\left(\frac{r}{r_0}\right),$$

$$R_4^{(6_L)}(r) = \frac{61}{5913600}\frac{r^2}{r_0^2} + \mathcal{O}\left(\frac{r}{r_0}\right),$$

$$R_6^{(6_L)}(r) = \frac{1}{563200}\frac{r^2}{r_0^2} + \mathcal{O}\left(\frac{r}{r_0}\right).$$

$$\psi_6(r,\theta) = R_2^{(6)}(r)\Theta_2(\theta) + R_4^{(6)}(r)\Theta_4(\theta) + R_6^{(6)}(r)\Theta_6(\theta) + \dots$$

$$R_2^{(6)}(r) = -\frac{3}{22400}\frac{r^3}{r_0^3}\log\frac{r}{r_0} + \frac{3}{22400}\frac{r^2}{r_0^2}\log\frac{r}{r_0} + \dots + \mathcal{O}\left(\frac{r}{r_0}\log\frac{r}{r_0}\right),$$

$$R_4^{(6_L)}(r) = \frac{61}{5913600}\frac{r^2}{r_0^2}\log\frac{r}{r_0} + \dots + \mathcal{O}\left(\frac{r}{r_0}\log\frac{r}{r_0}\right),$$

$$R_6^{(6_L)}(r) = \frac{1}{563200}\frac{r^2}{r_0^2}\log\frac{r}{r_0} + \dots + \mathcal{O}\left(\frac{r}{r_0}\log\frac{r}{r_0}\right).$$

EXAMPLE: red terms

$$\alpha^6\log\alpha\Big(\underbrace{-\frac{3}{22400}\frac{r^3}{r_0^3}+\frac{3}{22400}\frac{r^2}{r_0^2}}_{\psi_{6_L}}\Big) + \alpha^6\Big(\underbrace{-\frac{3}{22400}\frac{r^3}{r_0^3}\log\frac{r}{r_0}+\frac{3}{22400}\frac{r^2}{r_0^2}\log\frac{r}{r_0}}_{\psi_6}\Big) \xrightarrow{r\to\alpha^{-1}\bar{r}} \alpha^3\log\alpha\left(\cancel{-\frac{3}{22400}\frac{\bar{r}^3}{r_0^3}}\right) + \alpha^4\log\alpha\left(\cancel{\frac{3}{22400}\frac{\bar{r}^2}{r_0^2}}\right) +$$

$$\alpha^3\left(-\frac{3}{22400}\frac{\bar{r}^3}{r_0^3}\log\frac{\bar{r}}{r_0}\right) + \alpha^4\left(\frac{3}{22400}\frac{\bar{r}^2}{r_0^2}\log\frac{\bar{r}}{r_0}\right) +$$

$$\alpha^3\log\alpha\left(\cancel{\frac{3}{22400}\frac{\bar{r}^3}{r_0^3}}\right) + \alpha^4\log\alpha\left(\cancel{-\frac{3}{22400}\frac{\bar{r}^2}{r_0^2}}\right)$$

$$\log(r/r_0) \xrightarrow{r\to\alpha^{-1}\bar{r}} \log(\bar{r}/r_0) - \log\alpha$$

ADDING SUBLEADING TERMS

$$\boxed{\alpha^6\left[\psi_6 + \log\alpha\ \psi_{6L}\right] \xrightarrow{r\to\alpha^{-1}\bar{r}} \alpha^3\bar{\psi}_3^{(\bar{r}^3)} + \alpha^4\bar{\psi}_4^{(\bar{r}^2)} + \mathcal{O}\left(\alpha^5\log\alpha\right)}$$

$$\psi_{6_L}(r,\theta) = R_2^{(6_L)}(r)\Theta_2(\theta) + R_4^{(6_L)}(r)\Theta_4(\theta) + R_6^{(6_L)}(r)\Theta_6(\theta)$$

$$R_2^{(6_L)}(r) = -\frac{3}{22400}\frac{r^3}{r_0^3} + \frac{3}{22400}\frac{r^2}{r_0^2} + \mathcal{O}\left(\frac{r}{r_0}\right),$$

$$R_4^{(6_L)}(r) = \frac{61}{5913600}\frac{r^2}{r_0^2} + \mathcal{O}\left(\frac{r}{r_0}\right),$$

$$R_6^{(6_L)}(r) = \frac{1}{563200}\frac{r^2}{r_0^2} + \mathcal{O}\left(\frac{r}{r_0}\right).$$

$$\psi_6(r,\theta) = R_2^{(6)}(r)\Theta_2(\theta) + R_4^{(6)}(r)\Theta_4(\theta) + R_6^{(6)}(r)\Theta_6(\theta) + \dots$$

$$R_2^{(6)}(r) = -\frac{3}{22400}\frac{r^3}{r_0^3}\log\frac{r}{r_0} + \frac{3}{22400}\frac{r^2}{r_0^2}\log\frac{r}{r_0} + \dots + \mathcal{O}\left(\frac{r}{r_0}\log\frac{r}{r_0}\right),$$

$$R_4^{(6_L)}(r) = \frac{61}{5913600}\frac{r^2}{r_0^2}\log\frac{r}{r_0} + \dots + \mathcal{O}\left(\frac{r}{r_0}\log\frac{r}{r_0}\right),$$

$$R_6^{(6_L)}(r) = \frac{1}{563200}\frac{r^2}{r_0^2}\log\frac{r}{r_0} + \dots + \mathcal{O}\left(\frac{r}{r_0}\log\frac{r}{r_0}\right).$$

EXAMPLE: red terms

$$\alpha^6\log\alpha\underbrace{\left(-\frac{3}{22400}\frac{r^3}{r_0^3} + \frac{3}{22400}\frac{r^2}{r_0^2}\right)}_{\psi_{6_L}} + \alpha^6\underbrace{\left(-\frac{3}{22400}\frac{r^3}{r_0^3}\log\frac{r}{r_0} + \frac{3}{22400}\frac{r^2}{r_0^2}\log\frac{r}{r_0}\right)}_{\psi_6} \xrightarrow{r\to\alpha^{-1}\bar{r}} \alpha^3\log\alpha\left(-\cancel{\frac{3}{22400}\frac{\bar{r}^3}{r_0^3}}\right) + \alpha^4\log\alpha\left(\cancel{\frac{3}{22400}\frac{\bar{r}^2}{r_0^2}}\right) +$$

$$\alpha^3\left(-\frac{3}{22400}\frac{\bar{r}^3}{r_0^3}\log\frac{\bar{r}}{r_0}\right) + \alpha^4\left(\frac{3}{22400}\frac{\bar{r}^2}{r_0^2}\log\frac{\bar{r}}{r_0}\right) + \longrightarrow \alpha^3\bar{\psi}_3^{(\bar{r}^3)} + \alpha^4\bar{\psi}_4^{(\bar{r}^2)}$$

$$\alpha^3\log\alpha\left(\cancel{\frac{3}{22400}\frac{\bar{r}^3}{r_0^3}}\right) + \alpha^4\log\alpha\left(-\cancel{\frac{3}{22400}\frac{\bar{r}^2}{r_0^2}}\right)$$

$$\log(r/r_0) \xrightarrow{r\to\alpha^{-1}\bar{r}} \log(\bar{r}/r_0) - \log\alpha$$

NON-PERTURBATIVE STRUCTURE

Kerr geometry is analytic in α, we therefore expect that also the magnetosphere is analytic!

EXAMPLE

$$\frac{1}{\alpha^2+1} + \alpha^2\sqrt{\overbrace{(\alpha^2)^{\alpha^4}}^{\text{For finite } \alpha \text{ the function is analytic}} - \frac{1}{\alpha^2+1}} = 1 - \alpha^2 + |\alpha|^3 + \alpha^4 \underbrace{- \frac{1}{2}|\alpha|^5 + |\alpha|^5}_{\text{The limit } \alpha \to 0 \text{ leads to non-analytic terms}}\log|\alpha| + \mathcal{O}(\alpha^6)$$