# Accessing the properties of deconfined matter in heavy-ion collisions

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# The SIM network

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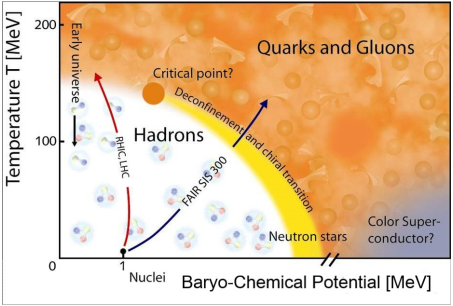
- Firenze: Francesco Becattini (PO), Francesco Bigazzi (PR INFN), Aldo Cotrone (PA), Maria Paola Lombardo (PR INFN), Andrea Palermo (PhD);
- LNS: Giuseppe Galesi (PhD), Vincenzo Greco (PO), Vincenzo Minissale (AdR), Salvatore Plumari (RTDA), Maria Lucia Sambataro (PhD);
- Torino: Wanda Alberico (PO), Andrea Beraudo (Ric INFN), Arturo De Pace (PR INFN), Marco Monteno (PR INFN), Marzia Nardi (Ric. INFN), Daniel Pablos (Ric INFN-Fellini)

# Heavy-ion collisions: exploring the QCD phase-diagram




QCD phases identified through the *order parameters*

- Polyakov loop $\langle L \rangle \sim e^{-\beta \Delta F_Q}$: energy cost to add an isolated color charge
- Chiral condensate $\langle \bar{q} q \rangle \sim$ effective mass of a “dressed” quark in a hadron

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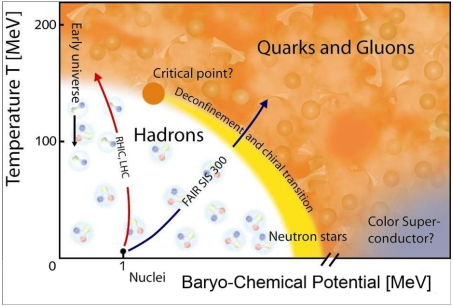


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Heavy-Ion Collision (HIC) experiments performed to study the transition

- From QGP (color deconfinement, chiral symmetry restored)
- to hadronic phase (confined, chiral symmetry broken)

NB $\langle\overline{q}q\rangle \neq 0$ responsible for most of the baryonic mass of the universe: *only ~35 MeV of the proton mass from* $m_{u/d} \neq 0$

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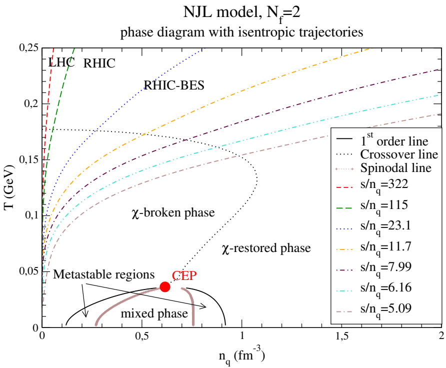


- Region explored at the LHC ($\sqrt{s_{\rm NN}} \approx 5$ TeV) and highest RHIC energy: *high-T/low-density* (early universe, $n_B/n_\gamma \sim 10^{-9}$)
- Higher baryon-density region accessible at lower $\sqrt{s_{\rm NN}} \approx 10$ GeV (Beam-Energy Scan at RHIC, fixed-target experiments)

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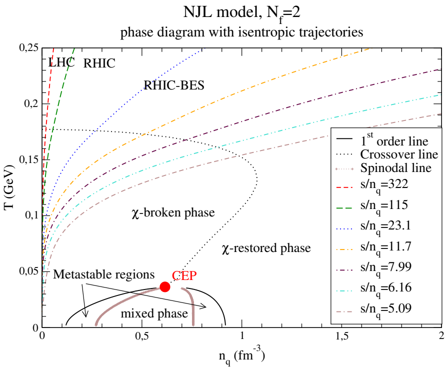


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**Is there a Critical End-Point in the QCD phase diagram?**

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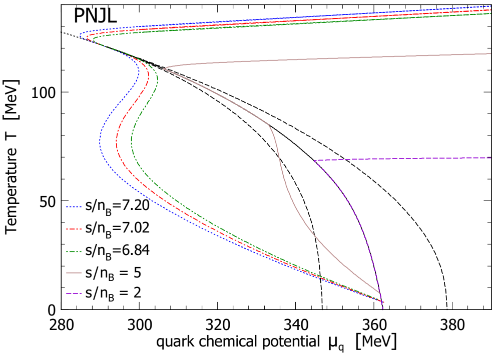


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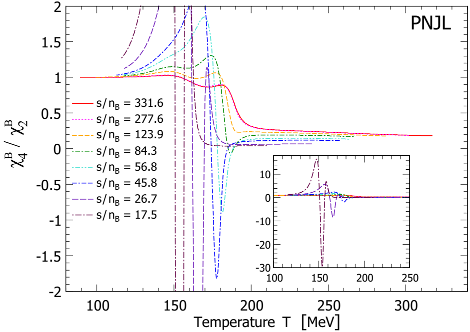
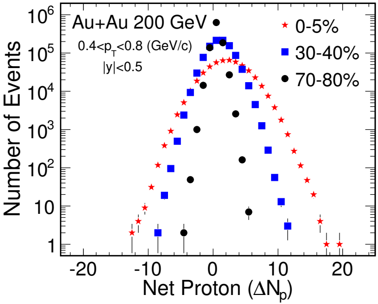
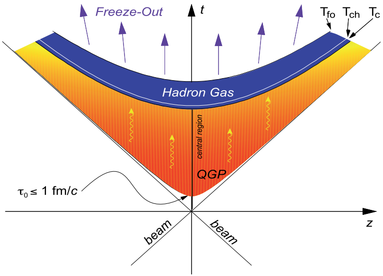
# Looking for signatures of the CEP

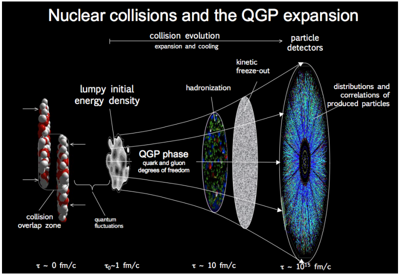



$\xi \to \infty$ at CEP should affect observables, e.g. ratio of cumulants of distributions of conserved charges (Mario Motta PhD thesis and M. Motta et al., Eur.Phys.J.C 80 (2020) 8, 770)

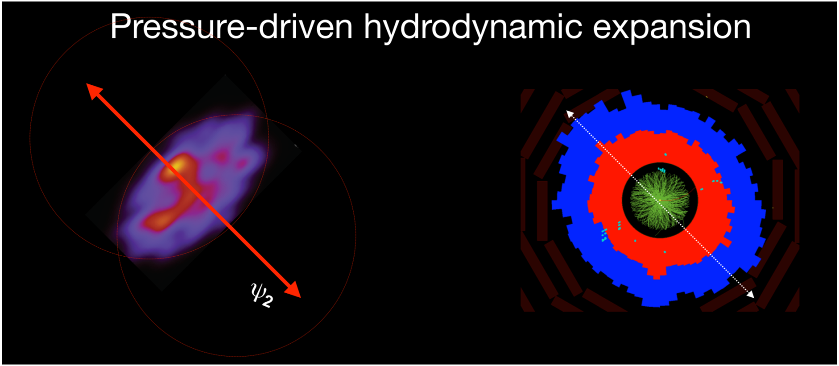
# Heavy-ion collisions: a cartoon of space-time evolution




- Soft probes (low-$p_T$ hadrons): collective behavior of the *medium*;
- Hard probes (high-$p_T$ particles, heavy quarks, quarkonia): produced in *hard pQCD processes* in the initial stage, allow to perform a tomography of the medium.

# A medium displaying a collective behavior




$$(\epsilon + P)\frac{dv^i}{dt} \underset{v \ll c}{=} -\frac{\partial P}{\partial x^i}$$

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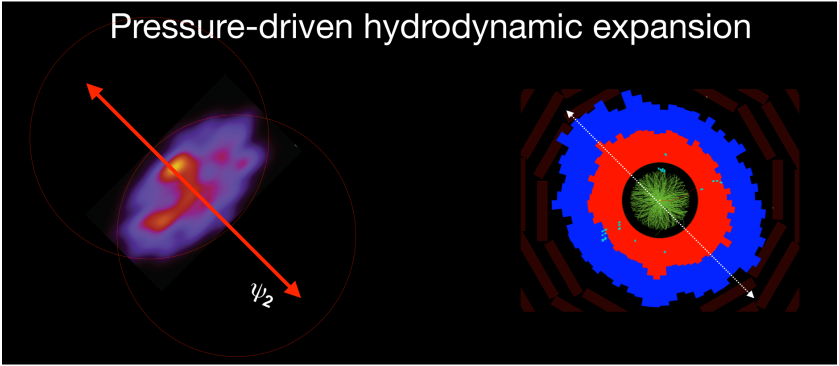


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NB picture relying on the condition $\lambda_{\mathrm{mfp}} \ll L$

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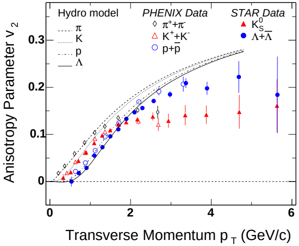


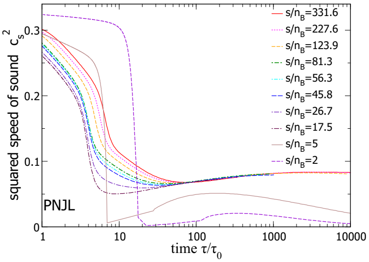


- Fourier expansion of azimuthal particle distribution

$$\frac{dN}{d\phi} = \frac{N}{2\pi}\left[1 + \sum_{n=1}^{\infty} 2v_n \cos[n(\phi - \psi_n)]\right]$$

$$v_n \equiv \langle \cos[n(\phi - \psi_n)] \rangle$$

- $\vec{\nabla} P = c_s^2 \vec{\nabla}\epsilon$: response to geometric deformation depends on the squared speed of sound (figure from M. Motta et al., Eur.Phys.J.C 80 (2020) 8, 770)

# Relativistic hydrodynamics: conceptual setup

When $\lambda_{\text{mfp}} \ll L$ only conservation laws matter:

$$\partial_\mu T^{\mu\nu} = 0 \quad + \quad \text{EoS} \quad P = P(\epsilon)$$

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- Relativistic Navier-Stokes first-order theory (violates causality)

$$\pi^{\mu\nu} = 2\eta \nabla^{<\mu} u^{\nu>}$$

with

$$\nabla^{<\mu} u^{\nu>} \equiv \frac{1}{2}(\nabla^\mu u^\nu + \nabla^\nu u^\mu) - \frac{1}{3}\Delta^{\mu\nu}(\nabla_\alpha u^\alpha)$$

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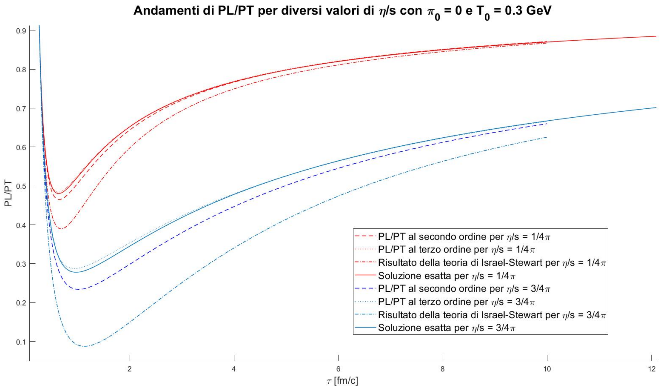
- Israel-Stewart second-order theory and further developments (respect causality): re-discovered and improved by heavy-ion community

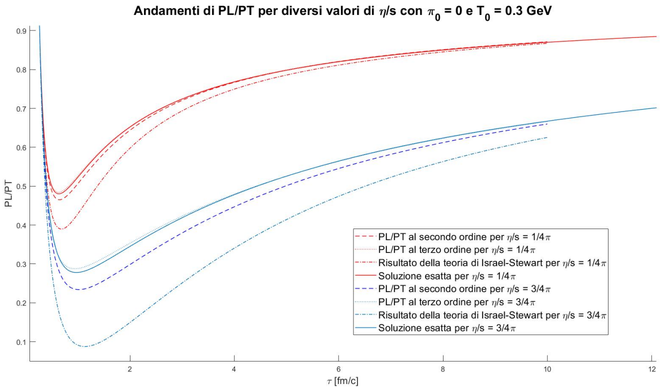
  $$\boxed{\dot{\pi}^{\mu\nu} = -\frac{1}{\tau_\pi}(\pi^{\mu\nu} - 2\eta \nabla^{<\mu} u^{\nu>})}$$

# Beyond the Israel-Stewart theory




One can perform a Chapman-Enskog expansion in powers of $\mathrm{Kn} = \lambda_{\mathrm{mfp}}/L$ and compare the results with the exact solution of the Boltzmann equation at fixed $\eta/s$ (bachelor thesis by Vittorio Larotonda).

# Beyond the Israel-Stewart theory

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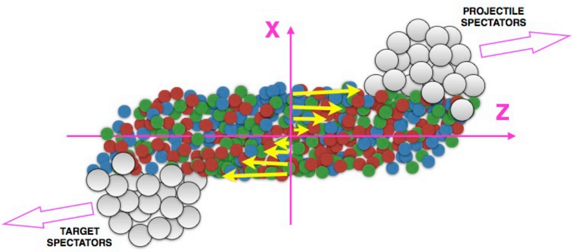
# Numerical implementation: development of ECHO-QGP

Major project which has involved the Universities and INFN sections of Firenze, Torino and Ferrara

- Eur.Phys.J. C73 (2013) 2524 Development of ECHO-QGP, first *public* relativistic viscous hydrodynamic code in 3+1 dimensions for the study of HIC's;
- Eur.Phys.J. C75 (2015) no.9, 406: study of $v_1$ of pions, vorticity and polarization of $\Lambda$ hyperons (recently measured by the STAR collaboration) in HIC's;
- Eur.Phys.J. C76 (2016) no.12, 659: first relativistic magneto-hydrodynamic code developed for the study of HIC's.

# Directed flow, vorticity and polarization




Participant nucleons deposit more energy along their direction of motion

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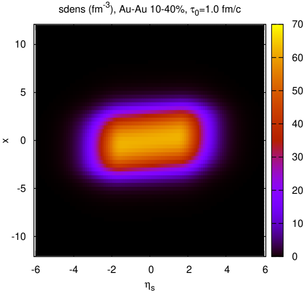


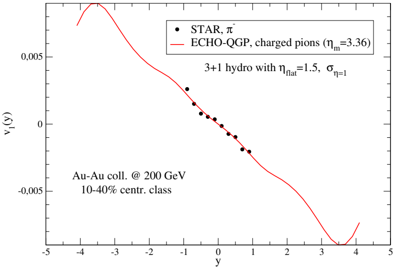


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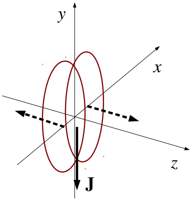


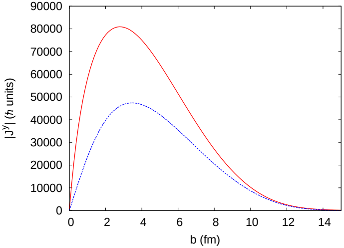


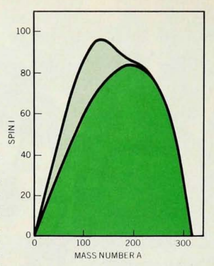


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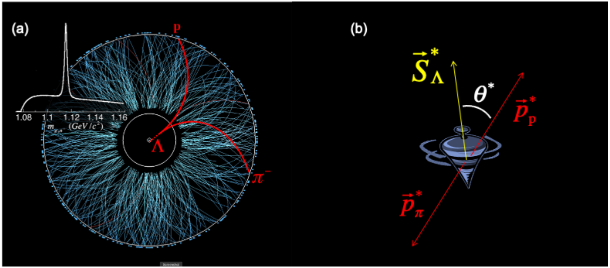
- Fireball deformation in the RP leads to $v_1 \equiv \langle \cos(\phi - \psi_{RP}) \rangle \neq 0$
- Enormous angular momentum ($|J_y| \sim 10^3 - 10^4 \hbar$) and vorticity $\vec{\omega} \equiv \frac{1}{2}(\vec{\nabla} \times \vec{v}) \sim 10^{22}\mathrm{s}^{-1}$ of the fireball partially transferred to polarization of produced particles via spin-orbit interaction

$$\widehat{\rho} \equiv \frac{1}{Z} \exp\left[ -(\widehat{H} - \boldsymbol{\omega} \cdot \widehat{\mathbf{J}} - \mu_Q \widehat{Q})/T \right]$$

(Eur.Phys.J. C75 (2015) no.9, 406)

# Vorticity and polarization: results




- Mean spin vector for spin 1/2 particles:

$$S^\mu(p) = -\frac{1}{8m}\epsilon^{\mu\rho\sigma\tau}p_\tau \frac{\int d\Sigma_\lambda p^\lambda n_F(1-n_F)\varpi_{\rho\sigma}}{\int d\Sigma_\lambda p^\lambda n_F}$$

where $\varpi_{\mu\nu} \equiv -\frac{1}{2}(\partial_\mu\beta_\nu - \partial_\nu\beta_\mu)$, with $\beta_\nu \equiv u_\nu/T$

- Polarization of $\Lambda$ hyperons $\sim 2\%$ measured through $\Lambda \to p\pi^-$

$$\frac{dN}{d\cos\theta^*} = \frac{1}{2}\left(1 + \alpha_\Lambda \vec{P}_\Lambda \cdot \hat{p}^*_p\right)$$

See e.g. F. Becattini et al., Lect.Notes Phys. 987 (2021)

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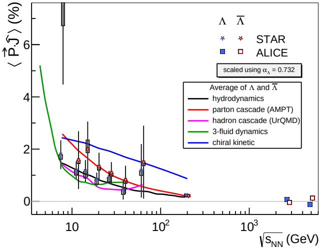


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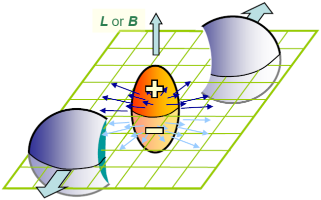
# The role of the magnetic field and axial anomaly




- Huge magnetic field ($B \sim 10^{15}$T) orthogonal to the reaction plane from

$$d_\mu(T^{\mu\nu}_{\rm matt} + T^{\mu\nu}_{\rm field}) = 0, \quad d_\mu F^{\mu\nu} = -J^\nu \quad \text{and} \quad d_\mu F^{\star\mu\nu} = 0$$

(Eur.Phys.J. C76 (2016) no.12, 659)

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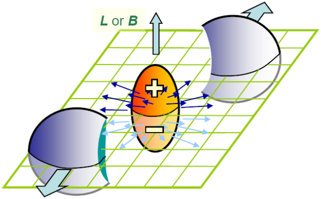


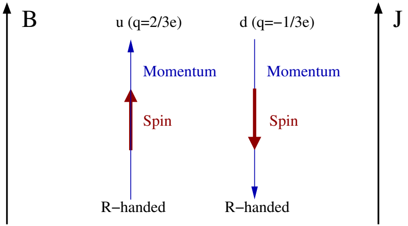


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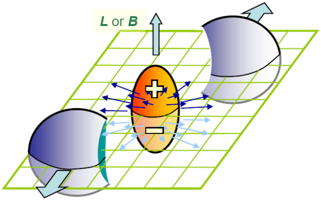


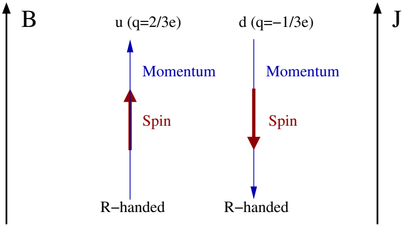


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- *Non-trivial topological configurations* of the colour field can lead, event by event, to an excess of quarks of a given chirality

$$\frac{d}{dt}(N_R - N_L) = -N_f \frac{g^2}{16\pi^2} \int d^3x\, \widetilde{F}^{\alpha\beta,a} F^a_{\alpha\beta} \neq 0$$

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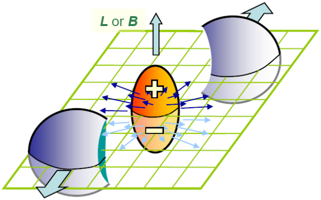


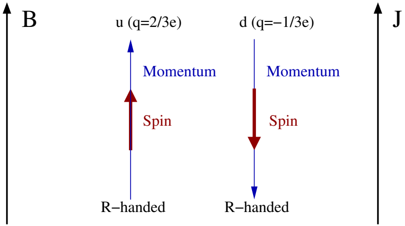


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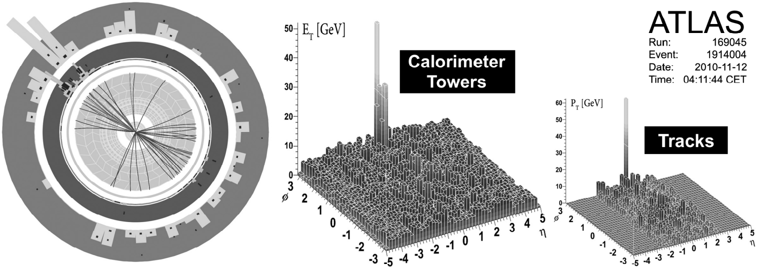
- $\vec{j} = \sigma_5 \vec{B}$: separation of opposite-charge particles wrt the reaction plane

# A medium inducing energy-loss to colored probes




Strong unbalance of di-jet events, visible at the level of the event-display itself, without any analysis: jet-quenching

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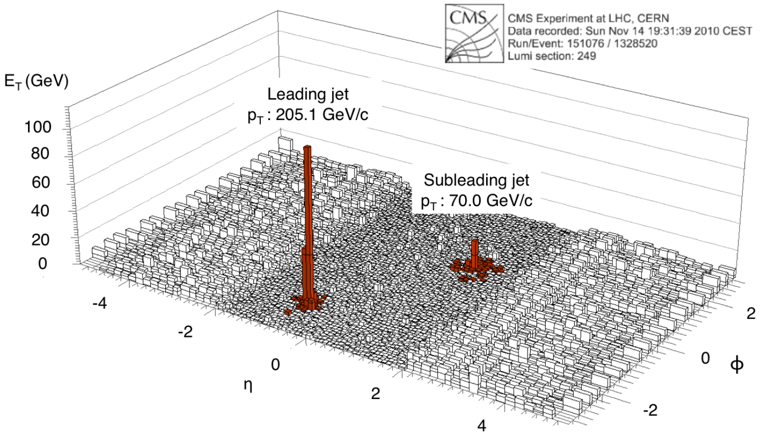


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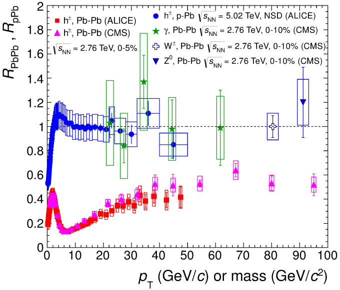


Medium-induced suppression of high-momentum hadrons and jets quantified through the *nuclear modification factor*

$$R_{AA} \equiv \frac{\left(dN^h/dp_T\right)^{AA}}{\langle N_{\mathrm{coll}}\rangle \left(dN^h/dp_T\right)^{pp}}$$

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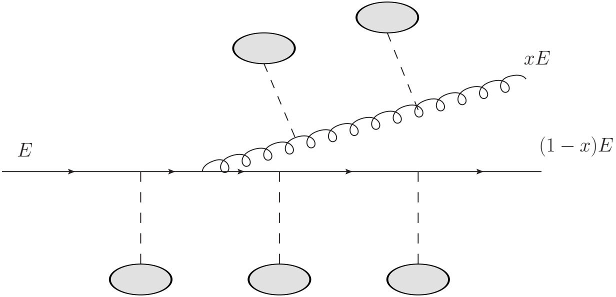


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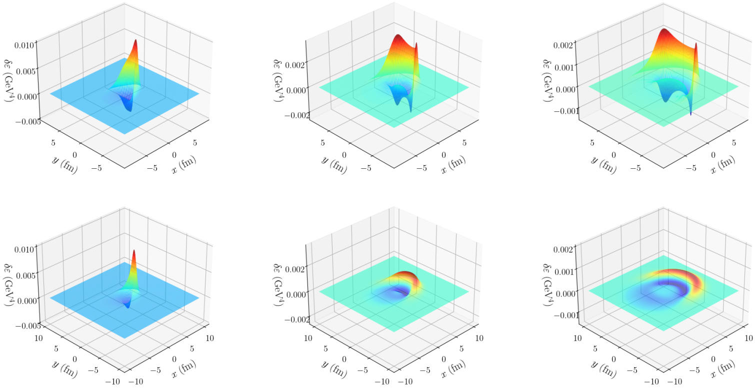
interpreted as energy carried away by radiated gluons

# How the medium responds to jets




Wake arising from jet propagation in an ideal and viscous medium studied in linearized hydrodynamics (Daniel Pablos et al., JHEP 05 (2021) 230)

# Heavy Flavour in the QGP: the conceptual setup

- Description of soft observables based on hydrodynamics, assuming to deal with a system close to local thermal equilibrium (no matter why): *collective behaviour* of the medium;

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NB At high-$p_T$ the interest in heavy flavor is no longer related to thermalization, but to the study of the mass and color charge dependence of jet-quenching (not addressed in this talk)

# Transport theory: the Boltzmann equation

Time evolution of HQ phase-space distribution $f_Q(t, \mathbf{x}, \mathbf{p})$:

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- Collision integral:

$$C[f_Q] = \int d\mathbf{k}[\underbrace{w(\mathbf{p}+\mathbf{k}, \mathbf{k}) f_Q(\mathbf{p}+\mathbf{k})}_{\text{gain term}} - \underbrace{w(\mathbf{p}, \mathbf{k}) f_Q(\mathbf{p})}_{\text{loss term}}]$$

  $w(\mathbf{p}, \mathbf{k})$: HQ transition rate $\mathbf{p} \to \mathbf{p} - \mathbf{k}$

# From Boltzmann to Fokker-Planck

Expanding the collision integral for *small momentum exchange*[1] (Landau)

$$C[f_Q] \approx \int d\mathbf{k} \left[ k^i \frac{\partial}{\partial p^i} + \frac{1}{2} k^i k^j \frac{\partial^2}{\partial p^i \partial p^j} \right] [w(\mathbf{p}, \mathbf{k}) f_Q(t, \mathbf{p})]$$

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$$A^i(\mathbf{p}) = \int d\mathbf{k}\, k^i w(\mathbf{p}, \mathbf{k}) \longrightarrow \underbrace{A^i(\mathbf{p}) = A(p)\, p^i}_{\text{friction}}$$

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Problem reduced to the *evaluation of three transport coefficients*, directly derived from the scattering matrix

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# Approach to equilibrium in the FP equation

The FP equation can be viewed as a continuity equation for the phase-space distribution of the kind $\partial_t \rho(t, \vec{p}) + \vec{\nabla}_p \cdot \vec{J}(t, \vec{p}) = 0$

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admitting a steady solution $f_{\rm eq}(p) \equiv e^{-E_p/T}$ when the current vanishes:

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One gets

$$A(p)p^i = \frac{B_1(p)}{TE_p}p^i - \frac{\partial}{\partial p^j}\left[\delta^{ij}B_0(p) + \hat{p}^i\hat{p}^j(B_1(p)-B_0(p))\right],$$

leading to the Einstein *fluctuation-dissipation* relation

$$A(p) = \frac{B_1(p)}{TE_p} - \left[\frac{1}{p}\frac{\partial B_1(p)}{\partial p} + \frac{d-1}{p^2}(B_1(p)-B_0(p))\right],$$

quite involved due to the *momentum dependence* of the transport coefficients (*measured* HQ's are relativistic particles!)

# The relativistic Langevin equation

The Fokker-Planck equation can be recast into a form suitable to follow the dynamics of each individual quark arising from the pQCD Monte Carlo simulation of the initial $Q\overline{Q}$ production: the Langevin equation

$$\frac{\Delta p^i}{\Delta t} = -\underbrace{\eta_D(p)p^i}_{\text{determ.}} + \underbrace{\xi^i(t)}_{\text{stochastic}} ,$$

with the properties of the noise encoded in

$$\langle\xi^i(\mathbf{p}_t)\rangle = 0 \quad \langle\xi^i(\mathbf{p}_t)\xi^j(\mathbf{p}_{t'})\rangle = b^{ij}(\mathbf{p})\frac{\delta_{tt'}}{\Delta t} \quad b^{ij}(\mathbf{p}) \equiv \kappa_L(p)\hat{p}^i\hat{p}^j + \kappa_T(p)(\delta^{ij} - \hat{p}^i\hat{p}^j)$$

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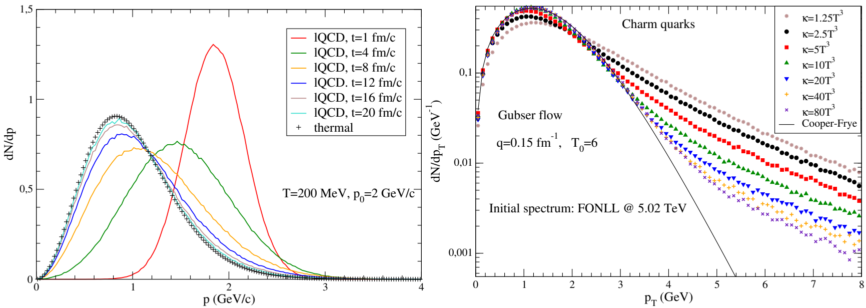
*Transport coefficients* related to the FP ones:

- *Momentum diffusion*: $\kappa_T(p) = 2B_0(p)$ and $\kappa_L(p) = 2B_1(p)$
- *Friction* term, in the *Ito pre-point discretization scheme*,

$$\eta_D^{\text{Ito}}(p) = A(p) = \frac{B_1(p)}{TE_p} - \left[ \frac{1}{p}\frac{\partial B_1(p)}{\partial p} + \frac{d-1}{p^2}(B_1(p) - B_0(p)) \right]$$

# Asymptotic approach to thermalization

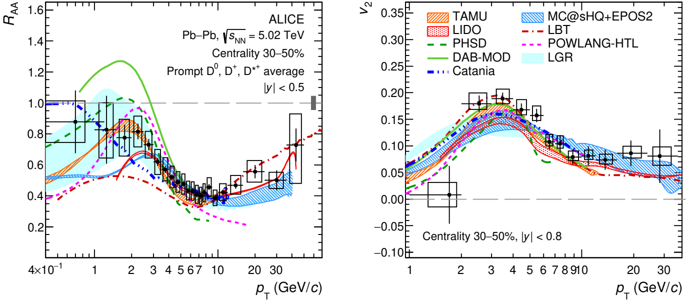



- Left panel: evolution in a static medium
- Right panel: decoupling from expanding medium at $T_{\rm FO} = 160$ MeV

For late times or for very large transport coefficients HQ's approach local kinetic equilibrium with the medium.

Figures adapted from Federica Capellino master thesis, awarded with *Milla Baldo Ceolin* and *Alfredo Molinari* INFN prizes.

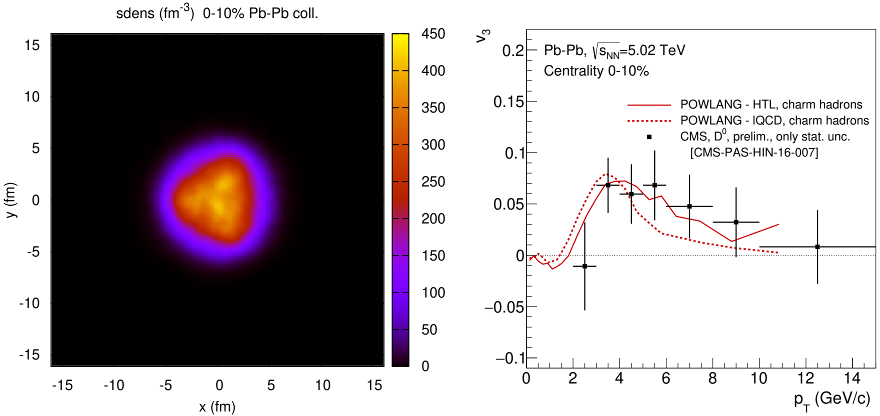
# Theory-to-data comparison: a snapshot of recent results




$$R_{\mathrm{AA}} \equiv \frac{dN/dp_T|_{\mathrm{AA}}}{\langle N_{\mathrm{coll}}\rangle \; dN/dp_T|_{\mathrm{pp}}} \qquad v_n \equiv \langle \cos[n(\phi - \Psi_n)] \rangle$$

In spite of their large mass, also the D-mesons turn out to be quenched and to have a sizable $v_2$. Does also charm reach local thermal equilibrium? Transport calculations are challenged to consistently reproduce this rich phenomenology.

# EBE fluctuations and $D$-meson $v_3$




Transport calculations carried out in JHEP 1802 (2018) 043, with hydrodynamic background calculated via the ECHO-QGP code (EPJC 73 (2013) 2524) starting from Glauber Monte-Carlo initial conditions.

Theory and experimental verification of brownian motion by Einstein (1905) and Perrin (1909)

From the vertical distribution of an emulsion

$$n(z) = n_0 e^{-(Mg/K_B T)z}$$

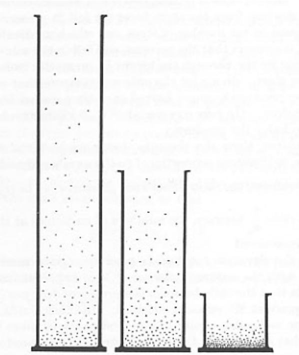


FIG. 3.

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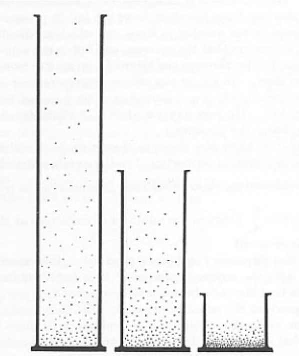

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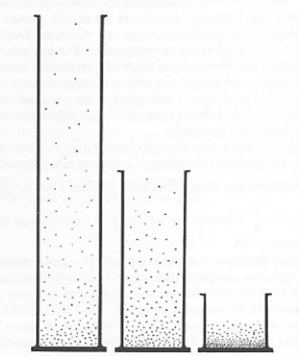
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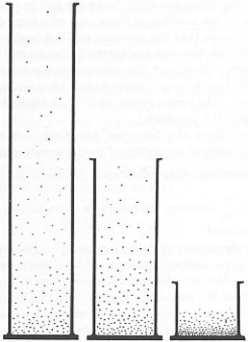

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One gets an expression for the diffusion coefficient

$$D = \frac{K_B T}{6\pi a \eta}$$

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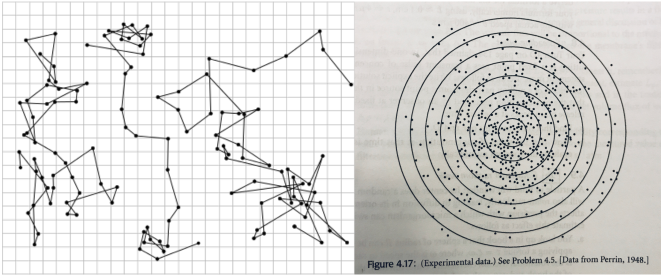


From the random walk of the emulsion particles (follow the motion along one direction!) one extracts the diffusion coefficient

$$< x^2 > \underset{t \to \infty}{\sim} 2Dt$$

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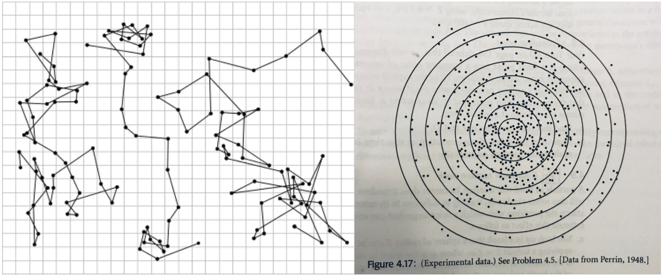

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Perrin obtained the values $\mathcal{N}_A \approx 5.5 - 7.2 \cdot 10^{23}$.

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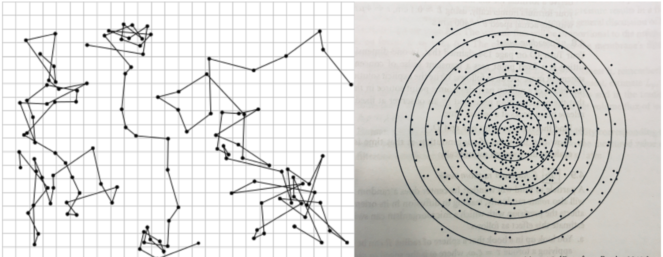

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Perrin obtained the values $\mathcal{N}_A \approx 5.5 - 7.2 \cdot 10^{23}$. We would like to extract HQ transport coefficients in the QGP with a comparable precision!

# HQ momentum diffusion: lattice-QCD

Getting the HQ momentum-diffusion coefficient requires to evaluate

$$\kappa = \frac{1}{3}\int_{-\infty}^{+\infty} dt \langle \xi^i(t)\xi^i(0)\rangle_{\rm HQ} = \frac{1}{3}\int_{-\infty}^{+\infty} dt\, \underbrace{\langle F^i(t)F^i(0)\rangle_{\rm HQ}}_{\equiv D^>(t)}$$

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$$\kappa \equiv \lim_{\omega\to 0} \frac{D^>(\omega)}{3} = \lim_{\omega\to 0} \frac{1}{3}\frac{\sigma(\omega)}{1 - e^{-\beta\omega}} \underset{\omega\to 0}{\sim} \frac{1}{3}\frac{T}{\omega}\sigma(\omega)$$

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From $D_E(\tau)$ one extracts the spectral density according to

$$D_E(\tau) = \int_0^{+\infty} \frac{d\omega}{2\pi}\frac{\cosh(\tau-\beta/2)}{\sinh(\beta\omega/2)}\sigma(\omega)$$

# HQ momentum diffusion: lattice-QCD

The direct extraction of the spectral density from the euclidean correlator

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is a ill-posed problem, since the latter is known for a limited set ($\sim 20$) of points $D_E(\tau_i)$, and one wishes to obtain a fine scan of the the spectral function $\sigma(\omega_j)$. A direct $\chi^2$-fit is not applicable.

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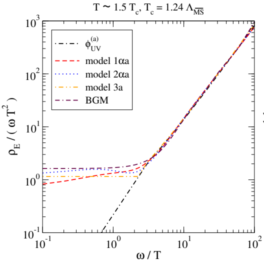
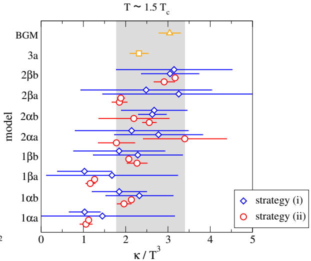
- Theory-guided ansatz for the behaviour of $\sigma(\omega)$ to constrain its functional form (A. Francis *et al.*, PRD 92 (2015), 116003)




From the different ansatz on the functional form of $\sigma(\omega)$ one gets a systematic uncertainty band:

$$\kappa/T^3 \approx 1.8 - 3.4$$

# From momentum broadening to spatial diffusion

In the *non-relativistic* limit an excess of HQ's initially placed at the origin will diffuse according to

$$\langle \vec{x}^2(t) \rangle \underset{t\to\infty}{\sim} 6 D_s t \quad \text{with} \quad D_S = \frac{2T^2}{\kappa}.$$

For a strongly interacting system spatial diffusion is very small! Theory calculations for $D_s$ have been collected (F. Prino and R. Rapp, JPG 43 (2016) 093002) and are often used by the experimentalists to summarize the difference among the various models

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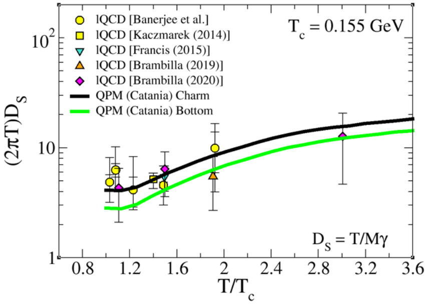


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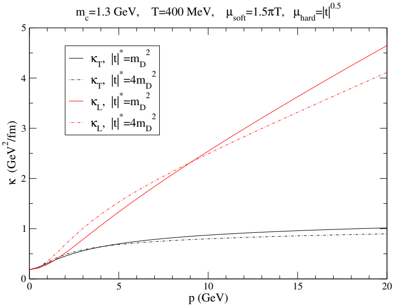


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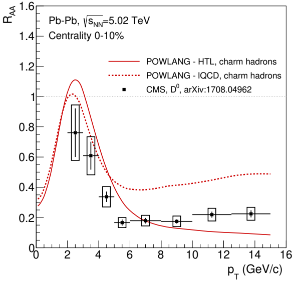


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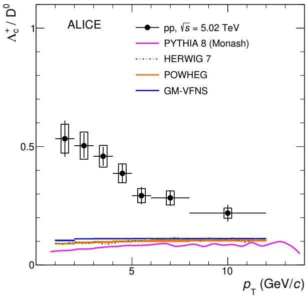
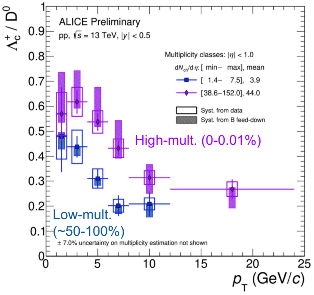
# Are *pp* collisions really a reference?




May a small fireball be produced also in (high-multiplicity) *pp* events?

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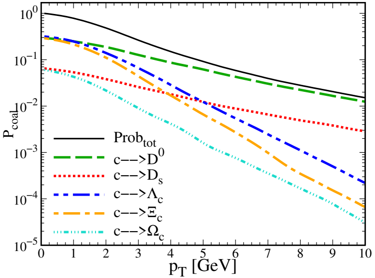


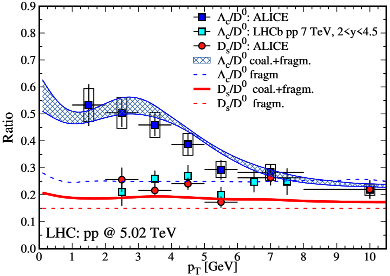


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- Models including the possibility of HQ coalescence with light thermal partons seem able to reproduce the data (V. Minissale et al., Phys.Lett.B 821 (2021) 136622)