Colour-kinematics duality, double copy, and homotopy algebras

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Flash review of CK duality and double copy

- n-points L-loops YM amplitude as sums of trivalent graphs

$$\mathcal{A}_{n,L}^{\mathrm{YM}} = \sum_i \int \prod_{l=1}^{L} \mathrm{d}^d p_l \frac{1}{S_i} \frac{C_i N_i}{D_i}$$

- i ranges over all trivalent L-loops graphs
- C_i: colour factor, composed of gauge group structure constants
- N_i: kinematic factor, composed of Lorentz-invariant contractions of polarisations and momenta

Flash review of CK duality and double copy

- Generalised gauge transformation

$$N_i \;\mapsto\; N_i + \Delta_i, \qquad \sum_i \int \prod_{l=1}^{L} \mathrm{d}^d p_l \frac{1}{S_i} \frac{C_i \Delta_i}{D_i} \;=\; 0$$

Bern–Carrasco–Johansson colour–kinematics duality (2008)

There is a choice of kinematic factors such that N_is obey the same algebraic relations (e.g., Jacobi identity) of the corresponding C_i

- True at tree-level, conjectured for loop-level
- If true, it would allow us to compute gravity amplitudes from YM ones

Flash review of CK duality and double copy

Yang–Mills double copy

If CK duality holds true, replacing the colour factor with a copy of the kinematic factor in $\mathcal{A}^{\mathrm{YM}}_{n,L}$ produces a $\mathcal{N}=0$ supergravity amplitude

$$\mathcal{A}^{\mathrm{YM}}_{n,L} = \sum_i \int \prod_{l=1}^{L} \mathrm{d}^d p_l \frac{1}{S_i} \frac{C_i N_i}{D_i} \rightarrow \mathcal{A}^{\mathcal{N}=0}_{n,L} = \sum_i \int \prod_{l=1}^{L} \mathrm{d}^d p_l \frac{1}{S_i} \frac{\tilde{N}_i N_i}{D_i}$$

- All-loop statement, the problem is then to validate CK duality at loop level
- Until now, on-shell scattering amplitude approach
- Homotopy algebras provide a natural setting for colour–kinematics factorisation and off-shell, Lagrangian double copy

Homotopy algebras

- Informally, homotopy algebras are generalisations of classical algebras (e.g., associative, Lie) where the respective structural identities (e.g., associativity, Jacobi identity) hold up to homotopies

Classical algebra	Homotopy algebra
Associative algebra	A_∞-algebra
Associative commutative algebra	C_∞-algebra
Lie algebra	L_∞-algebra

- Homotopy structures are ubiquitous in Physics: while homotopy algebras emerged in the context of string field theory, they were later recognized as underlying structures of every Lagrangian field theory

BV formalism

- Batalin–Vilkovisky (BV) formalism is the bridge between (quantum) field theories and homotopy algebras
- To quantize a classical theory means to make sense of the path integral

$$\int_{\mathfrak{F}} \mu_{\mathfrak{F}}(\Phi)\ e^{\frac{i}{\hbar}S[\Phi]}$$

- Standard approach: BRST formalism
- If the symmetries close off-shell, then BRST formalism is enough for quantization

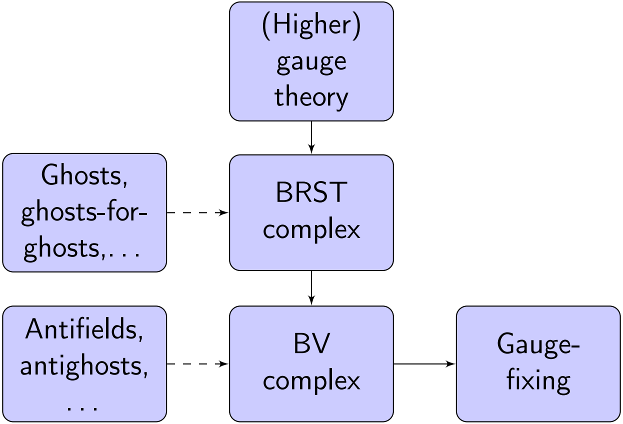
BV formalism

- In the case of open symmetries, BRST complex is a complex only up to e.o.m.
- The BV quantisation is a sophisticated machinery, that allows us to gauge-fix and quantize these complicated field theories
- We extend the BRST complex to a BV complex, doubling the field content of the theory

$$\mathfrak{F}_{\mathrm{BV}} \;=\; T^*[1]\mathfrak{F}_{\mathrm{BRST}}$$

- Fields Φ^A are local coordinates on $\mathfrak{F}_{\mathrm{BRST}}$, antifields Φ^+_A are fibre coordinates. As a cotangent bundle, $\mathfrak{F}_{\mathrm{BV}}$ comes with a natural symplectic structure and Poisson brackets $\{-,-\}$

BV formalism



BV formalism and L_∞-algebras

- The BV complex is dually described by an L_∞-algebra
- An L_∞-algebra is a differential graded vector space equipped with multibrackets (*higher products*), that satisfy a generalization of Jacobi identity
- Underlying every Lagrangian field theory is an L_∞-algebra that encodes the whole classical theory (symmetries, fields, equations of motion, Noether identities...)
- Every L_∞-algebra is equivalent (*quasi-isomorphic*) to a differential graded Lie algebra (*strictification theorem*)
- Analogous results for A_∞- and C_∞-algebras

BV$^{\blacksquare}$-algebras and CK duality

- Consider a differential graded Lie algebra $\mathfrak{L}^{\mathrm{st}}$ associated to Yang–Mills theory
- Factorize into the gauge algebra $\mathfrak{g}$ and a differential graded commutative algebra $\mathfrak{C}^{\mathrm{st}}$

$$\mathfrak{L}^{\mathrm{st}} = \mathfrak{g} \otimes \mathfrak{C}^{\mathrm{st}}$$

- $\mathfrak{C}^{\mathrm{st}}$ contains binary bracket m_2 of degree 0

$$m_2 : \text{field} \times \text{field} \to \text{antifield}$$

- Kinematic algebra: we want to define a new bracket of degree -1: extend $\mathfrak{C}^{\mathrm{st}}$ to a BV$^{\blacksquare}$-algebra

BV$^\blacksquare$-algebras and CK duality

A BV$^\blacksquare$-algebra is a graded vector space $\mathfrak{B}$, equipped with

- Hodge triple $(d, h, \blacksquare)$

$$\text{fields} \underset{h}{\overset{d}{\rightleftarrows}} \text{antifields} \qquad dh + hd = \blacksquare$$

- graded commutative product m_2 of degree 0
- a new product $\{-,-\}$ of degree -1

$$\{v_1, v_2\} = hm_2(v_1, v_2) - m_2(h(v_1), v_2) - (-1)^{|v_1|} m_2(v_1, h(v_2))$$

- and more technical stuff

BV$^{\blacksquare}$-algebras and CK duality

A strictified gauge field theory with differential graded Lie algebra $\mathfrak{L}^{\mathrm{st}}$ is manifestly CK-dual if and only if there is a factorisation $\mathfrak{L}^{\mathrm{st}} = \mathfrak{g} \otimes \mathfrak{C}^{\mathrm{st}}$ such that $\mathfrak{C}^{\mathrm{st}}$ can be enhanced to a BV$^{\blacksquare}$-algebra.

BV$^{\blacksquare}$-algebras and double copy

YM theory: $\mathfrak{L}^{\text{st}} = \mathfrak{g} \otimes \mathfrak{B}$. Naively: we want to replace $\mathfrak{g}$ with a copy of $\mathfrak{B}$

- In the factorization $\mathfrak{g} \otimes \mathfrak{C}$, $\mathfrak{g}$ can be thought as a BV$^{\blacksquare}$-algebra
- There is a notion of tensor product between two BV$^{\blacksquare}$-algebras

$$\mathfrak{L} \ \subset \ \mathfrak{B}_1 \otimes \mathfrak{B}_2$$

- Inspired by string theory: $\mathcal{H}_{\text{closed}} = \mathcal{H}_{\text{open}} \otimes \mathcal{H}_{\text{open}}$
- Section condition: states $|p, \ldots\rangle \otimes |p, \ldots\rangle$
- Level matching: $(b_0 - \tilde{b}_0)\,|\psi\rangle = 0$, $(L_0 - \tilde{L}_0)\,|\psi\rangle = 0$

$\mathrm{BV}^{\blacksquare}$-algebras and double copy

- String theory: Hodge triple $d = Q_{\mathrm{BRST}}$, $h = b_0$, $\blacksquare = L_0$
- Field theory: Hodge triple $d = m_1$, $h = [1]$, $\blacksquare = \square$
- Tensor product $\mathfrak{B}_1 \otimes \mathfrak{B}_2$: implementation of section condition and level matching
- A strictified field theory with differential graded Lie algebra $\mathfrak{L}^{\mathrm{st}}$ is manifestly CK-dual if and only if it factorises into a tensor product of $\mathrm{BV}^{\blacksquare}$-algebras

$\mathrm{BV}^{\blacksquare}$-algebras and double copy

Given two $\mathrm{BV}^{\blacksquare}$-algebras, tensor product combines them into a differential graded algebra: *syngamy*

$$
\begin{tikzcd}
 & \mathfrak{B}_1 \otimes \mathfrak{B}_3 & \\
 & \mathfrak{B}_1 \otimes \mathfrak{B}_4 & \mathfrak{B}_3 \otimes \mathfrak{B}_4 \arrow[ul] \arrow[l] \arrow[dl] \arrow[ddl] \\
\mathfrak{B}_1 \otimes \mathfrak{B}_2 \arrow[uur] \arrow[ur] \arrow[r] \arrow[dr] & \mathfrak{B}_2 \otimes \mathfrak{B}_3 & \\
 & \mathfrak{B}_2 \otimes \mathfrak{B}_4 &
\end{tikzcd}
$$

Outlook

- Mathematical formulation of kinematic algebra, CK duality and double copy in terms of homotopy algebras
- Lagrangian incarnation of CK duality and double copy
- Generalisation to the non-strict case: $\mathrm{BV}^{\blacksquare}_{\infty}$-algebras
- Toolbox to build and study a growing zoo of double copy theories
- Computational efficiency?
- Link to string theory

Thank you for listening!