Integrability as a new method for exact results on quasinormal modes of black holes

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Based on arXiv:1908.08030, arXiv:2112.11434 with Davide Fioravanti
and arXiv:22**.***** with also Hongfei Shu

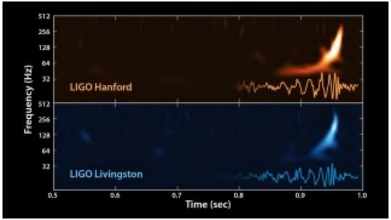
Outline

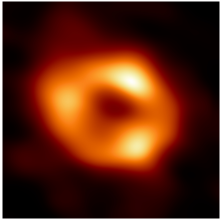
Introduction

Gravitational Phenomenology

- In the last few years, **gravitational waves** detections and **black hole imaging** have opened the doors of **gravitational phenomenology**. [Mayerson:2020]

- Finally, we can fully scientifically investigate whether real astrophysical black holes show **deviations from general relativity (GR)**, such as **horizon scale structure**.



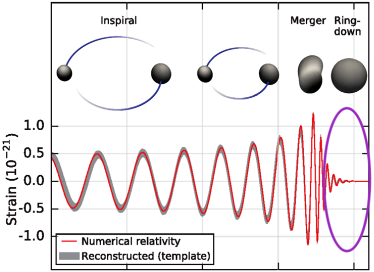


Could BHs hold the same surprises that the electron and the hydrogen atom did when they started to be experimentally probed?

[Cardoso,Pani:2017]

Colliding BHs and quasinormal modes

- A **black hole collision** can be divided in 3 phases: **inspiral, merger** and **ringdown**.



- The **quasinormal modes (QNMs)** are responsible for the **damped oscillations** appearing, for example, in the **ringdown phase** of two colliding BH and have a direct connection to **gravitational waves observations**.

Alternative models of BHs

- GR BHs present fundamental theoretical problems (e.g. information paradox).
- Also to solve such problems, theoretical models of **Exotic Compact Objects (ECOs)** in alternative theories of gravity have been developed. They have **horizon scale structure**.
- For subtype of ECOs, called **Clean Photosphere Objects (ClePhOs)**, the later stage ringdown signal shows a peculiar train of **echoes**, with **significant deviations from GR**.
- An example of ClePhoS are **fuzzballs** in **String Theory**, with neither horizon nor central singularity and which may solve also the information paradox.

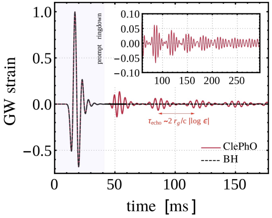


Figure: [Cardoso,Pani:2017]

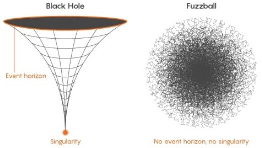


Figure: Cr. Quanta Magazine

From ($\mathcal{N} = 2$) gauge to gravity and back

Classical Seiberg-Witten curve and gauge periods

- The **classical Seiberg-Witten (SW) curve** and **SW differential** are defined as

$$\begin{aligned} y_{SW}^2 &= x^3 + c_2 x^2 + c_1 x + c_0 \\ \frac{\partial \lambda}{\partial u} &= \frac{\sqrt{2}}{8\pi} \frac{2u - (4 - N_f)x + C_0}{y_{SW}} \end{aligned} \tag{1}$$

- Define the **classical SW periods** by integrating over the cycles $\mathcal{A}, \mathcal{B}$ of the SW curve

$$\begin{aligned} a^{(0)}(u, m, \Lambda) &= \oint_{\mathcal{A}} \lambda(x, u, m, \Lambda)\, d\,x\,, \\ a_D^{(0)}(u, m, \Lambda) &= \oint_{\mathcal{B}} \lambda(x, u, m, \Lambda)\, d\,x\,. \end{aligned} \tag{2}$$

- From them one can compute the **SW prepotential** $\mathcal{F}^{(0)}(u, m, \Lambda)$.

Quantum Seiberg-Witten curve and gauge periods from resummation

- To compute **instanton contributions** spacetime is deformed by two complex parameters ϵ_1, ϵ_2 into the Ω-**background**.
- Interesting for the connection to gravity is the **Nekrasov-Shatashvili limit** $\epsilon_2 \to 0, \epsilon_1 = \hbar \neq 0$

$$-\hbar^2 \frac{d^2}{dy^2}\psi(y) + \left[\frac{\Lambda_2^2}{8}\cosh(2y) + \frac{\Lambda_2 m_1}{2}e^y + \frac{\Lambda_2 m_2}{2}e^{-y} + u\right]\psi(y) = 0\,, \tag{3}$$

  (here for $SU(2)$ $N_f = 2$ theory)
- We can define **quantum exact periods** by
  - **exact integrals** of $\mathcal{P}(y) = -i\frac{d}{dy}\ln\psi(y)$
  - or from **combinatorial calculus on Young Tableaux** of the gauge group representation.

A surprising application

- In the last two years, a **surprising connection** between $\mathcal{N} = 2$ supersymmetric (SUSY) $SU(2)$ gauge theories (Nekrasov-Shatashvili deformed) and black holes (BHs) perturbation theory has emerged [**arXiv:2006.06111**, 2105.04245, 2105.04483, arXiv:2109.09804].
- G. Aminov, A. Grassi and Y. Hatsuda first found that **quantization conditions on the gauge periods** a, A_D allow to **compute the (QNMs)** ω_n spectrum of black holes from gauge theory methods.

$$A_D(\hbar, u, m, \Lambda) = 2\pi \left(n + \frac{1}{2} \right) , \qquad n = 0, 1, 2, \ldots . \tag{4}$$

A fruitful new field

The importance of this result is manifold.

1. It constitutes a **novel analytic characterization of QNMs**, for which previously very few were known [arXiv:2006.06111].
2. In the increasingly growing outflow of research on this topic, it has already allowed to **find new results** for the BHs theory, such as:
    1. an **isospectral simpler equation** to the perturbation ODE [arXiv:2007.07906];
    2. improved **theoretical proofs of BHs stability** [arXiv:2105.13329];
    3. **more accurate computations** of observable quantities such as **Love numbers**, describing tidal deformations [arXiv:2105.04483];
    4. an **simpler interpretation of Chandrasekhar transformation** as exchange of gauge mass parameters [arXiv:2111.05857];
    5. precise determination of the **conditions of invariance under (Couch-Torrence) transformations** which exchange inner horizon and null infinity [arXiv:2203.14900].
3. It constitutes an **unexpected application of Supersymmetry**, which was originally thought to describe elementary particles, but has not yet been found by experimentalists.

Integrability for $\mathcal{N} = 2$ gauge theory

ODE/IM correspondence

- In this classic approach to integrability [**arXiv:9812211,9812247,9906219**], the Q **function** is typically the wronskian of the regular solutions at different singular points

$$Q = W[\psi_+, \psi_-] \qquad \psi_\pm(y) \to 0 \quad y \to \pm\infty \tag{5}$$

of some ODE, like (for self-dual Liouville IM or $SU(2)$ $N_f = 0$ gauge theory) [**arXiv:1908.08030**]:

$$-\frac{d^2}{dy^2}\psi + \left[e^{2\theta}(e^y + e^{-y}) + P^2\right]\psi = 0\,. \tag{6}$$

- This **innovates ODE/IM correspondence** itself because such ODEs have 2 **irregular singularities** rather than just 1 as usual. One can derive also T, Y **functions** as well as the **functional and integral equations** they satisfy.
- ODE/IM is an **elegant approach** to integrability which **allows to apply it to very different physical theories!**

Basic gauge-integrability identifications

- Using ODE/IM correspondence, we **connected** the **basic integrability functions** - the Baxter's Q, T and Y functions - to the gauge exact quantum periods a, a_D (from which the prepotential can be obtained). We proved relations like

$$\begin{aligned} Q(\theta, P) &= \exp 2\pi i a_D(\hbar, u, \Lambda_0) \\ T(\theta, P) &= 2\cos 2\pi a(\hbar, u, \Lambda_0) \end{aligned} \tag{7}$$

  under the parameters correspondence

$$\frac{\hbar}{\Lambda_0} = \frac{\epsilon_1}{\Lambda_0} = e^{-\theta} \qquad \frac{u}{\Lambda_0^2} = \frac{1}{2} P^2 e^{-2\theta} \tag{8}$$

- These for the self-dual Liouville IM and $SU(2)$ $N_f = 0$ gauge theory [**arXiv:1908.08030**] but also similar ones for the Perturbed Hairpin IM and $SU(2)$ $N_f = 1, 2$ [**Fioravanti,Gregori,Shu-to appear**] and $SU(3)$ $N_f = 0$ gauge theories [**arXiv:1909.11100**].

New results for both gauge theory and integrability

- This fundamental identification allowed us to find several **new interesting results for both sides** of this new kind of Integrability/Gauge correspondence, for instance:
    1. an exact non linear integral equation (**Thermodynamic Bethe Ansatz, TBA**) for the **gauge (dual) periods**;
    2. an interpretation of the integrability functional relations as **new exact** R**-symmetry relations for the periods**;
    3. **new formulas for the local integrals of motion** in terms of gauge periods.
- For instance, the **Baxter's** TQ **relation**

$$T(\theta, u) = \frac{Q(\theta - i\pi/2, -u) + Q(\theta + i\pi/2, -u)}{Q(\theta)} \tag{9}$$

turns out to be a new exact R-symmetry relation for the periods, reducing to the known asymptotic ones in the limit $\theta \to \infty$ (for the gauge periods expansion modes $a^{(n)}$, $a_D^{(n)}$)

$$a_D^{(n)}(-u) = i(-1)^n \left[-\mathrm{sgn}(\mathrm{Im}\, u) a_D^{(n)}(u) + a^{(n)}(u) \right] \tag{10}$$

Integrability for black holes

Mathematical definition of quasinormal modes

- **Perturbations of the BH** metric or fields turns out to be **solutions Φ of some PDEs** of the form

$$\left\{ + \frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2} + U(x) \right\} \Phi(t,x) = 0\,, \tag{11}$$

  (with coordinate x such that the BH horizon is put at $x \to -\infty$ and spacetime infinity at $x \to +\infty$)

- By ordinary DE techniques (Laplace tr. $\to$ non-hom. ODE $\to$hom. ODE) we can express the **perturbation ϕ as an expansion over some frequencies** ω_n

$$\Phi(t,x) = \sum_n e^{i\omega_n t} \mathrm{Res} \left(\frac{1}{W(s)} \right) \Bigg|_{\omega_n} \int_{-\infty}^{\infty} \Psi_-(\omega_n, x_<) \Psi_+(\omega_n, x_>) \mathcal{I}(\omega_n, x')\, dx'\,. \tag{12}$$

  ω_n are the **quasinormal modes (QNMs)** and we see that they are defined as the **zeros of wronskian of the fundamental regular solutions** at $x \to \pm\infty$ (**??**):

$$W[\Psi_+, \Psi_-](\omega_n) = 0\,, \qquad \Psi_\pm(x) \to 0 \quad x \to \pm\infty\,. \tag{13}$$

ODE for the perturbation of generalized RN BHs

- Line element for intersection of four stacks of D3-branes in type IIB supergravity:

$$ds^2 = -f(r)dt^2 + f(r)^{-1}[dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2)]\,, \tag{14}$$

with

$$f(r) = \prod_{i=1}^{4}\left(1 + \frac{\mathcal{Q}_i}{r}\right)^{-\frac{1}{2}}\,. \tag{15}$$

If the charges $\mathcal{Q}_i = Q = M$ are all equal, it leads to an **extremal Reissner Nordström (charged) BH** with $f(r) = 1 - \frac{2M}{r} + \frac{\mathcal{Q}^2}{r^2}$.
- After suitable changes of variable, the **ODE for the scalar perturbation** takes the form

$$-\frac{d^2}{dy^2}\psi + \left[e^{2\theta}(e^{2y} + e^{-2y}) + 2e^{\theta}(M_1 e^{y} + M_2 e^{-y}) + P^2\right]\psi = 0\,. \tag{16}$$

which turns out to be a **generalization** of the one for the **Perturbed Hairpin Integrable Model**.

Baxter's Q function and quasinormal modes

- The **regular solutions** of (16) at $y \to \pm\infty$ ($j = 1, 2$) have boundary conditions

$$\begin{aligned} \psi_{-,0}(y) &\simeq 2^{-\frac{1}{2}-M_2} e^{-(\frac{1}{2}+M_2)\theta+(\frac{1}{2}+M_2)y} e^{-e^{\theta-y}}, \quad \mathrm{Re}\, y \to -\infty\,. \\ \psi_{+,0}(y) &\simeq 2^{-\frac{1}{2}-M_1} e^{-(\frac{1}{2}+M_1)\theta-(\frac{1}{2}+M_1)y} e^{-e^{\theta+y}}, \quad \mathrm{Re}\, y \to +\infty\,. \end{aligned} \tag{17}$$

- The **Baxter's Q function** is defined **precisely as the wronskian of the regular solutions**

$$Q_{+,+}(\theta) = W[\psi_{+,0}, \psi_{-,0}]\,. \tag{18}$$

(We will use the notation $Q_{\pm,\pm} = Q(\theta, P, \pm M_1, \pm M_2)$, $Q_{\pm,\mp} = Q(\theta, P, \pm M_1, \mp M_2)$).

- Crucially, the **QNMs condition** (13) translates into

$$Q(\theta_n) = 0\,, \tag{19}$$

namely the **zeros of the Baxter's Q function** which are called **Bethe roots**.

Properties of Q function

- By properties of the ODE, we can write the QQ **system**

$$Q_{+,-}(\theta + \frac{i\pi}{2})Q_{-,+}(\theta - \frac{i\pi}{2}) = e^{-i\pi(M_1 - M_2)} + Q_{-,-}(\theta)Q_{+,+}(\theta)\,. \tag{20}$$

- an **integral formula for** Q

$$\ln Q_{+,+}(\theta) = \ln\left[-ie^{i\pi M_1} \lim_{y\to+\infty} \frac{\psi_{-,0}(y)}{\psi_{+,1}(y)}\right] \tag{21}$$

$$= \int_{-\infty}^{\infty} dy \left[\sqrt{2\cosh(2y)}\Pi(y) - 2e^{\theta}\cosh y - \left(\frac{M_1}{1+e^{-y/2}} + \frac{M_2}{1+e^{y/2}}\right)\right] + \left(\theta + \frac{1}{2}\ln 2\right)(M_1 - M_2) \tag{22}$$

where integrand is $\Pi(y) = \frac{1}{\sqrt{2\cosh(2y)}}\frac{d}{dy}\ln(\sqrt[4]{-2\cosh(2y)}\psi(y))$.

- From its $\theta \to +\infty$ expansion, we get the Local integrals of motion I_n.

Y function and quasinormal modes

- One can define a Y function as

$$Y_{+,+}(\theta) = e^{i\pi(M_1 - M_2)} Q_{+,+}(\theta) Q_{-,-}(\theta) \tag{23}$$

- From the QQ system it follows the Y **system** as

$$Y_{+,-}(\theta + \frac{i\pi}{2}) Y_{-,+}(\theta - \frac{i\pi}{2}) = [1 + Y_{+,+}(\theta)][1 + Y_{-,-}(\theta)] \,. \tag{24}$$

- Eventually, the QQ **system** written as

$$e^{i\pi(M_1 + M_2)} Q_{+,+}(\theta + \frac{i\pi}{2}) Q_{-,-}(\theta - \frac{i\pi}{2}) = 1 + Y_{+,-}(\theta) \,. \tag{25}$$

characterizes the **QNMs** with other **quantizations conditions**

$$Y_{+,-}(\theta_n - i\pi/2) = -1 \qquad Y_{-,+}(\theta_n + i\pi/2) = -1 \tag{26}$$

Thermodynamic Bethe Ansatz

- The Y **system can be inverted in the Thermodynamic Bethe Ansatz (TBA)** for $\varepsilon_{\pm,\pm}(\theta) = -\ln Y_{\pm,\pm}(\theta)$:

$$\begin{aligned}
\varepsilon_{\pm,\pm}(\theta) &= \frac{8\sqrt{\pi^3}}{\Gamma\left(\frac{1}{4}\right)^2} e^{\theta} \mp i\pi(M_1 - M_2) - \int_{-\infty}^{\infty} \frac{d\theta'}{2\pi} \frac{\ln[1+\exp(-\varepsilon_{\pm,\mp}(\theta'))] + \ln[1+\exp(-\varepsilon_{\mp,\pm}(\theta'))]}{\cosh(\theta-\theta')} \\
\varepsilon_{\pm,\mp}(\theta) &= \frac{8\sqrt{\pi^3}}{\Gamma\left(\frac{1}{4}\right)^2} e^{\theta} \mp i\pi(M_1 + M_2) - \int_{-\infty}^{\infty} \frac{d\theta'}{2\pi} \frac{\ln[1+\exp(-\varepsilon_{\pm,\pm}(\theta'))] + \ln[1+\exp(-\varepsilon_{\mp,\mp}(\theta'))]}{\cosh(\theta-\theta')}
\end{aligned} \tag{27}$$

- The **QNMs condition** in gravity variables **reads alternatively as**

$$\bar{Y}_{+,+}(\theta_{n'} - i\pi/2) = -1\,, \quad Q_{+,+}(\theta_n) = 0\,, \quad \bar{\varepsilon}_{+,+}(\theta_{n'} - i\pi/2) = -i\pi(2n'+1)\,. \tag{28}$$

- Through the quantization condition on $\bar{\varepsilon}_{+,+}$ we can **actually numerically compute QNMs from TBA**.

Comparison of methods of computation of quasinormal modes

n	l	TBA	Leaver	WKB
0	1	$\underline{0.869}623 - \underline{0.372}022i$	$\underline{0.86}8932 - \underline{0.372}859i$	$0.89642 - 0.36596i$
0	2	$\underline{1.477}990 - \underline{0.368}144i$	$\underline{1.477}888 - \underline{0.368}240i$	$1.4940 - 0.36596i$
0	3	$\underline{2.080}200 - \underline{0.367}076i$	$\underline{2.080}168 - \underline{0.367}097i$	$2.0916 - 0.36596i$
0	4	$\underline{2.680}363 - \underline{0.3666}37i$	$\underline{2.680}350 - \underline{0.3666}42i$	$2.6893 - 0.36596i$

Table: Comparison of QNMs in different methods ($n' = 0$, $\Sigma_1 = \Sigma_3 = 0.2$, $\Sigma_2 = 0.4$, $\Sigma_4 = 1$).

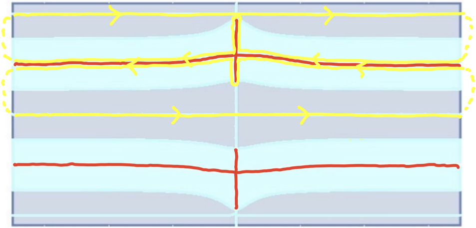
- Computing QNMs has been typically quite **laborious**, also because of their **few exact analytic characterizations**.
- The standard analytic method is the one with the **continued fractions** by Leaver we found **sometimes not applicable** (in its original form).
- Application of $\mathcal{N} = 2$ **gauge theory** is a new analytic characterization of QNMs, but it requires a **nontrivial re-summation procedure** for $\omega_n \sim \Lambda_n \sim 1$.
- Our **integrability** exact method is **direct and simple**, but now we have developed for just a **few models** (see below).

Quantum gauge B period from TBA

- Also for this $SU(2)$ $N_f = 2$ found a **relation between the pseudoenergy** $\epsilon = -\ln Y$ **and the gauge periods**

$$\varepsilon(\theta, -u, im_1, -im_2, \Lambda_2) = \frac{8\sqrt{2}\pi}{\Lambda_2} a_D(\hbar, u, m_1, m_2, \Lambda_2) \tag{29}$$

- We can prove it analytically by **Cauchy theorem relating the different integration contours** in the complex plane (in red the branch cuts).



- We see that from our formalism **it follows immediately that QNMs are given by quantization conditions on the gauge integral periods**

$$\frac{8\sqrt{2}\pi}{\Lambda_2} a_D(\hbar, u, m_1, m_2, \Lambda_2) = -i\pi(2n' + 1) \tag{30}$$

Other applications of integrability

- Through ODE/IM can also define also **Baxter's T, $\tilde{T}$ functions** as wronskians of solutions at $y \to \pm\infty$ and derive TQ and T periodicity functional relations.
- We have proven a relation between T and a gauge period.
- So we give a proof of the **alternative quantization condition for QNMs on** a for the $N_f = 0$ $SU(2)$ theory.
- Moreover we find that the **Couch-Torrence symmetry** corresponds to $T = \tilde{T}$ **self-duality**.
- We notice that **much of the BH theory seems to go in parallel to the ODE/IM correspondence** construction and its 2D integrable field theory interpretation, beyond the determination of QNMs.
- For instance also the **greybody factor** that parametrizes the **Hawking radiation** seems to be **ratio of** Q**s**.

Conclusions and perspectives

Conclusions

- We have shown how **2D integrable models** when studied in **ODE/IM correspondence** approach can find
  - a natural connection to (deformed) $\mathcal{N} = 2$ **supersymmetric gauge theory**
  - as well as to **black hole perturbation theory**
  - and shed light on the **relation** recently found **between the two**.
- This allows to
  - **find new results** an all three sides of the correspondence
  - at the **non-perturbative exact level**.
- In these new directions **much extension work in either breadth and depth** can still be done.

Present limitations and possible future developments

- What we have described holds for the **generalization of extremal charged BHs** (intersection of 4 stacks of D3 branes) and it corresponds to $SU(2)$ $N_f = 2, 1, 0$ **gauge theories**.
- However, from the **generality of construction** it is manifest that our method **should apply many other theories**. We can list
  1. the $SU(2)$ $N_f = (2, 0)$ gauge theory and the associated gravity counterparts, like D1D5 fuzzballs and CCLP 5D BHs;
  2. the $SU(2)$ $N_f = 3$ gauge theory and the general asymptotically flat Kerr-Newman BH;
  3. the $SU(2)$ $N_f = 4$ or $SU(2)$ quivers that is asymptotically AdS BHs.
- Besides **model generalization**, it is very intriguing to study the application of the integrability structure **beyond the determination of the QNMs** for the study of gravitational solutions (greybody factor, etc.).
- For **AdS BHs** the 3-fold correspondence Integrability/Gauge/Gravity could become 4-fold through **holographic duality to CFT**.

Thank you
for your attention!