Counting instantons at strong coupling

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Introduction

This talk is based on joint work with **I.Coman** and **J.Teschner**.

The main subject is the study of nonperturbative phenomena in 4d $\mathcal{N}=2$ QFT beyond the weak-coupling regime.

In this talk I will discuss a new approach to explore strongly-coupled dynamics, where standard techniques such as localization cannot be applied.

4d $\mathcal{N} = 2$ Yang-Mills

The $\mathcal{N} = 2$ Yang-Mills Lagrangian is ($\tau = \theta/2\pi + 4\pi i/g^2$ and $G = SU(2)$)

$$\begin{aligned}
\mathcal{L} &= \frac{1}{8\pi}\mathrm{Im}\Big(\int d^2\theta\, \tau\, W^\alpha W_\alpha + \int d^2\theta d^2\bar{\theta}\ 2\tau\, \Phi^\dagger e^{-2V} \Phi \Big) \\
&= \frac{1}{g^2}\mathrm{Tr}\Big(-\frac{1}{4} F_{\mu\nu}F^{\mu\nu} + g^2 \frac{\theta}{32\pi^2} F_{\mu\nu}\tilde{F}^{\mu\nu} + (D_\mu\phi)^\dagger (D^\mu\phi) - \frac{1}{2}[\phi^\dagger, \phi]^2 \\
&\qquad - i\ \lambda\sigma^\mu D_\mu\bar{\lambda} - i\ \bar{\psi}\bar{\sigma}^\mu D_\mu\psi - i\sqrt{2}[\lambda, \psi]\phi^\dagger - i\sqrt{2}[\bar{\lambda}, \bar{\psi}]\phi \Big)
\end{aligned}$$

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Coulomb branch: This theory has a moduli space of vacua $\mathcal{B}$, parameterized by $\phi \sim a\,\sigma_3$, where $SU(2)$ is spontaneously broken to $U(1)$.

The effective Lagrangian is governed by a 'prepotential' function $\mathcal{F}$

$$\mathcal{L} = \frac{1}{8\pi}\mathrm{Im}\Big(\int d^2\theta\, \mathcal{F}''(\Phi)\, W^{\alpha} W_{\alpha} + 2 \int d^2\theta d^2\bar{\theta}\, \mathcal{F}'(\Phi)\Phi^{\dagger} \Big)$$

where $\mathcal{F} = \mathcal{F}_{\mathsf{pert.}} + \mathcal{F}_{\mathsf{instanton}}$

$$\mathcal{F}_{\mathsf{pert.}} = \frac{i}{2\pi} a^2 \ln \frac{a^2}{\Lambda^2}$$
$$\mathcal{F}_{\mathsf{instanton}} = \sum_{k=1}^{\infty} \mathcal{F}_k \left(\frac{\Lambda}{a}\right)^{4k} a^2$$

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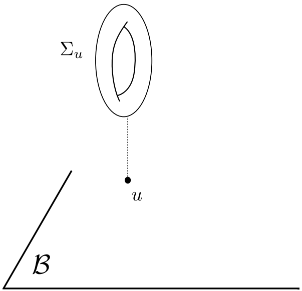
Instanton terms $\mathcal{F}_k$

- Not determined by symmetry alone
- Cannot be computed by perturbation theory

The Seiberg-Witten prepotential

A conjectural computation of $\mathcal{F}_k$ in terms of **periods** of a Riemann surface Σ [Seiberg Witten]



$$\gamma = (e, m) \in H_1(\Sigma_u, \mathbb{Z}) \qquad Z_\gamma(u) = \frac{1}{\pi} \oint_\gamma \lambda(u)$$

$$a := \frac{1}{\pi} \oint_\alpha \lambda = Z_e \qquad a_D := \frac{1}{\pi} \oint_\beta \lambda = Z_m$$

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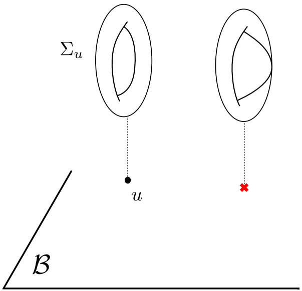

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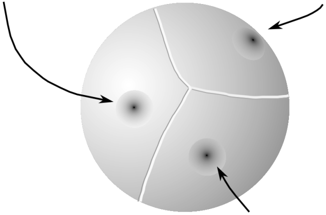
A key feature of this solution, is the presence of **singularities**

new massless d.o.f. $\leftrightarrow$ different low energy descriptions

The Seiberg-Witten solution has 2 different types of singularity on $\mathcal{B}$: [Figure from Lerche 9611190]

$$\tilde{\mathcal{F}}_D = \mathcal{F}_D(a - 2a_D)$$

$$\mathcal{F}(a) = \frac{ia^2}{\pi} \log\left[\frac{a^2}{\Lambda^2}\right] + \frac{a^2}{2\pi i} \sum_{\ell=1}^{\infty} c_\ell \left(\frac{\Lambda}{a}\right)^{4\ell}$$



$$\mathcal{F}_D(a_D) = \frac{{a_D}^2}{4\pi i} \log\left[\frac{a_D}{\Lambda}\right] - \frac{\Lambda^2}{2\pi i} \sum_{\ell=1}^{\infty} c_\ell^D \left(\frac{i a_D}{\Lambda}\right)^{\ell}$$

- A **weak-coupling** singularity, where $M_{W_\pm} = 0 \Rightarrow$ SU(2) Yang-Mills
- A **strong-coupling** singularity, where $M_{\text{monopole}} = 0 \Rightarrow U(1)_{\text{magnetic}}$ QED

Instanton counting

At **weak-coupling**, the Seiberg-Witten solution can be verified by **direct computation** of instanton contributions via localization in the Omega-background. [Nekrasov]

$$Z_{\textsf{inst}}(a,\hbar;q) = \sum_{Y_1,Y_2} q^{|Y_1|+|Y_2|} \prod_{i,j} \frac{a+\hbar(Y_{1,i}-Y_{2,j}+j-i)}{a+\hbar(j-i)}$$

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- Leading order in $\hbar$ recovers SW prepotential: $\lim_{\hbar\to 0} \ln Z_{\mathsf{inst}}(a,\hbar;q) = \frac{1}{\hbar^2}\mathcal{F}_{\mathsf{inst}}(a,\Lambda)$
- Z_{inst} contains much more: higher $\hbar$-orders $\sim$ gravitational couplings.
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Question: What is the analogue of Z_{inst} at **strong coupling**?

Geometrization of instanton partition functions

Main result: a geometric definition of Z_{inst} that recovers the weak coupling result from localization, but also extends to strong coupling where it agrees with $\mathcal{F}_{D,\text{inst}}$ as $\hbar \to 0$.

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Quantum curves:

$$\text{Seiberg-Witten curve } \Sigma \qquad \rightsquigarrow \qquad \overbrace{\underbrace{\left[\hbar^2 \partial_x^2 - q_\nu(x,\hbar)\right]}_{\mathcal{D}_\nu}}^{\text{differential operator}} \psi(x) = 0$$

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classical periods	$\rightsquigarrow$	quantum periods

Isomonodromic tau function

Quantum periods μ are transcendental functions of moduli ν of the quantum curve $\mathcal{D}_\nu$.

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But they are overparameterized: $\nu \sim (\mu, t)$. There are **isomonodromic deformations**

$$\mu(\nu_0) = \mu(\nu_1)$$

if ν_0, ν_1 are related by a non-autonomous Hamiltonian flow $\nu(t)$ defined by

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The **tau function** is defined, up to rescaling by monodromy data $\tau \to f(\mu) \cdot \tau$, by

$$\partial_t \log \tau(\mu, t) = H(\mu, t)$$

Tau function and instantons

The tau function **encodes** instanton partition functions. [Gamayun Iorgov Lisovyy]

$$\tau \begin{array}{l} \nearrow \\ \searrow \end{array} \begin{array}{ll} \tau^{(w)}(\sigma,\eta;\Lambda) = \sum_{n\in\mathbb{Z}} e^{4\pi\mathrm{i}\, n\,\eta}\, Z^{(w)}_{\text{pert}}(\sigma+n)\, Z^{(w)}_{\text{inst}}(\sigma+n,\Lambda) & \text{weak } (\Lambda\to 0) \\ \tau^{(s)}(\nu,\rho;\Lambda) = \sum_{n\in\mathbb{Z}} e^{4\pi\mathrm{i}\rho n}\, Z^{(s)}_{\text{pert}}(\nu+\mathrm{i}n,\Lambda)\, Z^{(s)}_{\text{inst}}(\nu+\mathrm{i}n,\Lambda) & \text{strong } (\Lambda\to\infty) \end{array}$$

Adopting specific **coordinates for monodromy data**, $\mu=(\sigma,\eta)$ gives a Fourier-expansion that recovers the weak-coupling Z_{inst} [Gavrylenko Lisovyy]

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Remarkably, there exists another set of coordinates (ν, ρ), that gives a different kind of Fourier-expansion valid at **strong coupling** [Its Lisovyy Tykhyy]

Turned around, this can be taken as a **definition** of Z_{inst} at strong coupling!

[Bonelli Lisovyy Maruyoshi Sciarappa Tanzini]

Instanton counting from tau functions

Our goals:

1. Explain this observation.

2. Turn it into a systematic definition of instanton counting in strong-coupling regimes, valid for all theories of class $S[A_1]$.

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Key questions:

- What defines the distinguished coordinates (σ, η), (ν, ρ)?
- How are $\tau^{(w)}$ and $\tau^{(s)}$ related exactly?
- How does this example generalize to other 4d $\mathcal{N} = 2$ QFTs?

The general picture

We outline a prescription to quantize Seiberg-Witten curves of general theories of class $S[A_1]$.

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Coordinates and charts: [Gaiotto Moore Neitzke] [Hollands Neitzke]

- Patches in $\mathcal{B} \times \mathbb{C}^*_\hbar$ defined by WKB Stokes graphs for $\mathcal{D}_\nu$.
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Normalization of τ in each chart: [Coman PL Teschner]

- Charts define a **line bundle**, transition functions are defined by coordinates
  $\chi(x'+1, x) = e^{-2\pi i y'}\chi(x', x), \qquad \chi(x', x+1) = e^{2\pi i y}\chi(x', x)\,.$
- τ is a **section**, which **renormalizes** from **weak to strong** coupling

$$\tau'(x', y'; \Lambda) = \chi(x', x)\tau(x, y; \Lambda)\,.$$

Results

Our construction defines, for any class $\mathcal{S}[A_1]$ theory:

1. **Charts** $\mathcal{U}_r \subset \mathcal{B} \times \mathbb{C}^*$ covering the entire moduli space/dynamical range of the theory.

2. Distinguished **coordinates** (x_r, y_r), together with relations among charts.

3. A **line bundle** over $\mathcal{B} \times \mathbb{C}^*_\hbar$, with given transition functions.

4. A distinguished section τ, with appropriate **renormalization** across charts

$$\tau^{(r')} = \chi_{r',r}(\mu) \cdot \tau^{(r)}$$

5. A **chart-wise Fourier decompositions** induced by distinguished coordinates

$$\tau^{(r)}(x_r, y_r, \Lambda) \quad \rightarrow \quad \boxed{Z^{(r)}_{\text{inst}}(x_r, \Lambda)}.$$

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In conclusion, this is our definition of Z_{inst} at strong coupling, and beyond.

It recovers results of [Gamayun Iorgov Lisovyy] [Gavrylenko Lisovyy] [Its Lisovyy Tykhyy] [Coman Pomoni Teschner] [...].
But it also extends to all other patches of moduli space, for the entire class $S[A_1]$ theories.

Thank You.