

Three-loop four-particle QCD amplitudes with Caola, Chakraborty, Gambuti, Manteuffel, Tancredi

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Presentation plan

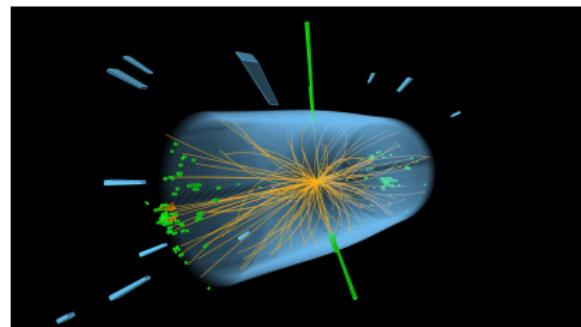
1 Motivation

2 Computation

3 Results

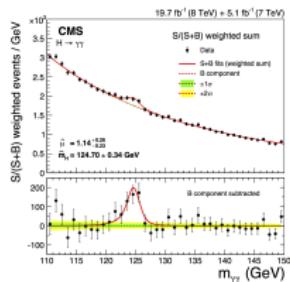
Motivation

precision **LHC** measurements

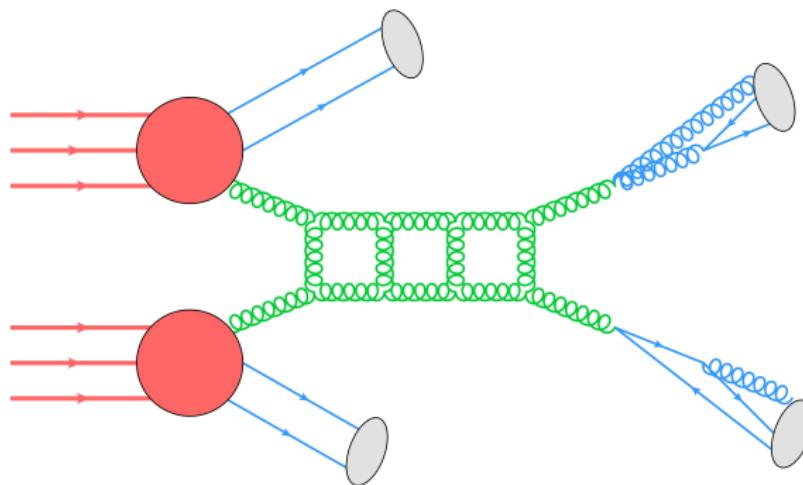


Higgs analysis

multiloop scattering amplitudes



Factorization theorem



hadronic cross section =

Parton Distribution Functions \otimes hard scattering \otimes real radiation \otimes hadronization
 \uparrow our interest

Towards 3-loop revolution

3-loop amplitude milestones

t

- ⌚ 1→1 QCD [[Tarasov et al. PRLB 1980](#)]
- ⌚ 2→1 QCD [[Moch et al. arXiv:0508055](#)]
- ⌚ 2→2 SYM [[Henn, Mistlberger arXiv:1608.00850](#)]

t

first 3-loop 2→2 QCD results

- ⌚ $q\bar{q} \rightarrow \gamma\gamma$ [[Caola, Manteuffel, Tancredi arXiv:2011.13946](#)]
- ⌚ $q\bar{q} \rightarrow q\bar{q}$ [[Caola, Chakraborty, Gambuti, Manteuffel, Tancredi arXiv:2108.00055](#)]
- ⌚ $gg \rightarrow \gamma\gamma$ [[PB, Caola, Manteuffel, Tancredi arXiv:2111.13595](#)]
- ⌚ $gg \rightarrow gg$ [[Caola, Chakraborty, Gambuti, Manteuffel, Tancredi arXiv:2112.11097](#)]

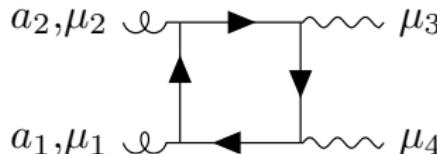
challenge = complexity

# diagrams	0L	1L	2L	3L
$q\bar{q} \rightarrow \gamma\gamma$	2	10	143	2922
$q\bar{q} \rightarrow q\bar{q}$	1	9	158	3584
$gg \rightarrow \gamma\gamma$	0	6	138	3299
$gg \rightarrow gg$	4	81	1771	48723

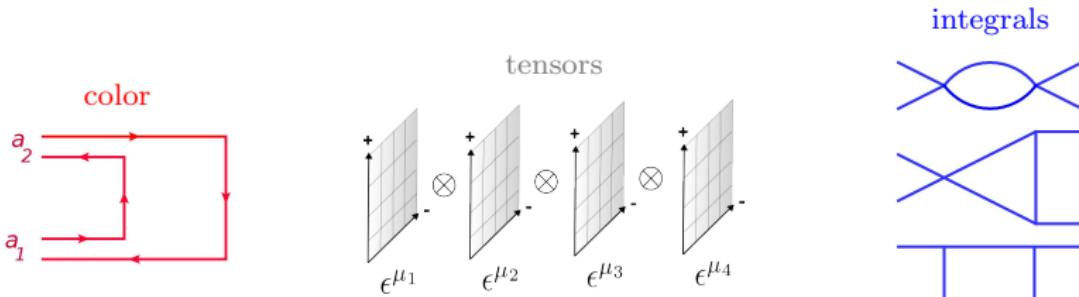
Amplitude structure

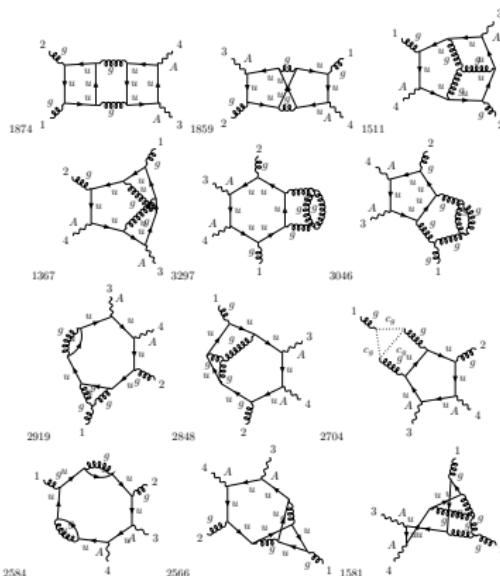
consider 1-loop example diagram for

$$g(p_1) + g(p_2) \rightarrow \gamma(-p_3) + \gamma(-p_4)$$



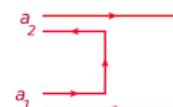
$$= g_s^4 \text{Tr}\{T^{a_1} T^{a_2}\} \int \frac{d^d k}{(2\pi)^d} \frac{\text{tr} \left[(\not{k}) \not{\epsilon}_1 (\not{k} + \not{p}_1) \not{\epsilon}_2 (\not{k} + \not{p}_{12}) \not{\epsilon}_3 (\not{k} + \not{p}_{123}) \not{\epsilon}_4 \right]}{(k)^2 (k + p_1)^2 (k + p_{12})^2 (k + p_{123})^2}$$



3-loop $gg \rightarrow \gamma\gamma$  $\times 275$ pages = 3299 diagrams

	1L	2L	3L
# diagrams	6	138	3299
# integral families	1	2	3
# integrals	209	20935	4370070
# MIs	6	39	486
Qgraf result [kB]	4	90	2820
amplitudes before IBPs [kB]	276	54364	19734644
amplitudes after IBPs [kB]	12	562	304409
expanded amplitudes [kB]	136	380	1195

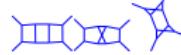
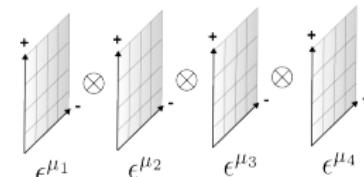
complexity summary

color structure $\text{Tr}\{T^{a_1} T^{a_2}\} = \frac{1}{2} \delta^{a_1 a_2}$

Computation

overcome complexity with recently proposed new ideas

- managing tensor structures
 $\#$ independent structures = $\#$ helicity amplitudes
& project out unphysical ($d - 4$)-dim structures
 $\Rightarrow 8$ physical tensors
- reducing integrals to a minimal set
finite field reconstruction
& syzygy constraints
 $\Rightarrow \mathbf{4370070} \xrightarrow{IBP} 486$ integrals
- evaluating integrals
differential equations in canonical Master Basis
& regularity requirement for boundary conditions
 $\Rightarrow 486 \rightarrow \mathbf{1}$ integral



Making maximal use of physical constraints allows for
simple way of obtaining all but 1 integrals.

The bubble needs to be evaluated with traditional techniques.

All-plus helicity amplitude

finite part in the **simplest** helicity configuration yields

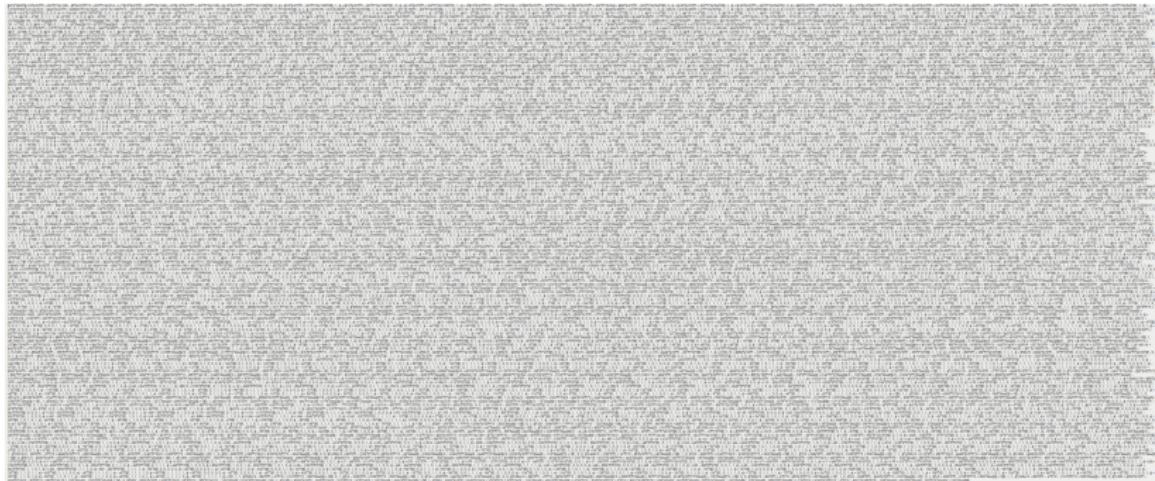
$$\begin{aligned}
 f_{++++}^{(3,\text{fin})} &= \Delta_1(x) n_f^{V_2} C_A^2 + \Delta_2(x) n_f^{V_2} C_A C_F + \Delta_3(x) n_f n_f^{V_2} C_A + \Delta_4(x) (n_f^V)^2 C_A + \Delta_5(x) n_f^{V_2} C_F^2 + \Delta_6(x) (n_f^V)^2 C_F + \Delta_7(x) n_f n_f^{V_2} C_F + \Delta_8(x) n_f^2 n_f^{V_2} \\
 &\quad + \{(x) \leftrightarrow (1-x)\}, \\
 \Delta_1(x) &= -\frac{23L_1(L_1+2i\pi)}{9x^2} + \frac{32L_1(L_1+2i\pi)-46(L_1+i\pi)}{9x} - \frac{17}{36}L_0^2 - \frac{19}{36}L_0L_1 + \frac{1}{9}L_0 - 2i\pi L_0 + \frac{1}{288}\pi^4 \\
 &\quad - \frac{373}{72}\zeta_3 - \frac{185}{72}\pi^2 + \frac{4519}{324} + \frac{1}{2}i\pi\zeta_3 + \frac{11}{144}i\pi^3 + \frac{157}{12}i\pi + \frac{43}{9}L_0x - \frac{7}{9}x^2 ((L_0-L_1)^2 + \pi^2), \\
 \Delta_2(x) &= \frac{8L_1(L_1+2i\pi)}{3x^2} + \frac{16(L_1+i\pi)-8L_1(L_1+2i\pi)}{3x} - \frac{1}{3}L_0^2 + \frac{5}{6}L_0L_1 + \frac{17}{3}L_0 + i\pi L_0 - \frac{5}{12}\pi^2 - \frac{199}{6} - 8i\pi - \frac{16}{3}L_0x + \frac{4}{3}x^2 ((L_0-L_1)^2 + \pi^2), \\
 \Delta_3(x) &= \frac{L_1(L_1+2i\pi)}{18x^2} + \frac{2(L_1+i\pi)-L_1(L_1+2i\pi)}{18x} - \frac{1}{36}L_0^2 + \frac{1}{36}L_0L_1 - \frac{1}{9}L_0 - \frac{61}{36}\zeta_3 + \frac{475}{432}\pi^2 - \frac{925}{324} - \frac{1}{72}i\pi^3 - \frac{175}{54}i\pi + \frac{2}{9}L_0x + \frac{1}{36}x^2 ((L_0-L_1)^2 + \pi^2), \\
 \Delta_4(x) &= -\frac{5L_1(L_1+2i\pi)}{4x^2} + \frac{L_1(L_1+2i\pi)-8(L_1+i\pi)}{2x} + \frac{1}{4}L_0^2 - \frac{1}{4}L_0L_1 - 2L_0 - 6\zeta_3 + \frac{1}{8}\pi^2 - \frac{1}{2} + 4L_0x - x^2 ((L_0-L_1)^2 + \pi^2), \\
 \Delta_5(x) &= -\frac{L_1(L_1+2i\pi)}{x^2} + \frac{L_1(L_1+2i\pi)-2(L_1+i\pi)}{x} - \frac{1}{2}L_0^2 - i\pi L_0 + \frac{39}{4} + i\pi + 2L_0x - \frac{1}{2}x^2 ((L_0-L_1)^2 + \pi^2), \\
 \Delta_6(x) &= \frac{10L_1(L_1+2i\pi)}{3x^2} + \frac{32(L_1+i\pi)-4L_1(L_1+2i\pi)}{3x} - \frac{2}{3}L_0^2 + \frac{2}{3}L_0L_1 + \frac{16}{3}L_0 + 16\zeta_3 - \frac{1}{3}\pi^2 + \frac{4}{3} - \frac{32}{3}L_0x + \frac{8}{3}x^2 ((L_0-L_1)^2 + \pi^2), \\
 \Delta_7(x) &= \frac{5L_1(L_1+2i\pi)}{3x^2} + \frac{10(L_1+i\pi)-8L_1(L_1+2i\pi)}{3x} + \frac{2}{3}L_0^2 + \frac{1}{3}L_0L_1 - \frac{10}{3}L_0 + 2i\pi L_0 + 4\zeta_3 - \frac{\pi^2}{6} + 5 - 3i\pi - \frac{10}{3}L_0x + \frac{1}{3}x^2 ((L_0-L_1)^2 + \pi^2), \\
 \Delta_8(x) &= -\frac{23}{216}\pi^2 + \frac{5}{27}i\pi,
 \end{aligned}$$

where $L_0 = \ln(x)$, $L_1 = \ln(1-x)$.

-- ++ helicity amplitude

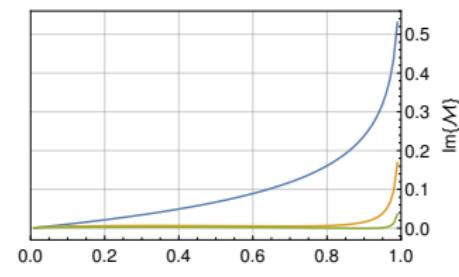
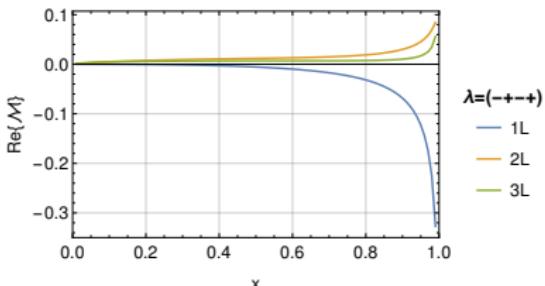
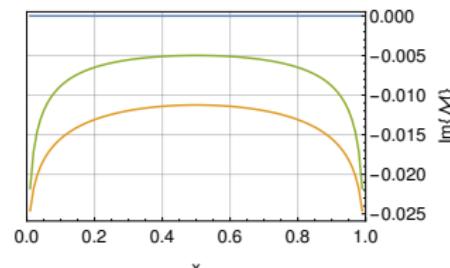
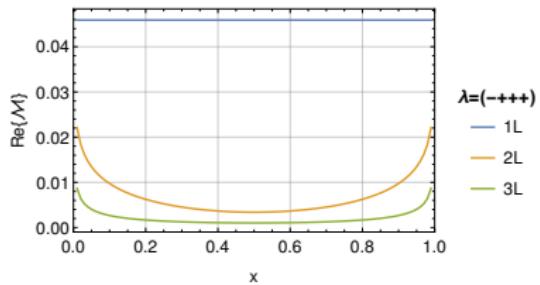
compact result even for the most complicated helicity configuration

$$f_{--++}^{(3,\text{fin})} =$$



Kinematic dependence

$$0 < x = -\frac{t}{s} < 1$$



results can be evaluated numerically in $[\mu\text{s}]$

Outlook

future directions

phenomenological

formal

- NNLO $gg \rightarrow \gamma\gamma$
differential **cross section**
- interesting **background**
for Higgs production
- nontrivial signal-background interference
to constrain **Higgs width**

- can we understand
amplitude **simplicity** ?
- why is only **1**
boundary integral enough ?
- can we unveil
hidden amplitude **structure** ?

THANK YOU

Appendix : kinematics and color

kinematics : 4-point process \Rightarrow 1 dimensionless variable

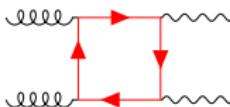
(Regge, forward) $0 < x = -\frac{t}{s} < 1$ (backward)

$$\text{color decomposition} \quad : \quad \mathcal{A} = \delta^{a_1 a_2} (4\pi\alpha) \left(\frac{\alpha_s}{2\pi}\right)^3 c_i A_i$$

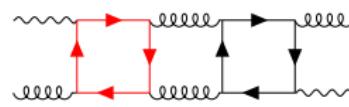
where \mathcal{C}_i is a degree=3 monomial in $\{C_A, C_F, n_f, n_f^V, n_f^{V_2}\}$

3 closed fermion **loop** types

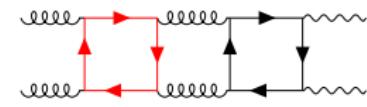
$$n_f^{V_2} = \sum_f Q_f^2$$



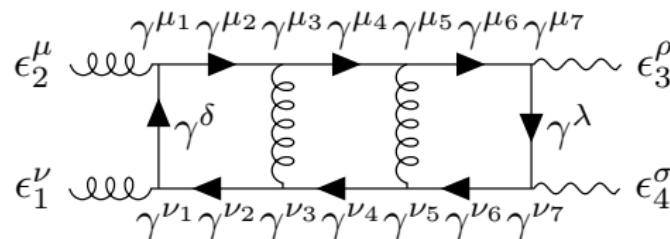
$$n_f^V = \sum_f Q_j$$



$$n_f = \sum_f 1$$



Appendix : tensors in $d=4-2\epsilon$ dimensions



$\Sigma(\text{diagrams}) \# \text{Lorentz indices} > \# \text{all invariant structures}$

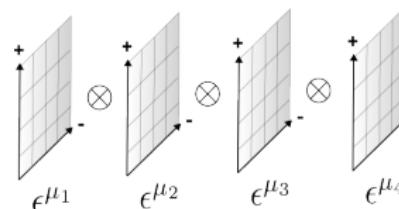
$$\begin{aligned} \#T_i &= 138 \text{ (Lorentz invariant tensors)} - 81 \text{ (by transversality } \epsilon_i \cdot p_i = 0) \\ &\quad - 47 \text{ (by gauge fixing } \epsilon_i \cdot p_{i+1} = 0) \\ &= 10 \text{ (independent in } d \text{ dimensions)} \end{aligned}$$

$$\begin{aligned} T_i = (p_1 \cdot \epsilon_2 p_1 \cdot \epsilon_3 p_2 \cdot \epsilon_4 p_3 \cdot \epsilon_1, & \quad \epsilon_3 \cdot \epsilon_4 p_1 \cdot \epsilon_2 p_3 \cdot \epsilon_1 \quad \epsilon_2 \cdot \epsilon_4 p_1 \cdot \epsilon_3 p_3 \cdot \epsilon_1, \\ \epsilon_2 \cdot \epsilon_3 p_2 \cdot \epsilon_4 p_3 \cdot \epsilon_1, & \quad \epsilon_1 \cdot \epsilon_4 p_1 \cdot \epsilon_2 p_1 \cdot \epsilon_3, \quad \epsilon_1 \cdot \epsilon_3 p_1 \cdot \epsilon_2 p_2 \cdot \epsilon_4, \\ \epsilon_1 \cdot \epsilon_2 p_1 \cdot \epsilon_3 p_2 \cdot \epsilon_4, & \quad \epsilon_1 \cdot \epsilon_2 \epsilon_3 \cdot \epsilon_4, \quad \epsilon_1 \cdot \epsilon_4 \epsilon_2 \cdot \epsilon_3, \quad \epsilon_1 \cdot \epsilon_3 \epsilon_2 \cdot \epsilon_4) \end{aligned}$$

Appendix : tensors in 4 dimensions

recent loop-universal **claim** in the 'tHV scheme [Peraro, Tancredi [arXiv:2012.00820](https://arxiv.org/abs/2012.00820)] :
 # tensors indpt in 4-dim = # indpt helicity states (here = $2^4/2 = 8$)

$$A = \sum_{i=1}^{10} \mathcal{F}_i T_i = \sum_{i=1}^8 \overline{\mathcal{F}}_i \overline{T}_i$$



orthogonalization : projects out $\overline{T}_9, \overline{T}_{10}$ from the **physical** 4-dim subspace

$$\sum_{pol} \overline{T}_i^\dagger \overline{T}_j = \left(\begin{array}{c|c} 8 \times 8 \text{ (4-dim)} & 0 \\ \hline 0 & 2 \times 2 \text{ (-2e-dim)} \end{array} \right)$$

gain : 1-1 **correspondence** between form factors and helicity amplitudes

$$\overline{\mathcal{F}}_i \iff A_{\vec{\lambda}_i}$$

Appendix : tensor projectors

resulting tensors live purely in the **unphysical** -2ϵ -dim subspace

$$\bar{T}_i = T_i - \sum_{j=1}^8 (\mathcal{P}_j T_i) \bar{T}_j, \quad i = 9, 10, \quad \sum_{\text{pol}} \mathcal{P}_i \bar{T}_j = \delta_{ij}$$

$$\begin{aligned}\bar{T}_9 &= T_9 - \frac{1}{3} \left(-\frac{2\bar{T}_1}{su} - \frac{\bar{T}_6}{s} - \frac{\bar{T}_2 + \bar{T}_3 + 2\bar{T}_4 - 2\bar{T}_5 - \bar{T}_6 - \bar{T}_7}{t} + \frac{\bar{T}_3}{u} + \bar{T}_8 \right) \\ \bar{T}_{10} &= T_{10} - \frac{1}{3} \left(\frac{4\bar{T}_1}{su} + \frac{2\bar{T}_6}{s} - \frac{\bar{T}_2 - \bar{T}_4 - 2\bar{T}_3 + 2\bar{T}_6 + \bar{T}_5 - \bar{T}_7}{t} - \frac{2\bar{T}_3}{u} + \bar{T}_8 \right)\end{aligned}$$

and they **vanish** ($\forall \epsilon$) for each fixed helicity configuration

Appendix : tensor projectors

$$\mathcal{P}_i = \sum_{k=1}^{10} (M^{-1})_{ik} \bar{T}_k^\dagger \quad M_{ij} = \sum_{pol} \bar{T}_i^\dagger \bar{T}_j \quad \sum_{pol} \epsilon_i^\mu \epsilon_i^{*\nu} = -g^{\mu\nu} + \frac{p_i^\mu q_i^\nu + q_i^\mu p_i^\nu}{p_i \cdot q_i}$$

$$M^{-1} = \begin{pmatrix} \frac{X^{(0)} + dX^{(1)}}{3(d-1)(d-3)t^2} & 0 & 0 \\ 0 & \frac{2}{(d-4)(d-3)} & \frac{1}{(d-4)(d-3)} \\ 0 & \frac{1}{(d-4)(d-3)} & \frac{2}{(d-4)(d-3)} \end{pmatrix},$$

$$X^{(0)} = \begin{pmatrix} -\frac{8}{s^2} + \frac{32}{su} - \frac{8}{u^2} & -\frac{2}{s} - \frac{2}{u} & \frac{4s}{u^2} - \frac{4}{s} - \frac{12}{u} & \frac{2}{s} + \frac{2}{u} & -\frac{2}{s} - \frac{2}{u} & -\frac{4u}{s^2} + \frac{12}{s} + \frac{4}{u} & \frac{2}{s} + \frac{2}{u} & -\frac{2s}{u} - \frac{2u}{s} - 7 \\ -\frac{2}{s} - \frac{2}{u} & -2 & \frac{s}{u} + 2 & -1 & 1 & -\frac{u}{s} - 2 & -1 & -t \\ \frac{4s}{u^2} - \frac{4}{s} - \frac{12}{u} & \frac{s}{u} + 2 & -\frac{2s^2}{u^2} + \frac{4s}{u} + 4 & -\frac{s}{u} - 2 & \frac{s}{u} + 2 & -\frac{2s}{u} - \frac{2u}{s} - 5 & -\frac{s}{u} - 2 & \frac{s^2}{u} + 3s + 2u \\ \frac{2}{s} + \frac{2}{u} & -1 & -\frac{s}{u} - 2 & -2 & -1 & \frac{u}{s} + 2 & 1 & t \\ -\frac{2}{s} - \frac{2}{u} & 1 & \frac{s}{u} + 2 & -1 & -2 & -\frac{u}{s} - 2 & -1 & -t \\ -\frac{4u}{s^2} + \frac{12}{su} + \frac{4}{u} & -\frac{u}{s} - 2 & -\frac{2s}{u} - \frac{2u}{s} - 5 & \frac{u}{s} + 2 & -\frac{u}{s} - 2 & -\frac{2u^2}{s^2} + \frac{4u}{s} + 4 & \frac{u}{s} + 2 & -\frac{u^2}{s} - 2s - 3u \\ \frac{2}{s} + \frac{2}{u} & -1 & -\frac{s}{u} - 2 & 1 & -1 & \frac{u}{s} + 2 & -2 & t \\ -\frac{2s}{u} - \frac{2u}{s} - 7 & -t & \frac{s^2}{u} + 3s + 2u & t & -t & -\frac{u^2}{s} - 2s - 3u & t & t^2 \end{pmatrix},$$

$$X^{(1)} = \begin{pmatrix} \frac{12}{s^2} - \frac{12}{su} + \frac{3(2+d)}{t^2} + \frac{12}{u^2} & \frac{3}{t} & -\frac{6s}{u^2} + \frac{3}{t} + \frac{3}{u} & -\frac{3}{t} & \frac{3}{t} & \frac{6u}{s^2} - \frac{3}{s} - \frac{3}{t} & -\frac{3}{t} & 0 \\ \frac{2}{t} & 3 & 0 & 0 & 0 & 0 & 0 & 0 \\ -\frac{6s}{u^2} + \frac{3}{t} + \frac{3}{u} & 0 & \frac{3s^2}{u^2} & 0 & 0 & 0 & 0 & 0 \\ -\frac{3}{t} & 0 & 0 & 3 & 0 & 0 & 0 & 0 \\ \frac{3}{t} & 0 & 0 & 0 & 3 & 0 & 0 & 0 \\ \frac{6u}{s^2} - \frac{3}{s} - \frac{3}{t} & 0 & 0 & 0 & 0 & \frac{3u^2}{s^2} & 0 & 0 \\ -\frac{3}{t} & 0 & 0 & 0 & 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Appendix : helicity amplitudes

evaluate tensors at fixed helicity configuration
spinor weights

$$A_{\vec{\lambda}} = \sum_i \bar{\mathcal{F}}_i \bar{T}_{\vec{\lambda}} = \mathcal{S}_{\vec{\lambda}} f_{\vec{\lambda}}$$

$$\begin{aligned} s_{++++} &= \frac{[12][34]}{\langle 12 \rangle \langle 34 \rangle}, & s_{-+++} &= \frac{\langle 12 \rangle \langle 14 \rangle [24]}{\langle 34 \rangle \langle 23 \rangle \langle 24 \rangle}, & s_{+-+-+} &= \frac{\langle 21 \rangle \langle 24 \rangle [14]}{\langle 34 \rangle \langle 13 \rangle \langle 14 \rangle}, & s_{++-+} &= \frac{\langle 32 \rangle \langle 34 \rangle [21]}{\langle 14 \rangle \langle 21 \rangle \langle 23 \rangle}, \\ s_{+++-} &= \frac{\langle 42 \rangle \langle 43 \rangle [23]}{\langle 13 \rangle \langle 21 \rangle \langle 23 \rangle}, & s_{--++} &= \frac{\langle 12 \rangle [34]}{[12] \langle 34 \rangle}, & s_{-+-+} &= \frac{\langle 13 \rangle [24]}{[13] \langle 24 \rangle}, & s_{+--+} &= \frac{\langle 23 \rangle [14]}{[23] \langle 14 \rangle} \end{aligned}$$

little group scalars

$$f_{++++} = \frac{t^2}{4} \left(\frac{2\bar{\mathcal{F}}_6}{u} - \frac{2\bar{\mathcal{F}}_3}{s} - \bar{\mathcal{F}}_1 \right) + \bar{\mathcal{F}}_8 \left(\frac{s}{u} + \frac{u}{s} + 4 \right) + \frac{t}{2} (\bar{\mathcal{F}}_2 - \bar{\mathcal{F}}_4 + \bar{\mathcal{F}}_5 - \bar{\mathcal{F}}_7),$$

$$f_{-+++} = \frac{t^2}{4} \left(\frac{2\bar{\mathcal{F}}_3}{s} + \bar{\mathcal{F}}_1 \right) + t \left(\frac{\bar{\mathcal{F}}_8}{s} + \frac{1}{2} (\bar{\mathcal{F}}_4 + \bar{\mathcal{F}}_6 - \bar{\mathcal{F}}_2) \right),$$

$$f_{+-+-} = -\frac{t^2}{4} \left(\frac{2\bar{\mathcal{F}}_6}{u} - \bar{\mathcal{F}}_1 \right) + t \left(\frac{\bar{\mathcal{F}}_8}{u} - \frac{1}{2} (\bar{\mathcal{F}}_2 + \bar{\mathcal{F}}_3 + \bar{\mathcal{F}}_5) \right),$$

$$f_{++-+} = \frac{t^2}{4} \left(\frac{2\bar{\mathcal{F}}_3}{s} + \bar{\mathcal{F}}_1 \right) + t \left(\frac{\bar{\mathcal{F}}_8}{s} + \frac{1}{2} (\bar{\mathcal{F}}_6 + \bar{\mathcal{F}}_7 - \bar{\mathcal{F}}_5) \right),$$

$$f_{+++-} = -\frac{t^2}{4} \left(\frac{2\bar{\mathcal{F}}_6}{u} - \bar{\mathcal{F}}_1 \right) + t \left(\frac{\bar{\mathcal{F}}_8}{u} + \frac{1}{2} (\bar{\mathcal{F}}_4 + \bar{\mathcal{F}}_7 - \bar{\mathcal{F}}_3) \right),$$

$$f_{--++} = -\frac{t^2}{4} \bar{\mathcal{F}}_1 + \frac{1}{2} t (\bar{\mathcal{F}}_2 + \bar{\mathcal{F}}_3 - \bar{\mathcal{F}}_6 - \bar{\mathcal{F}}_7) + 2\bar{\mathcal{F}}_8,$$

$$f_{-+-+} = t^2 \left(\frac{\bar{\mathcal{F}}_8}{su} - \frac{\bar{\mathcal{F}}_3}{2s} + \frac{\bar{\mathcal{F}}_6}{2u} - \frac{\bar{\mathcal{F}}_1}{4} \right),$$

$$f_{+--+} = -\frac{t^2}{4} \bar{\mathcal{F}}_1 + \frac{1}{2} t (\bar{\mathcal{F}}_3 - \bar{\mathcal{F}}_4 + \bar{\mathcal{F}}_5 - \bar{\mathcal{F}}_6) + 2\bar{\mathcal{F}}_8$$

Appendix : integral reduction

simplifying 4×10^6 integrals **before** the Integration by Parts (IBP) reduction

- map redundant crossings onto the 6 independent ones
& remove scaleless integrals $\rightarrow \mathcal{O}(10)$ reduction
- map equivalent sectors into common family $\rightarrow \mathcal{O}(2)$ reduction

$$f_{\vec{\lambda}}(x) = \sum_{i=1}^{2 \times 10^5} C_{\vec{\lambda},i}(d; x) \mathcal{I}_i(x) \stackrel{IBP}{=} \sum_{i=1}^{486} C_{\vec{\lambda},i}(d; x) M_i(x)$$

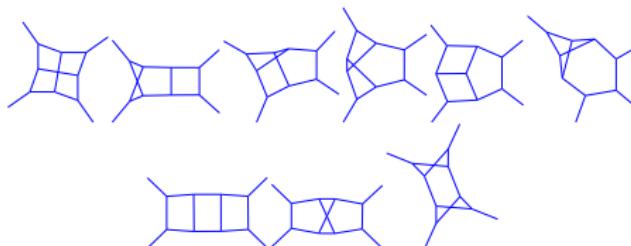
taming the complexity of the **IBP** reduction

- solve IBP system with Laporta algorithm [[Laporta arXiv:0102033](#)]
- use \mathbb{F}_p finite field arithmetic [[Manteuffel, Schabinger arXiv:1406.4513](#)]

$$C(d; x) = \#_j g_j(d; x), \quad \text{where} \quad g \sim \frac{\mathcal{N}(x, d)}{(d-\#)^p (x-\#)^q}$$
- impose syzygy constraints [[Gluza et al. arXiv:1009.0472](#)] [[Peraro arXiv:1905.08019](#)]

$$v_j^\mu \frac{\partial}{\partial l_j^\mu} D_i + b_i D_i = 0$$
- partial fraction in d and x

Appendix : Master Integrals



- differential equation [[Henn, Mistlberger et al. arXiv:2002.09492](#)]

$$d\vec{M}(\epsilon; x) = \epsilon a(x) \vec{M}(\epsilon; x) dx$$

- solved perturbatively by Harmonic Polylogarithms (**HPLs**)

$$G(\alpha_n, \dots, \alpha_1; x) = \int_0^x \frac{dz}{z - \alpha_n} G(\alpha_{n-1}, \dots, \alpha_1; z), \quad G(\underbrace{0, \dots, 0}_{n \text{ times}}; x) \equiv \frac{\ln^n x}{n!}, \quad \alpha_i \in \{0, 1\}$$

- claim : can relate **all** boundary conditions $\vec{M}_0(\epsilon)$ to a **single** overall normalization (3L sunrise  if

require $\lim_{s_{ij} \rightarrow 0} \vec{M}(\epsilon; x) \rightarrow s^{a_{s_{ij}}} \epsilon \vec{M}_{0,s_{ij}}$ regular

Appendix : boundary Master Integrals

procedure summary

- canonical matrix $a(x)$ with 3 poles $x = 0, 1, \infty$

$$\frac{d}{dx} \vec{M}(\epsilon; x) = \epsilon a(x) \vec{M}(\epsilon; x)$$

- iterative solution with arbitrary boundary constants c

$$M_i(\epsilon, x) \sim \epsilon^n c_{i,n} \otimes HPLs(x)$$

- 6 crossed solutions i.e. $x \rightarrow \left\{x, 1-x, \frac{1}{1-x}, \frac{x-1}{x}, \frac{x}{x-1}, \frac{1}{x}\right\}$
careful analytic continuation $s_{ij} + i\epsilon$: cross only 1 branch cut at a time
e.g. $x - i\epsilon \rightarrow \frac{1}{x} \pm i\epsilon \Rightarrow x \rightarrow -x \rightarrow -\frac{1}{x}$
- 3 UV regularity conditions
 $\lim_{x \rightarrow 0, 1, \infty} M_i(\epsilon; x)$ regular \Rightarrow linear **relations** between constants $c_{i,n}$
- solve **recursively** to 1 order higher to fix all but 1 boundary coefficient $\epsilon^n c_{sunrise,n}$
- mapping between topologies

e.g. $I_{2,0,0,0,0,0,0,2,0,0,0,2,0,1}^{(PL)} = I_{2,0,0,0,0,0,0,0,2,0,0,0,2,1,0}^{(NPL)}$



Appendix : finite part

coupling renormalization $Z[\alpha] = 1 - \frac{\beta_0}{\epsilon} \left(\frac{\alpha_s}{2\pi} \right) + \left(\frac{\beta_0^2}{\epsilon^2} - \frac{\beta_1}{2\epsilon} \right) \left(\frac{\alpha_s}{2\pi} \right)^2 + \mathcal{O}(\alpha_s^3)$

\overline{MS} scheme $S_\epsilon \mu_0^{2\epsilon} \alpha_{s,b} = \mu^{2\epsilon} \alpha_s(\mu) Z[\alpha_s(\mu)], \quad S_\epsilon = (4\pi)^\epsilon e^{-\gamma_E \epsilon}$

beta function $\beta_0 = \frac{11}{6} C_A - \frac{2}{3} T_F n_f, \quad \beta_1 = \frac{17}{6} C_A^2 - T_F n_f \left(\frac{5}{3} C_A + C_F \right)$

finite part $f_{\vec{\lambda}}^{(3,\text{fin})} = f_{\vec{\lambda}}^{(3)} - \mathcal{I}_2 f_{\vec{\lambda}}^{(1)} - \mathcal{I}_1 f_{\vec{\lambda}}^{(2)}$

IR operators [[Catani arXiv:9802439](#)]

$$\mathcal{I}_1(\epsilon) = -\frac{e^{i\pi\epsilon} e^{\gamma_E \epsilon}}{\Gamma(1-\epsilon)} \left(\frac{C_A}{\epsilon^2} + \frac{\beta_0}{\epsilon} \right)$$

$$\mathcal{I}_2(\epsilon) = -\frac{1}{2} \mathcal{I}_1(\epsilon) \left(\mathcal{I}_1(\epsilon) + \frac{2\beta_0}{\epsilon} \right) + \frac{e^{-\gamma_E \epsilon} \Gamma(1-2\epsilon)}{\Gamma(1-\epsilon)} \left(\frac{\beta_0}{\epsilon} + K \right) \mathcal{I}_1(2\epsilon) + 2 \frac{e^{\epsilon\gamma_E}}{\Gamma(1-\epsilon)} H_g$$

with

$$K = \left(\frac{67}{18} - \frac{\pi^2}{6} \right) C_A - \frac{10}{9} n_f T_F$$

$$H_g = \frac{1}{2\epsilon} \left[\left(\frac{\zeta(3)}{4} + \frac{5}{24} + \frac{11\pi^2}{288} \right) C_A^2 + T_F n_f \left(\frac{C_F}{2} - \left(\frac{29}{27} + \frac{\pi^2}{72} \right) C_A \right) + \frac{10}{27} T_F^2 n_f^2 \right]$$