CP violation and flavor invariants in the seesaw effective field theory

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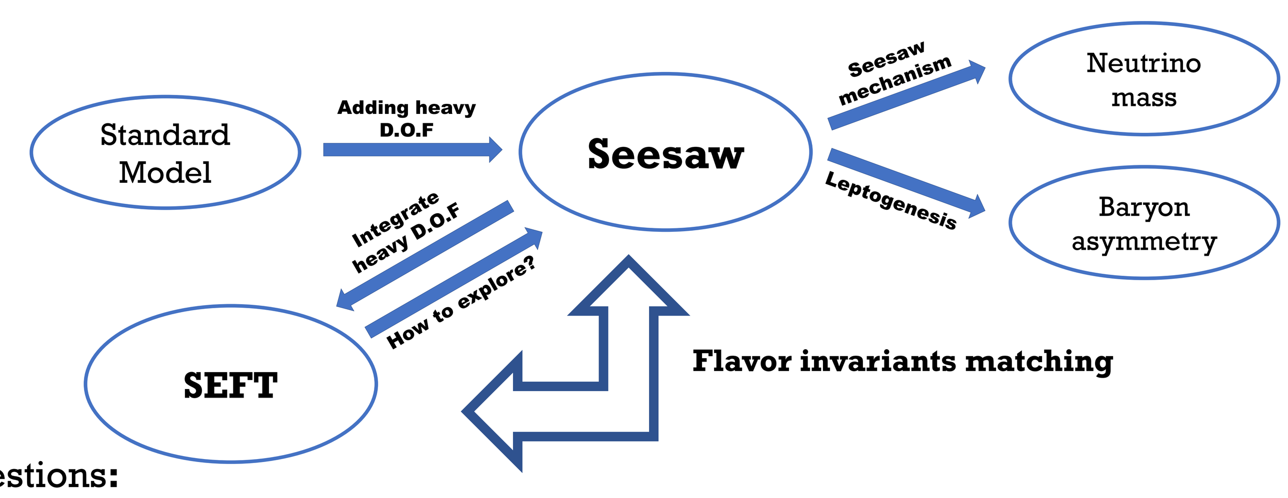
Based on 2203.00574 and 2203.10121

Seesaw effective field theory (SEFT)

Seesaw models provide a simple and natural explanation for both the tiny neutrino masses and the cosmological matter-antimatter asymmetries. The seesaw scale is usually very high and one can integrate out heavy degrees of freedom to obtain the seesaw effective field theory (SEFT).



relevant questions:
1) How does the UV theory affect the observables at low energies?
2) How can we know about the full seesaw model from low-energy experiments?
There have been many previous efforts in studying the relation between full seesaw model and low-energy SEFT in the literature[1]. Here we investigate this problem from a new point of view: the **invariant theory**[2,3].

Invariant theory and Hilbert series

In the particle physics, given the Lagrangian, many observables can be calculated from the coupling parameters in the Lagrangian

$$\mathcal{L}\left(c_1, c_2, \ldots\right) \xrightarrow{\text{QFT}} \mathcal{O}\left(c_1, c_2, \ldots\right) \xrightarrow[c_i \neq c_i^{\prime}]{\text{flavor trans.}} \mathcal{O}\left(c_1^{\prime}, c_2^{\prime}, \ldots\right)$$

The parameters in the Lagrangian are *not* invariant under the transformations in the flavor space. However, we know that the observables should be independent of the chosen basis. Therefore, it is well motivated to introduce some quantities that are *polynomials* of parameters in the Lagrangian and are invariant under the flavor-basis transformations (**flavor invariants**). Then one can use them to describe physical observables (e.g., CP asymmetries)

$$\mathcal{I}_i\left(c_1, c_2, \ldots\right) \xrightarrow{\text{flavor trans.}} \mathcal{I}_i\left(c_1^{\prime}, c_2^{\prime}, \ldots\right) \Rightarrow \mathcal{A}_{\rm CP}\left(\mathcal{I}_i\right)$$

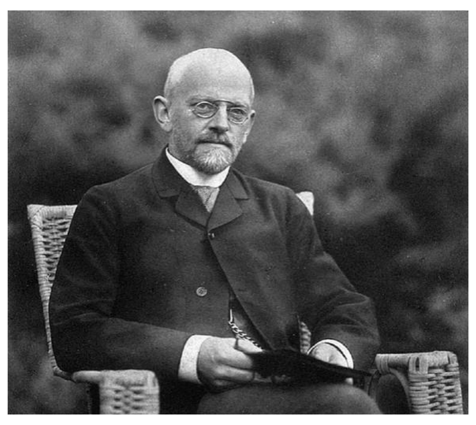
All the flavor invariants form a *ring* since the addition and multiplication of any two invariants are also invariants. The main task is to find out all the '**basic invariants**' and any invariant in the ring can be decomposed as the *polynomial* of these generators

$$\text{generators}: \left\{\mathcal{I}_1, \mathcal{I}_2, \ldots, \mathcal{I}_n\right\} \xrightarrow[\text{decomposition}]{\text{polynomial}} \forall \text{ invariant } \mathcal{I},\ \mathcal{I}=\mathcal{P}\left(\mathcal{I}_1, \mathcal{I}_2, \ldots, \mathcal{I}_n\right)$$

This can be accomplished using the powerful tool of **Hilbert series** (HS) in the invariant theory[4]

$$\mathscr{H}(q) \equiv \sum_{k=1}^{\infty} c_k q^k=\frac{1+a_1 q+\cdots+a_{l-1} q^{l-1}+a_l q^l}{\prod_{i=1}^r\left(1-q^{d_i}\right)}, \quad a_k=a_{l-k}$$

$q: |q|<1$, labeling degrees of the building blocks
c_k : # linearly-independent invariants at degree k
r : # independent physical parameters in the theory



The HS is defined as the **generating function** of invariants and has encoded all information about the algebraic structure of the invariant ring. It can be computed following the standard procedure:

$$\boxed{\text{Representations of building blocks} \xrightarrow[\text{decomposition}]{\text{tensor}} \text{Plethystic Exponential} \xrightarrow[\text{formula}]{\text{Molien-Weyl}} \text{Hilbert series}}$$

Below we will calculate the HS in the SEFT and in the full seesaw model and explicitly construct all the basic invariants.

Flavor invariants matching

To one's surprise, there are exactly equal number of CP-odd and CP-even basic invariants in the SEFT and in the full seesaw model, both are 6 and 12. Therefore, **the ring of the invariants in the SEFT and that in the full seesaw model share an equal number of generators**.

$$\boxed{\text{\# CP-odd (-even) basic invariants in SEFT=\# CP-odd (-even) basic invariants in seesaw}}$$

Furthermore, one can establish a direct connection between the two sets of generators by using the matching conditions of Wilson coefficients. For example, the 6 CP-odd basic invariants in the SEFT (cf. TABLE I) can be written as the **linear combinations** of the 6 CP-odd basic invariants in the full seesaw model (cf. TABLE II)[2,3]

$$\begin{aligned}
\mathcal{I}_{121}^{(2)} &= \frac{2}{\left(I_{002}^2-I_{004}\right)^2}\left[I_{242}^{(2)} I_{022}-I_{044} I_{220}+I_{262} I_{002}-I_{244} I_{020}\right], \\
\mathcal{I}_{221} &= \frac{2}{\left(I_{002}^2-I_{004}\right)^2}\left[I_{242}^{(2)} I_{222}+I_{244} I_{220}+I_{462} I_{002}-I_{444} I_{020}\right], \\
\mathcal{I}_{122} &= \frac{2}{\left(I_{002}^2-I_{004}\right)^3}\left\{I_{242}^{(2)}\left[3 I_{022}^2+2 I_{040}\left(I_{002}^2-I_{004}\right)-4 I_{020} I_{002} I_{022}\right]+I_{244}\left(3 I_{020} I_{022}-2 I_{042}\right)\right. \\
&\quad \left.+I_{044}\left(4 I_{020} I_{222}-I_{220} I_{022}-2 I_{242}^{(1)}\right)+I_{262}\left[3 I_{002} I_{022}-I_{020}\left(I_{002}^2+3 I_{004}\right)\right]\right\}, \\
\mathcal{I}_{240} &= \frac{1}{\left(I_{002}^2-I_{004}\right)^2}\left[3 I_{242}^{(2)}\left(I_{022} I_{220}-I_{020} I_{222}\right)-I_{044} I_{220}^2+I_{262}\left(3 I_{002} I_{220}-2 I_{222}\right)-2 I_{244} I_{020} I_{220}\right. \\
&\quad \left.+I_{462}\left(2 I_{022}-3 I_{002} I_{020}\right)+I_{444} I_{020}^2\right], \\
\mathcal{I}_{141} &= \frac{2}{\left(I_{002}^2-I_{004}\right)^3}\left\{I_{242}^{(2)} I_{020} I_{022}^2+I_{044} I_{020}\left(I_{022} I_{220}-2 I_{242}^{(1)}\right)+I_{244} I_{020}\left(I_{020} I_{022}-2 I_{042}\right)\right. \\
&\quad \left.+I_{262}\left[I_{002} I_{020} I_{022}+I_{040}\left(I_{004}-I_{002}^2\right)\right]\right\}, \\
\mathcal{I}_{042} &= \frac{2}{\left(I_{002}^2-I_{004}\right)^3} I_{044}\left(I_{020}^2-I_{040}\right)^2 .
\end{aligned}$$

Flavor invariants in seesaw and SEFT

The relevant Lagrangian in the type-I seesaw model is given by

$$\mathcal{L}_{\rm seesaw}=-\overline{\ell_{\rm L}} Y_l H l_{\rm R}-\overline{\ell_{\rm L}} Y_\nu \widetilde{H} N_{\rm R}-\frac{1}{2} \overline{N_{\rm R}^{\rm C}} M_{\rm R} N_{\rm R}+\text{h.c.}\,.$$

If the mass scale of the RH neutrinos $\Lambda=\mathcal{O}(M_{\rm R})$ is much heavier than the electroweak scale, one can integrate RH-neutrino fields to obtain the effective theory. To the order of $\mathcal{O}(1/\Lambda^2)$ one obtains

$$\mathcal{L}_{\rm SEFT}=-\overline{\ell_{\rm L}} Y_l H l_{\rm R}-\left(\frac{C_5}{2 \Lambda} \overline{\ell_{\rm L}} \widetilde{H} \widetilde{H}^{\rm T} \ell_{\rm L}^{\rm C}+\text{h.c.}\right)+\frac{C_6}{\Lambda^2}\left(\overline{\ell_{\rm L}} \widetilde{H}\right) i \not{\partial}\left(\widetilde{H}^{\dagger} \ell_{\rm L}\right).$$

At the tree-level matching, the Wilson coefficients turn out to be

$$C_5=-Y_\nu Y_{\rm R}^{-1} Y_\nu^{\rm T}, \quad C_6=Y_\nu\left(Y_{\rm R}^{\dagger} Y_{\rm R}\right)^{-1} Y_\nu^{\dagger}, \quad \text{with } Y_{\rm R} \equiv M_{\rm R}/\Lambda .$$

Therefore, the building blocks for the construction of flavor invariants in the SEFT are $\{X_l, C_5, C_6\}$ with the symmetry group U($n_{\rm L}$), while the building blocks in the full seesaw model are $\{Y_l, Y_\nu, Y_{\rm R}\}$ with the symmetry group U($n_{\rm L}$)⊗U($n_{\rm R}$). Here $X_l \equiv Y_l Y_l^\dagger$, $n_{\rm L}$ and $n_{\rm R}$ are the generations of active neutrinos and RH neutrinos. Below we focus on the case of $n_{\rm L}$ = $n_{\rm R}$ =2 for the purpose of illustration. The HS in the SEFT and in the full seesaw model can be calculated using the Molien-Weyl formula[5]

$$\mathscr{H}_{\rm SEFT}(q)=\frac{1+3 q^4+2 q^5+3 q^6+q^{10}}{(1-q)^2\left(1-q^2\right)^4\left(1-q^3\right)^2\left(1-q^4\right)^2}, \quad \mathscr{H}_{\rm seesaw}(q)=\frac{1+q^6+3 q^8+2 q^{10}+3 q^{12}+q^{14}+q^{20}}{\left(1-q^2\right)^3\left(1-q^4\right)^5\left(1-q^6\right)\left(1-q^{10}\right)}.$$

It is found that the denominators of the HS in the SEFT and that of the full seesaw model share exactly the same number of factors, which means there are same number of physical parameters in the SEFT and in the full seesaw model (both are 10 for $n_{\rm L}$ = $n_{\rm R}$ =2). Furthermore, from the HS, one can explicitly construct all the basic flavor invariants in the ring as shown below[2,3]:

flavor invariants	degree	CP parity
$\mathcal{I}_{100} \equiv \mathrm{Tr}\left(X_l\right) \quad (*)$	1	+
$\mathcal{I}_{001} \equiv \mathrm{Tr}\left(C_6\right) \quad (*)$	1	+
$\mathcal{I}_{200} \equiv \mathrm{Tr}\left(X_l^2\right) \quad (*)$	2	+
$\mathcal{I}_{101} \equiv \mathrm{Tr}\left(X_l C_6\right)$	2	+
$\mathcal{I}_{020} \equiv \mathrm{Tr}\left(X_5\right) \quad (*)$	2	+
$\mathcal{I}_{002} \equiv \mathrm{Tr}\left(C_6^2\right) \quad (*)$	2	+
$\mathcal{I}_{120} \equiv \mathrm{Tr}\left(X_l X_5\right) \quad (*)$	3	+
$\mathcal{I}_{021} \equiv \mathrm{Tr}\left(C_6 X_5\right) \quad (*)$	3	+
$\mathcal{I}_{220} \equiv \mathrm{Tr}\left(X_l G_{l5}\right) \quad (*)$	4	+
$\mathcal{I}_{121}^{(1)} \equiv \mathrm{Tr}\left(G_{l5} C_6\right)$	4	+
$\mathcal{I}_{121}^{(2)} \equiv \mathrm{Im}\,\mathrm{Tr}\left(X_l X_5 C_6\right)$	4	−
$\mathcal{I}_{040} \equiv \mathrm{Tr}\left(X_5^2\right) \quad (*)$	4	+
$\mathcal{I}_{022} \equiv \mathrm{Tr}\left(C_6 G_{56}\right) \quad (*)$	4	+
$\mathcal{I}_{221} \equiv \mathrm{Im}\,\mathrm{Tr}\left(X_l G_{l5} C_6\right)$	5	−
$\mathcal{I}_{122} \equiv \mathrm{Im}\,\mathrm{Tr}\left(C_6 G_{56} X_l\right)$	5	−
$\mathcal{I}_{240} \equiv \mathrm{Im}\,\mathrm{Tr}\left(X_l X_5 G_{l5}\right)$	6	−
$\mathcal{I}_{141} \equiv \mathrm{Im}\,\mathrm{Tr}\left(X_5 C_6 G_{l5}\right)$	6	−
$\mathcal{I}_{042} \equiv \mathrm{Im}\,\mathrm{Tr}\left(C_6 X_5 G_{56}\right)$	6	−

TABLE I. Summary of the basic flavor invariants along with their degrees and CP parities in the case of two-generation leptons in the SEFT, where the subscripts of the invariants denote the degrees of $X_l \equiv Y_l Y_l^\dagger$, C_5 and C_6, respectively. We have also defined $X_5 \equiv C_5 C_5^\dagger$, $G_{l5} \equiv C_5 X_l^* C_5^\dagger$ and $G_{56} \equiv C_5 C_6^* C_5^\dagger$ that transform adjointly under the flavor transformation. There are in total 12 CP-even basic invariants and 6 CP-odd basic invariants. Note that the 10 primary invariants are labeled with "(*)" in the first column.

flavor invariants	degree	CP parity
$I_{200} \equiv \mathrm{Tr}\left(X_l\right)$	2	+
$I_{020} \equiv \mathrm{Tr}\left(X_\nu\right)$	2	+
$I_{002} \equiv \mathrm{Tr}\left(X_{\rm R}\right)$	2	+
$I_{400} \equiv \mathrm{Tr}\left(X_l^2\right)$	4	+
$I_{220} \equiv \mathrm{Tr}\left(X_l X_\nu\right)$	4	+
$I_{040} \equiv \mathrm{Tr}\left(X_\nu^2\right)$	4	+
$I_{022} \equiv \mathrm{Tr}\left(\tilde{X}_\nu X_{\rm R}\right)$	4	+
$I_{004} \equiv \mathrm{Tr}\left(X_{\rm R}^2\right)$	4	+
$I_{222} \equiv \mathrm{Tr}\left(X_{\rm R} G_{l\nu}\right)$	6	+
$I_{042} \equiv \mathrm{Tr}\left(\tilde{X}_\nu G_{\nu{\rm R}}\right)$	6	+
$I_{242}^{(1)} \equiv \mathrm{Tr}\left(G_{l\nu} G_{\nu{\rm R}}\right)$	8	+
$I_{242}^{(2)} \equiv \mathrm{Im}\,\mathrm{Tr}\left(\tilde{X}_\nu X_{\rm R} G_{l\nu}\right)$	8	−
$I_{044} \equiv \mathrm{Im}\,\mathrm{Tr}\left(\tilde{X}_\nu X_{\rm R} G_{\nu{\rm R}}\right)$	8	−
$I_{442} \equiv \mathrm{Tr}\left(G_{l\nu} G_{l\nu{\rm R}}\right)$	10	+
$I_{262} \equiv \mathrm{Im}\,\mathrm{Tr}\left(\tilde{X}_\nu G_{l\nu} G_{\nu{\rm R}}\right)$	10	−
$I_{244} \equiv \mathrm{Im}\,\mathrm{Tr}\left(X_{\rm R} G_{l\nu} G_{\nu{\rm R}}\right)$	10	−
$I_{462} \equiv \mathrm{Im}\,\mathrm{Tr}\left(\tilde{X}_\nu G_{l\nu} G_{l\nu{\rm R}}\right)$	12	−
$I_{444} \equiv \mathrm{Im}\,\mathrm{Tr}\left(X_{\rm R} G_{l\nu} G_{l\nu{\rm R}}\right)$	12	−

TABLE II. Summary of the basic flavor invariants along with their degrees and CP parities in the case of two-generation leptons in type-I seesaw model. The subscripts of the invariants denote the degrees of Y_l, Y_ν and $Y_{\rm R}$, respectively. We have also defined some building blocks that transform adjointly under the flavor transformation: $X_l \equiv Y_l Y_l^\dagger$, $X_\nu \equiv Y_\nu Y_\nu^\dagger$, $\tilde{X}_\nu \equiv Y_\nu^\dagger Y_\nu$, $X_{\rm R} \equiv Y_{\rm R}^\dagger Y_{\rm R}$, $G_{l\nu} \equiv Y_\nu^\dagger X_l Y_\nu$, $G_{\nu{\rm R}} \equiv Y_{\rm R}^\dagger \tilde{X}_\nu^* Y_{\rm R}$ and $G_{l\nu{\rm R}} \equiv Y_{\rm R}^\dagger G_{l\nu}^* Y_{\rm R}$. There are in total 12 CP-even basic invariants and 6 CP-odd basic invariants.

Connections between CPV at high- and low-energies

The matching conditions in terms of flavor invariants serve as a bridge to link the observables at high energies to those at low energies in a **basis- and parametrization-independent** way. For example, one can relate the CP asymmetries in the leptogenesis to those in the oscillation experiments[2,3]

$$\epsilon_1=\mathcal{R}_1\left[I_{\rm even}\right] \mathcal{I}_{121}^{(2)}+\mathcal{R}_2\left[I_{\rm even}\right] \mathcal{I}_{240},$$

where ϵ_1 is the CP asymmetry in the decay of the lightest RH neutrino, $\mathcal{R}_i\left[I_{\rm even}\right]$ are rational functions of the CP-even basic invariants in the full seesaw model in TABLE II, $\mathcal{I}_{121}^{(2)}$ and $\mathcal{I}_{240}$ are two CP-odd invariants in the SEFT in TABLE I. In addition, it can be shown that

$$\mathcal{A}_{\nu\nu}=\mathcal{F}_{\nu\nu} \mathcal{I}_{121}^{(2)}, \quad \mathcal{A}_{\nu\bar{\nu}}=\mathcal{F}_{\nu\bar{\nu}} \mathcal{I}_{240},$$

where $\mathcal{A}_{\nu\nu}$ and $\mathcal{A}_{\nu\bar{\nu}}$ are CP asymmetries in neutrino-neutrino and neutrino-antineutrino oscillations, respectively, while $\mathcal{F}_{\nu\nu}$ and $\mathcal{F}_{\nu\bar{\nu}}$ are functions of CP-even primary invariants in TABLE I.
If CP violation is absent in both $\nu\nu$ and $\nu\bar{\nu}$ oscillations, then the CP asymmetries in RH neutrino decays also vanish. Conversely, if CP violation is measured at low energies either in $\nu\nu$ or in $\nu\bar{\nu}$ oscillations, then CP asymmetries may exist in the leptogenesis, i.e.,

$$\boxed{\begin{array}{l}
\text{CP conservation at low energies} \Rightarrow \mathcal{I}_{121}^{(2)}=\mathcal{I}_{240}=0 \Rightarrow \text{no CP asymmetries in leptogenesis} \\
\text{CPV in oscillation experiments} \Rightarrow \mathcal{I}_{121}^{(2)} \text{ or } \mathcal{I}_{240} \neq 0 \Rightarrow \text{CPV may exist in leptogenesis}
\end{array}}$$

Summary

- Flavor invariants are free of basis and more natural to be used to describe physical observables.
- The SEFT with only one d=5 and one d=6 operator is already adequate to incorporate all physical information about the full seesaw model.
- Invariant theory provides a basis- and parametrization-independent way to connect observables at low- and high-energies. Precise measurements in the low-energy experiments about C_5 and C_6 can help explore the full seesaw model and determine high-energy observables such as the CP asymmetries in the decays of RH neutrinos.

Reference

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