From quantum to classical theories:

the origin of the ODE/IM correspondence

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Outline

We start from a quantum integrable field theory (e.g. sine-Gordon), characterised by functional relations (e.g. Baxter's *TQ*-system) between operators

We associate to this quantum (integrable) model a classical model: PDEs (Lax pair), ODEs. Tool: Marchenko equation Marchenko '55

Opposite arrow with respect to ODE/IM Dorey,Tateo; Bazhanov, Lukyanov, Zamolodchikov '98;Gaiotto-Moore-Neitzke '08,'09; Lukyanov, Zamolodchikov '10 in which:

One starts from ODEs $-\frac{d^2}{dx^2}\psi(x)+\left(\frac{l(l+1)}{x^2}+x^{2M}\right)\psi(x)=E\psi(x)$: connection coefficients between different pairs of solutions are eigenvalues of Q-operators of CFT minimal models

Generalisation: from PDEs $(\partial_w+V)\Psi=(\partial_{\bar{w}}+\bar{V})\Psi=0$: connection coefficients between different pairs of solutions are eigenvalues of Q-operators of sine-Gordon model.

Functional relations

Define a quantum integrable model: example of sine-Gordon on a cylinder

$$\mathcal{L} = \frac{1}{16\pi}\left[(\partial_t\varphi)^2 - (\partial_x\varphi)^2\right] + 2\mu\cos\beta\varphi\,, \quad \varphi(x+R,t) = \varphi(x,t)$$

Different k-vacua: $\varphi \to \varphi + 2\pi/\beta \Rightarrow |\Psi_k\rangle \to e^{2\pi i k}|\Psi_k\rangle$. Infinite number of conserved charges $I_n, \bar{I}_n$. They appear in asymptotic expansion at $\theta \to \pm\infty$ of $\hat{Q}_\pm(\theta)$ ($\pm$ sign of k). Vacuum eigenvalue of $\hat{Q}_\pm = Q_\pm$. Properties of $Q_\pm$.

- Entire quasi-periodic functions: $Q_\pm(\theta + i\tau) = e^{\pm i\pi\left(l+\frac{1}{2}\right)}Q_\pm(\theta)$, $l = 2|k| - 1/2$, quasi-period $\tau = \pi/(1-\beta^2)$
- TQ-system

$$T(\theta)Q_\pm(\theta) = e^{\mp i\pi\left(l+\frac{1}{2}\right)}Q_\pm(\theta + i\pi) + e^{\pm i\pi\left(l+\frac{1}{2}\right)}Q_\pm(\theta - i\pi)$$

  where

$$T(\theta) = \frac{i}{2\cos\pi l}\left[e^{-2i\pi l}Q_+(\theta+i\pi)Q_-(\theta-i\pi) - e^{2i\pi l}Q_+(\theta-i\pi)Q_-(\theta+i\pi)\right]$$

- Asymptotic behaviour $\ln Q_\pm(\theta) \simeq -w_0 e^{\theta} - \bar{w}_0 e^{-\theta}\,, \quad w_0 = -\frac{MR}{4\cos\frac{\pi\beta^2}{2(1-\beta^2)}}$
- Extensions: Homogeneous sine-Gordon model (many masses)

From functional to integral equations

- Encode the properties of Q in an integral equation

$$Q_\pm(\theta + i\tau/2) = q_\pm(\theta) \pm \int_{-\infty}^{+\infty} \frac{d\theta'}{4\pi} \tanh\frac{\theta-\theta'}{2} T\left(\theta' + i\frac{\tau}{2}\right) e^{-w_0(e^\theta + e^{\theta'}) - \bar{w}_0(e^{-\theta}+e^{-\theta'})} \cdot$$

$$\cdot e^{\pm(\theta-\theta')l} Q_\pm\left(\theta' + i\frac{\tau}{2}\right), \quad q_\pm(\theta) = C e^{\pm\frac{i\pi}{4} \pm \left(\theta + \frac{i\pi}{2}\right)l} e^{-w_0 e^\theta - \bar{w}_0 e^{-\theta}}$$

- The TQ-system holds due to the property (of the kernel):

$$\lim_{\epsilon\to 0^+}\left[\tanh\left(x + \frac{i\pi}{2} - i\epsilon\right) - \tanh\left(x - \frac{i\pi}{2} + i\epsilon\right)\right] = 2\pi i \delta(x), \quad x \in \mathbb{R}.$$

- Define the functions $X_\pm(\theta)$: $q_\pm(\theta)X_\pm(\theta) = Q_\pm(\theta + i\tau/2)$
- Make $w_0, \bar{w}_0$ dynamical: $w_0 \to -iw'$, $\bar{w}_0 \to i\bar{w}'$
- Integral equation satisfied by $X_\pm(w', \bar{w}'|\theta)$, $\lambda = e^\theta$:

$$X_\pm(w', \bar{w}'|\theta) = 1 \pm \int_0^{+\infty} \frac{d\lambda'}{4\pi\lambda'} \frac{\lambda - \lambda'}{\lambda + \lambda'} T(\lambda' e^{\frac{i\tau}{2}}) e^{-2iw'\lambda' + 2i\frac{\bar{w}'}{\lambda'}} X_\pm(w', \bar{w}'|\theta')$$

Getting a Marchenko equation

- Let us define the Fourier of $X_{\pm}-1$

$$K_{\pm}(w',\xi;\bar{w}')=\int_{-\infty-i\epsilon}^{+\infty-i\epsilon}d\lambda e^{i(\xi-w')\lambda}[X_{\pm}(w',\bar{w}'|\theta)-1]\,.$$

- Let us take the Fourier transform of the integral equation for $X_{\pm}$. We get

$$K_{\pm}(w',\xi;\bar{w}')\pm F(w'+\xi;\bar{w}')\pm\int_{w'}^{+\infty}\frac{d\xi'}{2\pi}K_{\pm}(w',\xi';\bar{w}')F(\xi'+\xi;\bar{w}')=0\,,\quad \xi>w'\,,$$

  with $F(x;\bar{w}')=i\int_0^{+\infty}d\lambda' e^{-ix\lambda'+2i\frac{\bar{w}'}{\lambda'}}T(\lambda' e^{i\frac{\tau}{2}})$.
- This has the structure of a Marchenko equation (quantum inverse scattering: from S-matrix and bound states to Schroedinger equation)
- For the ODE (Schroedinger) we are going to construct such data are encoded in T, vacuum eigenvalue of the transfer matrix a quantum integrable model.

From Marchenko to Schroedinger

Standard procedure:

- Define the wave function $\psi_\pm(w',\bar{w}'|\theta)=e^{-iw'\lambda+i\frac{\bar{w}'}{\lambda}}X_\pm(w',\bar{w}'|\theta)$,

$$X_\pm(w',\bar{w}'|\theta)-1=\int_{w'}^{+\infty}\frac{d\xi}{2\pi}e^{-i(\xi-w')\lambda}K_\pm(w',\xi;\bar{w}')\,,\quad \lambda=e^\theta$$

- Differentiate (twice) and use Marchenko equation: we get

$$\frac{\partial^2}{\partial w'^2}\psi_\pm(w',\bar{w}'|\theta)+e^{2\theta}\psi_\pm(w',\bar{w}'|\theta)=u_\pm(w';\bar{w}')\psi_\pm(w',\bar{w}'|\theta)\,,$$

  with potentials

$$u_\pm(w';\bar{w}')=-2\frac{d}{dw'}\frac{K_\pm(w',w';\bar{w}')}{2\pi}\,,$$

- Explicit solution of Marchenko equation gives access to the potential and the (Jost) wave function. The potential is

$$u_\pm(w';\bar{w}')=\mp\partial_{w'^2}\eta+(\partial_{w'}\eta)^2\,,\quad \eta=\ln\det(1+\hat{V})-\ln\det(1-\hat{V})$$

$$V(\theta,\theta')=\frac{T\left(\theta+i\frac{\tau}{2}\right)}{4\pi}\frac{e^{-2iw'e^\theta+2i\bar{w}'e^{-\theta}}}{\cosh\frac{\theta-\theta'}{2}}$$

Wave function and first Lax

- The wave function is $\psi_\pm(w',\bar{w}'|\theta) = X_\pm(w',\bar{w}'|\theta) e^{-iw'\lambda + i\frac{\bar{w}'}{\lambda}}$,

$$X_\pm(w',\bar{w}'|\theta) = -2 \mp \int \frac{d\theta'}{4\pi} e^{\frac{\theta-\theta'}{2}} V(\theta,\theta') X_\pm(w',\bar{w}'|\theta')$$

and satisfies the '$T\psi$-system' (extension of TQ-system)

$$T\left(\theta + i\frac{\tau}{2}\right)\psi_\pm(w',\bar{w}'|\theta) = \mp i\psi_\pm(w',\bar{w}'|\theta+i\pi) \pm i\psi_\pm(w',\bar{w}'|\theta-i\pi)\,.$$

- To summarise, we have obtained two Schroedinger equations

$$\frac{\partial^2}{\partial w'^2}\psi_\pm(w',\bar{w}'|\theta) + e^{2\theta}\psi_\pm(w',\bar{w}'|\theta) = u_\pm(w';\bar{w}')\psi_\pm(w',\bar{w}'|\theta)\,,$$

- Introduce $D_\eta = \partial_w + \frac{1}{2}\partial_w\eta\,\sigma^3 - e^{\theta+\eta}\sigma^+ - e^{\theta-\eta}\sigma^-$.

$$\mathbf{D} = \begin{pmatrix} D_\eta & 0 \\ 0 & D_{-\eta} \end{pmatrix}\,, \quad \Psi = \begin{pmatrix} e^{\frac{\theta+\eta}{2}}\psi_+ \\ e^{-\frac{\theta+\eta}{2}}(\partial_w + \partial_w\eta)\psi_+ \\ e^{\frac{\theta-\eta}{2}}\psi_- \\ e^{-\frac{\theta-\eta}{2}}(\partial_w - \partial_w\eta)\psi_- \end{pmatrix}$$

- The first order matrix equation $\mathbf{D}\Psi = 0$ is equivalent to Schroedinger equations in w'.

Second Lax

- Introduce the 'conjugate' differential equation

$$\frac{\partial^2}{\partial \bar{w}'^2}\psi_\pm^{bar}(w',\bar{w}'|\theta) + e^{-2\theta}\psi_\pm^{bar}(w',\bar{w}'|\theta) = \bar{u}_\pm(w',\bar{w}')\psi_\pm^{bar}(w',\bar{w}'|\theta)\,.$$

- Introduce $\bar{D}_\eta = \partial_{\bar{w}} - \frac{1}{2}\partial_{\bar{w}}\eta\,\sigma^3 - e^{-\theta+\eta}\sigma^- - e^{-\theta-\eta}\sigma^+$ and

$$\bar{\mathbf{D}} = \begin{pmatrix} \bar{D}_\eta & 0 \\ 0 & \bar{D}_{-\eta} \end{pmatrix}\,, \quad \Psi^{bar} = \begin{pmatrix} e^{\frac{\theta-\eta}{2}}(\partial_{\bar{w}} + \partial_{\bar{w}}\eta)\psi_+^{bar} \\ e^{-\frac{\theta-\eta}{2}}\psi_+^{bar} \\ e^{\frac{\theta+\eta}{2}}(\partial_{\bar{w}} - \partial_{\bar{w}}\eta)\psi_-^{bar} \\ e^{-\frac{\theta+\eta}{2}}\psi_-^{bar} \end{pmatrix}$$

- The first order matrix equation $\bar{\mathbf{D}}\Psi^{bar} = 0$ is equivalent to Schroedinger equations in $\bar{w}'$.
- Jost solutions are the functions $\psi_\pm^{bar}(w',\bar{w}'|\theta) = \bar{\psi}_\pm(w',\bar{w}'|-\bar{\theta})$.
- Property: $\psi_\pm^{bar}(w',\bar{w}'|\theta) = \psi_\mp(w',\bar{w}'|\theta)e^{\mp\eta(w,\bar{w})}$. It follows that $\Psi = -e^{\theta}\Psi^{bar}$. From $[\mathbf{D},\bar{\mathbf{D}}]\Psi = 0$ we get

$$\partial_w\partial_{\bar{w}}\eta = 2\sinh 2\eta\,,$$

i.e. that η satisfies the classical sinh-Gordon equation.

Conformal limit

- Potentials $u_{\pm}(w', \bar{w}')$ of Schroedinger equations are complicated functions (Fredholm determinants)
- Simplifications occur in the conformal limit, when masses $(w_0) \to 0$, $\bar{w}' \to 0$ and w' scales as

$$\frac{dw'}{dx} = \sqrt{p(x)} e^{-\theta} \quad \theta \to +\infty$$

with p a polynomial (θ 'rapidity').

- Then, the new wave function $\psi^{cft}(x) = \psi_+(w')p(x)^{-\frac{1}{4}}$ satisfies the ODE

$$-\frac{d^2}{dx^2}\psi^{cft}(x) + \left(p(x) + \frac{l(l+1)}{x^2}\right)\psi^{cft}(x) = 0$$

which agrees and generalises ODEs considered by Dorey and Tateo and Bazhanov, Lukyanov, Zamolodchikov.

Special case

- Particular case: $\beta^2 = 2/3, l = 0$ which imply $T = 1$
- Now $\eta = \ln\det(1+\hat{V}) - \ln\det(1-\hat{V})$, with

$$V(\theta,\theta') = \frac{e^{-2iw' e^{\theta} + 2i\bar{w}' e^{-\theta}}}{4\pi\cosh\frac{\theta-\theta'}{2}}$$

- The field η depends only on $t = 4\sqrt{w'\bar{w}'}$, $w' = t/4e^{i\varphi}$ and the sinh-Gordon equation reduces to the Painlevè III_3 equation:

$$\frac{1}{t}\frac{d}{dt}\left(t\frac{d}{dt}\eta(t)\right) = \frac{1}{2}\sinh 2\eta(t)$$

- The wave functions $\psi_{\pm}(t|\theta+i\varphi)$ depend on $t, \theta+i\varphi$ only. They satisfy differential equations in t and θ. This means that $Q_{\pm}(\theta) = \psi_{\pm}(t = 4w_0|\theta)$ satisfy also differential equations (in θ).

Summary and Perspectives

- We have given an explanation for the occurrence of the ODE/IM correspondence. The idea is that functional relations (*TQ*-system) between eigenvalues of operators $\hat{T}$, $\hat{Q}$ of a quantum integrable model are equivalent to a Marchenko equation. From a Marchenko equation one gets Schroedinger equations by a standard procedure.
- We have proved this in the case of vacuum eigenvalues of $\hat{T}$, $\hat{Q}$ for (Homogeneous) sine-Gordon model. Since the *TQ*-system holds for operators, construction of (the at present unknown) Schroedinger equations corresponding to excited states eigenvalues of $\hat{T}$, $\hat{Q}$ is feasible
- More in general: *TQ*-system is ubiquitous in quantum integrable models, since it is equivalent to Bethe Ansatz. In principle, we can derive Schroedinger equations corresponding to generic states of a generic quantum model (e.g. spin chain).