

## Modular symmetries and the flavor problem

Davide Meloni
Dipartimento di Matematica e Fisica, Roma Tre

## The Standard Model of Particle Physics

Left-handed


Right-handed


Scalar sector

## The Flavor Problem

Mass hierarchies

$m_{d} \ll m_{s} \ll m_{b}, \frac{m_{d}}{m_{s}}=5.02 \times 10^{-2}$,
$m_{u} \ll m_{c} \ll m_{t}, \frac{m_{u}}{m_{c}}=1.7 \times 10^{-3}$,
$\frac{m_{s}}{m_{b}}=2.22 \times 10^{-2}, m_{b}=4.18 \mathrm{GeV} ;$
$\frac{m_{c}}{m_{t}}=7.3 \times 10^{-3}, m_{t}=172.9 \mathrm{GeV} ;$

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Fermion mixing


Leptonic PMNS mixing matrix

all mixing are large but the 13 element

almost a diagonal matrix

## Suggested solutions

* Smallness of
neutrino masses:


## See-saw


$\mathcal{M}=\left[\begin{array}{ll}\boldsymbol{m}_{M}^{L} & \boldsymbol{m}_{D} \\ \boldsymbol{m}_{D} & \boldsymbol{m}_{M}^{R}\end{array}\right]$

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m_{\text {light }} \sim \frac{m_{D}^{2}}{M_{M}^{R}}
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* Hierarchical Pattern

Froggatt-Nielsen mechanism

$$
L \sim \overline{\Psi_{L}} H \Psi_{R}\left(\frac{\theta}{\Lambda}\right)^{n} \rightarrow e^{\left(-q_{L}+q_{H}+q_{R}+n * q_{\theta}\right)}
$$

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No clue on mixing!

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Works better for small mixing

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## Too many O(1) coefficients

Works better for small mixing

* mixing angles
elegant explanation:
non-Abelian
discrete flavour symmetries


Complicated scalar sector

$$
m_{\text {light }} \sim \frac{m_{D}^{2}}{M_{M}^{R}}
$$

No clue on mixing!

## Modular Symmetry

We start from

$$
\Gamma(N)=\left\{\left(\begin{array}{ll}
a & b \\
c & d
\end{array}\right) \in S L(2, Z),\left(\begin{array}{ll}
a & b \\
c & d
\end{array}\right)=\left(\begin{array}{ll}
1 & 0 \\
0 & 1
\end{array}\right)(\operatorname{Mod} N)\right\}
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the group of $2 \times 2$ matrices with integer entries modulo N and determinant equals to one modulo N

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$\Gamma(\mathbf{N}), \mathbf{N}>=\mathbf{2}$ are infinite normal subgroups of $\Gamma$
the group $\Gamma(N)$ acts on the complex variable $\tau(\operatorname{lm} \tau>0)$

$$
\gamma \tau=\frac{a \tau+b}{c \tau+d}
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## Modular Symmetry

Important observation for $\mathrm{N}=1$ : a transformation characterized by parameters $\{a, b, c, d\}$ is identical to the one defined by $\{-a,-b,-c,-d\}$
$\Gamma(\mathbf{1})$ is isomorphic to $\operatorname{PSL}(\mathbf{2}, \mathbf{Z})=\mathbf{S L}(\mathbf{2}, \mathbf{Z}) /\{ \pm \mathbf{1}\}=\Gamma$
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since 1 and -1 cannot be distinguished

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Finite Modular Group:

$$
\Gamma_{N}=\frac{\bar{\Gamma}}{\bar{\Gamma}(N)}
$$

## Modular Symmetry

Generators of $\Gamma_{\mathrm{N}}$ : elements S and T satisfying

$$
\begin{aligned}
& S^{2}=1, \quad(S T)^{3}=1, \quad T^{N}=1 \\
& S=\left(\begin{array}{cc}
0 & 1 \\
-1 & 0
\end{array}\right), \quad T=\left(\begin{array}{cc}
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corresponding to:

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\stackrel{S}{\rightarrow}-\frac{1}{\tau} \quad \tau \xrightarrow{T} \tau+1
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relevant for model building:
for $\mathrm{N} \leq 5$, the finite modular groups $\Gamma_{\mathrm{N}}$ are isomorphic to non-Abelian discrete groups

$$
\Gamma_{2} \simeq \mathrm{~S}_{3} \quad \Gamma_{3} \simeq \mathrm{~A}_{4} \quad \Gamma_{4} \simeq \mathrm{~S}_{4} \quad \Gamma_{5} \simeq \mathrm{~A}_{5}
$$

Then the question is: why Modular Symmetry?

## Modular Forms

Modular Forms:
holomorphic functions of the complex variable $\tau$ with well-defined transformation properties under the group $\Gamma(\mathbf{N})$

$$
f(\gamma \tau)=(c \tau+d)^{2 k} f(\tau), \quad \gamma=\left(\begin{array}{ll}
a & b \\
c & d
\end{array}\right) \in \Gamma(N) \quad 2 \mathrm{k}=\text { weigth, } \mathrm{N}=\text { level }
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$$

$$
K<0:
$$


no modular forms linear space of finite dimension

$\longrightarrow$| $N$ | $d_{2 k}(\Gamma(N))$ |
| :---: | :---: |
| 2 | $k+1$ |
| 3 | $2 k+1$ |
| 4 | $4 k+1$ |
| 5 | $10 k+1$ |
| 6 | $12 k$ |
| 7 | $28 k-2$ |

## Model Building

## Key points:

1. Modular forms of weight $2 k$ and level $N \geq 2$ are invariant, up to the factor $(\mathrm{ct}+\mathrm{d})^{2 \mathrm{k}}$ under $\Gamma(\mathbf{N})$ but they transform under $\Gamma_{\mathrm{N}}$ !

$$
f_{i}(\gamma \tau)=(c \tau+d)^{2 k} \rho(\gamma)_{i j} f_{j}(\tau)
$$

representative element of $\Gamma_{\mathrm{N}}$

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$$

unitary representation of $\Gamma_{\mathrm{N}}$ representative element of $\Gamma_{N}$
2. in addition, one assumes that the fields of the theory $\chi_{i}$ transforms nontrivially under $\Gamma_{N}$

$$
\chi(x)_{i} \rightarrow(c \tau+d)^{-k_{i}} \rho(\gamma)_{i j} \chi(x)_{j}
$$

not modular forms ! No restrictions on ki

## Model Building

Building blocks:

1. Modular forms and fields: $L_{e f f} \in f(\tau) \times \phi^{(1)} \ldots \phi^{(n)}$

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1. Modular forms and fields: $L_{e f f} \in f(\tau) \times \phi^{(1)} \ldots \phi^{(n)}$
2. Invariance under modular transformation requires:

$$
\begin{gathered}
2 k=\Sigma_{i} k_{i} \\
\rho_{f} \otimes \rho_{\chi_{1}} \otimes \ldots \otimes \rho_{\chi_{n}} \supset I
\end{gathered}
$$

To start playing the game:

Can someone give me the Modular Forms?

## Model Building

## Long list from S.T. Petcov, Bethe Forum, University of Bonn, 04/05/2022

For $\left(\Gamma_{3} \simeq A_{4}\right)$, the generating (basis) modular forms of weight 2 were shown to form a 3 of $A_{4}$ (expressed in terms of log derivatives of Dedekind $\eta$-function $\eta^{\prime} / \eta$ of 4 different arguments).
F. Feruglio, arXiv:1706.08749

For $\left(\Gamma_{2} \simeq S_{3}\right)$, the two basis modular forms of weight 2 were shown to form a 2 of $S_{3}$ (expressed in terms of $\eta^{\prime} / \eta$ of 3 different arguments).
T. Kobayashi, K. Tanaka, T.H. Tatsuishi, arXiv:1803.10391

For $\left(\Gamma_{4} \simeq S_{4}\right)$, the 5 basis modular forms of weight 2 were shown to form a 2 and a $3^{\prime}$ of $S_{4}$ (expressed in terms of $\eta^{\prime} / \eta$ of 6 different arguments).
J. Penedo, STP, arXiv:1806.11040

For ( $\Gamma_{5} \simeq A_{5}$ ), the 11 basis modular forms of weight 2 were shown to form a 3, a $3^{\prime}$ and a 5 of $A_{5}$ (expressed in terms of Jacobi theta function $\theta_{3}(z(\tau), t(\tau))$ for 12 different sets of $z(\tau), t(\tau))$.

```
P.P. Novichkov et al., arXiv:1812.02158; G.-J. Ding et al., arXiv:1903.12588
```

Multiplets of higher weight modular forms have been also constructed from tensor products of the lowest weight 2 multiplets:
i) for $N=4$ (i.e., $S_{4}$ ), multiplets of weight 4 (weight $k \leq 10$ ) were derived in arXiv:1806.11040 (arXiv:1811.04933);
ii) for $N=3$ (i.e., $A_{4}$ ) multiplets of weight $k \leq 6$ were found in arXiv:1706.08749;
iii) for $N=5$ (i.e., $A_{5}$ ), multiplets of weight $k \leq 10$ were derived in arXiv:1812.02158.

## Model Building

## Constructing the Modular Forms

Crucial observation:
if $\quad g(\tau) \rightarrow e^{i \alpha}(c \tau+d)^{k} g(\tau)$
then $\quad \frac{d}{d \tau} \log [g(\tau)] \rightarrow(c \tau+d)^{2} \frac{d}{d \tau} \log [g(\tau)]+k c(c \tau+d)$
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this term prevents of having a modular form of weight $\mathbf{2 k}=\mathbf{2}$
The inhomogeneous term can be removed if we combine several $f_{i}(\tau)$ with weights $k_{i}$

$$
\frac{d}{d \tau} \Sigma_{i} \log \left[g_{i}(\tau)\right] \quad \rightarrow \quad(c \tau+d)^{2} \frac{d}{d \tau} \Sigma_{i} \log \left[g_{i}(\tau)\right]+\left(\Sigma_{i} k_{i}\right) c(c \tau+d)
$$

$$
\text { with } \quad \sum_{i} k_{i}=0
$$

## A case study: $\Gamma_{2} \sim S_{3}$

## Let us find the functions $\mathbf{f}(\tau)$ !

The group $S_{3}$ contains $1+1^{\prime}+2$

two independent modular forms can fit into a doublet of $S_{3}$

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The group $\mathrm{S}_{3}$ contains $1+1^{\prime}+2$

| $N$ | $d_{2 k}(\Gamma(N))$ |
| :---: | :---: |
| 2 | $k+1$ |

$\Longleftarrow$ two independent modular forms can fit into a doublet of $\mathrm{S}_{3}$

$$
\underline{\text { Dedekind eta functions }} \quad \eta(\tau)=q^{1 / 24} \prod_{n=1}^{\infty}\left(1-q^{n}\right) \quad q \equiv e^{i 2 \pi \tau}
$$

$$
\mathrm{S}: \eta(-1 / \tau)=\sqrt{-i \tau} \eta(\tau) \quad, \quad \mathrm{T}: \eta(\tau+1)=e^{i \pi / 12} \eta(\tau)
$$


$\eta^{24}$ is a modular form of weight 12

## A case study: $\Gamma_{2} \sim S_{3}$

Constructing the Modular Forms

the system is closed under modular transformation


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## Constructing the Modular Forms

$$
\eta(2 \tau)
$$

$$
\eta(\tau / 2)
$$

the system is closed under modular transformation
candidate modular form

$$
\begin{array}{r}
Y(\alpha, \beta, \gamma)=\frac{d}{d \tau}[\alpha \log \eta(\tau / 2)+\beta \log \eta((\tau+1) / 2)+\gamma \log \eta(2 \tau)] \\
\alpha+\beta+\gamma=0
\end{array}
$$

## A case study: $\Gamma_{2} \sim S_{3}$

## Constructing the Modular Forms

Equations to be satisfied:

$$
\binom{Y_{1}(-1 / \tau)}{Y_{2}(-1 / \tau)}=\tau^{2} \rho(S)\binom{Y_{1}(\tau)}{Y_{2}(\tau)}, \quad\binom{Y_{1}(\tau+1)}{Y_{2}(\tau+1)}=\rho(T)\binom{Y_{1}(\tau)}{Y_{2}(\tau)}
$$

representation of generators

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\begin{aligned}
& \rho(S)=\frac{1}{2}\left(\begin{array}{cc}
-1 & -\sqrt{3} \\
-\sqrt{3} & 1
\end{array}\right), \quad \rho(T)=\left(\begin{array}{cc}
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\end{array}\right) \\
& (\rho(S))^{2}=\mathbb{I}, \quad(\rho(S) \rho(T))^{3}=\mathbb{I}, \quad(\rho(T))^{2}=\mathbb{I}
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$$

$$
Y_{1}(\alpha, \beta, \gamma) \sim Y(1,1,-2) \quad Y_{2}(\alpha, \beta, \gamma) \sim Y(1,-1,0)
$$

$$
\begin{aligned}
& Y_{1}(\tau)=\frac{i}{4 \pi}\left(\frac{\eta^{\prime}(\tau / 2)}{\eta(\tau / 2)}+\frac{\eta^{\prime}((\tau+1) / 2)}{\eta((\tau+1) / 2)}-\frac{8 \eta^{\prime}(2 \tau)}{\eta(2 \tau)}\right) \\
& Y_{2}(\tau)=\frac{\sqrt{3} i}{4 \pi}\left(\frac{\eta^{\prime}(\tau / 2)}{\eta(\tau / 2)}-\frac{\eta^{\prime}((\tau+1) / 2)}{\eta((\tau+1) / 2)}\right)
\end{aligned}
$$

## A case study: $\Gamma_{2} \sim S_{3}$

How to predict the Neutrino mass matrix (from the Weinberg operator, wrong path...)
For a satisfactory model, we ask:

1. small number of operators $\rightarrow$ predictability
2. no new scalar fields beside Higgs(es) $\rightarrow$ symmetry breaking dictated by the vev of $\tau$

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|  | $\mathrm{S}_{3}$ | $\mathrm{SU}(2)$ | $\mathrm{k}_{\mathrm{i}}$ |
| :--- | :--- | :--- | :--- |
| $\mathrm{L}_{\mathrm{e} \mu}=(\mathrm{e}, \mu)$ | 2 | 2 | -1 |
| $\mathrm{~L}_{\tau}$ | 1 | 2 | -1 |
| $\mathrm{H}_{\mathrm{u}}$ | 1 | 2 | 0 |

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using one power of $Y$ (modular form of lowest weight)

|  | $\mathrm{S}_{3}$ | $\mathrm{SU}(2)$ | $\mathrm{k}_{\mathrm{i}}$ |
| :--- | :--- | :--- | :--- |
| $\mathrm{L}_{\mathrm{e} \mu}=(\mathrm{e}, \mu)$ | 2 | 2 | -1 |
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| $\mathrm{H}_{\mathrm{u}}$ | 1 | 2 | 0 |

$$
L=h_{u}^{2}\left[a\left(\left(L_{e \mu} L_{e \mu}\right)_{2}, Y\right)_{1}+b L_{\tau}\left(L_{e \mu} Y\right)_{1}\right]
$$

$$
m_{v}=\left(\begin{array}{ccc}
a Y_{2} & a Y_{1} & b Y_{1} / 2 \\
a Y_{1} & -a Y_{2} & b Y_{2} / 2 \\
b Y_{1} / 2 & b Y_{2} / 2 & 0
\end{array}\right)
$$

## A case study: $\Gamma_{2} \sim S_{3}$

How to predict the Neutrino mass matrix (from the Weinberg operator, wrong path...)

Mass matrix against the experimental data

$$
m_{v}=\left(\begin{array}{ccc}
a Y_{2} & a Y_{1} & b Y_{1} / 2 \\
a Y_{1} & -a Y_{2} & b Y_{2} / 2 \\
b Y_{1} / 2 & b Y_{2} / 2 & 0
\end{array}\right)
$$

| $\sin ^{2} \theta_{12} / 10^{-1}$ | $2.97_{-0.16}^{+0.17}$ |
| :---: | :---: |
| $\sin ^{2} \theta_{13} / 10^{-2}$ | $2.15_{-0.07}^{+0.07}$ |
| $\sin ^{2} \theta_{23} / 10^{-1}$ | $4.25_{-0.15}^{+0.21}$ |
| $\delta_{C P} / \pi$ | $1.38_{-0.20}^{+0.23}$ |
| $r$ | $2.92_{-0.11}^{+0.10} \times 10^{-2}$ |

5 observables, 2 complex parameters: $\mathrm{a} / \mathrm{b}$ and $\tau \rightarrow$ very difficult task!

$$
\text { large } \chi^{2} \text { of } O(100) \text { mainly driven by } \theta_{13}
$$

## Conclusions

Modular symmetries offer an alternative way for model building

Yukawa couplins dictated by modular forms
unified description of quarks and leptons
symmetry breaking by the vev of tau only

A lot to do:
mass hierarchy
more pheno: leptogenesis, LFV...

## Backup slides

## Kahler potential

Under $\Gamma: \quad\left\{\begin{array}{l}\tau \rightarrow \frac{a \tau+b}{c \tau+d} \\ \varphi^{(I)} \rightarrow(c \tau+d)^{-k_{I}} \rho^{(I)}(\gamma) \varphi^{(I)}\end{array}\right.$

Tte invariance of the action requires the invariance of the superpotential $w(\Phi)$ and the invariance of the Kahler potential up to a Kahler transformation:

Kahler potential:

$$
\sum_{I}(-i \tau+i \bar{\tau})^{-k_{I}}\left|\varphi^{(I)}\right|^{2}
$$

modular invariant kinetic terms

$$
\frac{h}{\langle-i \tau+i \bar{\tau}\rangle^{2}} \partial_{\mu} \bar{\tau} \partial^{\mu} \tau+\sum_{I} \frac{\partial_{\mu} \bar{\varphi}^{(I)} \partial^{\mu} \varphi^{(I)}}{\langle-i \tau+i \bar{\tau}\rangle^{k_{I}}}
$$

## Some definitions

a normal subgroup (also known as an invariant subgroup or self-conjugate subgroup) is a subgroup which is invariant under conjugation by members of the group of which it is a part: a subgroup $N$ of the group $G$ is normal in $G$ if and only if $\left(\mathrm{g} \mathrm{n} \mathrm{g}^{-1}\right) \in N$ for all $g \in G$ and $n \in N$
$\Gamma(\mathbf{N}), \mathbf{N}>=\mathbf{2}$ are infinite normal subgroups of $\Gamma$, called principal congruence subgroups
the group $\Gamma(N)$ acts on the complex variable $\tau(\operatorname{lm} \tau>0)$

$$
\gamma \tau=\frac{a \tau+b}{c \tau+d}
$$

And it can be shown that the upper half-plane is mapped to itself under this action. The complex variable is henceforth restricted to have positive imaginary part

## Some definitions

Modular Functions and Modular Forms
J. S. Milne

Definition 0.2. A holomorphic function $f(z)$ on $\mathbb{H}$ is a modular form of level $N$ and weight 2 k if
(a) $f(\alpha z)=(c z+d)^{2 k} \cdot f(z)$, all $\alpha=\left(\begin{array}{ll}a & b \\ c & d\end{array}\right) \in \Gamma(N)$;
(b) $f(z)$ is "holomorphic at the cusps".

Fundamental domain of $\tau$ on $\operatorname{SL}(2, Z)$ : connected open subset such that no two points of $D$ are equivalent under $\operatorname{SL}(2, Z)$


Theorem 2.12. Let $D=\{z \in \mathbb{H}| | z|>1,|\Re(z)|<1 / 2\}$.
(a) $D$ is a fundamental domain for $\Gamma(1)=\mathrm{SL}_{2}(\mathbb{Z})$; moreover, two elements $z$ and $z^{\prime}$ of $\bar{D}$ are equivalent under $\Gamma(1)$ if and only if
(i) $\Re(z)= \pm 1 / 2$ and $z^{\prime}=z \pm 1$, (then $z^{\prime}=T z$ or $z=T z^{\prime}$ ), or
(ii) $|z|=1$ and $z^{\prime}=-1 / z=S z$.

## A case study: $\Gamma_{2} \sim S_{3}$

## Constructing the Modular Forms

Under T:

$$
Y(\alpha, \beta, \gamma) \rightarrow Y(\gamma, \beta, \alpha)
$$

Under S:

$$
Y(\alpha, \beta, \gamma) \rightarrow \tau^{2} Y(\gamma, \alpha, \beta)
$$

representation of generators

$$
\begin{gathered}
\rho(S)=\frac{1}{2}\left(\begin{array}{cc}
-1 & -\sqrt{3} \\
-\sqrt{3} & 1
\end{array}\right), \quad \rho(T)=\left(\begin{array}{cc}
1 & 0 \\
0 & -1
\end{array}\right) \\
(\rho(S))^{2}=\mathbb{I}, \quad(\rho(S) \rho(T))^{3}=\mathbb{I}, \quad(\rho(T))^{2}=\mathbb{I}
\end{gathered}
$$

## A case study: $\Gamma_{2} \sim S_{3}$

q-expansion of the Modular Forms

$$
\begin{aligned}
& Y_{1}(\tau)=\frac{1}{8}+3 q+3 q^{2}+12 q^{3}+3 q^{4} \cdots \\
& Y_{2}(\tau)=\sqrt{3} q^{1 / 2}\left(1+4 q+6 q^{2}+8 q^{3} \cdots\right) \\
& Y_{1}(\tau) \gg Y_{2}(\tau) \quad \text { for } \operatorname{lm}(\tau) \gg 1
\end{aligned}
$$

## Weinberg operators for $\Gamma_{2} \sim S_{3}$

Neutrino mass matrices from the Weinberg operator

|  | $\mathrm{S}_{3}$ | $\mathrm{SU}(2)$ | $\mathrm{k}_{\mathrm{i}}$ |
| :--- | :--- | :--- | :--- |
| $\mathrm{L}_{\mathrm{e} \mu}=(\mathrm{e}, \mu)$ | 2 | 2 | $\mathrm{k}_{\mathrm{e} \mu}$ |
| $\mathrm{L}_{\tau}$ | 1 | 2 | $\mathrm{k}_{\tau}$ |
| $\mathrm{H}_{\mathrm{u}}$ | 1 | 2 | 0 |

Case a) $\quad\left(L_{e \mu}^{2}\right)_{1} \otimes\left(Y^{2}\right)_{1},\left(Y^{3}\right)_{1}, \ldots,\left(Y^{n}\right)_{1}$
$-2 k_{e \mu}+2 n=0, \quad n=2 \ldots$

Case b) $\quad\left(L_{e \mu}^{2}\right)_{2} \otimes Y,\left(Y^{2}\right)_{2},\left(Y^{3}\right)_{2}, \ldots,\left(Y^{n}\right)_{2}$
$-2 k_{e \mu}+2 n=0, \quad n=1 \ldots$

Case c) $\left(L_{e \mu} L_{\tau}\right)_{2} \otimes Y,\left(Y^{2}\right)_{2},\left(Y^{3}\right)_{2}, \ldots,\left(Y^{n}\right)_{2}$
$-k_{e \mu}-k_{e \tau}+2 n=0, \quad n=1 \ldots$

Case d) $\quad\left(L_{\tau}\right)^{2} \otimes\left(Y^{2}\right)_{1},\left(Y^{3}\right)_{1}, \ldots,\left(Y^{n}\right)_{1}$ $-2 k_{e \tau}+2 n=0, \quad n=2 \ldots$

## Weinberg operators for $\Gamma_{2} \sim S_{3}$

Neutrino mass matrices from the Weinberg operator

$$
(n=1)
$$

Case b) $\quad\left(L_{e \mu}^{2}\right)_{2} \otimes Y,\left(Y^{2}\right)_{2},\left(Y^{3}\right)_{2}, \ldots,\left(Y^{n}\right)_{2}$

$$
-2 k_{e \mu}+2 n=0, \quad n=1 \ldots
$$

Case c) $\left(L_{e \mu} L_{\tau}\right)_{2} \otimes Y,\left(Y^{2}\right)_{2},\left(Y^{3}\right)_{2}, \ldots,\left(Y^{n}\right)_{2} \quad \longrightarrow \quad-k_{e \mu}-k_{e \tau}+2 n=0, \quad n=1 \ldots$

Solutions:

$$
\left[k_{e \mu}=1 \quad k_{e \tau}=0\right] \quad\left[\begin{array}{lll}
k_{e \mu} & =0 & \left.k_{e \tau}=2\right] \quad\left[k_{e \mu}=1\right.
\end{array} k_{e \tau}=1\right]
$$

$$
m_{v}=\left(\begin{array}{ccc}
b Y_{2} & b Y_{1} & c Y_{1} / 2 \\
b Y_{1} & -b Y_{2} & c Y_{2} / 2 \\
c Y_{1} / 2 & c Y_{2} / 2 & 0
\end{array}\right)
$$

## Weinberg operators for $\Gamma_{2} \sim S_{3}$

Neutrino mass matrices from the Weinberg operator

$$
(n=2)
$$

Case a) $\quad$ Case c) $\quad-2 k_{e \mu}+4=0 \quad-k_{e \mu}-k_{e \tau}+4=0$

Case b)

$$
-2 k_{e \mu}+4=0
$$

$$
\text { Case d) } \quad-2 k_{e \tau}+4=0
$$

Solutions:

$$
\left[\begin{array}{ll}
k_{e \mu} & =2
\end{array} k_{e \tau}=2\right] \quad\left[\begin{array}{ll}
k_{e \mu} & =2
\end{array} k_{e \tau} \neq 2\right] \quad\left[\begin{array}{ll}
k_{e \mu} \neq 2 & k_{e \tau}
\end{array}=2\right]
$$

$$
m_{v}=\left(\begin{array}{lcc}
(a+b) y_{1}^{2}+(a-b) y_{2}^{2} & 2 b y_{1} y_{2} & c y_{1} y_{2} \\
* & (a-b) y_{1}^{2}+(a+b) y_{2}^{2} & 1 / 2 c\left(y_{1}^{2}-y_{2}^{2}\right) \\
* & * & d\left(y_{1}^{2}+y_{2}^{2}\right)
\end{array}\right)
$$

## A case study: $\Gamma_{2} \sim S_{3}$

Dedekind eta functions $\quad \eta(\tau)=q^{1 / 24} \prod_{n=1}^{\infty}\left(1-q^{n}\right) \quad q \equiv e^{i 2 \pi \tau}$
Under $\mathbf{T}:\left\{\begin{array}{lll}\eta(2 \tau) & \rightarrow & e^{i \pi / 6} \eta(2 \tau) \\ \eta(\tau / 2) & \rightarrow & \eta((\tau+1) / 2) \\ \eta((\tau+1) / 2) & \rightarrow & e^{i \pi / 12} \eta(\tau / 2)\end{array}\right.$

$$
\text { Unders: }\left\{\begin{aligned}
\eta(2 \tau) & \rightarrow \sqrt{-i \tau / 2} \eta(\tau / 2) \\
\eta(\tau / 2) & \rightarrow \sqrt{-2 i \tau} \eta(2 \tau) \\
\eta\left(\frac{(\tau+1)}{2}\right) & \rightarrow e^{-i \pi / 12} \sqrt{-i \tau(\sqrt{3}-i)} \eta\left(\frac{(\tau+1)}{2}\right)
\end{aligned}\right.
$$

$$
\operatorname{Id}\left[a_{-}, b_{-}\right]:=\{\{\operatorname{Mod}[a, b], 0\},\{0, \operatorname{Mod}[a, b]\}\}
$$

$$
\begin{aligned}
& \operatorname{Id}[-1,2] \quad\left(\begin{array}{ll}
1 & 0 \\
0 & 1
\end{array}\right) \\
& \operatorname{Id}[-1,3] \quad\left(\begin{array}{ll}
2 & 0 \\
0 & 2
\end{array}\right)
\end{aligned}
$$

