The Eikonal Exponentiation and Gravitational Waves

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Theories of Fundamental Interactions
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Based on
2008.12743, 2101.05772, 2104.03256,
2105.04594, 2203.11915, 2204.02378
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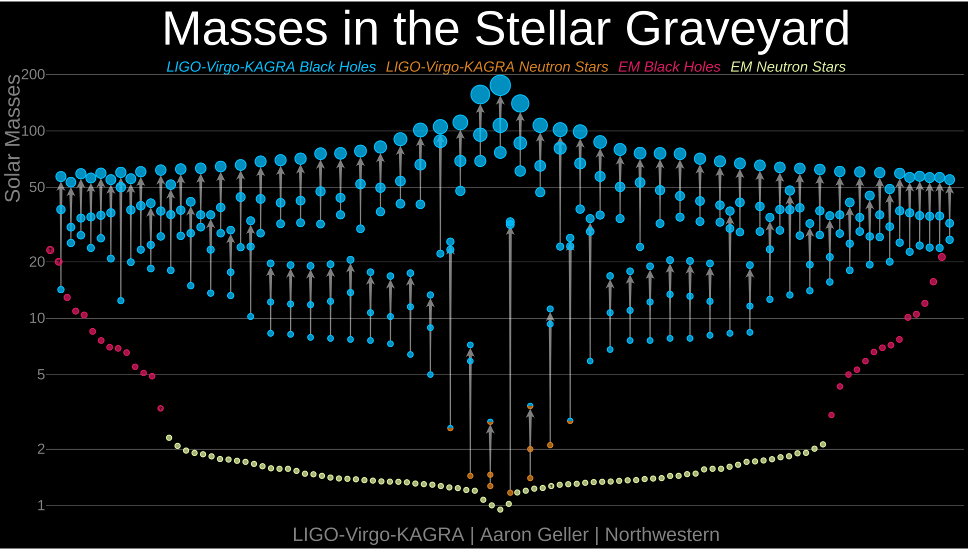
Outline

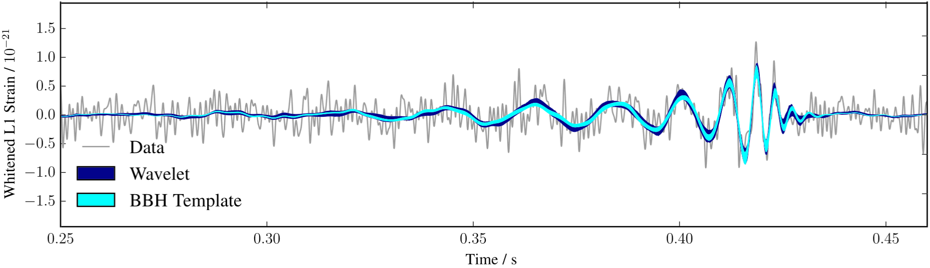
Outline

Gravitational Wave Astronomy



Waveform Templates

[LIGO Scientific Collaboration '16]



Inspiral
Weak gravity

Merger
Strong gravity

Ringdown
Small oscillations

Analytical Approximation Methods

- Post-Newtonian (PN): expansion "for small G and small v"

$$\frac{2Gm}{rc^2} \sim \frac{v^2}{c^2} \ll 1$$

- Post-Minkowskian (PM): expansion "for small G"

$$\frac{2Gm}{rc^2} \ll 1 \,, \qquad \text{generic } \frac{v}{c}$$

- Self-Force: expansion in the near-probe limit $m_2 \ll m_1$ or

$$\nu = \frac{m_1 m_2}{(m_1 + m_2)^2} \ll 1 \,.$$

General Relativity from Scattering Amplitudes

Idea

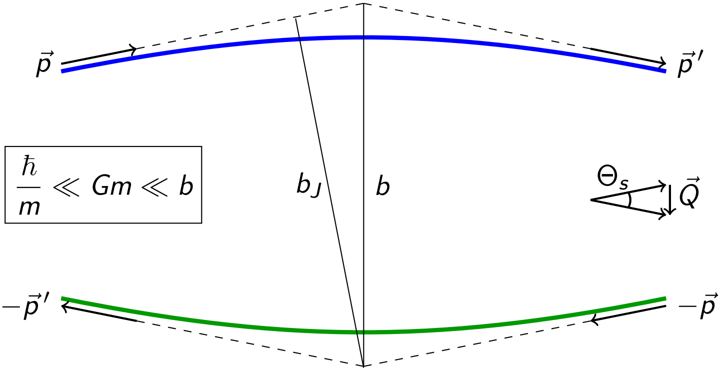
Extract the PM gravitational dynamics from scattering amplitudes.

- Weak-coupling expansion ↔ PM expansion

Weak-coupling:	$\mathcal{A}_0 = \mathcal{O}(G)$	$\mathcal{A}_1 = \mathcal{O}(G^2)$	$\mathcal{A}_2 = \mathcal{O}(G^3)$	$\mathcal{A}_3 = \mathcal{O}(G^4)$
PM:	1PM	2PM	3PM This talk	4PM State of the art (conservative)

- Lorentz invariance ↔ generic velocities
- Study scattering events, then export to bound trajectories (V_{eff}, analytic continuation...)

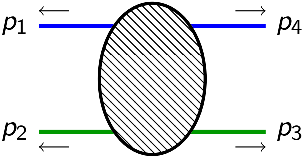
Post-Minkowskian (PM) Scattering



$$\boxed{\frac{\hbar}{m} \ll Gm \ll b}$$

$$Gm^2 \underset{CL}{\gg} \hbar\,, \qquad \frac{Gm}{b} \underset{\text{PM}}{\ll} 1\,, \qquad \begin{aligned} s &= E^2 = m_1^2 + 2m_1 m_2\,\sigma + m_2^2\,, \\ \sigma &= 1/\sqrt{1-v^2}\,. \end{aligned}$$

The Elastic Eikonal



$$s = -(p_1 + p_2)^2 = E^2\,,$$
$$t = -(p_1 + p_4)^2 = -q^2\,.$$

- From q to b: Fourier transform

$$\tilde{\mathcal{A}}(s,b) = \frac{1}{4Ep}\int \frac{d^{D-2}q}{(2\pi)^{D-2}}\, e^{ib\cdot q}\mathcal{A}(s,q)\,, \qquad \boxed{1 + i\tilde{\mathcal{A}}(s,b) = e^{2i\delta(s,b)}}$$

with $2\delta = 2\delta_0 + 2\delta_1 + 2\delta_2 + \cdots \sim Gm^2\left(1 + \frac{Gm}{b} + \left(\frac{Gm}{b}\right)^2 + \cdots\right)$

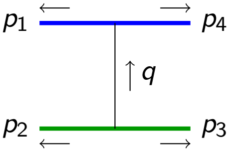
- From b to Q: the stationary-phase approximation gives

$$\int d^{D-2}b\, e^{-ib\cdot Q} e^{i2\delta(s,b)} \implies \boxed{Q_\mu = \frac{\partial\,\mathrm{Re}\,2\delta}{\partial b^\mu}}$$

with $\Theta_s \sim \frac{Q}{p} \sim \frac{Gm}{b} + \left(\frac{Gm}{b}\right)^2 + \left(\frac{Gm}{b}\right)^3 + \cdots$

Example: the 1PM Eikonal

- Tree-level amplitude in $D = 4 - 2\epsilon$ dimensions



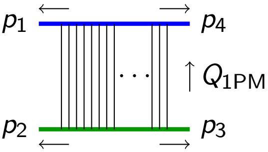
$$\mathcal{A}_0(s,q) = \frac{32\pi G m_1^2 m_2^2 (\sigma^2 - \frac{1}{2-2\epsilon})}{q^2} + \cdots$$

$$\tilde{\mathcal{A}}_0(s,b) = \frac{4 G m_1 m_2 (\sigma^2 - \frac{1}{2-2\epsilon})}{2\sqrt{\sigma^2-1}} \frac{\Gamma(-\epsilon)}{(\pi b^2)^{-\epsilon}} \,.$$

- Matching to the eikonal exponentiation [Kabat, Ortiz '92; Bjerrum-Bohr et al.'18]

$$e^{2i\delta_0} \underset{\text{“small } G\text{”}}{\rightarrow} 1 + i\tilde{\mathcal{A}}_0 \implies 2\delta_0 = \tilde{\mathcal{A}}_0 \,.$$

- From $Q = \partial_b 2\delta$, we obtain the leading-order deflection



$$Q_{1\text{PM}} = \frac{4 G m_1 m_2 \left(\sigma^2 - \frac{1}{2}\right)}{b\sqrt{\sigma^2-1}}$$

$$\Theta_s = \frac{4 G E \left(\sigma^2 - \frac{1}{2}\right)}{b(\sigma^2-1)} \,.$$

Outline

The 3PM Eikonal [2008.12743, 2101.05772, 2104.03256]

[Related work at 3PM: Bern al.'19; Parra-Martinez, Ruf, Zeng '20; Damour '20, Herrmann et al. '21, Bjerrum-Bohr et al.'21; Brandhuber et al.'21]

- Eikonal phase:

$$\mathrm{Re}\, 2\delta_2 = \frac{4 G^3 m_1^2 m_2^2}{b^2} \left[\frac{s\left(12\sigma^4 - 10\sigma^2 + 1\right)}{2 m_1 m_2 \left(\sigma^2 - 1\right)^{\frac{3}{2}}} - \frac{\sigma\left(14\sigma^2 + 25\right)}{3\sqrt{\sigma^2 - 1}} - \frac{4\sigma^4 - 12\sigma^2 - 3}{\sigma^2 - 1}\, \mathrm{arccosh}\sigma \right] + \mathrm{Re}\, 2\delta_2^{\mathrm{RR}}$$

  with

$$\mathrm{Re}\, 2\delta_2^{\mathrm{RR}} = \frac{G}{2} Q_{1\mathrm{PM}}^2 \mathcal{I}(\sigma)\,, \quad \mathcal{I}(\sigma) \equiv \frac{8 - 5\sigma^2}{3(\sigma^2 - 1)} + \frac{\sigma\left(2\sigma^2 - 3\right)}{(\sigma^2 - 1)^{3/2}}\, \mathrm{arccosh}\, \sigma\,.$$

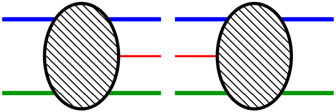
- Infrared divergent exponential suppression:

$$\mathrm{Im}\, 2\delta_2 = \frac{1}{\pi}\left[-\frac{1}{\epsilon} + \log(\sigma^2 - 1)\right] \mathrm{Re}\, 2\delta_2^{\mathrm{RR}} + \cdots$$

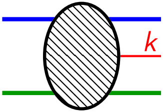
3PM Radiation Reaction from Soft Theorems [2101.05772]

- Analyticity: $i\log(1-\sigma^2-i0) = i\log(\sigma^2-1)+\pi$
- Unitarity: $\text{Im}\,2\delta_2 = [\text{Im}\,\tilde{\mathcal{A}}_2]_{3p.c.}$ and

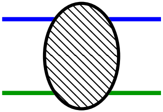
$$[\text{Im}\,2\mathcal{A}]_{3p.c.} = \int d(\text{LIPS})$$



- Soft theorem: [Weinberg '64,'65]



$$\mu\nu \sim \left[\sum_n \frac{\kappa\, p_n^\mu p_n^\nu}{p_n\cdot k}\right]$$



For $\text{Im}\,2\delta_2$ this gives $\frac{1}{\pi}\text{Re}\,2\delta_2^{\text{RR}}$ times

$$\int_0^{\omega_{\max}b} \frac{2\,d\omega}{\omega^{1+2\epsilon}} \sim -\frac{1}{\epsilon} + 2\log(\omega_{\max}b) \sim -\frac{1}{\epsilon} + \log(\sigma^2-1)\,.$$

Smoothness of $\mathrm{Re}\,2\delta_2$ at High Energy

The IR divergence in $\mathrm{Im}\,2\delta_2$ determines $\mathrm{Re}\,2\delta_2^{RR}$

$$\mathrm{Re}\,2\delta_2^{RR} = \lim_{\epsilon\to 0}\left[-\pi\epsilon\,\mathrm{Im}\,2\delta_2\right].$$

At high energy, as $\sigma \to \infty$ and $s \sim 2m_1 m_2 \sigma$, i.e. in the massless limit, the complete eikonal phase is smooth

$$\mathrm{Re}\,2\delta_2 \sim Gs\,\frac{\Theta_s^2}{4}\,, \qquad \Theta_s \sim \frac{4G\sqrt{s}}{b}$$

in agreement with [Amati, Ciafaloni, Veneziano '90].

Outline

Zero-Frequency Limit of the Energy Emission Spectrum

- ZFL of the energy emission spectrum:

$$W \equiv \lim_{\omega \to 0} \frac{dE_{\text{rad}}}{d\omega} = \lim_{\epsilon \to 0} \left[-4\epsilon \, \text{Im} \, 2\delta_2 \right].$$

- At high energy, as $\sigma \to \infty$ and $s \sim 2 m_1 m_2 \sigma$, the energy radiated in the window $\Delta\omega = 1/b$ divided by the incoming energy $\sqrt{s}$ is

$$\frac{\Delta E_{\text{rad}}}{E} = \frac{W \Delta\omega}{\sqrt{s}} \approx \Theta_s^3 \log \sigma \,.$$

The system can emit more energy that it initially has!

G-expansion VS ultrarelativistic (UR) limit

$$Q \approx \sqrt{s}\, \Theta_s \,, \qquad \frac{Q}{\sqrt{s}} \approx \Theta_s \ll 1 \,, \qquad \frac{Q}{m} \approx \frac{\sqrt{s}}{m} \Theta_s$$

Eikonal Operator in the ZFL [2204.02378]

[Soft dressing: Bloch, Nordsieck '37; Thirring, Touschek '51; Weinberg '65; Mirbabayi, Porrati '16, Choi, Akhoury '17; Arkani-Hamed et al.20. Operator exponentiation: Damgaard, Planté, Vanhove '21; Cristofoli et al.'12. Classical soft theorems: Laddha, Sen '18; Sahoo, Sen '18; Saha, Sahoo, Sen '19; Sahoo, Sen '21.]

Operator dressing of the elastic eikonal in b space

$$S_{s.r.} = e^{\int_{\vec{k}} \left[f^{\mu\nu}(k) a^{\dagger}_{\mu\nu}(k) - f^{*\mu\nu}(k) a_{\mu\nu}(k) \right]} e^{i \operatorname{Re} 2\delta} .$$

- $f^{\mu\nu}(k) = F^{\mu\nu}_{TT}(k)$ comes from Weinberg's soft theorem ($\kappa = \sqrt{8\pi G}$)

$$F^{\mu\nu}(k) = \sum_n \frac{\kappa\, p_n^\mu p_n^\nu}{p_n \cdot k - i0} ,$$

  and $\int_{\vec{k}} = \int \frac{d^D k}{(2\pi)^D} 2\pi\delta(k^2)\theta(k^0)\theta(\Lambda - k^0)$, with Λ the cutoff.

- Key identification: $p_1 + p_4 = Q = -p_2 - p_3$ with

$$Q_\mu = e^{-i \operatorname{Re} 2\delta} \left(-i \frac{\partial}{\partial b^\mu} \right) e^{i \operatorname{Re} 2\delta} = \frac{\partial \operatorname{Re} 2\delta}{\partial b^\mu} .$$

Using the Soft Eikonal Operator

- Infrared divergences: $\langle 0|S_{s.r.}|0\rangle = e^{2i\delta}$,

$$\mathrm{Im}\,2\delta = \frac{1}{2}\int_{\vec{k}} F^{\mu\nu}\left(\eta_{\mu\rho}\eta_{\nu\sigma} - \tfrac{1}{D-2}\,\eta_{\mu\nu}\eta_{\rho\sigma}\right)F^{\rho\sigma}\,.$$

- Probability of n soft emissions and average number of soft quanta

$$\mathcal{P}_n = \frac{1}{n!}\int_{\vec{k}_1}\cdots\int_{\vec{k}_n}\langle 0|S^\dagger_{s.r.}|n\rangle\langle n|S_{s.r.}|0\rangle = \frac{1}{n!}\,\left[2\,\mathrm{Im}\,2\delta\right]^n e^{-2\,\mathrm{Im}\,2\delta}\,,$$

$$\langle 0|S^\dagger_{s.r.}N\,S_{s.r.}|0\rangle = \mathcal{N}\,,\qquad \mathcal{N} = 2\,\mathrm{Im}\,2\delta\,.$$

- Energy and momentum in the ZFL:

$$\langle 0|S^\dagger_{s.r.}P^\mu S_{s.r.}|0\rangle = \mathcal{P}^\mu_{\mathrm{rad}}\,,\qquad W \equiv \lim_{\omega\to 0}\frac{dE_{\mathrm{rad}}}{d\omega} = \lim_{\epsilon\to 0}\left[-4\epsilon\ \mathrm{Im}\,2\delta\right].$$

ZFL of the Spectrum for a $2 \to 2$ Process

Let us define $\sigma_Q = \sigma - \dfrac{Q^2}{2m_1 m_2}$, so that

$$s = m_1^2 + 2m_1 m_2\,\sigma + m_2^2\,, \quad t = -Q^2\,, \quad u = m_1^2 - 2m_1 m_2\,\sigma_Q + m_2^2\,.$$

ZFL of the spectrum, with an exact dependence on σ, Q and $m_{1,2}$

$$W = \frac{4G}{\pi}\Bigg\{2m_1 m_2\left(\sigma^2 - \tfrac{1}{2}\right)\frac{\operatorname{arccosh}\sigma}{\sqrt{\sigma^2-1}} - 2m_1 m_2\left(\sigma_Q^2 - \tfrac{1}{2}\right)\frac{\operatorname{arccosh}\sigma_Q}{\sqrt{\sigma_Q^2-1}}$$

$$+\sum_{j=1,2}\left[\frac{m_j^2}{2} - m_j^2\left(\left(1+\frac{Q^2}{2m_j^2}\right)^2 - \frac{1}{2}\right)\frac{\operatorname{arccosh}\left(1+\frac{Q^2}{2m_j^2}\right)}{\sqrt{\left(1+\frac{Q^2}{2m_j^2}\right)^2-1}}\right]\Bigg\}.$$

$$= \lim_{\epsilon\to 0}\left[-4\epsilon\ \mathrm{Im}\,2\delta\right]$$

PM Expansion at Low and High Energy

- Standard PM regime: When $Q^2 \ll m_{1,2}^2$, we recover the previous results. *Bonus:* W, $\text{Im}\, 2\delta$ up to $\mathcal{O}(G^5)$ for any background hard process, including the spinning case [Alessio, Di Vecchia '22].

- Sharp convergence radius $\boxed{Q < 2m_{1,2}}$ [Kovacs, Thorne '77,'78; D'Eath '78].

- UR regime: Expanding for $m_{1,2}^2 \ll Q^2 \ll s$, because $Q \sim \frac{\sqrt{s}}{2}\Theta_s$ and $\Theta_s \sim \frac{4G\sqrt{s}}{b} \ll 1$, [Sahoo, Sen '21]

$$W \sim \frac{4G}{\pi}\left(\frac{\Theta_s}{2}\right)^2\left[\log\left(\frac{2}{\Theta_s}\right)^2 + 1\right] \implies \frac{\Delta E_{\text{rad}}}{E} \approx \Theta_s^3 \log\frac{1}{\Theta_s}\,.$$

Finite energy emission! But of course not analytic in G.

Warm-Up: Memory Effect [2203.11915,2204.02378]

[Kovacs, Thorne '77,'78; Strominger, Zhiboedov '14; Jakobsen et al.'21; Mougiakakos, Riva, Vernizzi '21; Cristofoli et al.'21]

- Waveform: $\kappa \langle 0 | S^\dagger_{s.r.} H_{\mu\nu}(x) S_{s.r.} | 0 \rangle = W_{\mu\nu}(x)$ with

$$H_{\mu\nu}(x) = \int_{\vec{k}} \left[a_{\mu\nu}(k) e^{ikx} + a^\dagger_{\mu\nu}(k) e^{-ikx} \right].$$

- Send $r \to \infty$ for fixed u, $\hat{x}$, letting $p_n = \eta_n(E_n, \vec{k}_n)$,

$$W^{\mu\nu} \sim \frac{2G}{r} \sum_n \theta\left(\eta_n u\right) \frac{(p_n^\mu p_n^\nu)_{TT}}{E_n - \vec{k}_n \cdot \hat{x}}.$$

- The $-i0$ fixes the u-independent ST ambiguity

[cf. Damour '20; Veneziano, Vilkovisky '22]

$$\int_{-\infty}^{+\infty} \frac{d\omega}{i2\pi} \frac{e^{-i\omega u}}{-\eta_n \omega - i0} = \int_{-\infty}^{+\infty} \frac{d\omega}{i2\pi} \frac{e^{i\omega \eta_n u}}{\omega - i0} = \theta\left(\eta_n u\right).$$

$\mathcal{J}^{\rm sc}_{\alpha\beta}$ for a Massless Scalar [2203.11915]

- Charge associated to Lorentz transformations:

$$iJ^{\rm sc}_{\alpha\beta} = \frac{1}{2}\int_{\vec{k}} \left[a^\dagger(k) k_{[\alpha} \frac{\partial a(k)}{\partial k^{\beta]}} - k_{[\alpha} \frac{\partial a^\dagger(k)}{\partial k^{\beta]}} a(k) \right].$$

- Classical contribution due to gravitons with $\omega < \Lambda$:
$\mathcal{J}^{\rm sc}_{\alpha\beta} = \langle 0 | S^\dagger_{s.r.} J^{\rm sc}_{\alpha\beta} S_{s.r.} | 0 \rangle$ gives

$$i\mathcal{J}^{\rm sc}_{\alpha\beta} = \frac{1}{2}\int_{\vec{k}} \left(f^* k_{[\alpha} \frac{\partial f}{\partial k^{\beta]}} - k_{[\alpha} \frac{\partial f^*}{\partial k^{\beta]}} f \right), \qquad f = \sum_n \frac{g_n}{p_n \cdot k - i0}$$

- The $-i0$ prescription is important, and the result localizes to $\omega = 0$,

$$\int_{-\Lambda}^{\Lambda} \frac{\omega d\omega}{(-\eta_n \omega + i0)(-\eta_m \omega - i0)} = -\frac{i\pi}{2}(\eta_n - \eta_m)$$

$$\int_{-\Lambda}^{\Lambda} \frac{\omega d\omega}{(-\eta_m \omega - i0)^2} = -i\pi\eta_m .$$

- Total angular momentum/mass dipole operator:

$$iJ_{\alpha\beta} = \int_{\vec{k}} a^{\dagger}_{\mu\nu}(k)\left(P^{\mu\nu,\rho\sigma}\, k_{[\alpha}\frac{\overset{\leftrightarrow}{\partial}}{\partial k^{\beta]}} + 2\eta^{\mu\rho}\delta^{\nu}_{[\alpha}\delta^{\sigma}_{\beta]}\right) a_{\rho\sigma}(k)$$

with $P^{\mu\nu,\rho\sigma} = \frac{1}{2}\left(\eta^{\mu\rho}\eta^{\nu\sigma} + \eta^{\mu\sigma}\eta^{\nu\rho} - \eta^{\mu\nu}\eta^{\rho\sigma}\right)$.

- Classical average: $\mathcal{J}_{\alpha\beta} = \langle 0|S^{\dagger}_{s.r.} J_{\alpha\beta} S_{s.r.}|0\rangle$

$$i\mathcal{J}_{\alpha\beta} = \int_{\vec{k}} F^{*}_{\mu\nu}\left[\left(\eta^{\mu\rho}\eta^{\nu\sigma} - \frac{1}{D-2}\eta^{\mu\nu}\eta^{\rho\sigma}\right) k_{[\alpha}\frac{\overset{\leftrightarrow}{\partial}}{\partial k^{\beta]}} + 2\eta^{\mu\rho}\delta^{\nu}_{[\alpha}\delta^{\sigma}_{\beta]}\right] F_{\rho\sigma}$$

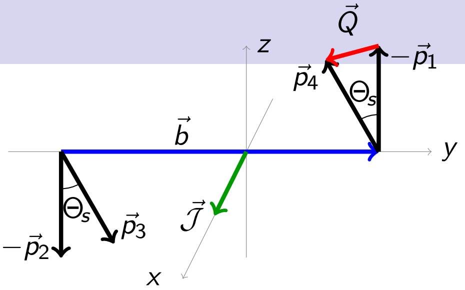
in agreement with [Manohar, Ridgway, Shen '22].

Angular momentum/mass dipole loss due to gravitons

$$\mathcal{J}^{\alpha\beta} = \frac{G}{2}\sum_{n,m}\left[\left(\sigma_{nm}^2 - \frac{1}{2}\right)\frac{\frac{\sigma_{nm}\operatorname{arccosh}\sigma_{nm}}{\sqrt{\sigma_{nm}^2-1}} - 1}{\sigma_{nm}^2 - 1} - \frac{2\sigma_{nm}\operatorname{arccosh}\sigma_{nm}}{\sqrt{\sigma_{nm}^2-1}}\right](\eta_n - \eta_m)\, p_n^{[\alpha} p_m^{\beta]}\,.$$

Angular Momentum Loss

The formula captures the loss of mechanical angular momentum $\vec{\mathcal{J}}$ completely up to $\mathcal{O}(G^2)$ [Damour '20; Jakobsen et al.'21; Mougiakakos, Riva, Vernizzi '21; Gralla, Lobo '21; Manohar, Ridgway, Shen '22]



Analyticity vs Linear Response:

$$\mathcal{J}^{yz} \sim \frac{4p}{Q} \lim_{\epsilon\to 0}[-\pi\epsilon \,\mathrm{Im}\, 2\delta] + \mathcal{O}\left(G^4\right)$$

ensures that/explains why [2104.03256]

$$\Theta^{\mathrm{RR}}_{3\mathrm{PM}} = -\frac{1}{p}\frac{\partial\, \mathrm{Re}\, 2\delta_2^{\mathrm{RR}}}{\partial b} = \frac{2}{pb}\lim_{\epsilon\to 0}\left[-\pi\epsilon\, \mathrm{Im}\, 2\delta_2\right]$$

agrees with the linear-response link [Bini, Damour '12; Damour '20]

$$\Theta^{\mathrm{RR}}_{3\mathrm{PM}} \simeq -\frac{1}{2p}\frac{\partial \Theta_{1\mathrm{PM}}}{\partial b}\mathcal{J}^{yz} \simeq \frac{Q}{2p^2 b}\mathcal{J}^{yz}\,.$$

Angular Momentum Loss

- The formula captures the loss of mechanical angular momentum to $\mathcal{O}(G^2)$ also for spinning particles, for generic spin alignments [Alessio, Di Vecchia '22].

- It also captures the $\mathcal{O}(G^n)$ loss due to zero-frequency gravitons attached to the elastic process. Cross-checked to $\mathcal{O}(G^3)$ against [Manohar, Ridgway, Shen '22].

- In the high-energy limit $m_i^2 \ll Q^2 = s \sin^2 \frac{\Theta_s}{2}$,

$$\mathcal{J}^{yz} \sim 2Gs \sin\Theta_s \log \frac{\cos\frac{\Theta_s}{2}}{\sin\frac{\Theta_s}{2}}$$

  and for small Θ_s

$$\mathcal{J}^{yz} \sim Gs\Theta_s \log \frac{4}{\Theta_s^2} \implies \frac{\mathcal{J}^{yz}}{pb} \approx \Theta_s^2 \log \frac{1}{\Theta_s} \,.$$

Mass-Dipole Loss

- We find an $\mathcal{O}(G^2)$ loss for the *ty* component

$$\frac{\mathcal{J}_{ty}}{b\,(E_1 - E_2)} \sim \frac{\mathcal{J}_{yz}}{2bp}$$

in agreement with [Manohar, Ridgway, Shen '22]. Solving

$$\Delta\,(b_1 E_1 - b_2 E_2) = -\mathcal{J}_{ty}\,, \qquad \Delta\,(b_1 + b_2)\,p = -\mathcal{J}_{yz}$$

yields

$$\Delta b_1 p = \Delta b_2 p = -\mathcal{J}_{yz}/2\,.$$

- There is an $\mathcal{O}(G^3)$ loss for the *tz* component

$$\frac{\mathcal{J}_{tz}}{b\,(E_1 - E_2)} \sim \frac{\Theta_s}{8}\frac{\mathcal{J}_{yz}}{bp}\,.$$

- Our formula and [Manohar, Ridgway, Shen '22] do not find some $\mathcal{O}(G)$ and $\mathcal{O}(G^2)$ terms in the mass-dipole loss in [Gralla, Lobo '21].

More on Linear Response and Eikonal Operator

- If we define

$$\langle \Delta P_\alpha \rangle \equiv -i S_{s.r.}^\dagger \frac{\partial S_{s.r.}}{\partial b^\alpha}$$

we obtain

$$Q_\alpha^{\mathrm{RR}} \equiv \langle \Delta P_\alpha \rangle - Q_\alpha = -\frac{i}{2} \int_{\mathbf{k}} \left[f_{\mu\nu}^*(k) \frac{\partial f^{\mu\nu}(k)}{\partial b^\alpha} - \frac{\partial f_{\mu\nu}^*(k)}{\partial b^\alpha} f^{\mu\nu}(k) \right].$$

- $f_{\mu\nu}$ depends on b via Q itself and the momenta, and we obtain

$$Q_\alpha^{\mathrm{RR}} = -G \sum_{\substack{n=1,2 \\ m=3,4}} \left[\left(\sigma_{nm}^2 - \tfrac{1}{2}\right) \frac{\sigma_{nm}\Delta_{nm} - 1}{\sigma_{nm}^2 - 1} - 2\sigma_{nm}\Delta_{nm} \right] p_n \cdot \frac{\partial p_m^\mu}{\partial b^\alpha}.$$

- This quantity satisfies $Q^{\mathrm{RR}} = \frac{Q}{2pb} \mathcal{J}^{yz}$ to leading PM order, in agreement with the above discussion.

Summary and Outlook

- Elastic eikonal: $2\delta_2$ is under control and $\mathrm{Re}\,2\delta_2$ is smooth at high energy thanks to radiation reaction.
- Operator eikonal $S_{s.r.}$ gives a unitary description for soft radiation:
  - yields the ZFL of the energy emission spectrum W
  - predicts/subtracts divergent part of $\mathrm{Im}\,2\delta_2$
  - is crucial to restore smoothness at high energy
- Including the $-i0$ prescription, $\mathcal{J}_{\alpha\beta}$ due to the "zero-frequency gravitons", or rather to static field effects.
- The $-i0$ is also crucial to reproduce the radiation reaction *from* the eikonal operator [cf. Cristofoli et al.'21]

For the future:

- Beyond the ZFL (in progress...)
- Subleading soft theorem
- Learning more about RR effects at $\mathcal{O}(G^4)$ [cf. Manohar, Ridgway, Shen '22].