MECHANICAL NOISE IN GW DETECTORS MODELING METHODS

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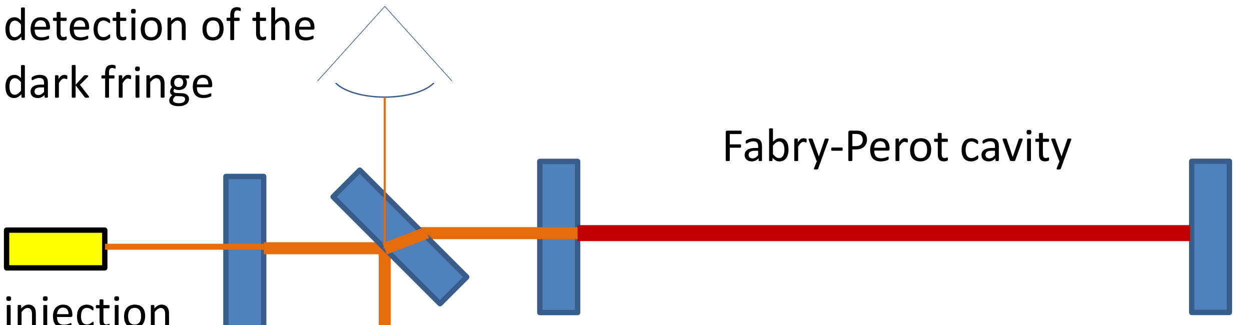
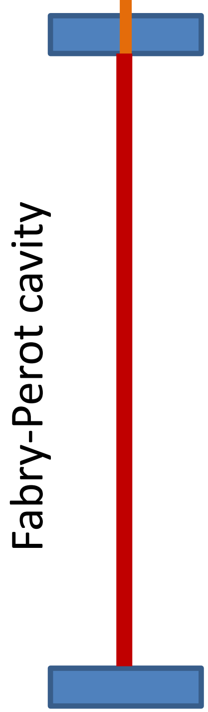
OUTLINE: Tutorial for people who want to produce their own mechanical noise models and noise projections.

Schematic view of GW detector

Michelson interferometer





The reflections of two Fabry-Perot cavities are kept in distructive interference configuration.

One can imagine the sensible element as a couple of compact bodies (Test Masses) kept at a distance **L** as large as possible

The field **h(t)** of a GW is a space-time strain. The distance between the test masses is perturbed and a length variation **s(t)=Lh(t)** is generated for each cavity. $\mathbf{s_D}$ (the differential component) is detected as a power fluctuation at the dark port.

GW detection sensitivity
readout noise

The power is detected by a photodiode generating an electrical signal, sampled at a certain frequency (20 KHz for VIRGO). The output is a sequence of calibrated data:

$$\boxed{\mathbf{p(t_i)=s_D(t_i)+n_i}}$$

where **n** is the intrinsic error of the measurement (**readout noise**), given by a stochastic process connected to the measurement itself (for instance: shot noise).

The sensitivity of the detector is the minimum amplitude of a GW producing an output which can be distinguished from the noise.

The first obvious characterization of the noise is its standard deviation, or **rms** (root mean square).

For a given time series lasting **T**, sampled at time intervals **dt**: (**1/dt=f_s** sampling frequency; **T/dt =** number of samples in the time series)

$$rms(n) = \sqrt{\frac{dt}{T}\sum n_i^2}$$

GW detection sensitivity mechanical noise

The readout noise is not the only random process disturbing the measurement. The biggest problem come from the fact that a residual motion of the test masses, not due to GW disturbance, is alway present. A more complete expression of the detector output is:

$\mathbf{p(t_i)=s_D(t_i)+x_D(t_i)+n_i}$ **x: elongation of the cavities due to TM motion**

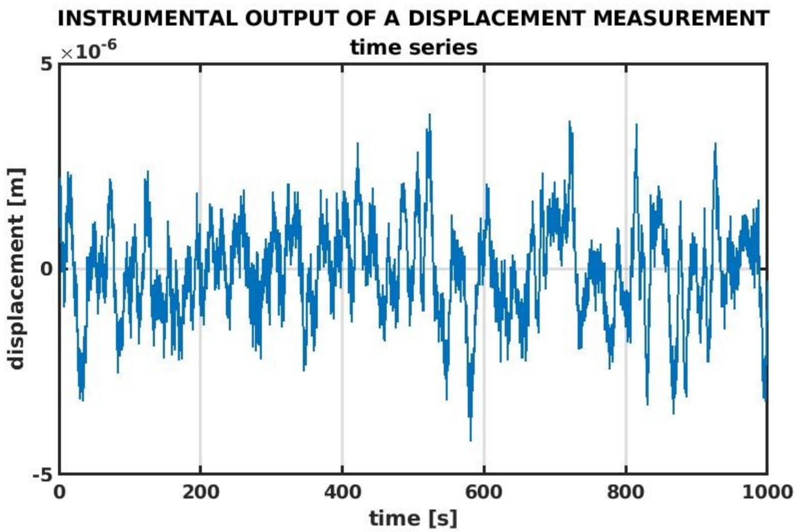
The time evolution of **x** is a random process determined by the superposition of several possible effects. Here is a short list of noises, with their technical names:

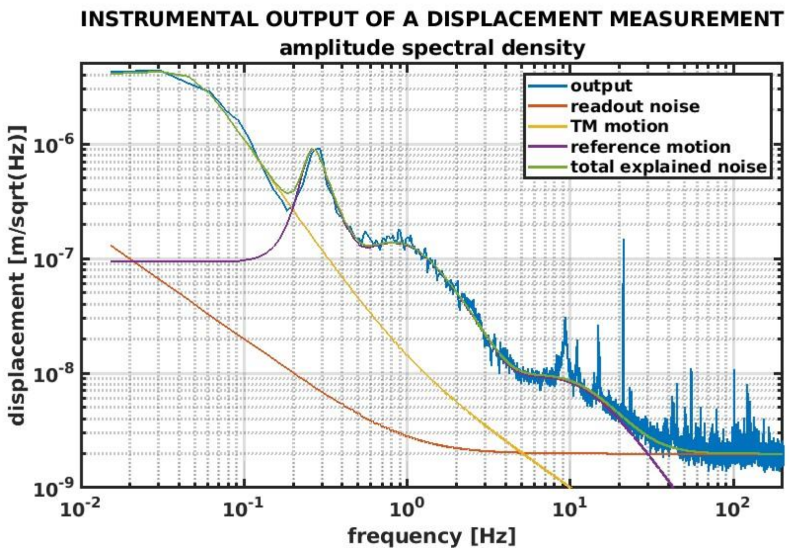
- Interaction with the environment
  - natural vibration of the ground (**seismic noise**)
  - actions on the TM or TM suspension for control purpose (**control noise**)
  - coupling of the TM with external fields or devices (**environmental noise**)
- internal motion
  - brownian motion of the TM due to its own thermal agitation or to the thermal agitation of TM suspension (**thermal noise**)

Signal and noise in frequency domain

Having models of the expected GW perturbation, together with a statistical description of the noise, allows the scientist to separate as much as possible the signal from the noise, optimizing in such a way the detector sensitivity. This is done by means of mathematical methods, the working instruments of data analysis.

The simplest and most popular analysis is the one which converts the time series of data into the *frequency domain*. It has a large application, not only to the analysis of GW data. An example of displacement measurement on the suspension upper stage will be used to show how **the components of signal and noise become visible**.





THIS IS NOT A GW DETECTOR !!!

Amplitude spectral density

The transformation of an instrumental output time series to frequency domain is given by the computation of the Power Spectral Density (PSD). Another possible representation is the PSD square root, called Amplitude Spectral Density (ASD). The output become a sequence of data: $\mathbf{s(f_i)}$, for each frequency up to $\mathbf{f_{max}=f_s/2}$. The resolution $\mathbf{df=f_{i+1}-f_i=1/T}$ is the inverse of the total observation time $\mathbf{T}$. Important properties of the ASD are:

$$rms(s) = \sqrt{\sum s_i^2 df}$$

For a limited band f_{init}:f_{end} ➔ $$brms(s) = \sqrt{\sum_{f_{init}}^{f_{end}} s_i^2 df}$$

If two uncorrelated noises $\mathbf{s_A}$ $\mathbf{s_B}$ are added together to compose a total noise:

$$\mathbf{s_{tot}(t)=s_A(t)+s_B(t)}$$

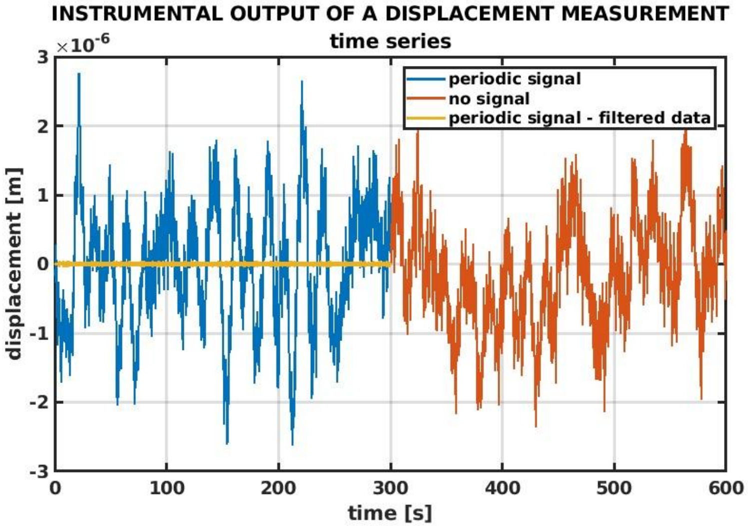
the ASD of the total noise is the quadratic sum of its components

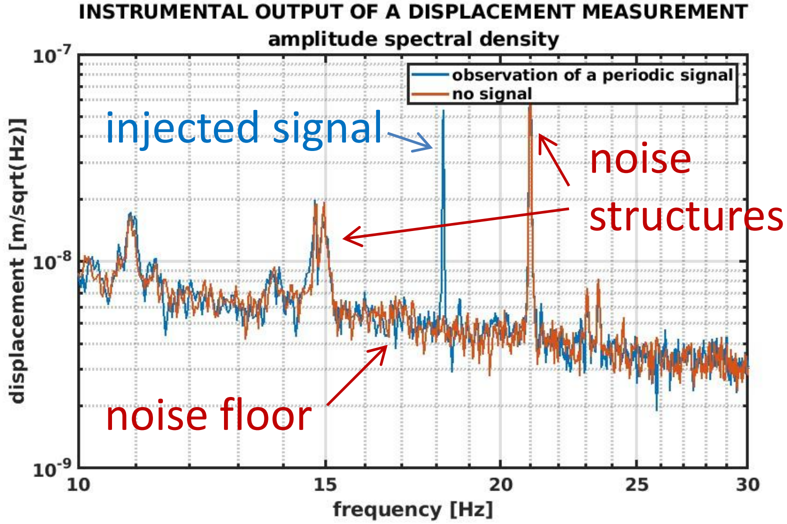
$$s_{tot}(f_i) = \sqrt{s_A^2(f_i) + s_B^2(f_i)}$$

Observation of a peridic signal

For a periodic signal, if τ is the period, the frequency **$f=1/\tau$** determines the only sample in the frequency series containing information. A periodic signal could be totally invisible in time domain and perfectly visible in frequency domain.

A time domain sequence containing the signal can be obtained applying a narrow band-pass filter around the frequency of the signal. This means that almost all the noise can be eliminated, apart from the noise having the same frequency content of the signal.

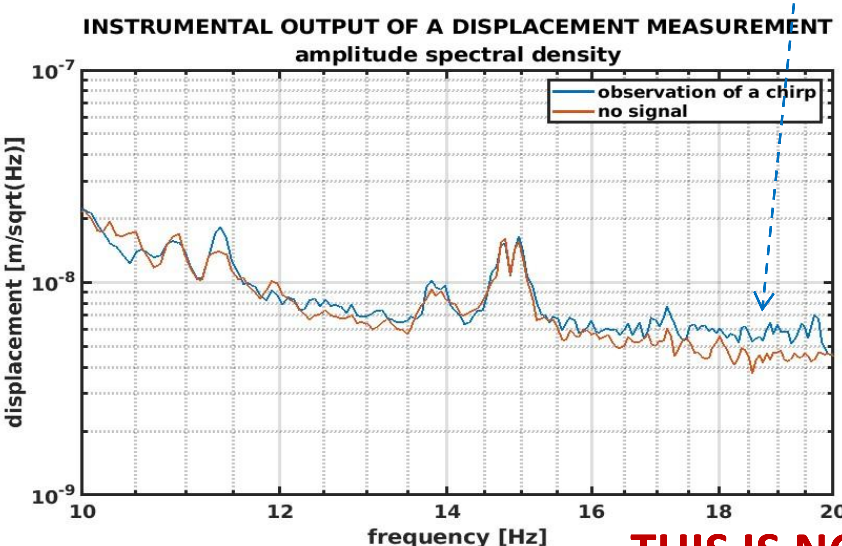


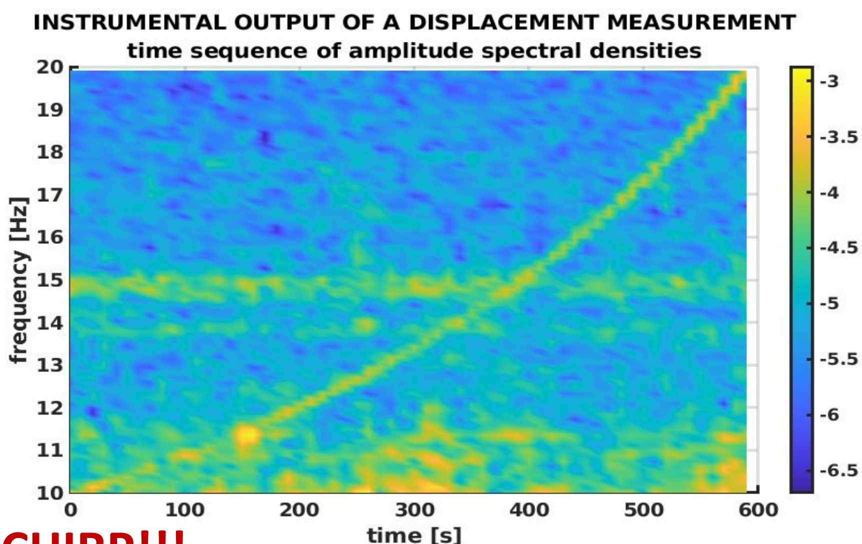


Observation of a chirp

More compex signals like the *chirps*, generated by the revolution of a binary system close to the coalescence, have the information distributed over a limited interval of frequency: the perturbation appears in the data as a periodic signal at a certain frequency, but the frequency changes in time as the revolution become faster.

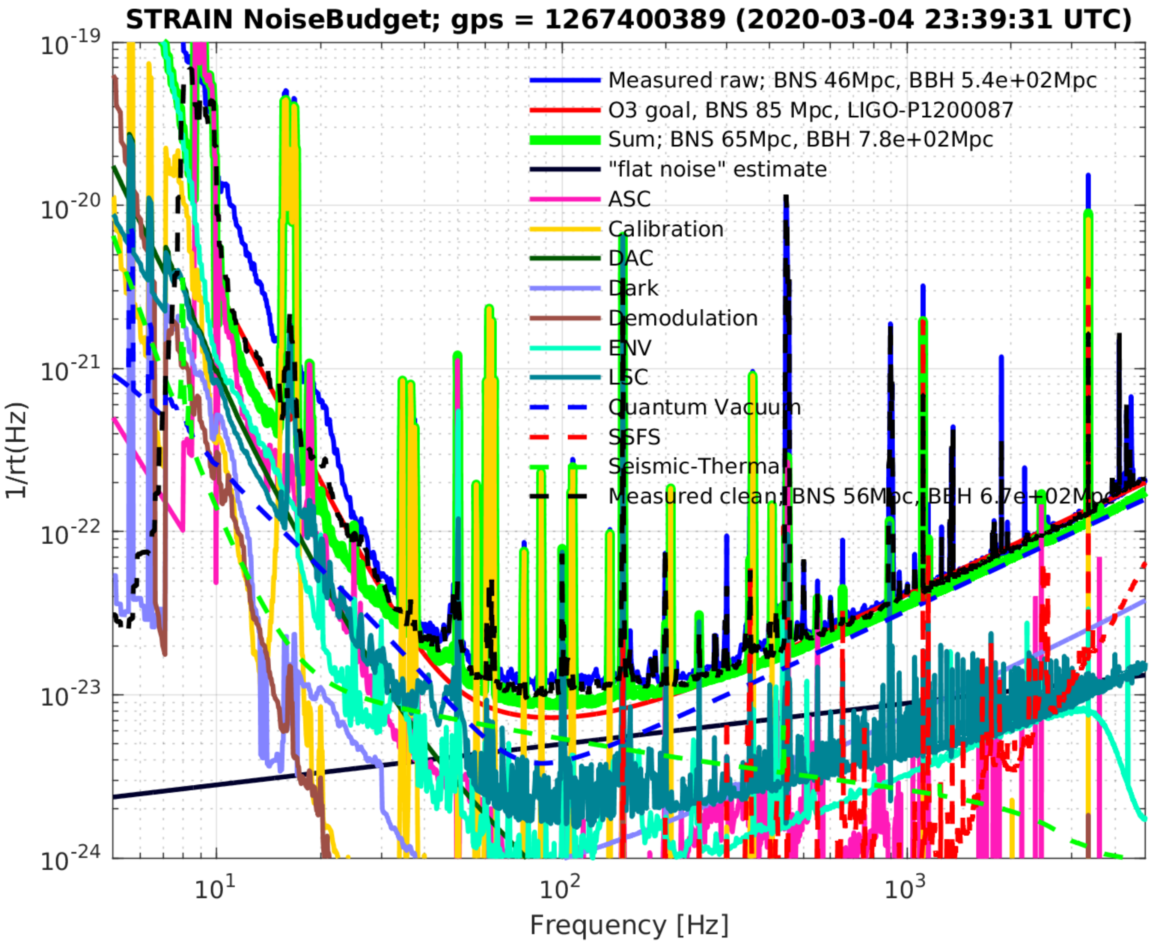
A disturbance at variable frequency has been generated on the same local apparatus of measurement, mantaining a similar amplitude as the previus example.
In the overall ASD the signal is barely visible: it do not give a clear information about the signal shape. It is better to divide the time sequence in short slices and plot the ASD for each slice in a **colored** representation, called **time-frequency diagram**.





THIS IS NOT A CHIRP!!!

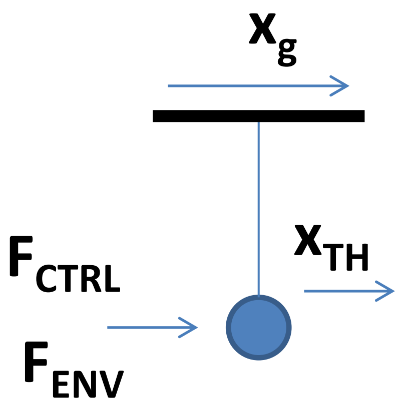
GW detector sensitivity: noise budget



This is an example of strain noise budget computed during the observation run O3. Each curve is an estimation of a noise contribution from a given source. The estimation can be based on data on-line coming from some monitoring channel, or can be an off-line projection based on a theoretical model.

Mechanical noise

The mechanical noise in a GW detector can be described starting from a simple representation: Test Mass is a pendulum hanged to the ground through a thin wire.



The ground motion $\mathbf{x_g}$ is partially transmitted to the suspended mass.

The wire itself generates a TM brownian motion $\mathbf{x_{TH}}$ due to its thermal agitation.

The TM needs to be controlled, in order to keep the detector in its working point. The current sent to the actuator, which produces the control force $\mathbf{F_{CTRL}}$, have also a stocastic component, to be taken in account as a source of residual motion of the test mass. Other components of the external force, $\mathbf{F_{ENV}}$, might come from electromagnetic coupling of the actuation system to the environment.

Modeling mechanical noise

The evaluation of Test Mass motion induced by a stochastic process needs two steps:

1. A model of the disturbance (displacement, force or some other quantity);
2. A model of the deterministic relationship between the mechanical quantities in two points of the suspension:

INPUT: application point of the disturbance ---- **OUTPUT**: TM motion

INPUT
ASD of a stochastic process mechanically coupled to the TM

→

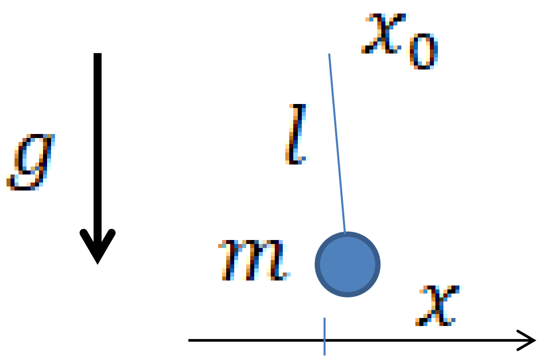
Transfer Function
Transfer mechanism from IN to OUT

→

OUTPUT
TM displacement noise generated by the process in INPUT

The modeling of the **TF** can be obtained by the use of the motion equations. In the best case, things move in the regime of the small oscillation and we are interested only in the stationary conditions on motion. The relationship we are looking for is expected to be linear, and will be modeled under that hypotesis.

Transmission of ground motion



$$m\ddot{x}(t) = -\frac{mg}{l}\left[x(t) - x_0(t)\right]$$

The model is valid for small deviations from the rest position. The approximation allows to assume the force acting on the test mass to be linearly dependent on its position.

The time evolution of test mass position can be derived applying a linear operator to the time evolution of the suspension point. This operator is what we call **Transfer Function**.

$$x(t) = \boldsymbol{TF}\big(x_0(t)\big)$$

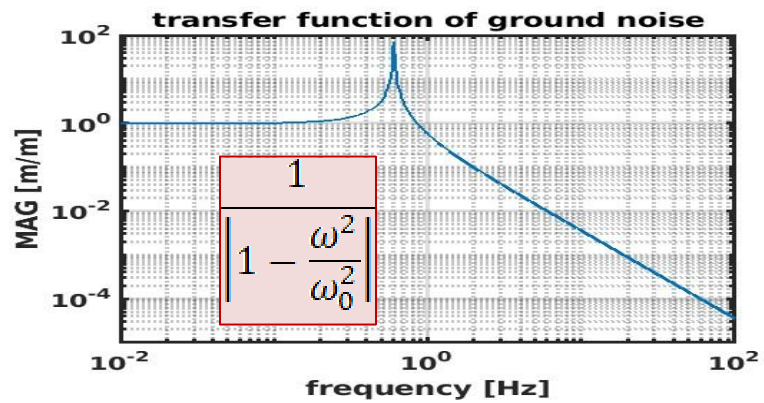
Transfer function in frequency domain

In frequency domain the time evolution is replaced by its fourier transform and the derivative is replaced by an algebraic operation. The **TF** become a multiplicative operator in the complex domain (imaginary part appears in case of damped system). <u>Its magnitude, applied to the ASD of the input, gives the ASD of the output.</u>

$$\dot{x}(t) \rightarrow i\omega x(f) \qquad \omega = 2\pi f$$

$$-\omega^2 x(f) = -\frac{g}{l}\left[x(f) - x_0(f)\right]$$

$$\boxed{x(f) = \frac{1}{1 - \frac{\omega^2}{\omega_0^2}} x_0(f)} \qquad \omega_0^2 = \frac{g}{l}$$

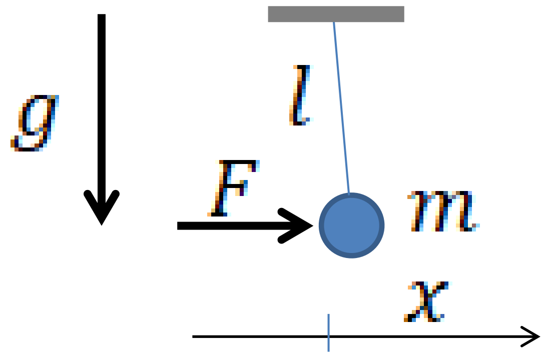


The pendulum has a resonance at a certain frequency, at which the transmission of the ground motion is enhanced. Above that frequency, the transmission is filtered. The length of the wires suspending the test mass is the only relevant parameter.

$$\boxed{l = 0.7\ m \longrightarrow f_0 = 0.6\ Hz}$$

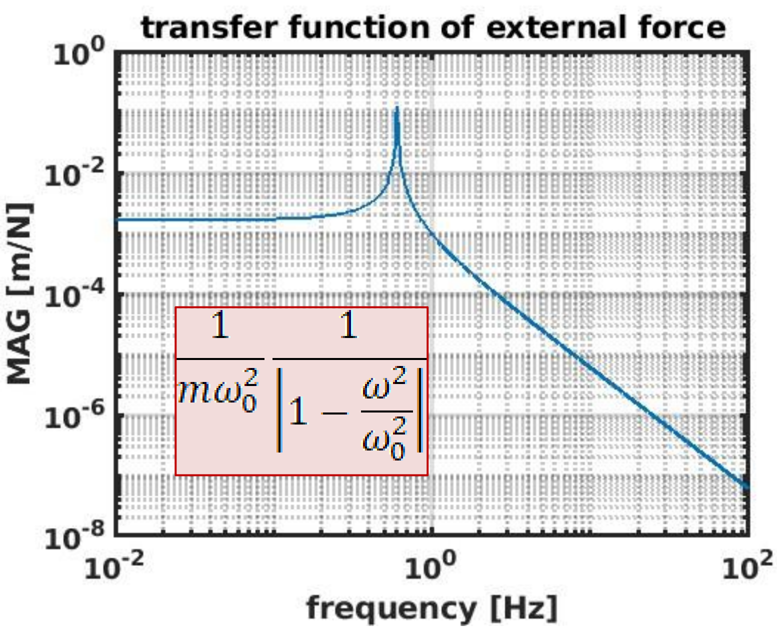
The longer the pendulum, the lower the frequency, the higher the efficiency of the suspension as filter of ground motion at high frequency.

Transfer function of an external force



$$-\omega^2 x(f) = -\frac{g}{l}x(f) + \frac{1}{m}F(f)$$

$$\boxed{x(f) = \frac{1}{1-\frac{\omega^2}{\omega_0^2}} \cdot \frac{F(f)}{m\omega_0^2}}$$

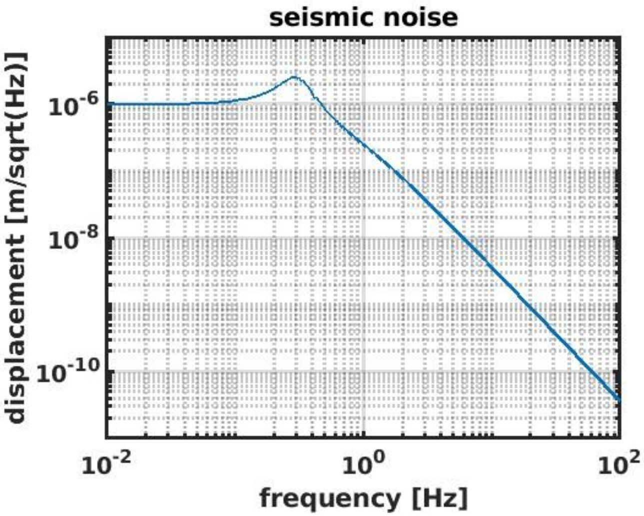


The pendulum oscillation amplifies the effect of an external force at the resonance frequency. The transfer function has the same shape as in the previous case: a peak at f_0. The inverse of the constant value at low fequency is the horizontal stiffness of the elastic link. For TM mass 42 kg:

$$k = m\omega_0^2 = 600\ N/m$$

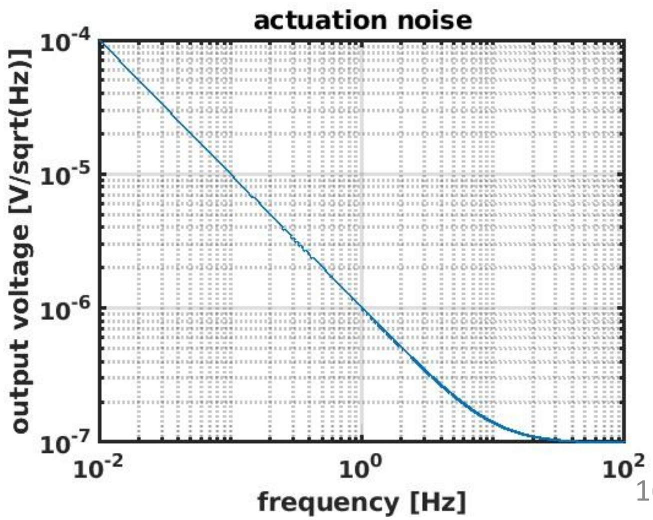
Modeling the noise sources

The evaluation of test mass residual motion requires the knowledge of the processes shaking the TM suspension system in INPUT.



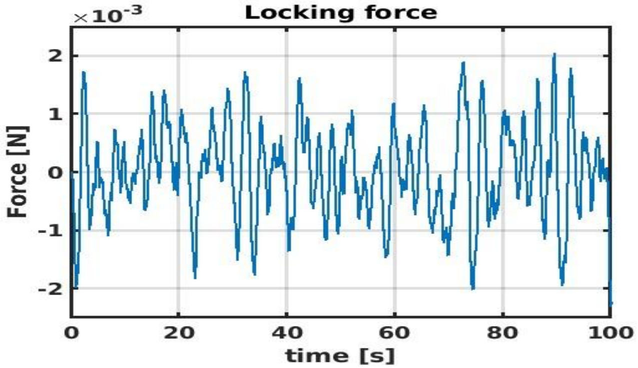
The ASD of the ground motion can be easily measured putting a seismometer close to the pendulum suspension point.
Here, a simple model is taken into account, just to give an idea of the amplitude

One noisy component of the force acting directly on the TM come from the digital board driving the control system. Here is shown a rough model of the noise ASD for a Digital Analog Converter providing 10 V in output.



Test Mass noise budget

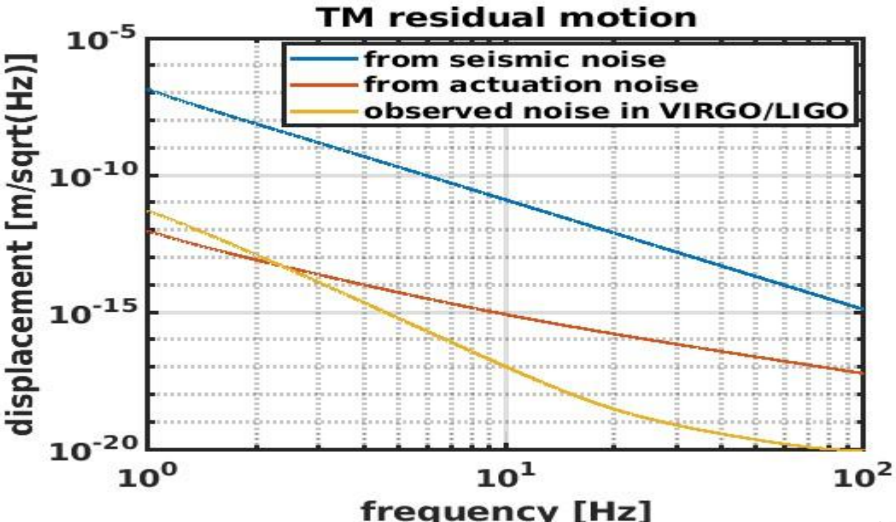
The projection of actuation noise requires one more step: the definition of the actuation gain **gA**, which converts the model of DAC noise from Volt to Newton.
This definition passes through the evaluation of the maximum force required in order to compensate the test mass residual motion due to seismic noise.
By using the model of seismic noise and the response of the TM suspension to the external force, a possible time evolution of the control force can be deduced:



$$gA = \frac{k * maxS}{maxV} * SafetyFactor$$

$$gA = \frac{2 \cdot 10^{-3}[N]}{10[V]} * 5 = 10^{-3}\left[\frac{N}{V}\right]$$

the actuation noise is converted in Newton
the noise models are converted in TM motion



Dissipation: viscous damping

In the simple description of the pendulum motion shown so far, only an elastic term of force has been taken into account.

In a more complete model, a force proportional to the speed is present:

$$F_v(t) = -\gamma \dot{x}(t)$$

This force, the viscous damping, is just one example of a more general kind of interactions: the **dissipative forces**.

Anelasticity associated to deformation of materials is another kind of dissipation, more relevant for the mechanics of the GW detector test mass.

The common characteristic of the dissipative forces is that the integral of the work is not null at the end of a closed path.

In the case of the harmonic oscillation, at the end of a cycle a fraction of the elastic energy is lost. The inverse of this fraction is called **QUALITY FACTOR**.

Dissipation in frequency domain

In frequency domain, a viscous force is a term proportional to the displacement, with a purely immaginary coefficient:

$$F_v = -i\omega\gamma x(f)$$

This is actually a characteristic common to all the dissipative forces: the difference is how this coefficient depends on the frequency.

The dissipative term associated to the strain of a material is frequency indipendent. Usually it is included in the definition of the stiffness, adding an imaginary fraction: the loss angle:

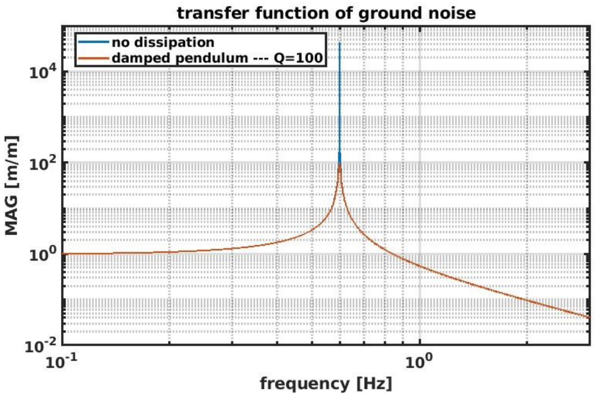
$$F_{el}(f) = -k(1 + i\emptyset)x(f)$$

The quality factor Q is the inverse of the loss angle $Q = \frac{1}{\emptyset}$

Dissipation and thermal noise

A small dissipation does not change the TF of the pendulum. A change is visible at the resonance: divergence disappears, peak height is equal to **Q**

$$\frac{1}{\left|1-\frac{\omega^2}{\omega_0^2}-i\,\frac{\omega}{\omega_0 Q}\right|}=\frac{1}{\sqrt{\left(1-\frac{\omega^2}{\omega_0^2}\right)^2+\frac{\omega^2}{\omega_0^2 Q^2}}}$$



The main effect associated to a dissipation, having an impact on the test mass residual motion, is the **thermal noise**: a stochastic component of the force acts continuously, determining a fluctuation of the mechanical quantities.

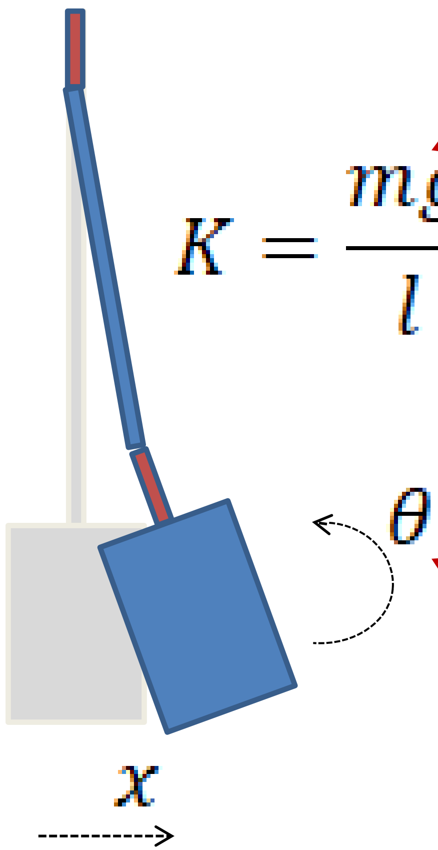
FLUCTUATION DISSIPATION THEOREM

The power spectral density of the test mass displacement is proportional to the imaginary part of its response to an external force:

$$x_{TH}^2(f)=\frac{4k_bT}{\omega}\cdot Imag(TF_{F\to x}(f))$$

T: temperature in Kelvin
k_B: Boltzmann constant

Thermal noise in a physical pendulum



$$K = \frac{mg}{l} + \varepsilon$$

In the model described above, the suspension of the TM is an ideal pendulum. **Its elastic reaction is purely gravitational**. No dissipation is associated to this term. This model is too simple.

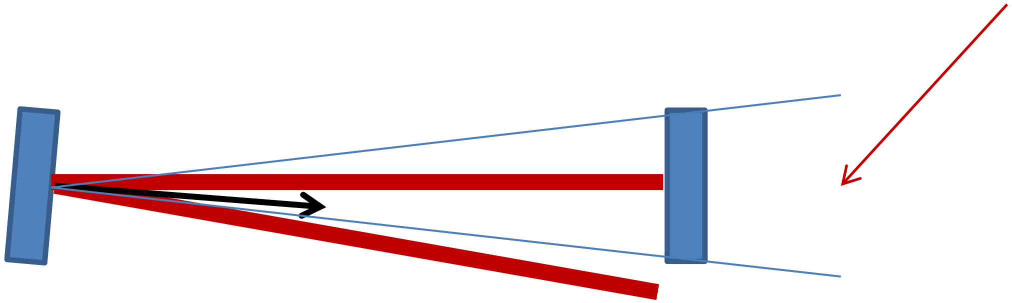
The wire suspending the TM has its own elasticity, the wire deformation induces an additional stiffness.

A heavy body can be suspendend to a very thin wire, having a very low strain during the oscillation. The correction to the mechanical model is negligible, but the anelastic componet has a role in the TM residual motion, because generates thermal noise.

The deformation of the wire is always associated to a rotation. In a model describing the pendulum motion, the test mass angular degrees of freedom should be taken into account. This is not so easy.

Alignment of a Fabry Perot cavity

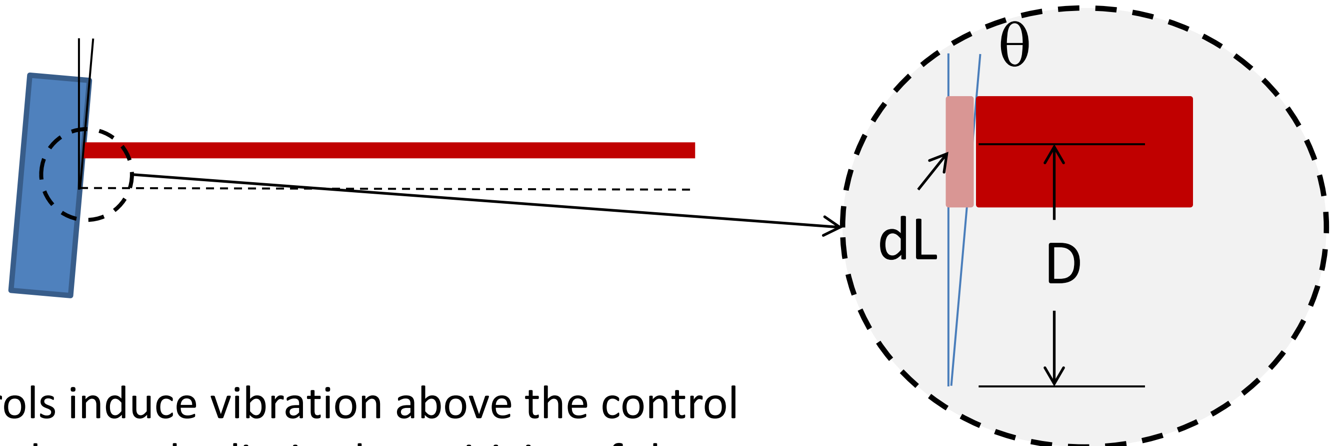
So far we have talked about test mass as a point moving along one direction, because we are thinking about an interferometer sensitive to the length variation of the cavities. Things are more complex: the orientation of the test mass is as critical as the position.
TM is an extended body: it is a disk of diameter 35 cm, thickness 20 cm.
One TM is seen by the other, 3 km far away, under an angle of **100 μRad**



The cavity can stay in resonance if the beam arriving on a TM from the other has a very small angle of incidence on the reflecting surface. A rotation larger than 25 μRad send the reflecion out of the cavity . But we are dealing with Fabry Perot high finesse cavities: the beam has to stay inside for 100 or more reflections. This bring the requirement of TM orientation accuracy to **0.1 μRad**.

TM angular control and angular noise

The requirement on the test mass orientation can be fulfilled only applying an angular control. This is a solution which introduces a new problem: the angular noise.



Controls induce vibration above the control band, due to the limited sensitivity of the sensor in loop. A TM angular vibration is converted in length noise by a displacement **D** of the beam with respect to the center of rotation.

$$dL(t)=D\theta(t)$$

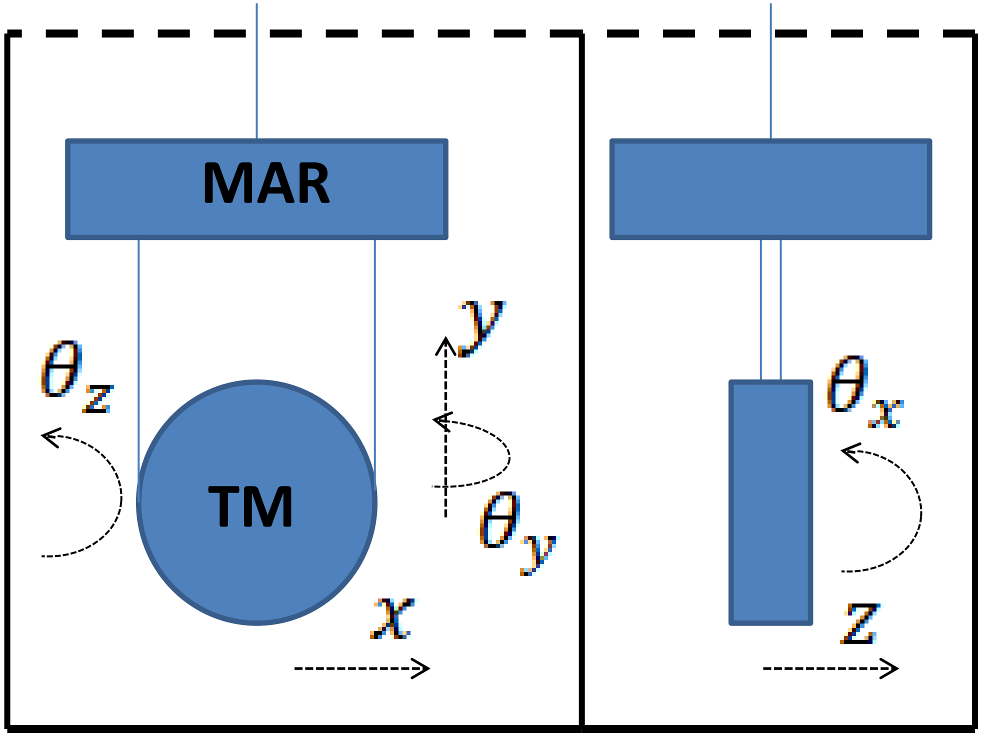
$D=10^{-4}$ m typical off-centering

$\theta=10^{-15}$ rad/sqrt(Hz)
requirement for angular noise

TM suspension: the payload

A controlled stage needs to be added above the test mass: the **Marionetta**.
It is a stage of pendulum hanged to a single wire. TM is hanged to it by four wires.
Through the control of MAR, TM can be displaced and rotated avoiding a direct action on TM itself. The whole system is called **PAYLOAD**

The complexity of this system requires MAR and TM to be described as extended bodies.



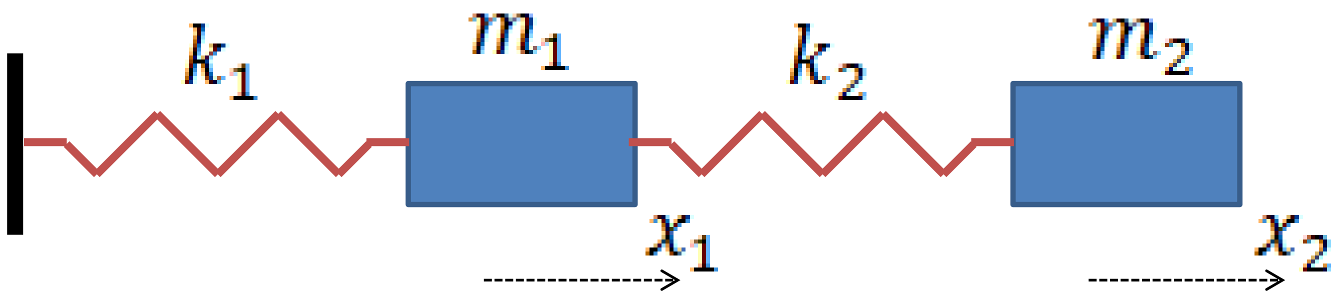
z θ_x θ_y are the d.o.f.s requiring a precise control of the working point and a low residual motion.

Mechanical models treating those d.o.f.s are useful for developing controls and predicting the noise transmission.

MAR d.o.f.s needs to be included in the model.

Two bodies elastically connected

As a starting point, let's assumes MAR and TM to be point-like, moving along one dimension. Elastic links connect MAR to the ground and TM to MAR.



$$X = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$F = \begin{bmatrix} F_1 \\ F_2 \end{bmatrix}$$

$$m_1 \ddot{x}_1 = -k_1 x_1 - k_2 (x_1 - x_2) + F_1$$

$$m_2 \ddot{x}_2 = -k_2 (x_2 - x_1) + F_2$$

$$m = \begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix}$$

$$k = \begin{bmatrix} k_1 + k_2 & -k_2 \\ -k_2 & k_2 \end{bmatrix}$$

$$\longrightarrow \quad -m\omega^2 X = -kX + F$$

Matricial expression of TF in frequency domain

$$X = (-\omega^2 + m^{-1} k)^{-1} m^{-1} F$$

Normal modes

The key element of the transfer function is the inverse of the matrix:

$$\boxed{\omega^2 - m^{-1}k}$$

One can recognize the same structure of the pendulum TF, even if in matricial form. The denominator of each element is the determinant of the matrix, a second order polinomial in the variable ω^2 (or N order, extending to N oscillators).
Its factorization:

$$\frac{1}{\omega^2 - \omega_1^2} \cdot \frac{1}{\omega^2 - \omega_2^2} \qquad \omega_1^2, \omega_2^2 = eig(m^{-1}k)$$

says that the transmission of a double oscillator is the product of two single oscillators. The two poles are the **resonance frequencies of the oscillation normal modes**.

The matricial expression of the TF provides 4 components: we can derive the motion of MAR or TM for a given force on MAR or TM. This information is useful, because actually two actuation systems are installed: one for MAR and one for TM.

The same peaks are present in all the TF components: normal modes are shared by MAR and TM coordinates

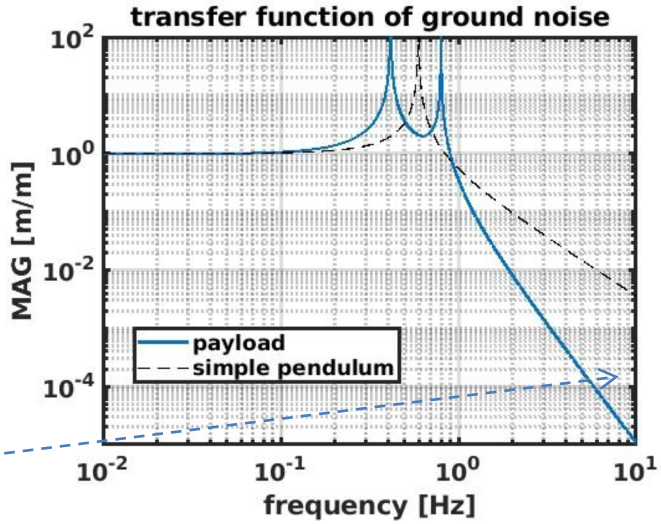
Longitudinal transfer functions

m1=105 kg ; m2=42 kg

k1=1250N/m ; k2=600 N/m

f1=0.41 Hz ; f2=0.79 Hz

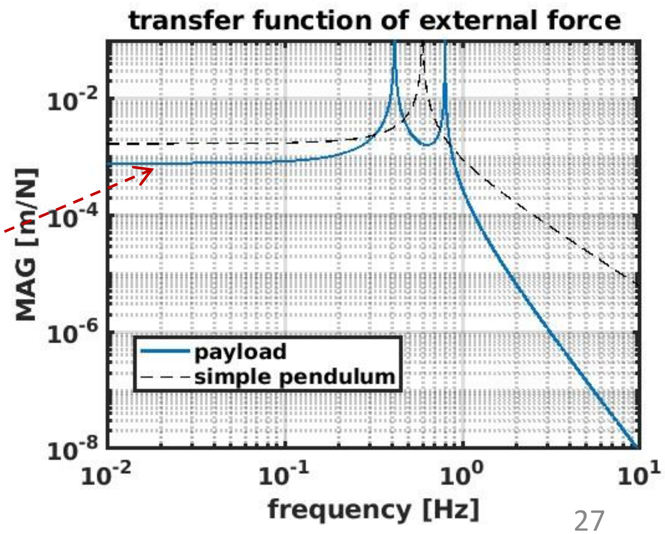
The transmission of seismic noise is filtered with a slope 1/f4, instead of 1/f2



At 10 Hz the attenuation is a factor of 500 larger

The response ot TM to an external force applied on MAR has the same two resonances and the same slope of the seismic transfer function.
The stiffness seen at the actuation point is about the double than in the case of the single pendulum.
The actuation gain has to be rescale accordingly.

$$gA = 2 \cdot 10^{-3} \left[\frac{N}{V}\right]$$



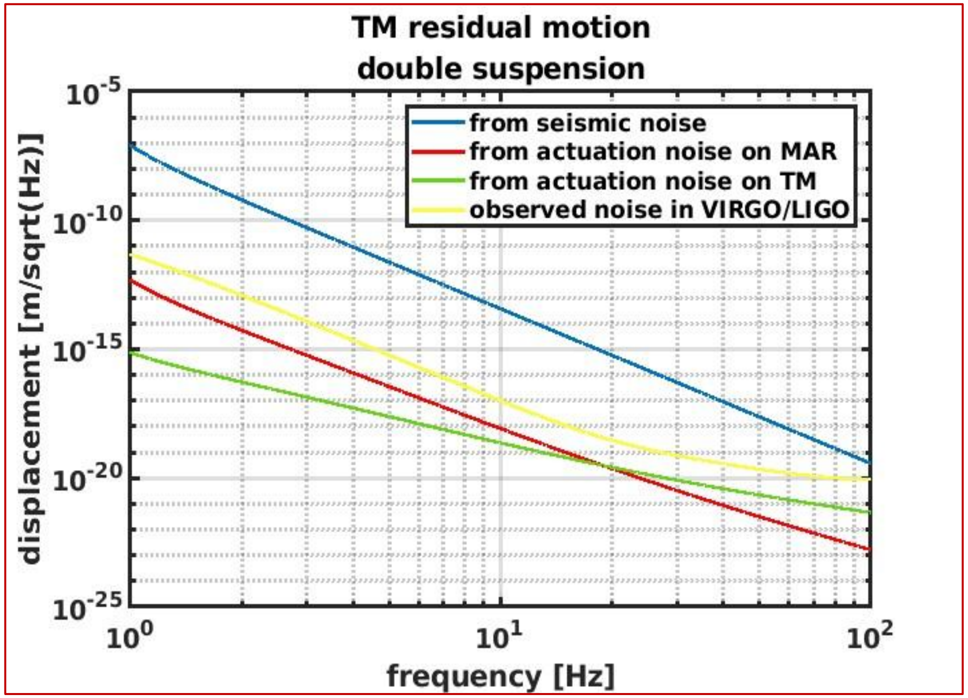
Payload noise budget

➢**Actuation noise**: here we are close to have a realistic projection, because the double pendulum is a good representation of the mechanics seen by the actuation system.

➢**Seismic noise**: it is still far from being compliant: a double pendulum filtering is too low. Actually, the payload is suspended to an additional multi-stage pendulum.

The noise component from the actuation on TM is 1000 times smaller than in the case of the simple pendulum: most of the force is applied to the marionette.

The amplification of the actuation board is *shaped*, taking into account that most of the required force is confined at frequency below 1 Hz, where the residual seismic motion is largely dominant.

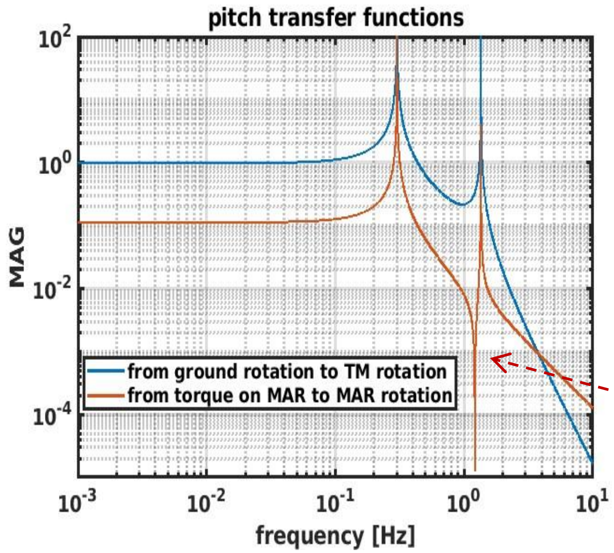


Angular transfer functions

The double pendulum equations can be used as well to describe the rotational motion of the payload.

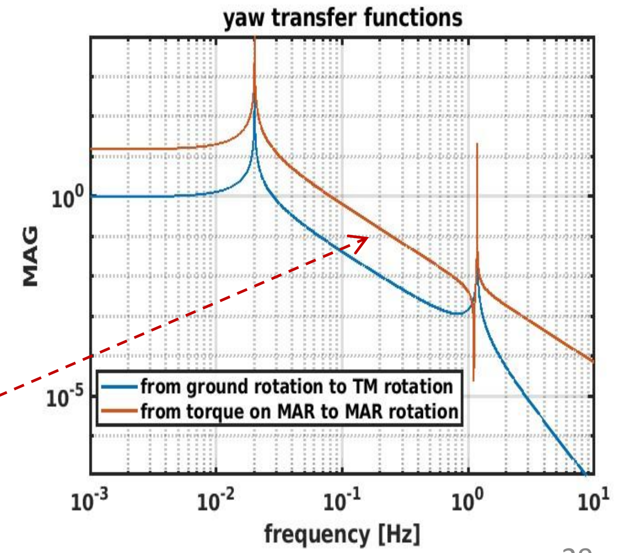
- Moments of inertia take the place of the masses
- Reactions of the constraints to rotations are assumed as pure torques.

Models for wires angular stiffness will be shown soon; here, the values have been choosen in order to reproduce resonances similar to the experimental ones.



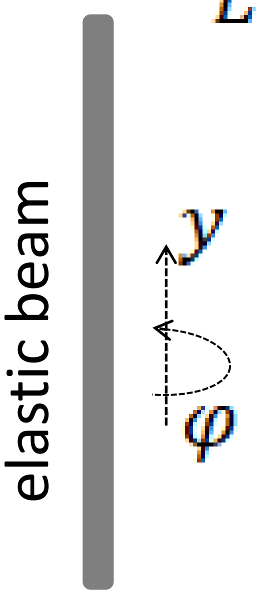
Angular controls acts on MAR using sensors of MAR rotation. The mechanical response involved in the control design is

$M_{MAR} \rightarrow \theta_{MAR}$



Modeling the stiffness of a suspension

The basic component of a suspension is the **WIRE**. It is a long and thin, elastic and massive cylinder, known as **elastic beam**.



L length d thickness ρ density E Young modulus

σ Poisson ratio

$$S = \frac{\pi d^2}{4}$$ area of the section

$$I = \frac{\pi d^4}{64} \qquad I_\varphi = 2I$$ moments of inertia of the section

$$k_y = \frac{ES}{L}$$ longitudinal stiffness

$$k_\varphi = \frac{EI}{L(1+\sigma)}$$ torsional stiffness

longitudinal and angular stiffness of a beam

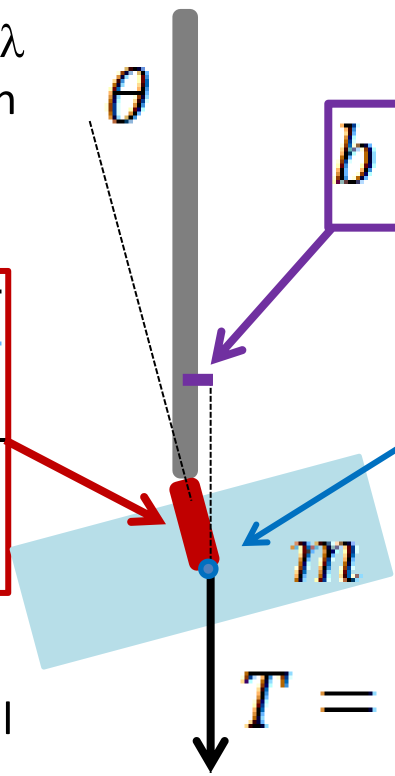
A thin elastic beam under tension changes dramatically its dynamical behaviour. When a rotation of an extremity is applied, a deformation is induced only in a short final segment. The upper point of this segment is the ***bending point***.

The bending lenght λ is small if the section is small and the tension is high

$$\boxed{\lambda = \sqrt{\frac{EI}{T}}}$$

In our working conditions the typical bending lengths are:

1 – 10 mm



$$\boxed{b = \lambda\theta}$$

the body is attached to the wire in its center of mass

$$T = mg$$

$$M = \lambda T\theta$$

torque exerted by the beam

$$k_t = \frac{T}{L - 2\lambda}$$

transversal stiffness

$$k_\theta = \lambda T$$

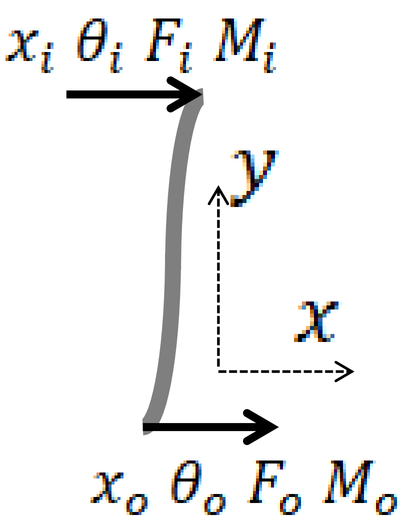
angular stiffness

Deformation of the elastic beam

Following this way, the stiffness of the payload wires can be derived.
A complication come from the fact that **the wires are not attached to the center of masses**: an additional effort is required.

But, what about mass effects (violin vibration) and thermal noise?

Let's try to improve the model, facing the fact that the wire has its dynamical response when the pendulum oscillates and forces are applied.



$$EIx''''(y) - Tx''(y) = -\rho S\ddot{x}(y)$$
$$EIx'''(0) - Tx'(0) = F_i$$
$$EIx'''(L) - Tx'(L) = F_o$$
$$EIx''(0) = M_i$$
$$EIx''(L) = M_o$$

Impedance matrix of the elastic beam

Somebody solved the motion equation.

Here, a linear, frequency dependent connection among coordinates and actions at the ends of the beam is represented in a matricial form, called **IMPEDANCE**.

$$
Z_W = \begin{bmatrix}
\frac{\lambda_1^2 c_3 + \lambda_3^2 c_1}{\sqrt{4\gamma^4+\tau^2}} & \frac{L(\lambda_3 s_3 + \lambda_1 s_1)}{\sqrt{4\gamma^4+\tau^2}} & \frac{L^3(-\lambda_1 s_3 + \lambda_3 s_1)}{EI\gamma^2\sqrt{4\gamma^4+\tau^2}} & \frac{L^2(c_3 - c_1)}{EI\sqrt{4\gamma^4+\tau^2}} \\
\frac{\gamma^2(-\lambda_1 s_3 + \lambda_3 s_1)}{L\sqrt{4\gamma^4+\tau^2}} & \frac{\lambda_3^2 c_3 + \lambda_1^2 c_1}{\sqrt{4\gamma^4+\tau^2}} & \frac{L^2(-c_3 + c_1)}{EI\sqrt{4\gamma^4+\tau^2}} & -\frac{L(\lambda_3 s_3 + \lambda_1 s_1)}{EI\sqrt{4\gamma^4+\tau^2}} \\
\frac{EI\gamma^2(\lambda_1^3 s_3 + \lambda_3^3 s_1)}{L^3\sqrt{4\gamma^4+\tau^2}} & \frac{EI\gamma^4(-c_3 + c_1)}{L^2\sqrt{4\gamma^4+\tau^2}} & \frac{\lambda_1^2 c_3 + \lambda_3^2 c_1}{\sqrt{4\gamma^4+\tau^2}} & \frac{\gamma^2(\lambda_1 s_3 - \lambda_3 s_1)}{L\sqrt{4\gamma^4+\tau^2}} \\
\frac{EI\gamma^4(c_3 - c_1)}{L^2\sqrt{4\gamma^4+\tau^2}} & \frac{EI(\lambda_3^3 s_3 - \lambda_1^3 s_1)}{L\sqrt{4\gamma^4+\tau^2}} & -\frac{L(\lambda_3 s_3 + \lambda_1 s_1)}{\sqrt{4\gamma^4+\tau^2}} & \frac{\lambda_3^2 c_3 + \lambda_1^2 c_1}{\sqrt{4\gamma^4+\tau^2}}
\end{bmatrix}
$$

$$
\begin{aligned}
c_3 &= \cos(\lambda_3) \\
s_3 &= \sin(\lambda_3) \\
c_1 &= \cosh(\lambda_1) \\
s_1 &= \sinh(\lambda_1)
\end{aligned}
\qquad \lambda_1 = \sqrt{\frac{b+\tau}{2}} \qquad \lambda_3 = \sqrt{\frac{b-\tau}{2}} \qquad \tau = \frac{TL^2}{EI} \qquad \gamma^4 = \frac{\omega^2 \rho S L^4}{EI} \qquad b = \sqrt{\tau^2 + 4\gamma^4}
$$

Impedance matrix formalism

A mechanical system, in linear approximation and frequency domain, is represented by a frequency dependent matrix, connecting coordinates and actions in two points of the system, called ***input*** and ***output***.

$$Z \equiv \begin{bmatrix} A & B \\ C & D \end{bmatrix}$$ **IMPEDANCE**

IDENTITIES

$$AD^t - BC^t = I$$

$$AB^t = BA^t \quad CD^t = DC^t$$

Valid for all the elementary and composed impedances used here

$$\begin{pmatrix} X_{out} \\ F_{out} \end{pmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{pmatrix} X_{in} \\ F_{in} \end{pmatrix}$$

The coordinates can be up to 6: three displacements and three rotations.

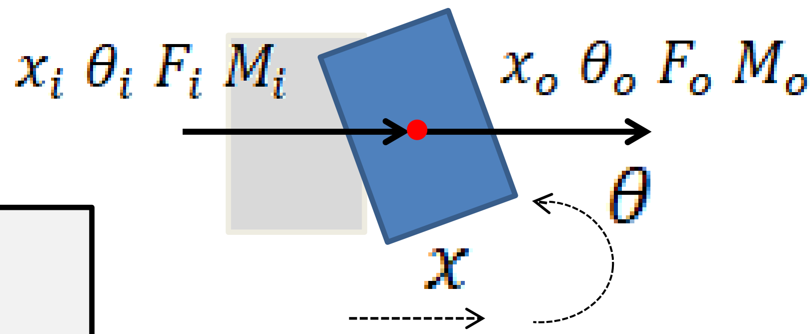
The actions can be up to 6: three forces and three torques.

Actions are received in input and exerted in output.

The impedance Z and its elements A,B,C,D are not the *transfer functions* defined before. The TFs can be computed from the elements when a constraint is imposed.

Impedance of a rigid body

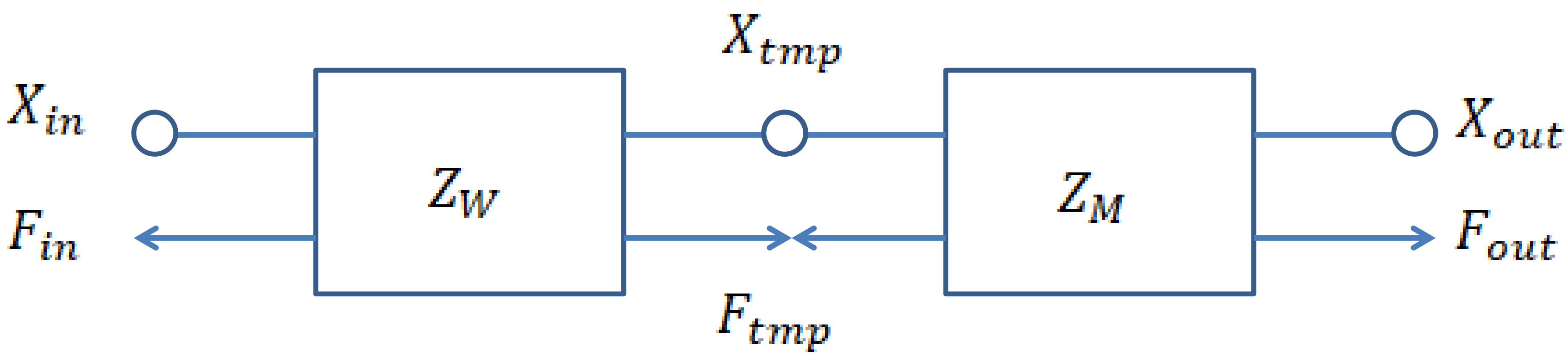
The simplest system we can represent by an impedance is a rigid body.
In view of describing the pendulum in this formalism, let’s consider a one-dimensional displacement and a one-dimensional rotation. Mass **m** and momentum of inertia **J** around the rotational axis are the only parameters required for the representation.



$$
\boxed{
\begin{aligned}
x_o &= x_i \\
-m\omega^2 x_i &= F_i - F_o \\
\theta_o &= \theta_i \\
-J\omega^2\theta_i &= M_i - M_o
\end{aligned}
}
\rightarrow Z_M = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ m\omega^2 & 0 & 1 & 0 \\ 0 & J\omega^2 & 0 & 1 \end{bmatrix}
$$

Series connection

The mechanical system represented by an impedance is not isolated. Actions in input and output are included, meaning that the interactions between two or more systems can be represented. The big power of the representation is that **the impedance of a complex system can be derived by the impedances of its elements**. As the impedance is known, the information is equivalent to the solution of the equations of motion.



$$\begin{pmatrix} X_{tmp} \\ F_{tmp} \end{pmatrix} = Z_W \begin{pmatrix} X_{in} \\ F_{in} \end{pmatrix} \qquad \begin{pmatrix} X_{out} \\ F_{out} \end{pmatrix} = Z_M \begin{pmatrix} X_{tmp} \\ F_{tmp} \end{pmatrix}$$

$$\longrightarrow \begin{pmatrix} X_{out} \\ F_{out} \end{pmatrix} = Z_M Z_W \begin{pmatrix} X_{in} \\ F_{out} \end{pmatrix}$$

Transfer functions in impedance formalism

A variable of the system (X_{out}, for instance), is uniquely defined as a function of another variable (X_{in}, for instance), only if one of the four variables is fixed.
For instance:

$$F_{out} = 0$$

means that the output point is 'suspended', i.e. it is not touched by an external force.
In this case X_{in} and F_{in} are no more indipendent:

$$0 = CX_{in} + DF_{in} \longrightarrow F_{in} = -D^{-1}CX_{in}$$

$$\longrightarrow \boxed{X_{out} = (A - BD^{-1}C)X_{in}}$$ defining $TF_{X \to X}$

or, **applying the identities** previously shown: $\boxed{X_{out} = {}^{t}D^{-1}X_{in}}$

The contribution to the motion coming from an external force applied to the output can be computed separately imposing: $X_{in} = 0 \qquad F_{ext} = -F_{out}$

$$\longrightarrow \quad X_{out} = BF_{in} \qquad F_{out} = DF_{in}$$

$$\longrightarrow \quad \boxed{X_{out} = -BD^{-1}F_{ext}}$$ defining $TF_{F \to X}$

2D transfer functions of the pendulum

We are ready to write an analytic expression of a pendulum impedance.

$$Z_P = Z_M Z_W$$

from which, the transmission from ground motion to TM motion and from external force to TM motion can be computed

$$\boxed{TF_{x\to x} = A_P - B_P D_P^{-1} C_P = {}^t D_P^{-1}}$$

$$\boxed{TF_{F\to x} = -B_P D_P^{-1}}$$

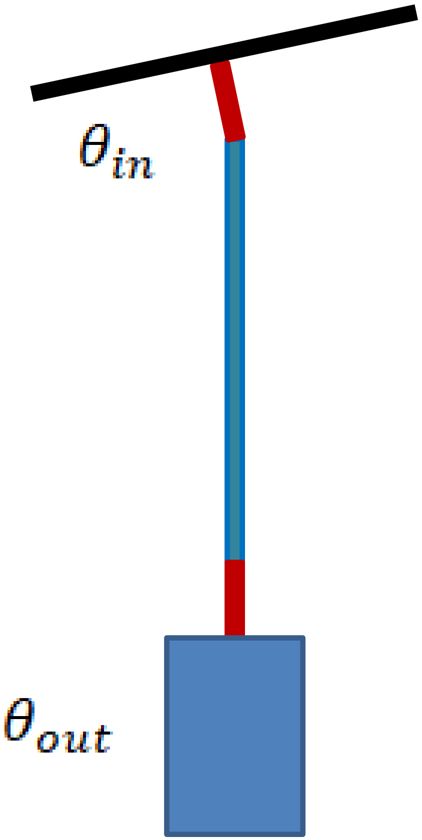
Everything is written in a matricial form. The explicit expressions are very very long, but for our purpose a computation frequency by frequency is enough. So the products can be computed after the numerical conversion for each frequency.

But here is a nasty surprise: for a long wire under big tension, the elements of Z_W are typically:

$$\cosh\,(L/\lambda) \sim e^{100}$$

The numerical precision of the computation will never be sufficient.

Numerical divergence: why, and what can we do?



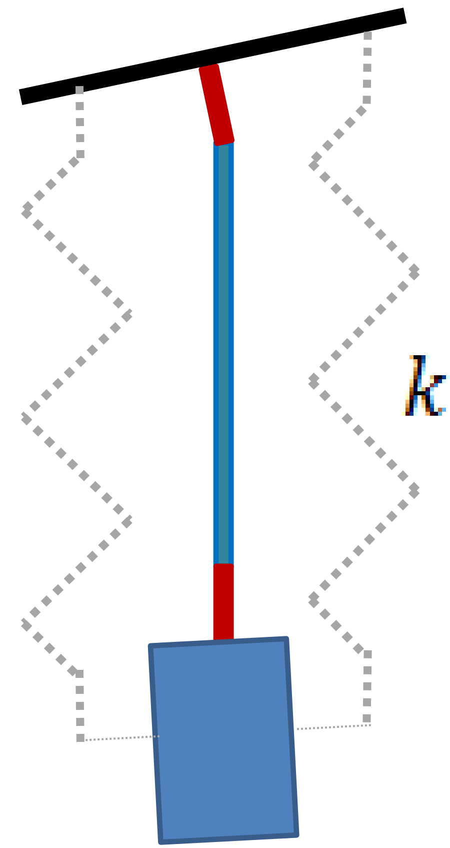
A short bending point means a fast decay of the angular deformation along the wire.

$$\theta_{in} = \cosh{(L/\lambda)}\theta_{out}$$

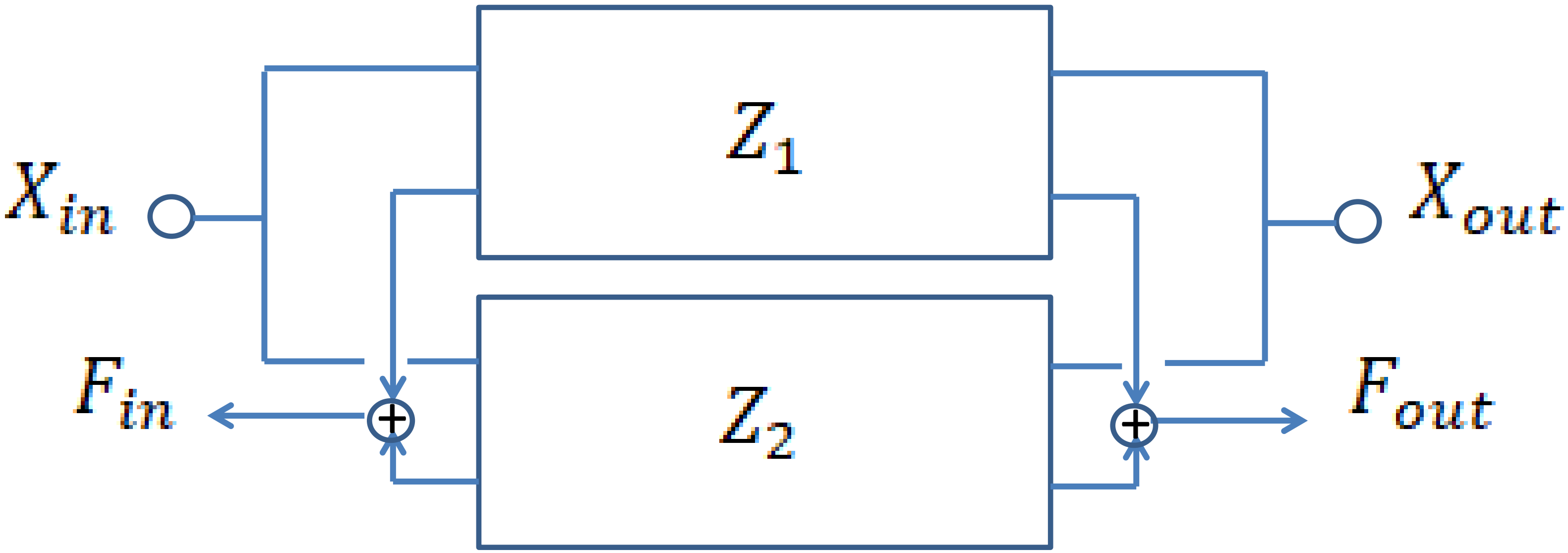
The impedance contains those terms in order to produce this decay.

A direct coupling between the upper and the lower body would be always larger and would avoid divergences. We have to convince ourself that a small angular coupling will perturb the system in a negligible way.

Then **we need to do a calculation**.



Parallel connection: $Z_{\|} = Z_1 \| Z_2$



$$A_{||} = A_1 + B_1(B_1 + B_2)^{-1}(A_2 - A_1)$$

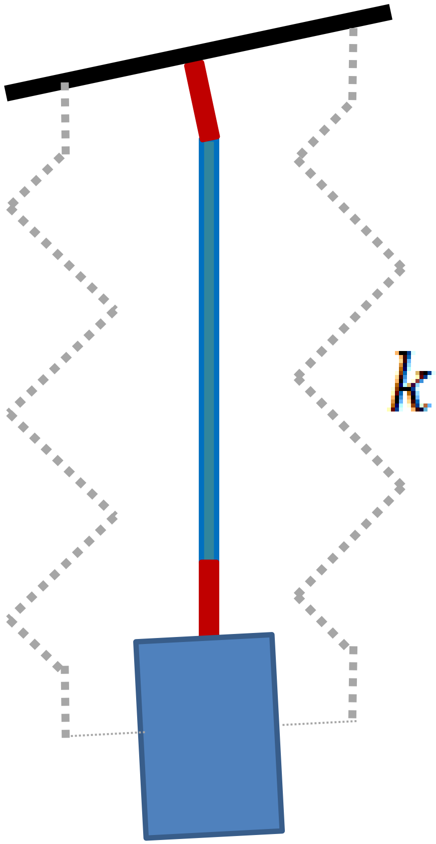
$$B_{||} = B_1(B_1 + B_2)^{-1}B_2$$

$$C_{||} = C_1 + C_2 + (D_1 - D_2)(B_1 + B_2)^{-1}(A_2 - A_1)$$

$$D_{||} = D_1 + (D_1 - D_2)(B_1 + B_2)^{-1}B_1$$

Bypassed beam

$$Z = Z_w \| \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1/k \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \rightarrow$$



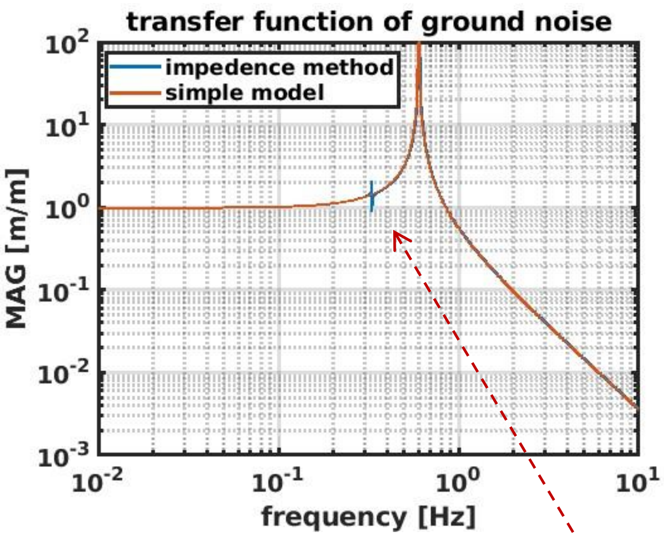
$$A = \begin{bmatrix} c_3 + \frac{\gamma^2 s_3}{\lambda_1^2} + \frac{EI\gamma^4}{Lk\lambda_1^3} & \frac{L(b + 2\gamma^2 s_3 + \tau c_3)}{b\lambda_1} + \frac{EI}{k} \\ \frac{EI\gamma^4}{L^2 k \lambda_1^2} & \frac{EI\lambda_1}{Lk} + 1 \end{bmatrix}$$

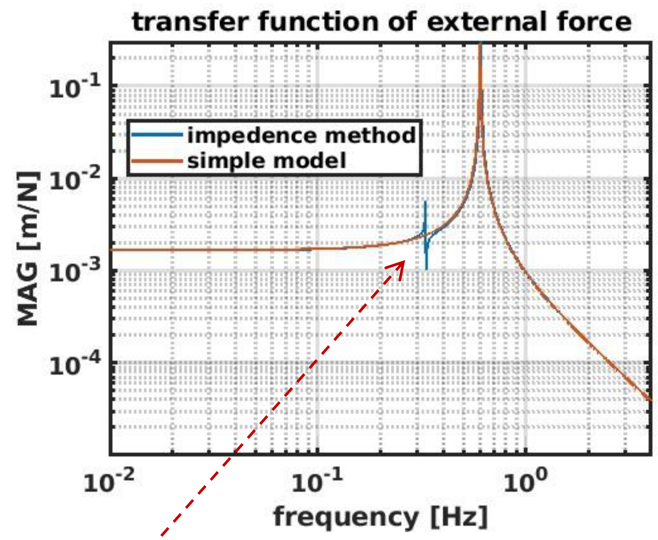
$$B = \begin{bmatrix} \frac{L^3(2c_3\gamma^2 - \tau s_3)}{EI\gamma^2 b\lambda_1} + \frac{L^2}{k\lambda_1^2} & -\frac{L}{\lambda_1 k} \\ \frac{L}{\lambda_1 k} & -\frac{1}{k} \end{bmatrix}$$

$$C = \begin{bmatrix} \frac{\gamma^2 b s_3}{L^2 \lambda_1} + \frac{EI\gamma^8}{L^3 k \lambda_1^4} & \frac{\gamma^2 s_3}{L} + \frac{\gamma^4(1 - c_3)}{L\lambda_1} + \frac{EI\gamma^4}{L^2 \lambda_1 k} \\ -\frac{\gamma^2 s_3}{L} - \frac{\gamma^4(1 - c_3)}{L\lambda_1} - \frac{EI\gamma^4}{L^2 \lambda_1 k} & -2\lambda_1 + \frac{2\gamma^4 c_3 - \gamma^2 s_3}{b\lambda_1} + \frac{EI\lambda_1^2}{Lk} \end{bmatrix} \frac{EI}{L}$$

$$D = \begin{bmatrix} c_3 + \frac{\gamma^2 s_3}{\lambda_1^2} + \frac{EI\gamma^4}{Lk\lambda_1^3} & -\frac{EI\gamma^4}{L^2 k \lambda_1^2} \\ -\frac{L(b + 2\gamma^2 s_3 + \tau c_3)}{b\lambda_1} - \frac{EI}{k} & \frac{EI\lambda_1}{Lk} + 1 \end{bmatrix}$$

Pendulum transfer functions impedance method

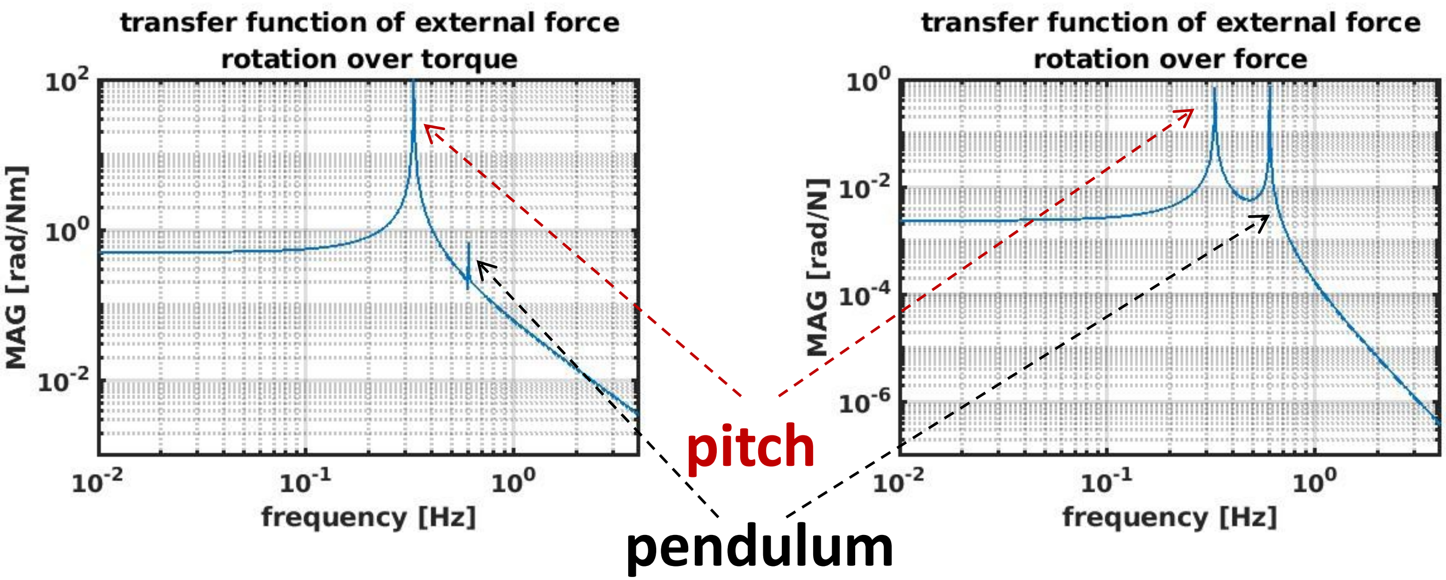




First of all, we can appreciate the fact that the bypassed beam impedance matrix works: the results of the simple model shown before are confirmed.

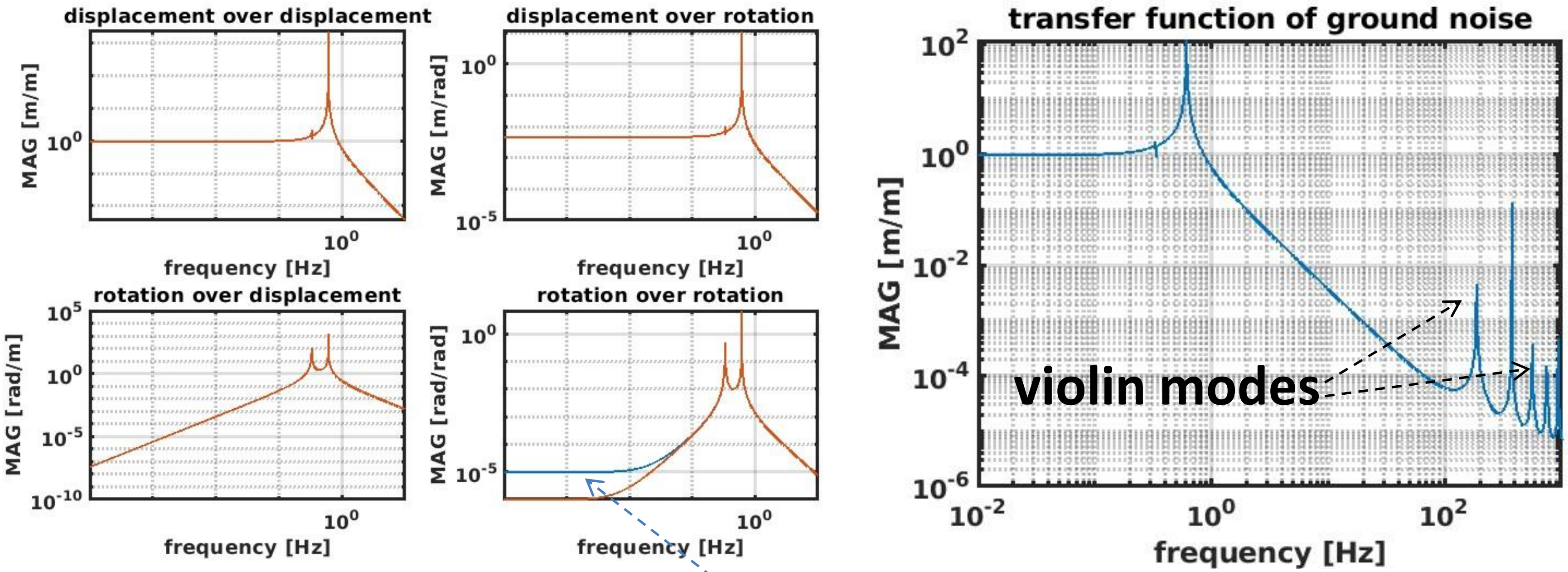
The 2D model produces a **structure at 0.3 Hz**. It is the first evidence of coupling with the rotational mode.

Pitch mode and cross-coupling



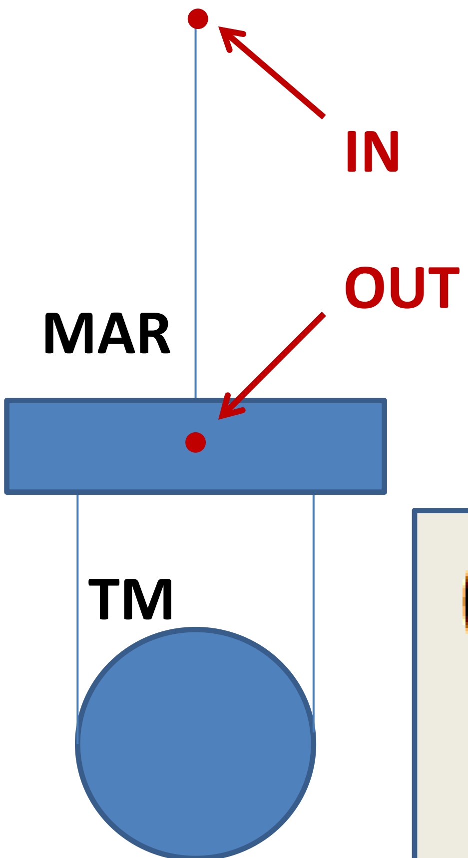
The cross-coupling between translation and rotation has a big effect on the transfer function from a force applied to the suspended mass and its rotation. The application point is the center of mass, nevertheless a torque is present in the reaction of the wire. The effect is the excitation of the pitch mode.

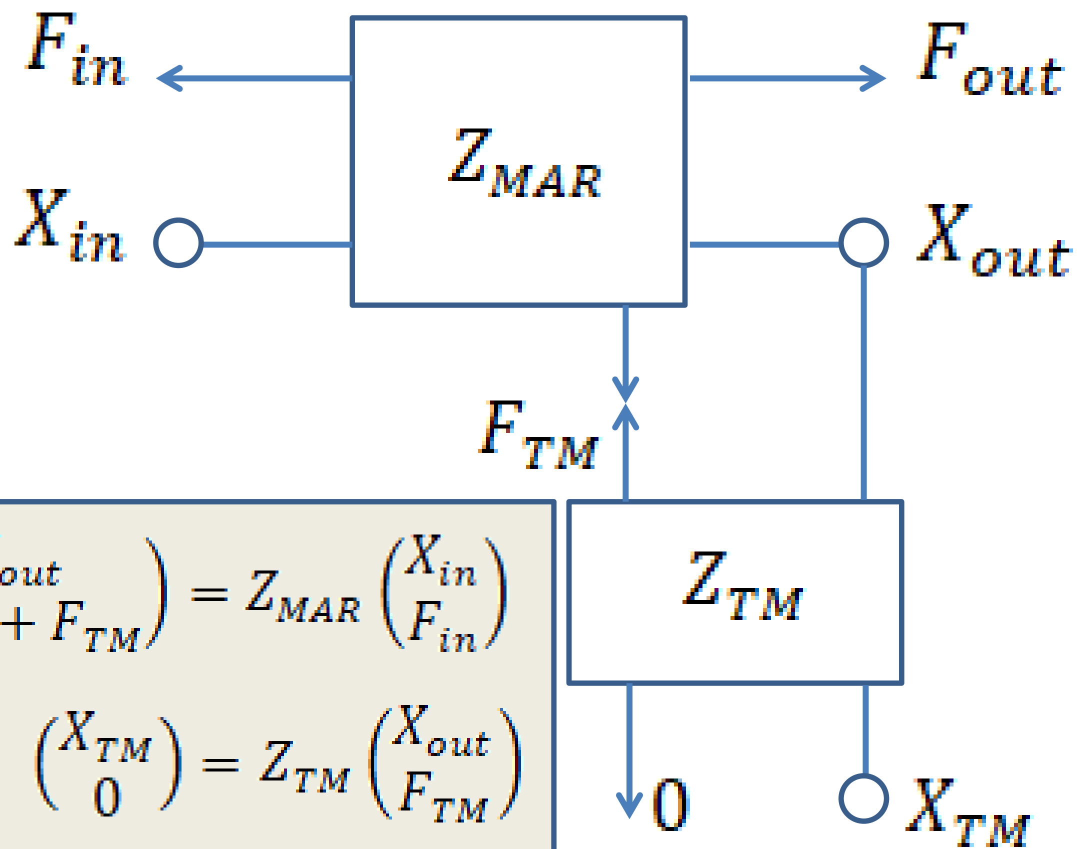
Effect of the bypass



The effect of the angular coupling added as a bypass can be tested comparing the results for different values of the stiffness. The only change regards the transfer function from ground rotation to mass rotation: a plateau is present at very low frequency, at different levels: the higher the stiffness, the higher the level. In principle, removing the stiffness the plateau would disappear; the slope would be f^2 down to zero. This defect is nothing with respect to have precise information about **rotations**, **cross-couplings** and **violin modes**.

Connection in derivation (floating)



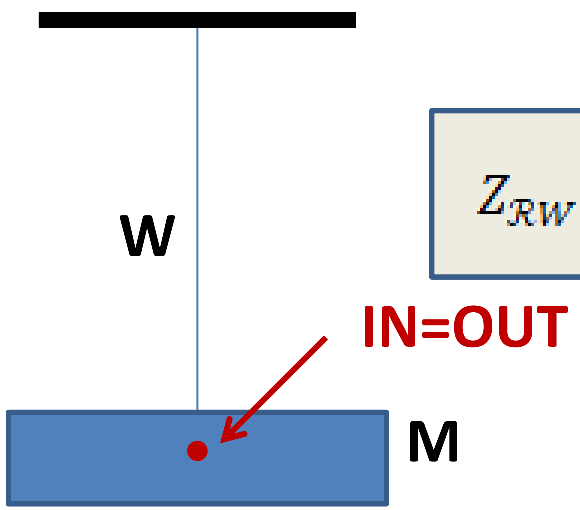


$$\begin{pmatrix} X_{out} \\ F_{out} + F_{TM} \end{pmatrix} = Z_{MAR} \begin{pmatrix} X_{in} \\ F_{in} \end{pmatrix}$$

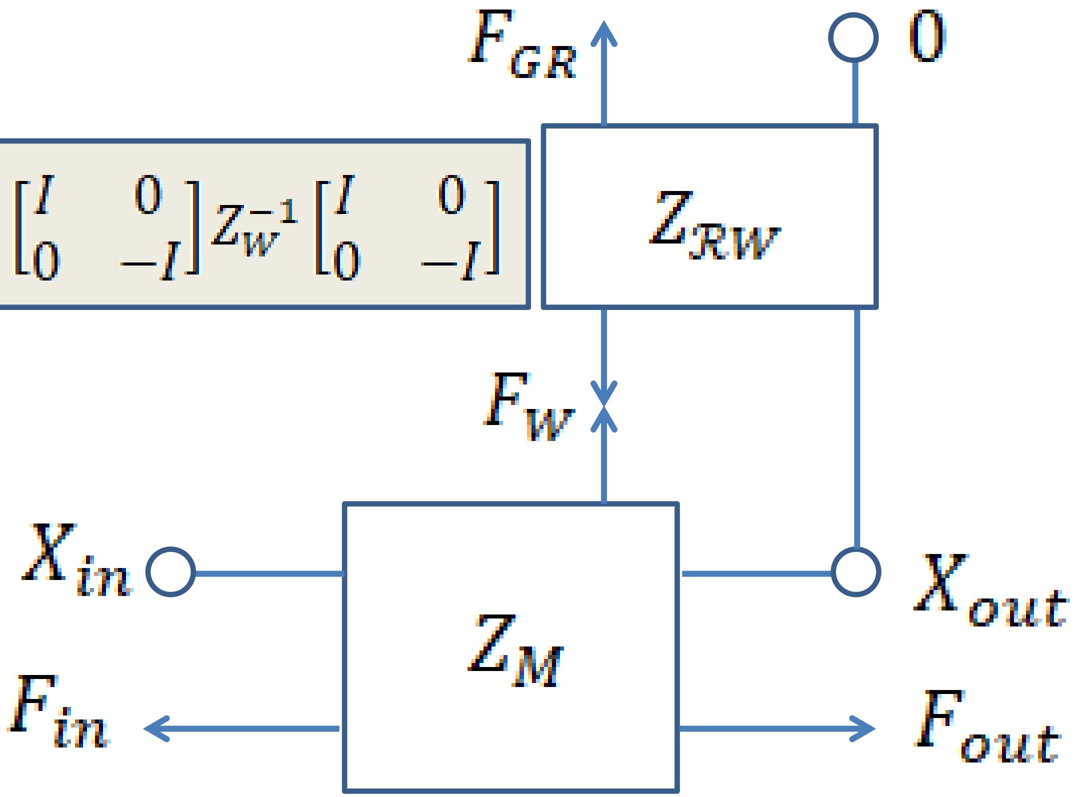
$$\begin{pmatrix} X_{TM} \\ 0 \end{pmatrix} = Z_{TM} \begin{pmatrix} X_{out} \\ F_{TM} \end{pmatrix}$$

$$Z_{MAR}^{\mathcal{F}} = \mathcal{F}(Z_{TM}) Z_{MAR} \quad \mathcal{F}(Z_{TM}) = \begin{bmatrix} 1 & 0 \\ D_{TM}^{-1} C_{TM} & 1 \end{bmatrix}$$

Connection in derivation (grounded)



$$Z_{\mathcal{R}W} = \begin{bmatrix} I & 0 \\ 0 & -I \end{bmatrix} Z_W^{-1} \begin{bmatrix} I & 0 \\ 0 & -I \end{bmatrix}$$

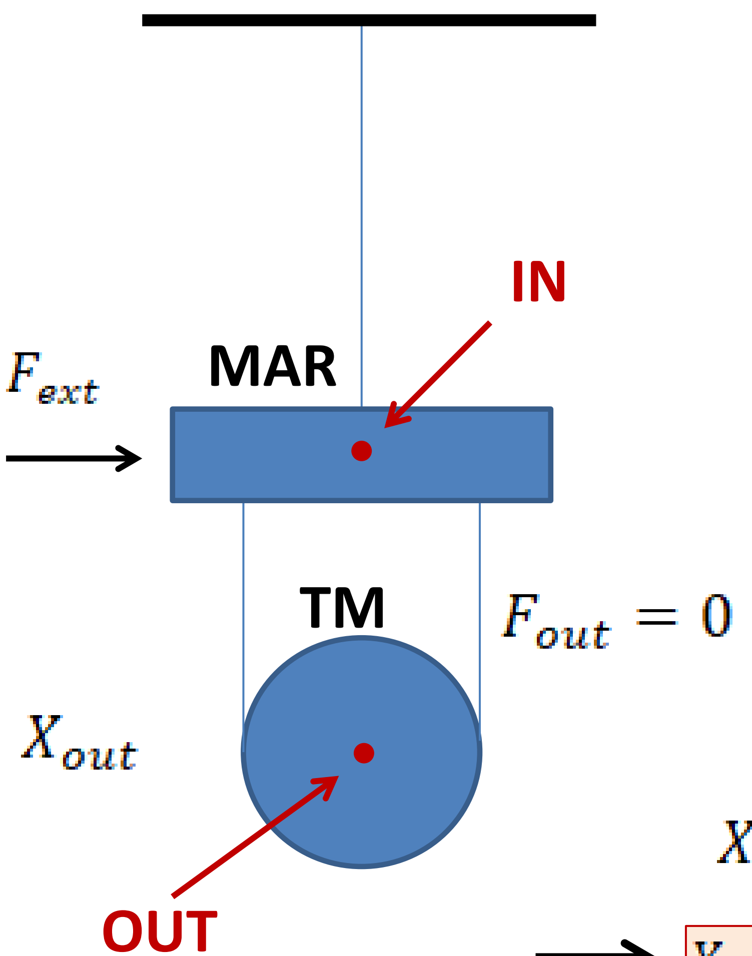


$$\begin{pmatrix} X_{out} \\ F_{out} + F_W \end{pmatrix} = Z_M \begin{pmatrix} X_{in} \\ F_{in} \end{pmatrix}$$

$$\begin{pmatrix} 0 \\ F_{GR} \end{pmatrix} = \mathcal{R}(Z_W) \begin{pmatrix} X_{out} \\ F_W \end{pmatrix}$$

$$Z_M^{\mathcal{G}} = \mathcal{G}(Z_{\mathcal{R}W}) Z_M \qquad \mathcal{G}(Z_{\mathcal{R}W}) = \begin{bmatrix} 1 & 0 \\ B_{\mathcal{R}W}^{-1} A_{\mathcal{R}W} & 1 \end{bmatrix}$$

Force applied to an intermediate body



The same system requires many different impedances to be described, depending on the choosen input and output.

The two points changes when a different transfer function needs to be computed.

$$Z = Z_M^{TM} Z_W^{TM} \mathcal{G}(Z_{RW}^{MAR}) Z_M^{MAR}$$

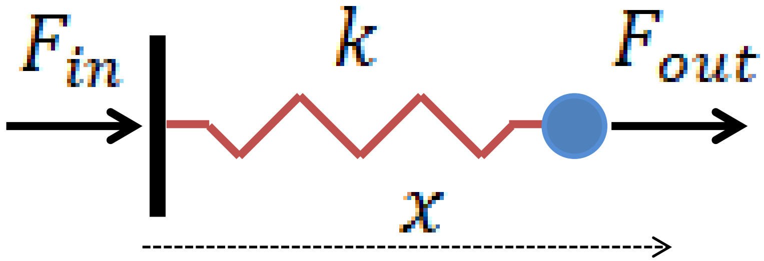
$$0 = CX_{in} + DF_{in}$$

$$\longrightarrow \quad X_{in} = -C^{-1} D F_{in}$$

$$X_{out} = AX_{in} + BF_{in} \qquad F_{ext} = F_{in}$$

$$\longrightarrow \quad \boxed{X_{out} = (B - AC^{-1}D)F_{ext} = -\,{}^{t}C^{-1}F_{ext}}$$

Impedance of an elastic element



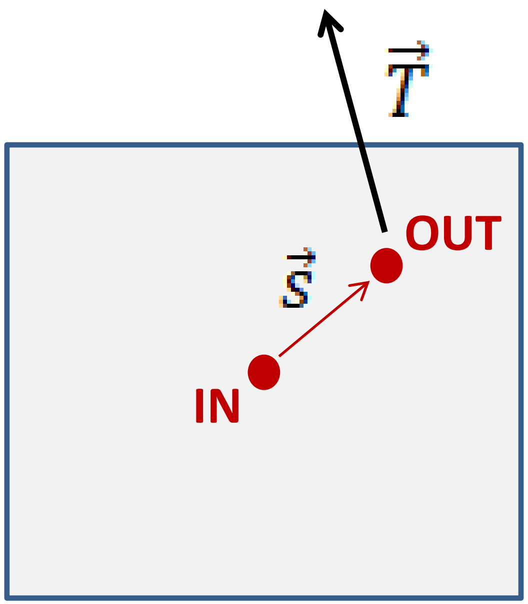
$$F_{in} = k(x_{in} - x_{out})$$
$$F_{out} = F_{in}$$

$$\rightarrow Z_E = \begin{bmatrix} 1 & -k^{-1} \\ 0 & 1 \end{bmatrix}$$

A wire is equivalent to a simple elastic element if the deformation is due to a longitudinal displacement or a rotation around the longitudinal axis

$$\begin{pmatrix} y \\ z \\ \theta_x \\ \theta_y \end{pmatrix}_{out} = Z_W^{4D} \begin{pmatrix} y \\ z \\ \theta_x \\ \theta_y \end{pmatrix}_{in} \qquad B_W^{4D} = \begin{bmatrix} -k_y^{-1} & 0 & 0 & 0 \\ 0 & & & 0 \\ 0 & & B_W & 0 \\ 0 & 0 & 0 & -k_\varphi^{-1} \end{bmatrix}$$

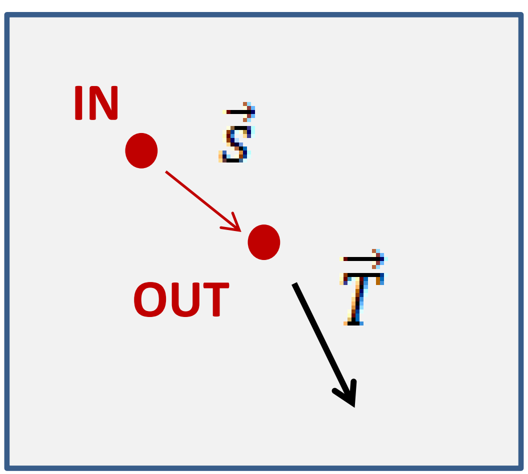
Rigid body clamped out of the CM



Coordinate transformation from one point to another in a rigid body: it has its own representation as an impedance

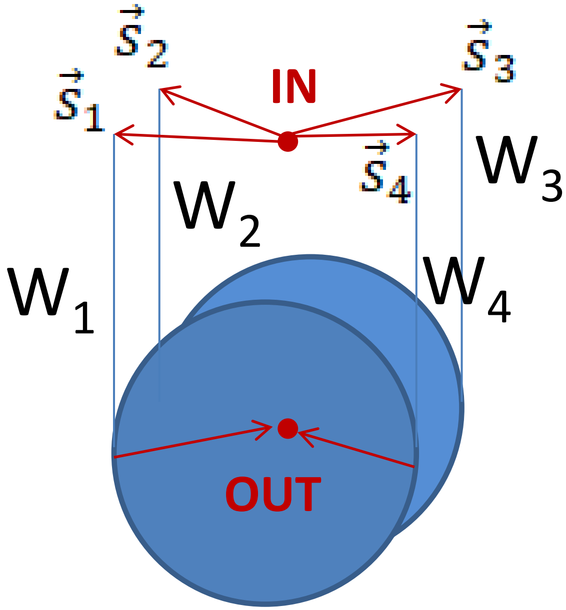
$$\vec{X}_{out} = \vec{X}_{in} + \vec{\theta}_{in} \otimes \vec{s}$$
$$\theta_{out} = \theta_{in}$$
$$F_{out} = F_{in}$$
$$\vec{M}_{out} = \vec{M}_{in} + \vec{F}_{in} \otimes \vec{s} + \vec{T} \otimes \vec{\theta}_{in} \otimes \vec{s}$$

$$\vec{s} = (x, y, z)$$
$$\vec{T} = (X, Y, Z)$$



$$Z_s = \begin{bmatrix} 1 & 0 & 0 & 0 & z & -y & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -z & 0 & x & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & y & -x & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & yY+zZ & -xY & -xZ & 0 & z & -y & 1 & 0 & 0 \\ 0 & 0 & 0 & -yX & xX+zZ & -yZ & -z & 0 & x & 0 & 1 & 0 \\ 0 & 0 & 0 & -zX & -zY & xX+yY & y & -x & 0 & 0 & 0 & 1 \end{bmatrix}$$

TM suspension



The main properties of the test mass suspension dynamic have been shown representing it as a simple pendulum, with an additional elasticity connecting the rotations. The representation is realistic enough, but now we are ready to do better.

We have all the elements to build the impedance matrix associated the a suspension made by 4 wires, connected at each side of the TM. This is the parallel connection of four impedances built in the same way.

The impedance of each wire is a series connection of three elements:

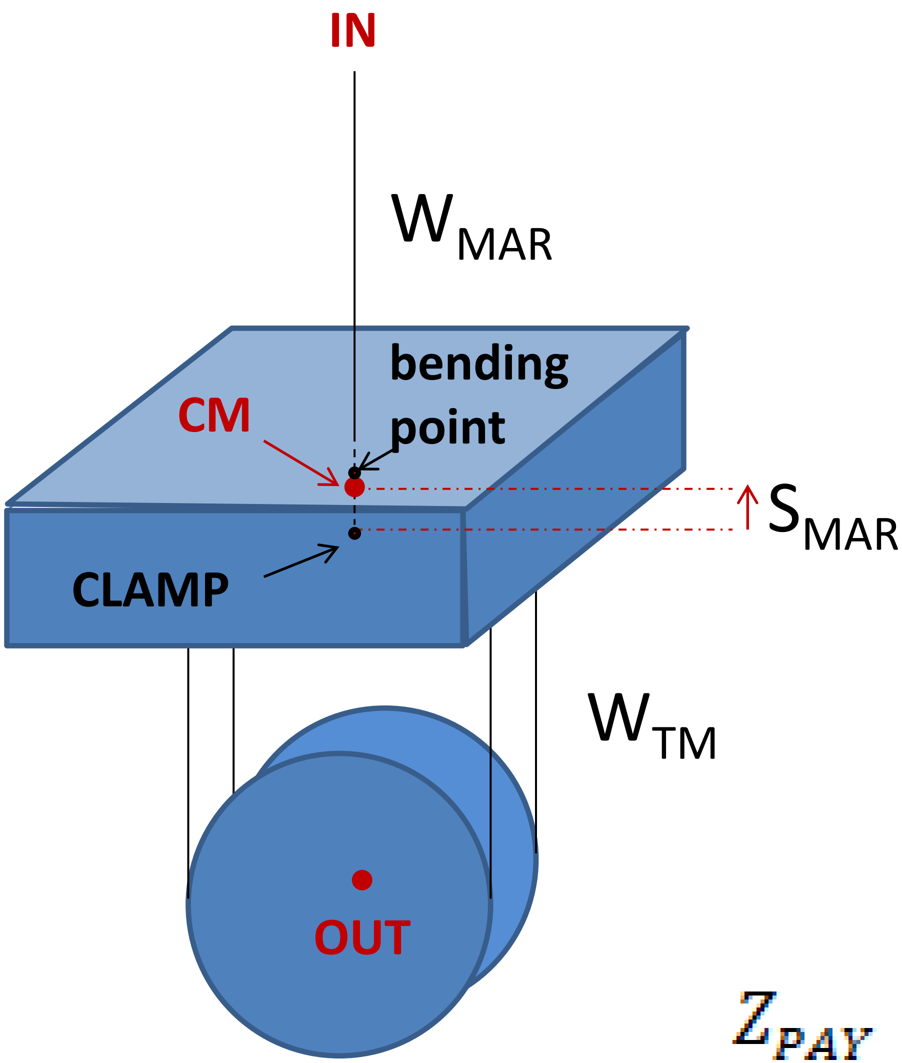
•The displacement from a common input to each upper clamping point

•A wire (one equal to each other or not; we can decide)

•The displacement from the lower clamp to the common output (TM center of mass)

$$Z_W^{TM} = Z_W^1 \| Z_W^2 \| Z_W^3 \| Z_W^4 \qquad Z_W^i = Z_{sOUT}^i Z_{Wi} Z_{sIN}^i$$

PAYLOAD



The impedance matrix of the payload is the series connection of two pendulums. The impedance of MAR suspension is simpler than TM, but the positioning of the wire clamp with respect to MAR CM needs to be taken into account.

There is just one parameter in that impedance: a vertical displacement. But this is very important for the design of the payload.

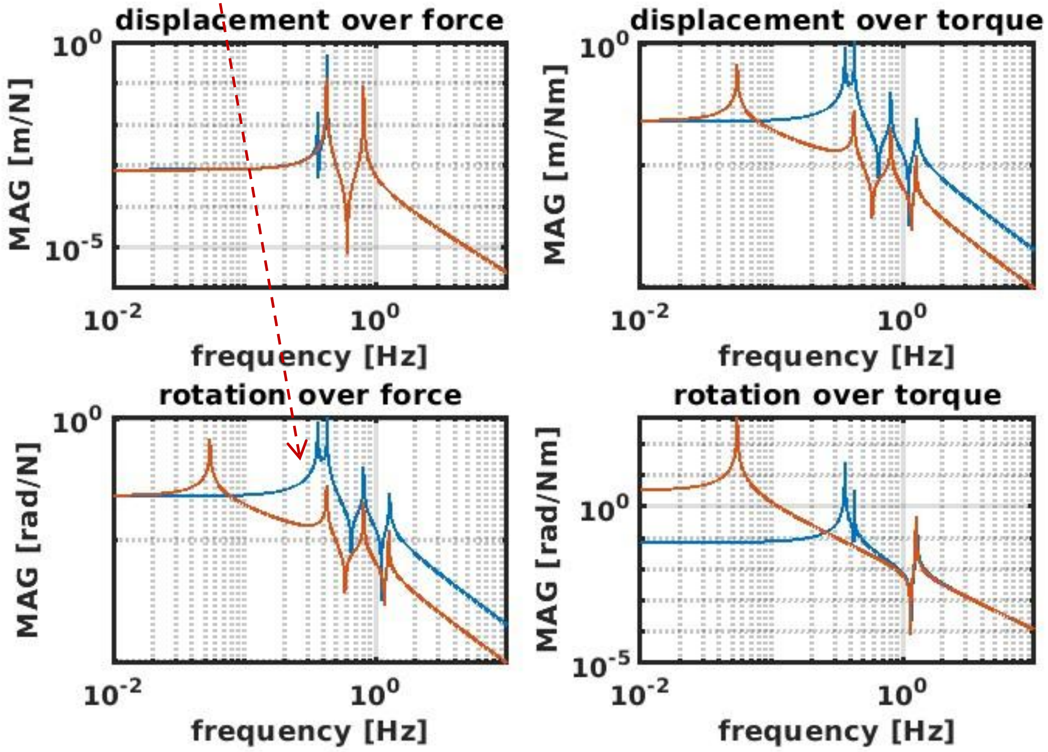
The best place to put the CM is above the clamp, and a little bit below the bending point. The pitch is softened by a negative component of gravitational stiffness.

$$Z_{PAY} = Z_M^{TM}\, Z_W^{TM}\, Z_M^{MAR}\, Z_S^{MAR}\, Z_W^{MAR}$$

Payload transfer function

Taking care of the marionette CM, the rotation induced by a longitudinal force can be minimized. The angular control requires less gain, becoming less noisy

MAR TF$_{F\rightarrow X}$



clamp on the center of mass

center of mass on the bendig point

MAR CM is 20 μm below the bending point, which is 870 μm above the clamp. This is not unrealistic: VIRGO payloads are all tuned with pitch frequency among 60 and 100 mHz

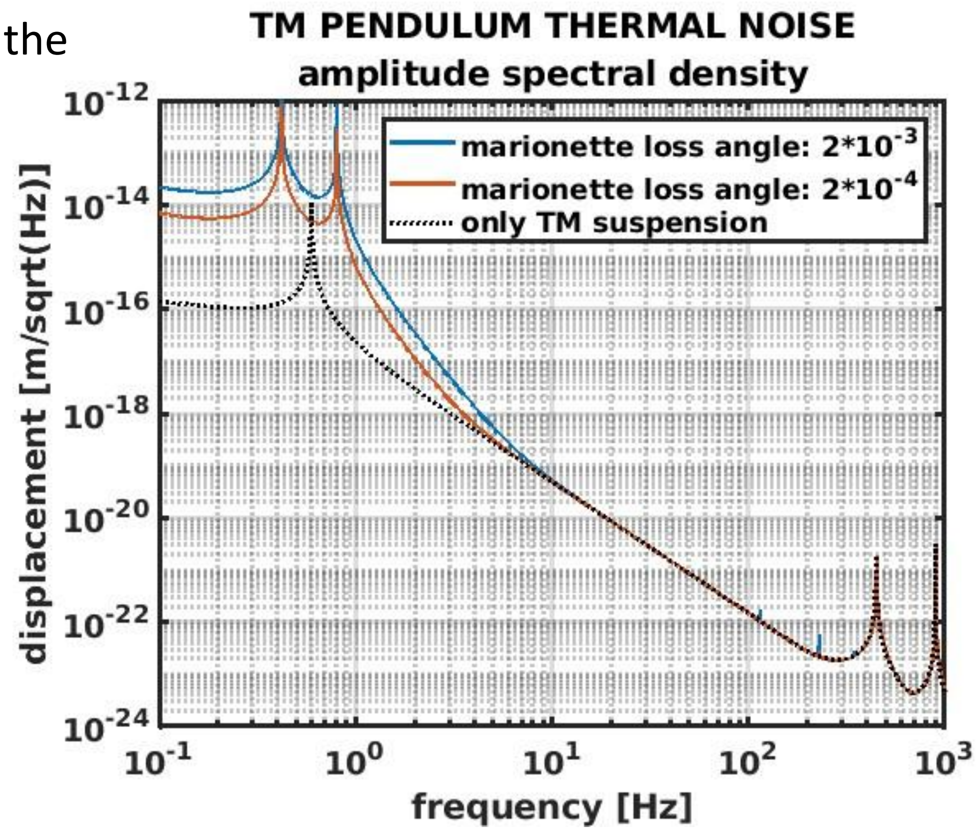
Thermal noise

The pendulum model is ready to be used for the computation of the thermal noise. A intrinsic dissipation can be included in the description of the wire elasticity: it is sufficient to add an imaginary part to the Young modulus

$$E(1 + i\emptyset) \quad \emptyset{:}\, loss\ angle$$

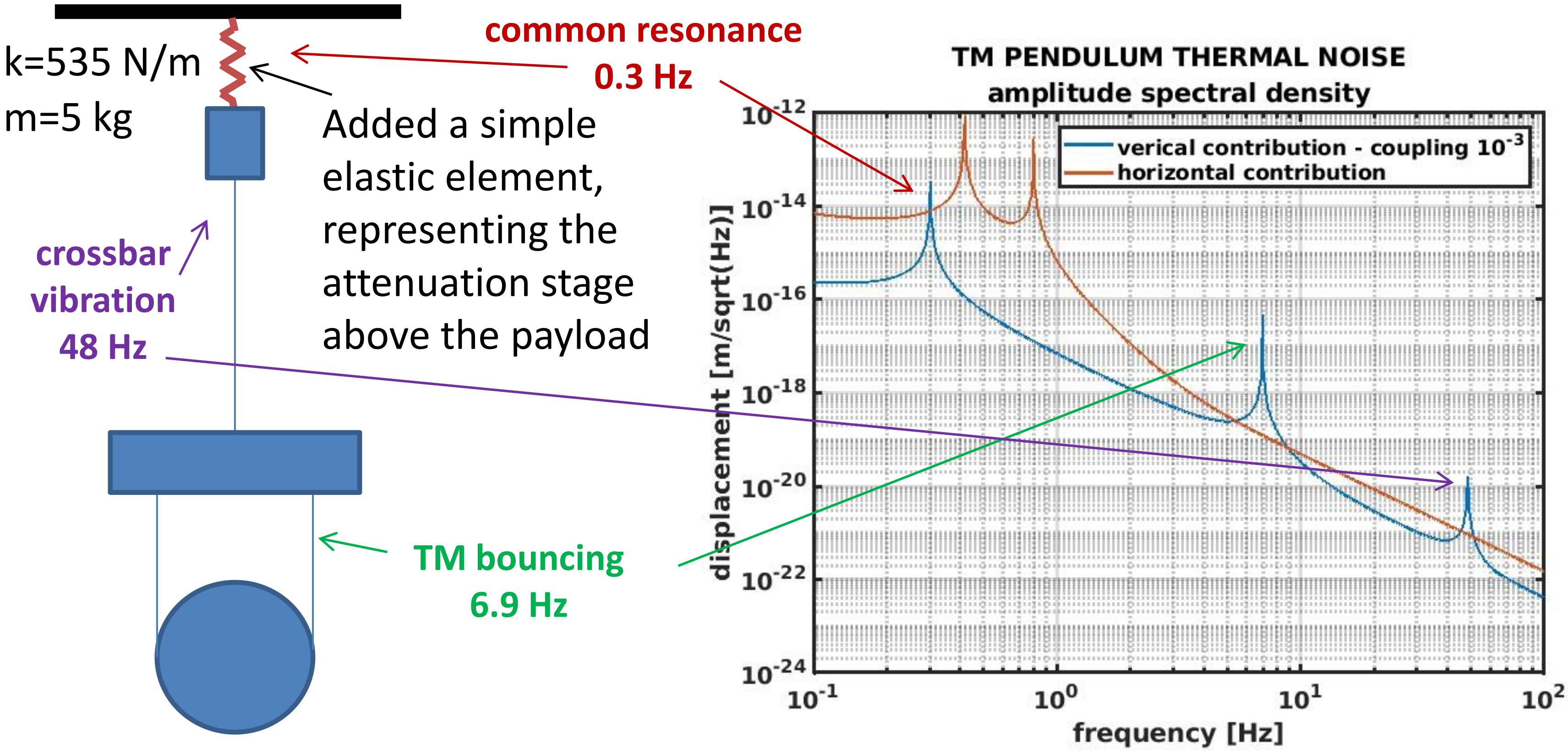
The thermal noise of a fused silica suspension is shown. A loss angle $3{\cdot}10^{-7}$ is used.
The marionette is suspended to a stainless steel wire
nominal loss angle: $2{\cdot}10^{-4}$
Loss angle a factor of 10 higher is still good: at 10 Hz, MAR contribution is negligible.

$$x_{TH} = \sqrt{\frac{4k_bT}{\omega} Imag(B_{PAY}D_{PAY}^{-1})}$$

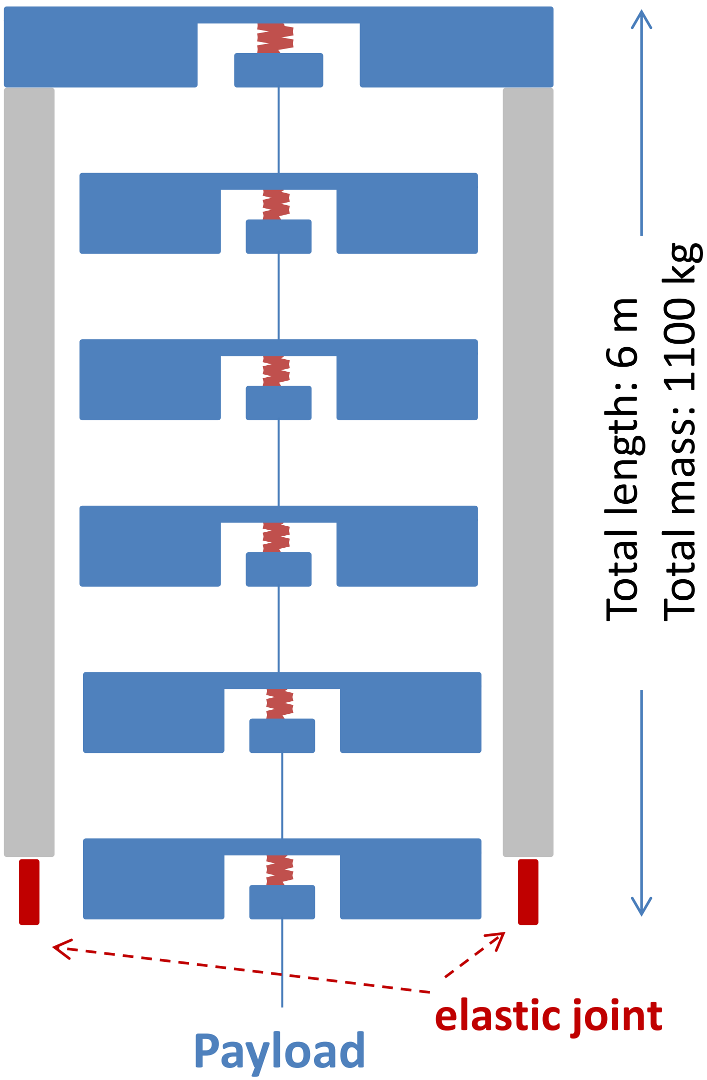


Vertical thermal noise

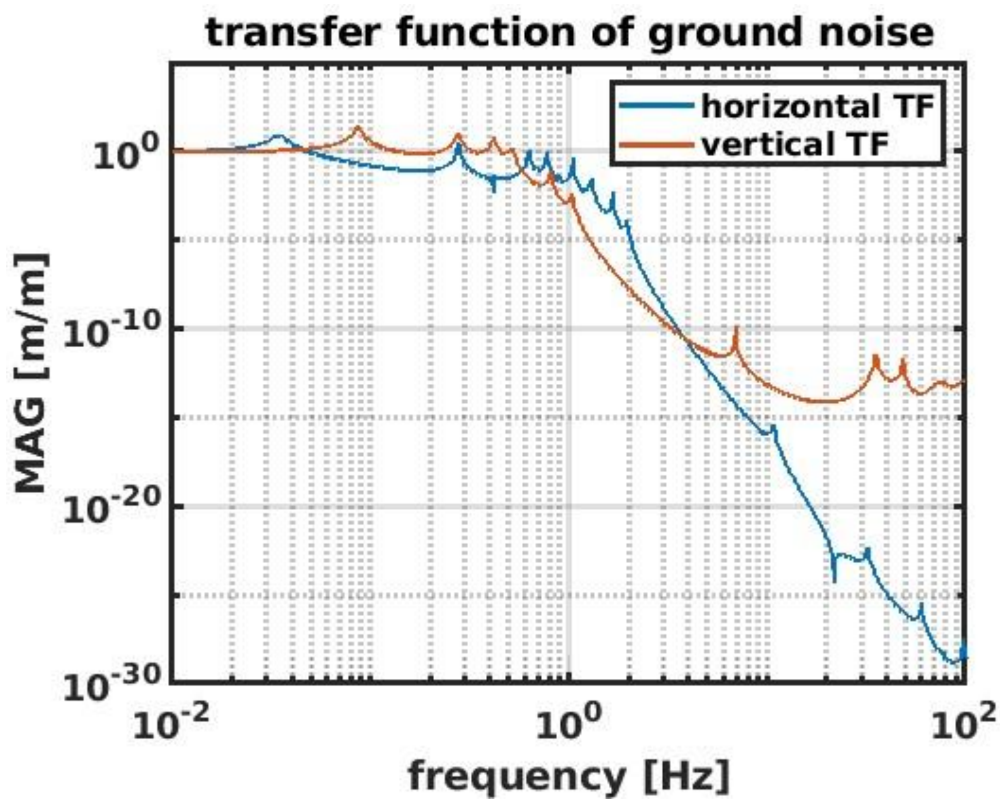
TM vertical motion is apparently orthogonal to the sensible d.o.f. This is true in a certain degree of approximation: it is impossible to guarantee that the cavity axis is perfectly orthogonal to the local gravity. A safety coupling 10^{-3} will be taken into account.



Seismic noise: the superattenuator



Six heavy bodies (m > 100 kg)
attached to long wires (l > 1 m)
The first is suspended to an inverted pendulum



Inverted pendulum

Total mass on the top

$$k = \frac{K_J}{l^2} - \frac{mg}{l}$$

Longitudinal stiffness

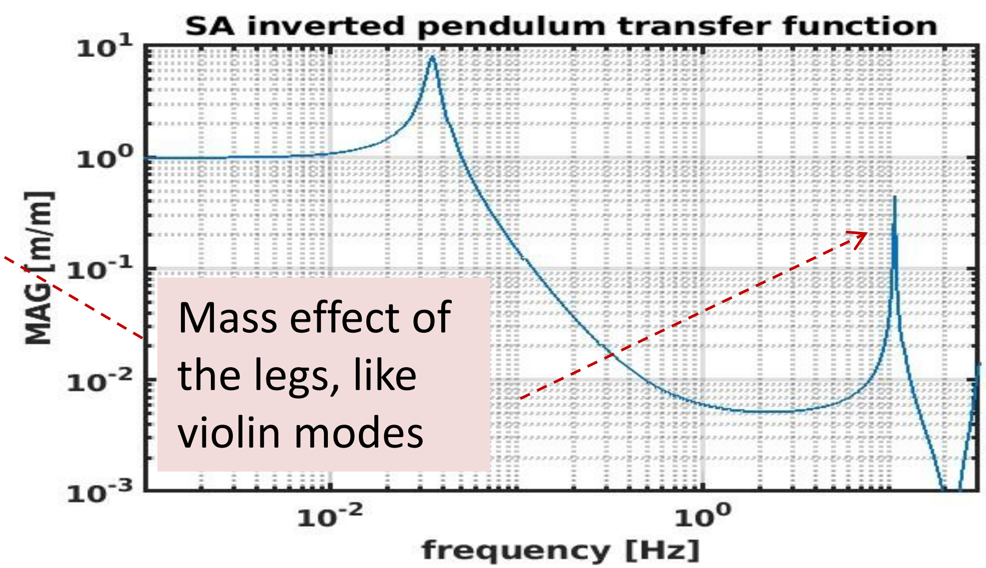
This quantity is not an input of the model. The impedance matrix does all the work.

Flexural stiffness K_J

The first stage of the superattenuator is an **inverted pendulum**.

This is a system designed to obtain an incredible reduction of the resonance frequency.

The gravitational part of the stiffness works with a negative sign. Parameters can be choosen in order to put the effective stiffness very close to zero.

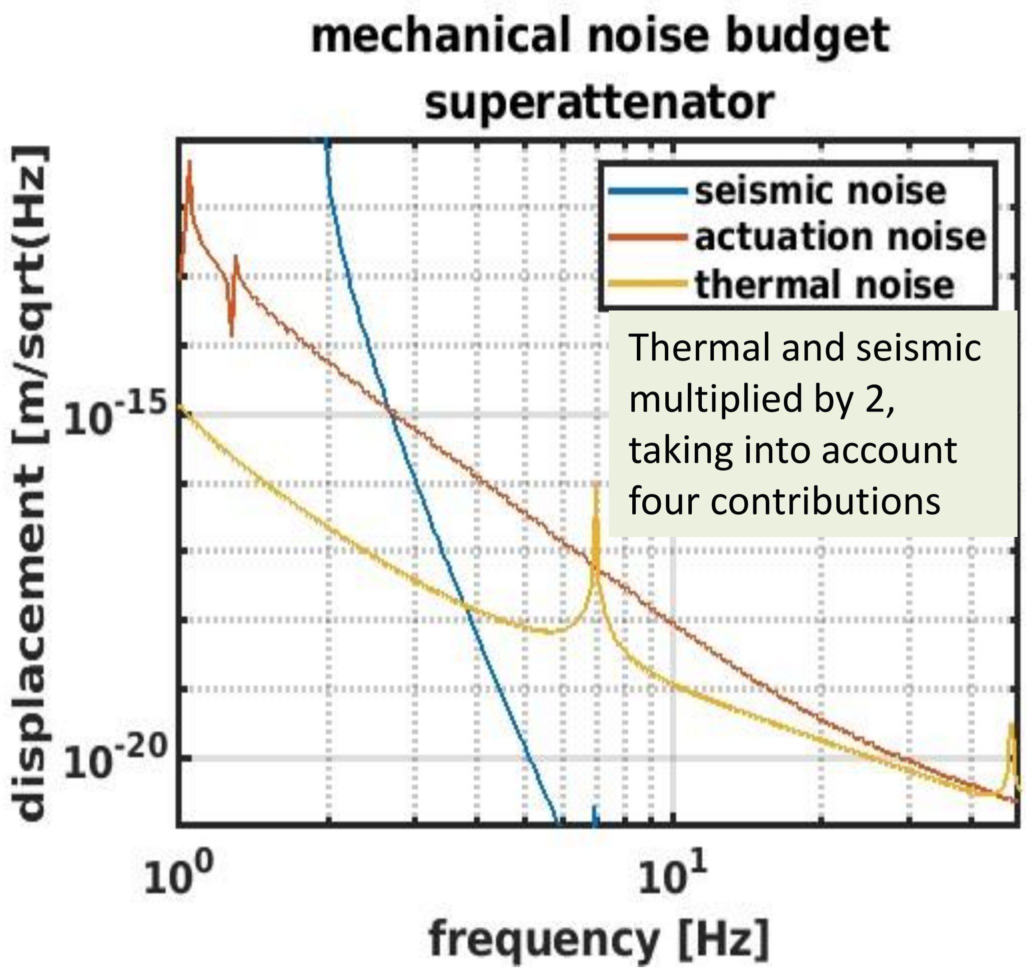


Mass effect of the legs, like violin modes

2° generation GW detectors mechanical noise budget

Thermal noise projection is a lower limit. It does not take into account possible additional dissipations not occurring in the materials, but due to defects at the level of the links.

Other kind of control noise, like angular control noise, is missing. This is something in the range of the actuation noise.



From the 2° to the 3° generation

With the current tecnology, the fundamental limit for the mechanical noise, due to thermal noise, is:

10^{-19} m/sqrt(Hz) @ 10 Hz **10^{-18} m/sqrt(Hz) @ 4 Hz**

Below 4 Hz the seismic noise 'wall' starts to be limiting.
Actually, the limitation is the control noise:

expected	measured
10^{-18} m/sqrt(Hz) @ 10 Hz	**10^{-17} m/sqrt(Hz)**
10^{-16} m/sqrt(Hz) @ 4 Hz	**10^{-15} m/sqrt(Hz)**

But this is considered 'technical noise'. Anyway, nobody knows the solution of the issue. Moreover, the best result obtained with the active detectors is ten times worse: the technical noise is much larger than expected one.

For the third generation, the goal is

10^{-20} m/sqrt(Hz) @ 10 Hz **10^{-19} m/sqrt(Hz) @ 4 Hz** **10^{-18} m/sqrt(Hz) @ 2 Hz**

Very ambicious.

Thermal noise reduction

Thermal noise needs to be reduced by a factor of 10. There are two possibilities:

- loss angle lower by a factor of 100
- temperature lower by a factor of 100

TM cooling technology is mature, but a few mechanical issue has to be faced:

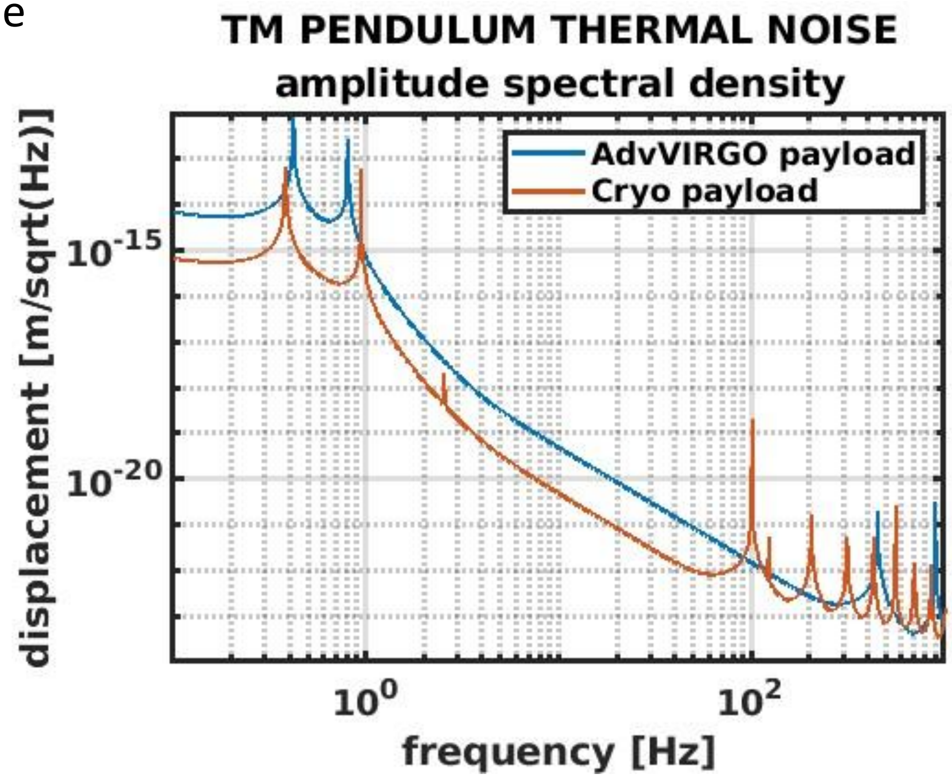
1. In order to allow the extraction of the heat, the fibers suspending the TM needs to be thick.
2. The heat needs to be brought out. Somewhere the payload needs to be touched by a conductive link. This introduces a seismic shortcut
3. The HeatLink adds losses. This introduces additional thermal noise

cryogenic payload

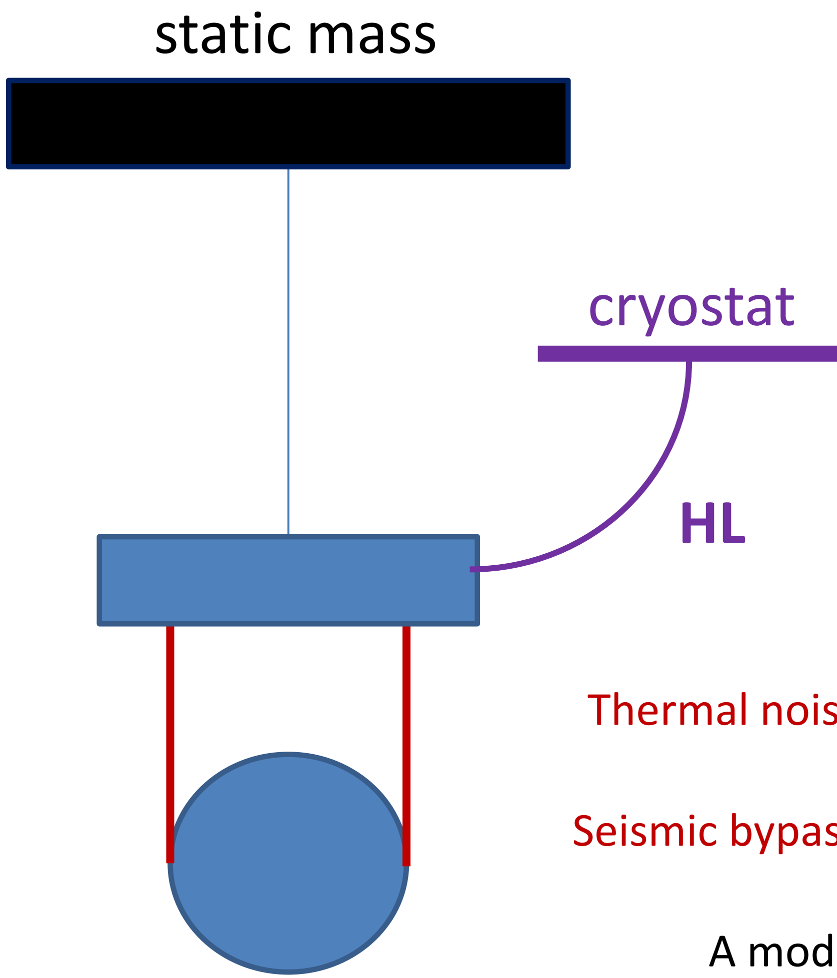
The design of a cryogenic payload under study involves the a 141 kg TM being suspended by 2 mm thick sapphire fibers. The thickness is supposed to be the minimal for cooling the TM, but it is larger than what is needed for that charge. This is not good for thermal noise: a worsening by a factor of 2 is estimated.

In the thermal noise computation done here, the factor of 10 reduction is reached if a significative reduction of the loss angle is assumed

parameters	AdvPAY	CryoPAY
material	fus. silica	sapphire
thickness	0.4 mm	2 mm
length	0.7 m	0.9 m
T	295 K	5 K
Loss angle	3.10-7	5.10-8
TM mass	42 kg	141 kg



Heat Link



Let's assume that the heat is brought out by a link between the cryostat and the marionette. An additional channel of vibration transmission is introduced. The same object is also an additional generator of thermal noise.

The thermal noise computation requires simply the replacement of MAR wire impedance with the parallel connection of MAR wire and HL impedance.

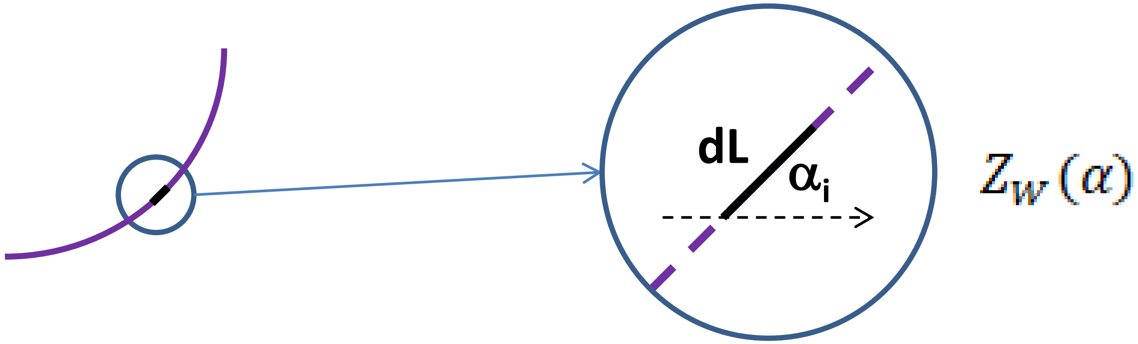
Thermal noise $\longrightarrow$ $Z_W^{MAR} \rightarrow Z_W^{MAR} \| Z_W^{HL}$

Seismic bypass $\longrightarrow$ $Z_{SEIS} = Z_M^{TM} \, Z_W^{TM} \, Z_M^{MAR} \, Z_W^{HL}$

A model of the HL impedance is needed.

Heat Link impedance

HL basic element is an aluminum fiber, very high conductivity. Its first mechanical requirement is to be soft: its tension needs to be null. It implies that the fiber will not have a straight disposition.

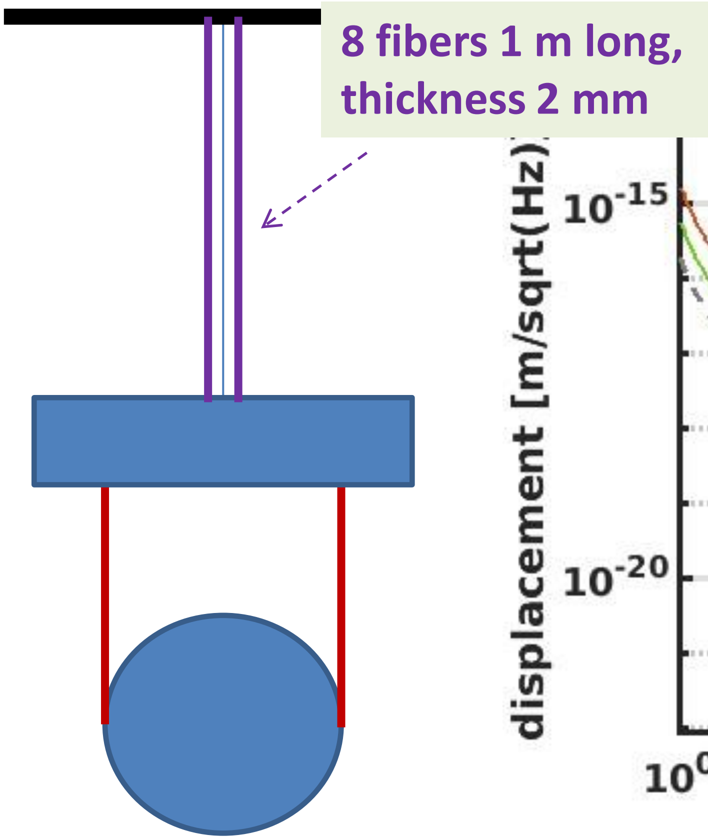


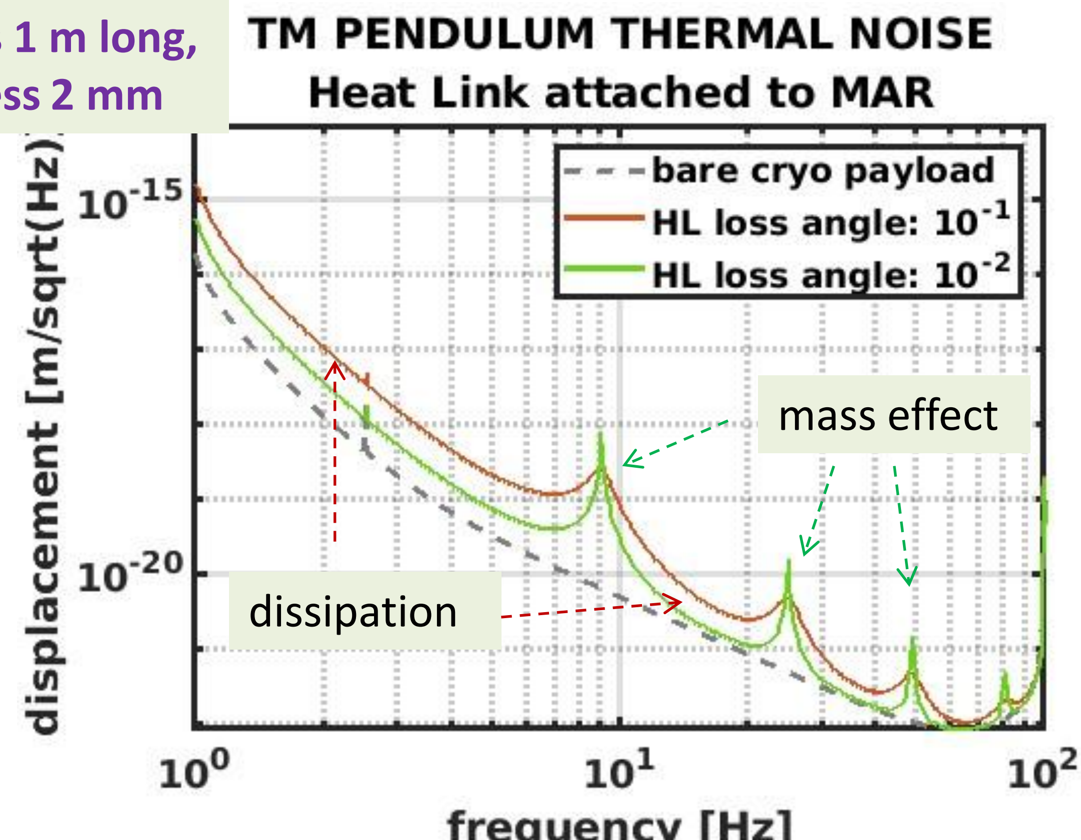
The impedance of a fiber with an arbitrary geometrical profile is not directly available, but a relatively simple computation con provide it: one can divide the fiber in a big number of short and straight fibers, connected in series one to each other. The impedance of each element is different from another because of an incremental rotation α_i. The total impedance is the product of all the elements.

$$Z_W^{HL} = \prod Z_W^i(\alpha)$$

Thermal noise of the Heat Link

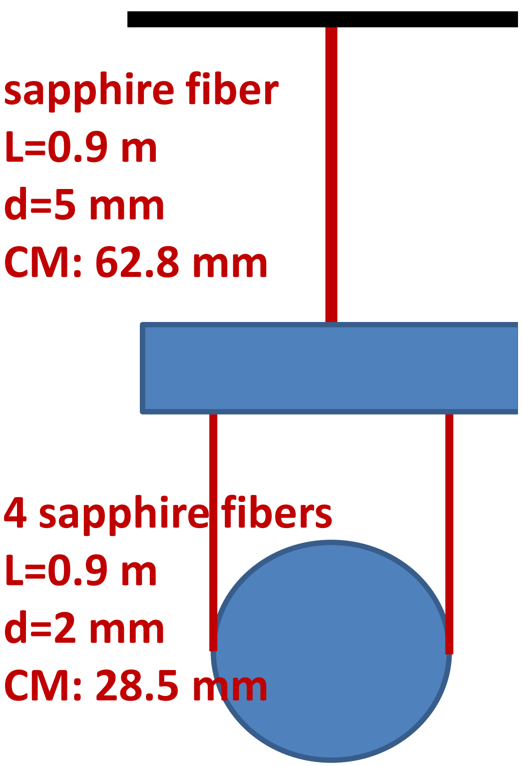
The computation of a HL with arbitrary profile has been done, but for semplicity a straight, uncharged, vertical fiber will be used for the next extimations. In a model taking into account only the transversal deformations, the difference is not very important.



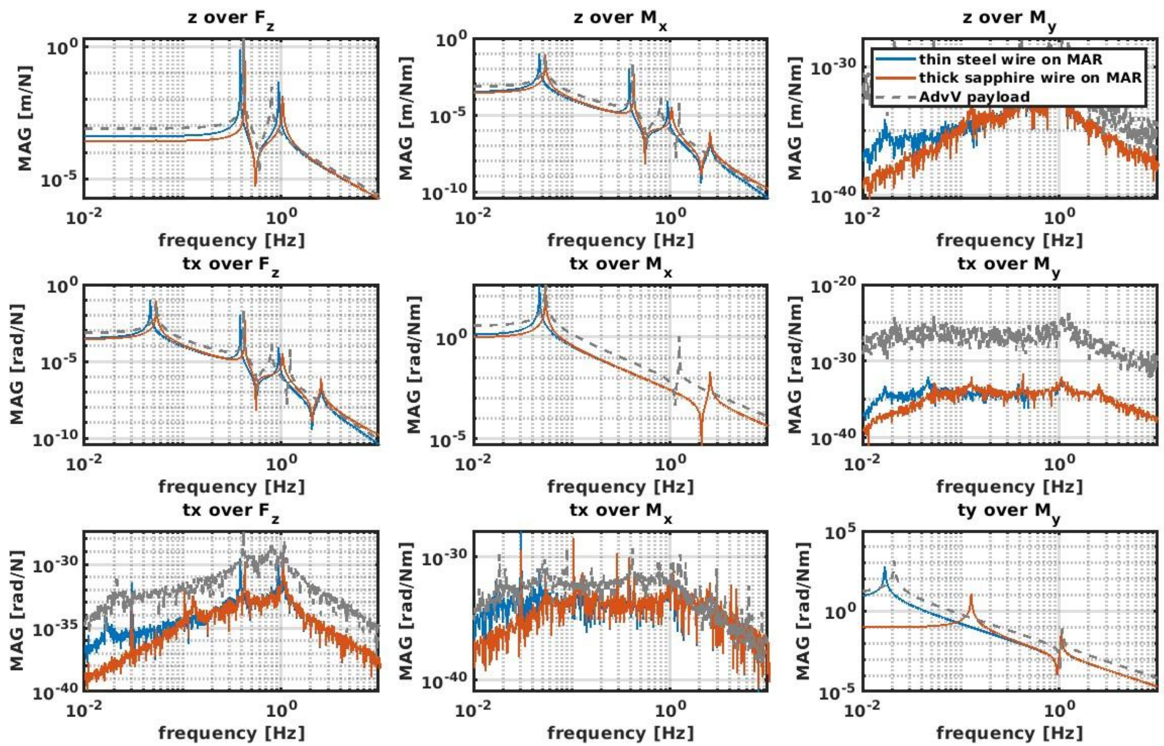


Conductive payload

The conclusion seems to be that a dissipative, uncharged HL cannot be directly connected to the marionette. The heat should pass through MAR suspension wire. From the point of view of the mechanical behaviour, this solution seems feasable, as long as the CM position is correctly managed.

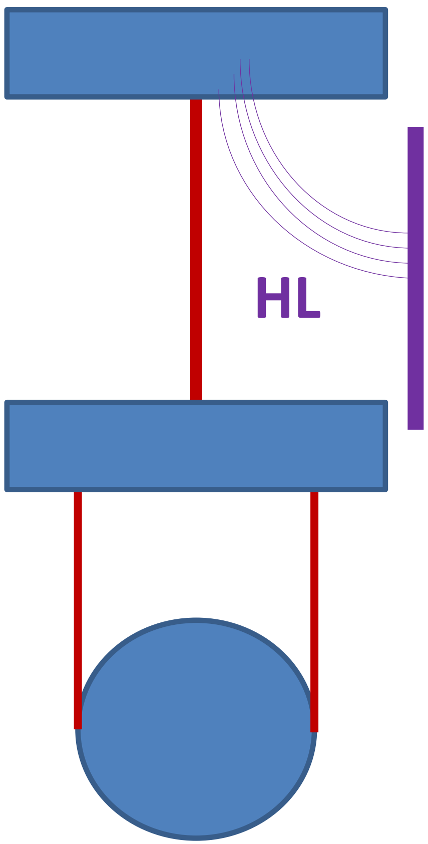


MAR $TF_{F \rightarrow X}$



Seismic bypass

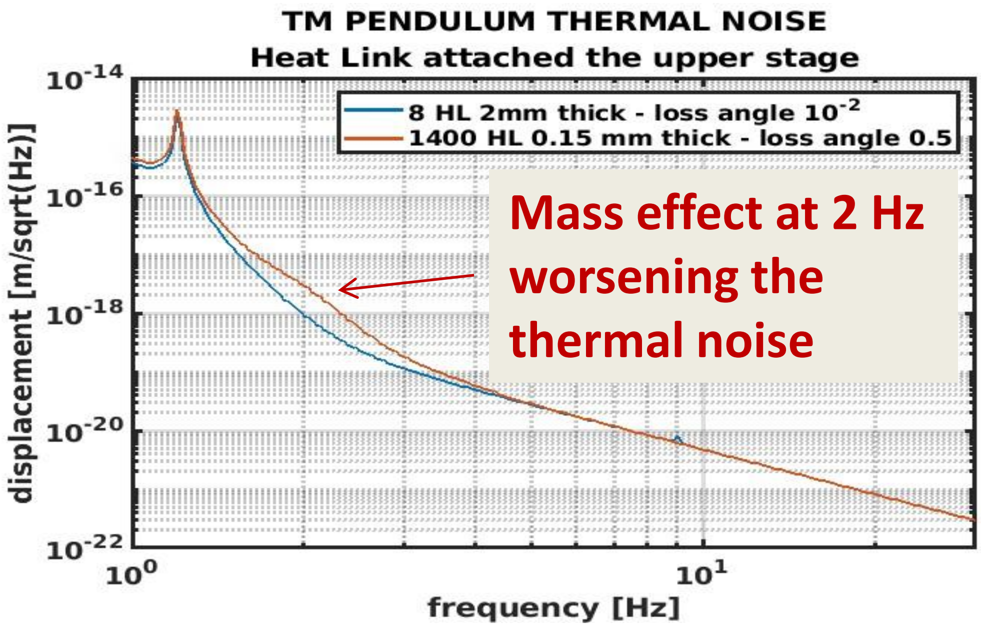
The HL needs to be attached to a stage above MAR, lowering the impact on thermal noise. In order to minimize the transmission of noise from the cryostat, a few thick links are not the best choice. It is better to distribute the given cross-section into a lot of thin wires.



$$Z_{HL} = Z_{HL}^{th} \| Z_{HL}^{th} \| \ldots$$

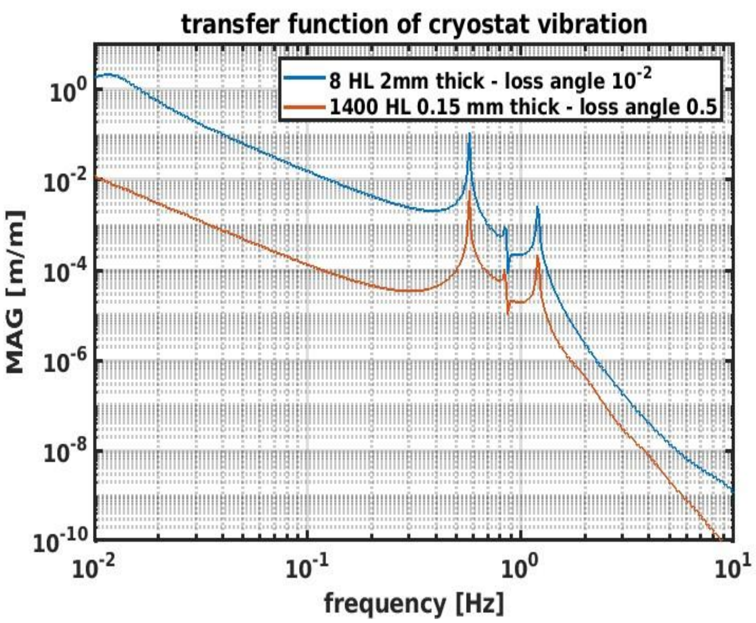
$$\boxed{\begin{array}{ll} A_{HL} = A_{HL}^{th} & B_{HL} = B_{HL}^{th}/N \\ C_{HL} = N \cdot C_{HL}^{th} & D_{HL} = D_{HL}^{th} \end{array}}$$

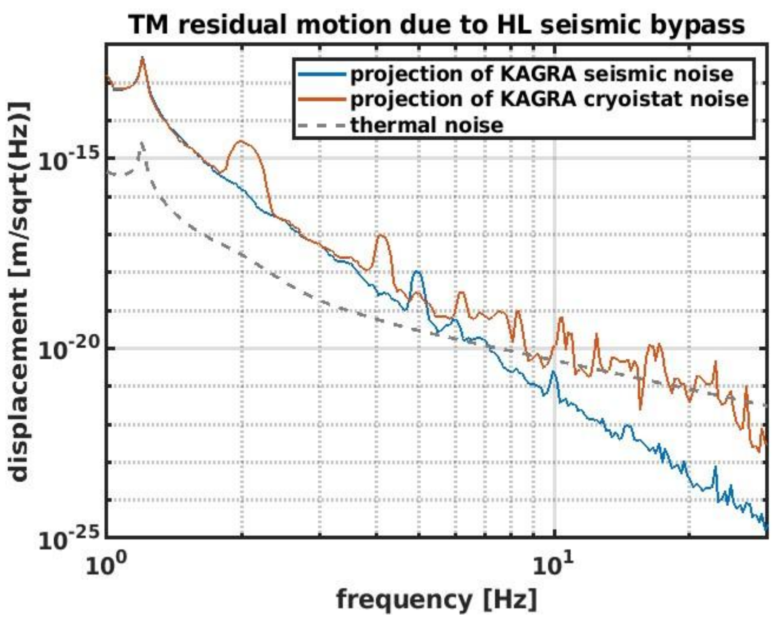
Indipendent links in parallel. Actually, HL is a beam of thin wires strongly interacting. No modeling of that: we only put a very high loss angle (0.5).



Seismic noise projection

Stiffness scales as d^4; section as d^2. The new total stiffness is a factor of 200 lower; the same is expected for the seismic transfer function. This is true at low frequency, but above 1 Hz the gap is reduced because of the mass effect, already seen in the thermal noise computation.
The transmitted seismic noise is above the thermal noise. The HeatLink should be attached to a cold mass more quiet than the ground.





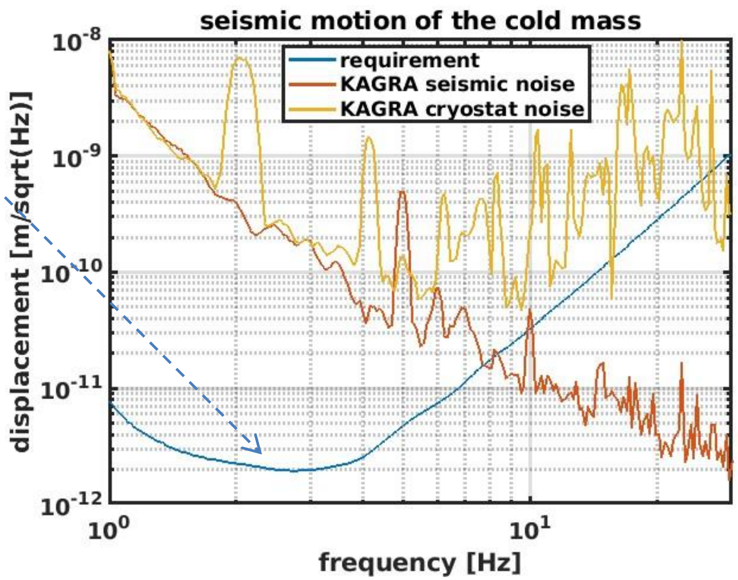
Requirement for cold mass vibration

The blue curve is obtained dividing the thermal noise by the seismic transfer function, both shown previously. An additional safety factor of 3 has been applied.

The result is the level of motion which should not be exceeded at the input side of the HeatLink, in order to have a residual seismic noise sufficiently lower than the thermal noise.

In the critical region, around 2-3 Hz, the expected seismic noise is 100 times larger.

A factor of 100 attenuation is not so large to be achieved: an inverted pendulum, like the one imployed as first stage of the superattenuator, can provide it.



Conclusion and perspective

✓ The impedance method has been explained. Its powerful in the modeling of complex systems has been shown in a few examples. No special mathematics or coding is required: everybody can redo the computation shown here, simply following the instructions. It is matter of writing algebrical formulas in matricial form and computing them frequency by frequency.

✓ An approximation is present in order to avoid numerical problems, but it has a negligible impact on the results. When the number of elements composing the system grows, the numerical problems could appear again. Coding optimization would be advisable in that case.

❑ An automatic way of translating any connection topology in the corrispondent formulas has not yet developed.

❑ The method has been already used also to represent controlled systems: a controller is associated to a specific impedance matrix. This is an interesting application to be better developed.

List of plots

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